Tutorial Letter 202/2/2013

Distribution theory II

STA2603

Semester 2

Department of Statistics

Solutions to Assignment 02

BAR CODE

Learn without limits.

UNISA university of south africa

Question 1

$$p_{XY}(x,y) = \begin{cases} c\dfrac{2^x 3^y}{x!y!}, & x = 0,1,2,\ldots;\ y = 0,1,2,\ldots \\ 0 & x, y < 0 \end{cases}$$

Note that this is a discrete distribution!

(a) If p_{XY} is a joint frequency function, then we must have

$$\sum_{x=0}^{\infty}\sum_{y=0}^{\infty} p_{XY}(x,y) = 1.$$

Here, we would get

$$\begin{aligned} \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} p_{XY}(x,y) &= \sum_{x=0}^{\infty}\sum_{y=0}^{\infty} c\frac{2^x 3^y}{x!y!} = c\left(\sum_{x=0}^{\infty}\frac{2^x}{x!}\right)\left(\sum_{y=0}^{\infty}\frac{3^y}{y!}\right) \\ &= c\cdot e^2\cdot e^3 = 3^5 c \end{aligned}$$

so we must have $c = e^{-5}$.

(b) The marginal frequency function of X is found by summing up the joint frequency function over all y–values:

$$\begin{aligned} P_X(x) &= \sum_{\text{all } y} P_{XY}(x,y) \\ &= \begin{cases} \displaystyle\sum_{y=0}^{\infty} e^{-5}\frac{2^x 3^y}{x!y!} = e^{-5}\frac{2^x}{x!}\sum_{y=0}^{\infty}\frac{3^y}{y!} = e^{-5}\cdot e^3\tfrac{2^x}{x!} = e^{-x}\tfrac{2^x}{x!}, & x = 0,1,2,\ldots \\ 0, \text{ elsewhere} \end{cases} \end{aligned}$$

[Alternatively we can note that $P_{XY}(x,y)$ factors as follows:

$$P_{XY}(x,y) = \left(e^{-2}\frac{2^x}{x!}\right)\cdot\left(e^{-3}\frac{3^y}{y!}\right)$$

This means that X *and* Y *are independent, and*

$$P_X(x) = \left(e^{-2}\frac{2^x}{x!}\right).]$$

(c) From the marginal frequency function found in (b), we see that X has the Poisson distribution with parameter $\lambda = 2$.

Question 2

$$f_{XY}(x,y) = \begin{cases} k(6-x-y), & 0 < x < 2,\ 0 < y < 4 \\ 0, & \text{elsewhere} \end{cases}$$

(a) Integrating the joint density function over all possible $x-$ and y–values must give 1. Here we have

$$\begin{aligned}
\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy &= \int_0^2\int_0^4 k(6-x-y)\,dydx \\
&= k\int_0^2 \left(6y - xy - \frac{1}{2}y^2\right)\Big|_0^4 dx = k\int_0^2 (24-4x-8)\,dx \\
&= k\int_0^2 (16-4x)\,dx = k\left(16x-2x^2\right)\Big|_0^2 = k(32-8) \\
&= 24k.
\end{aligned}$$

Therefore, to have

$$24k = 1$$

we should take

$$k = \frac{1}{24}.$$

(b) For $0 < x < 2,\ 0 < y < 4$, the distribution function is given by

$$\begin{aligned}
F_{XY}(x,y) &= \frac{1}{24}\int_0^x\int_0^y (6-u-v)\,dvdu \\
&= \frac{1}{24}\int_0^x \left(6v - uv - \frac{1}{2}v^2\right)\Big|_0^y du = \frac{1}{24}\int_0^x \left(6y - uy - \frac{1}{2}y^2\right) du \\
&= \frac{1}{24}\left(6yu - \frac{1}{2}y^2u - \frac{1}{2}yu^2\right)\Big|_0^x = \frac{1}{24}\left(6yx - \frac{1}{2}y^2x - \frac{1}{2}yx^2\right) \\
&= \frac{1}{48}xy(12-x-y).
\end{aligned}$$

