Question 1

If x = 4, find the value of the sum $\sum_{t=0}^{3}(5+\sqrt{x^t})$

Solution

- Substitute x = 4 into this equation: $\sum_{t=0}^{3}(5+\sqrt{4^t})$
- When t = 0, you get (5 + $\sqrt{4^0}$) = 5 + $\sqrt{1}$) = 5 + 1 = 5
- When t = 1, you get (5 + $\sqrt{4^1}$) = 5 + $\sqrt{4}$ = 5 + 2 = 7
- When t= 2, you get (5 + $\sqrt{4^2}$) = 5 + $\sqrt{16}$ = 5 + 4 = 9
- When t= 3, you get (5 + $\sqrt{4^3}$) = 5 + $\sqrt{64}$ = 5 + 8 = 13
- The sum of all these gives 5 + 7 + 9 + 13 = 34

Question 2

Solve for x in the following equation

$\frac{3x-2}{4}$ - $\frac{1}{4}$ $=\frac{1-5x}{2}$

Solution

- Left Hand Side gives: $\frac{3x-2-1}{4}=\frac{3x-3}{4}$
- Right Hand Side gives: $\frac{1-5x}{2}$
  - Multiplying top and bottom by 2 gives Right Hand Side gives: $\frac{2*(1-5x)}{2*2}=\frac{2-10x}{4}$

Now equate the left to the right and you get: $\frac{3x-3}{4}=\frac{2-10x}{4}$

Since the denominators are equal, it means that we can also equate the numerators. This gives 3x -3 = 2 -10x

Manipulate the equation to get 13x = 5

And therefore x = $\frac{5}{13}$

Question 3

Find r if $\sqrt{4r+3}-2=3$

Solution

Manipulate the equation to isolate the variable that you are trying to solve for

$\sqrt{4r+3}=3+2=5$

Square both sides to give: $\left(\sqrt{4r+3}\right)^2=5^2=25$

On the left: the square root and the power of 2 cancel each other out. We are left with just 4r + 3

Equating both sides gives 4r + 3 = 25

More manipulation gives: 4r = 25 – 3 = 22

Divide both sides by 4 and get r = $\frac{22}{4} = \frac{11}{2}$

Question 4

Find the value of the following expression:

$\mathbf{4}\frac{5}{8}$ - $3\frac{1}{4}$ + $1\frac{1}{5}$

Solution

- Convert the mixed fractions into improper fractions (i.e. where the numerator is bigger than the denominator)
  - $\mathbf{4}\frac{5}{8}$ becomes $\frac{37}{8}$
  - $3\frac{1}{4}$ becomes $\frac{13}{4}$
  - $1\frac{1}{5}$ becomes $\frac{6}{5}$
- Find common denominator
  - 8 and 4 already have a common denominator between them which is 8. Multiply this by 5 to get a common denominator amongst all 3 – you get 40
- Adjust numerators accordingly before adding them
  - Multiply 37 by 5, to get 185
  - Multiply 13 by 10, to get 130
  - Multiply 6 by 6, to get 48
  - Adding these up gives a numerator of 103 (185 – 130 + 48)
- The new expression of $\frac{103}{40}$
- Simplifying this gives us 2 $\frac{23}{40}$

Question 5

Simplify the following:

$$\frac{8^x * 8^{3x}}{2^{3x} * 4^{x+2}}$$

Solution

Denominator

- need the same base before you can add exponents
- **tip: $4 = 2^2$**
- therefore: $4^{x+2} = (2^2)^{x+2} = 2^{2x+4}$

- now that you have the same base, you can add the exponents (i.e. 3x + 2x + 4 = 5x +4)
- the denominator is therefore equals 2^{5x+4}

Numerator

- same base, therefore can add exponents (i.e. add x and 3x to give 4x)
- this gives 8^{4x}
- we know that 8 = 2*2*2 = 2^3
- therefore $8^{4x} = (2^3)^{4x} = 2^{12x}$

So you now have the following

$$\frac{2^{12x}}{2^{5x+4}}$$

This can be further simplified since you have the same base

- **Tip: division means you have to subtract the exponents**
- **12x – (5x + 4) = 12x -5x – 4 = 7x – 4**

So you now have the following

$$2^{7x-4}$$

This can be re-written as: $2^{7x} * 2^{-4}$

Remember that a negative exponent means the inverse function

- Therefore $2^{-4} = \frac{1}{2^4}$

You now have: $\frac{2^{7x}}{2^4}$

Since 2^4 = 2*2*2*2 = 16, this gives a final answer of $\frac{2^{7x}}{16}$

Question 6

Simplify the following expression:

$$\frac{x^{\frac{2}{5}} * x^{\frac{2}{3}}}{x^{-\frac{2}{4}}}$$

Solution

Denominator

- $\frac{2}{4} = \frac{1}{2}$

- Therefore $x^{-2/4} = x^{-1/2}$
- This is the same as $\frac{1}{x^{\frac{1}{2}}}$

Numerator

- Same base therefore can add exponents
- Must find common denominator first
- Tip – easiest is to multiply the two numbers 3 and 5 to give you 15 as the common denominator
- Adjust the numerators before you add them (i.e. multiply 2 by 3 to get 6; and multiply the other 2 by 5 to get 11). Adding 6 and 10 gives 16
- Therefore 2/5 + 2/3 = 16/15
- $x^{2/5} * x^{2/3} = x^{16/15}$