(c)

$$
\begin{aligned}
& P\left(1<X<2;\ 2<Y<3\right) \\
= & \int_1^2 \int_2^3 \left(\frac{1}{24}(6-x-y)\right) dydx = \int_1^2 \left(\frac{1}{24}\left(6y - xy - \frac{1}{2}y^2\right)\right)\Bigg|_2^3 dx \\
= & \frac{1}{24}\int_1^2 \left(\left(18-3x-\frac{9}{2}\right)-(12-2x-2)\right) dx \\
= & \frac{1}{24}\int_1^2 \left(\frac{7}{2}-x\right) dx = \frac{1}{24}\left(\frac{7}{2}x-\frac{1}{2}x^2\right)\Bigg|_1^2 \\
= & \frac{1}{24}\left(7-2-\frac{7}{2}+\frac{1}{2}\right) = \frac{2}{24} = \frac{1}{12}.
\end{aligned}
$$

Alternatively,

$$
\begin{aligned}
& P\left(1<X<2;\ 2<Y<3\right) \\
= & P\left(X<2,\ Y<3\right) - P\left(X<2,\ Y<2\right) - P\left(X<1,\ Y<3\right) + P\left(X<1,\ Y<2\right) \\
= & F(2,3) - F(2,2) - F(1,3) + F(1,2) \\
= & \frac{1}{48}\left(6(12-2-3) - 4(12-2-2) - 3(12-1-3) + 2(12-1-2)\right) \\
= & \frac{1}{48}(6.7 - 4.8 - 3.8 + 2.9) = \frac{1}{48}(42-56+18) = \frac{4}{48} = \frac{1}{12},
\end{aligned}
$$

again.

Question 3

$$
f_X(x) = \begin{cases} \frac{1}{x^2}, & 1<x \\ 0 & \text{elsewhere} \end{cases}
$$

(a) If X_1 and X_2 are independent, then their joint density function is

$$
f_{X_1,X_2}(x_1,x_2) = \begin{cases} \frac{1}{x_1^2}\cdot\frac{1}{x_2^2}, & x_1>1,\ x_2>1 \\ 0 & \text{elsewhere} \end{cases}
$$

(b) If

$$
U_1 = \frac{X_1}{X_1+X_2}, \qquad U_2 = X_1 + X_2
$$

then

$$
U_1 = \frac{X_1}{U_2} \qquad \therefore \qquad X_1 = U_1 U_2
$$

and

$$X_2 = U_2 - X_1 = U_2 - U_1 U_2.$$

(c)

$$\begin{aligned} J(x_1, x_2 \to u_1, u_2) &= \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} \\ \frac{\partial x_2}{\partial u_1} & \frac{\partial x_2}{\partial u_2} \end{vmatrix} = \begin{vmatrix} u_2 & u_1 \\ -u_2 & 1-u_1 \end{vmatrix} \\ &= |u_2(1-u_1) + u_1 u_2| \\ &= |u_2 - u_2 u_1 + u_1 u_2| = |u_2| = u_2. \end{aligned}$$

(d) If $X_1 > 1$ and $X_2 > 1$ then $U_2 = X_1 + X_2 > 2$,

and

$$U_1 = \frac{X_1}{X_1 + X_2} = \frac{1}{1 + \frac{X_2}{X_1}}$$

where $\frac{X_2}{X_1}$ can have any positive value, so $0 < U_1 < 1$.

(e) Applying the transformation method, we get

$$\begin{aligned} f_{U_1 U_2}(u_1, u_2) &= f_{X_1 X_2}(u_1 u_2,\ u_2 - u_1 u_2) \, |J(x_1, x_2 \to u_1, u_2)| \\ &= \frac{1}{(u_1 u_2)^2} \frac{1}{(u_2(1-u_1))^2} \cdot u_2 \\ &= \frac{1}{u_1^2 (1-u_1)^2 u_2^3}, \quad 0 < u_1 < 1,\ u_2 > 2. \end{aligned}$$

Question 4

Let $Y_1 = \dfrac{X_1}{X_1 + X_2}$ and let us define (similarly to Question 3 above) $Y_2 = X_1 + X_2$, in which case we will have $0 < Y_1 < 1,\ Y_2 > 0$
Then

$$X_1 = Y_1 Y_2, \qquad X_2 = Y_2(1 - Y_1)$$

$$J(x_1, x_2 \to y_1, y_2) = \begin{vmatrix} y_2 & y_1 \\ -y_2 & (1-y_2) \end{vmatrix} = |y_2(1-y_2) + (y_1 y_2)| = |y_2|$$

and applying the transformation method gives

$$\begin{aligned}
f_{Y_1Y_2}(y_1y_2) &= f_{X_1X_2}(y_1y_2,\ y_2(1-y_1))(y_2) \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(y_1y_2)^{\alpha-1}(y_2(1-y_1))^{\beta-1}e^{-y_1y_2-y_2(1-y_1)}y_2 \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y_1^{\alpha-1}(1-y_1)^{\beta-1}y_2^{\alpha-1}y_2y_2^{\beta-1}e^{-y_2} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y_1^{\alpha-1}(1-y_1)^{\beta-1}y_2^{\alpha+\beta-1}e^{-y_2},\ 0<y_1<1,\ \ y_2>0
\end{aligned}$$

The marginal density function of Y_1 is

$$\begin{aligned}
f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f_{Y_1Y_2}(y_1,y_2)\,dy_2 \\
&= \int_0^{\infty}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y_1^{\alpha-1}(1-y_1)^{\beta-1}y_2^{\alpha+\beta-1}e^{-y_2}dy_2 \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}y_1^{\alpha-1}(1-y_1)^{\beta-1}\underbrace{\int_0^{\infty}y_2^{\alpha+\beta-1}e^{-y_2}dy_2}_{=\Gamma(\alpha+\beta)} \\
&= \frac{\Gamma(\alpha+\beta)^2}{\Gamma(\alpha)\Gamma(\beta)}y_1^{\alpha-1}(1-y_1)^{\beta-1},\quad \text{for}\quad 0<y_1<1
\end{aligned}$$

Alternatively, we could for instance take $Y_1 = \dfrac{X_1}{X_1+X_2}$, $Y_2 = X_1$, in which case we will have $0<Y_1<1,\ Y_2>0$
and

$$X_1 = Y_2,\qquad X_2 = Y_2\left(\frac{1}{Y_1}-1\right)$$

$$J(x_1,x_2\to y_1,y_2) = \begin{vmatrix} 0 & y_1 \\ -\frac{y_2}{y_1^2} & \frac{1}{y_1}-1 \end{vmatrix} = \frac{y_2}{y_1^2},$$

and applying the transformation method gives

$$\begin{aligned}
f_{Y_1Y_2}(y_1y_2) &= f_{X_1X_2}\left(y_2,\ y_2\left(\frac{1}{y_1}-1\right)\right)\left(\frac{y_2}{y_1^2}\right) \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(y_2)^{\alpha-1}\left(y_2\left(\frac{1}{y_1}-1\right)\right)^{\beta-1}e^{-y_2-y_2\left(\frac{1}{y_1}-1\right)}\frac{y_2}{y_1^2} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(y_2)^{\alpha-1}\left(y_2\left(\frac{1}{y_1}-1\right)\right)^{\beta-1}e^{-y_2-y_2\left(\frac{1}{y_1}-1\right)}\frac{y_2}{y_1^2} \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(y_2)^{\alpha+\beta-1}(1-y_1)^{\beta-1}(y_1)^{-\beta-1}e^{\left(-\frac{y_2}{y_1}\right)},\ 0<y_1<1,\ \ y_2>0.
\end{aligned}$$

The marginal density function of Y_1 is

$$\begin{aligned}
f_{Y_1}(y_1) &= \int_{-\infty}^{\infty} f_{Y_1Y_2}(y_1, y_2)\,dy_2 \\
&= \int_{0}^{\infty} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} (y_2)^{\alpha+\beta-1}(1-y_1)^{\beta-1}(y_1)^{-\beta-1} e^{\left(-\frac{y_2}{y_1}\right)} dy_2 \\
&= \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}(1-y_1)^{\beta-1}(y_1)^{-\beta-1} \underbrace{\int_{0}^{\infty} (y_2)^{\alpha+\beta-1} e^{\left(-\frac{y_2}{y_1}\right)} dy_2}_{= \, y_1^{\alpha+\beta}\Gamma(\alpha+\beta)} \\
&= \frac{\Gamma(\alpha+\beta)^2}{\Gamma(\alpha)\Gamma(\beta)} y_1^{\alpha-1}(1-y_1)^{\beta-1}, \quad \text{for} \quad 0 < y_1 < 1
\end{aligned}$$

Question 5

$$f_X(x) = \begin{cases} e^{-x}, & 0 < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

(This is the exponential distribution with $\lambda = 1$). For the distribution function, we therefore have

$$F_X(x) = 1 - e^{-x}.$$

The density function of the 4–th order statistics in a sample of 4 is (as given on page 114 of the study guide, with $n = 4,\ n = 4$),

$$\begin{aligned}
f_4(x) &= \frac{4!}{(4-1)!\,(4-4)!} f_X(x)\,(F_X(x))^{4-1}[1-F(x)]^{4-4} \\
&= \frac{4!}{3!0!} e^{-x}\left(1-e^{-x}\right)^3 \\
&= 4e^{-x}\left(1-e^{-x}\right)^3,\ x > 0.
\end{aligned}$$

Therefore

$$\begin{aligned}
& P\left(Y_{(4)} > 3\right) \\
&= \int_3^{\infty} f_4(y)\,dy = \int_3^{\infty} 4e^{-y}\left(1-e^{-y}\right)^3 dy \\
&= 4e^{-9} - 6e^{-6} + 4e^{-3} - e^{12} \approx 0.18476.
\end{aligned}$$

Question 6

$$F(x,y) = \left(1 - e^{-x}\right)\left(1 - e^{-y}\right), \quad x > 0, \quad y > 0$$

Joint density function:

$$f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}F(x,y) = \begin{cases} e^{-x}e^{-y}, & x > 0,\ y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

[Note that the random variables here are independent, later with the exponential distribution with parameter $\lambda = 1$.]

The marginal density functions are

$$\begin{aligned} f_X(x) &= e^{-x}, \quad x > 0, \\ f_Y(y) &= e^{-y}, \quad y > 0. \end{aligned}$$

Question 7

(a) The marginal frequency functions are found by summing up over columns and rows, respectively:

		x 1	2	3	4	$p_Y(y)$
y	1	0.1	0.05	0.02	0.02	0.19
	2	0.05	0.2	0.05	0.02	0.32
	3	0.02	0.05	0.2	0.04	0.31
	4	0.02	0.02	0.04	0.1	0.18
	$p_X(x)$	0.19	0.32	0.31	0.18	1.0

(b) Conditional frequency of X given $Y = 1$: For any x,

$$P(X = x \mid Y = 1) = \frac{P(X = x,\ Y = 1)}{P(Y = 1)} = \frac{p_{XY}(x,1)}{p_Y(1)}$$

so

$$p_{X \mid Y=1}(x) = \begin{cases} \frac{0.1}{0.19} = \frac{10}{19} \approx 0.52632, & x = 1 \\ \frac{0.05}{0.19} = \frac{5}{19} \approx 0.26316, & x = 2 \\ \frac{0.02}{0.19} = \frac{2}{19} \approx 0.10526, & x = 3 \\ \frac{0.02}{0.19} = \frac{2}{19} \approx 0.10526, & x = 4 \end{cases}$$

(c)

$$P(Y = y \mid X = 2) = \frac{P(Y = y,\ X = 2)}{P(X = 2)} = \frac{p_{XY}(2, y)}{p_X(2)}$$

so

$$P_{Y \mid X=2}(y \mid X = 2) = \begin{cases} \frac{0.05}{0.32} = \frac{5}{32} \approx 0.15625, & y = 1 \\ \frac{0.2}{0.32} = \frac{20}{32} \approx 0.625, & y = 2 \\ \frac{0.05}{0.32} = \frac{5}{32} \approx 0.15625, & y = 3 \\ \frac{0.02}{0.32} = \frac{2}{32} \approx 0.0625, & y = 4 \end{cases}$$