Putting these together gives the following:

$$\frac{x^{\frac{16}{15}}}{x^{-\frac{1}{2}}}$$

Because we are dividing where the base is the same we can subtract the exponents,

giving $(\frac{16}{15} - (-\frac{1}{2}))$. This is the same as $\frac{16}{5} + \frac{1}{2}$

once again, we need a common denominator, which is 30 (multiplying 15 and 2)

we then adjust the numerators before adding them (multiply 16 by 2 giving 32 and multiply 1 by 15 giving 15). Adding 32 and 15 gives us 47

so we have $x^{47/30}$

$\frac{47}{30}$ is an improper fraction. This can therefore be further simplified by writing the exponent as a mixed number gives $1\frac{17}{30}$

Question 7

Denzel buys 300 outdoor refuse bags for R300 (VAT included). He sells them for R25.99 for a pack of 20. How much profit did he make?

Solution

This means each bag cost him R1. Therefore a pack of 20 would have cost him R20 (20 * R1)

How many bags of 20 in a total of 300 bags? = $\frac{300}{20}$ = 15 bags of 20

He sells the bag of 20 for R25.99. The difference between cost price and revenue received is R5.99 (R25.99 – R20)

Profit per bag of 20 is worked out to be R5.99

Profit on all 15 bags of 20 is then R89.85 (R5.99 * 15)

Question 8

Consider the below to be the monthly income of 10 drivers. Find the following:

- Average income
- Median of the incomes
- Mode of the incomes
- The standard deviation
- The quartile deviation

1 080.00
2 000.00
1 580.00
1 540.00
2 500.00
1 800.00
1 580.00
3 000.00
3 280.00
2 930.00

Solution

The variable x is the income, where x1 = 1 080, x2 = 2 000…..x10 = 2 930

X can take on 10 different values, therefore n= 10

To find **the average**, use the formula

$$\frac{1}{n}\sum_{i=1}^{n} x = \frac{1}{10}\sum_{i=1}^{10} x i$$

This gives $\frac{1}{10}$ (1080 + 2000 +1580 + 1540 +2500 + 1800 +1580 + 3000 + 3280 + 2930)

= $\frac{1}{10}$(21 290)

= 2 129

To get **the median**, you need to sort the data, giving the below list.

1 080.00
1 540.00
1 580.00
1 580.00
1 800.00
2 000.00
2 500.00
2 930.00
3 000.00
3 280.00

There are 2 middle values (1800 and 2000). Therefore you need to take the average of the two to be the median. Using the formula for arithmetic mean you should get $\frac{1}{2}\sum_{i=1}^{2} x\,i$

The median is therefore $\frac{1}{2}$ (1800 + 2000) = $\frac{1}{2}$ (3800) = 1 900

The mode is the number that occurs most often. In this case that income is 1 580

The variance is derived from the formula

$$\frac{1}{n-1}\sum_{10}^{1}(xi - \bar{x})^2$$

We have already solved for the arithmetic mean to be 2129, so we get:

$$\frac{1}{9}\sum_{10}^{1}(xi - 2129)^2$$

$= \frac{1}{9}$ [$(1080 – 2129)^2$ + $(1540 – 2129)^2$ + $(1580-2129)^2$ + $(1580 – 2129)^2$ + $(1800 – 2129)^2$ + $(2000 – 2129)^2$ + $(2500 – 2129)^2$ + $(2930 – 2129)^2$ + $(3000 – 2129)^2$ + $(3280 – 2129)^2$]

$=\frac{1}{9}$ [$(-1049)^2$ +$(-589)^2$ + $(-549)^2$ + $(-549)^2$ $(-329)^2$ + (-129)2 + $(371)^2$ + $(801)^2$ + $(871)^2$ + $(1151)^2$]

$= \frac{1}{9}[5\ 037\ 690\ ]$

= 559 743.33

Remember that **the standard deviation** is the square root of the variance ($\sqrt{559743.33}$) = 748.16

The quartile deviation formula is $\frac{Q3-Q1}{2}$

Q1 = 1580

Q3 = 2930

So Q3 – Q1 = 2930-1580 = 1350

Dividing this by 2 gives 675

<u>Question 9</u>

Given the set of data below, draw the box and whiskers plot

4, 17, 7, 14, 18, 12, 3, 16, 10, 4, 4, 11

<u>Solution</u>

First order the data

3, 4, 4, 4, 7, 10, 11, 12, 14, 16, 17, 18

Find values for Q1, Q2, Q3 and Q4

Q1 = $\frac{1}{2}$ (4 + 4) = 4

Q2 = $\frac{1}{2}$ (10 + 11) = 10.5

Q3 = = $\frac{1}{2}$ (14 + 16) = 15

…then draw

<u>Question 10</u>

The pollution level in the city centre at 06:00 is 25 parts per million particles and it grows linearly by 30 parts per million particles every hour. Determine the linear equation that relates the pollution y with elapsed time x.

Tip: The best way to solve such an equation is to look out for key words such as:

Mathematical representation	Descriptive words examples
Addition (+)	Sum; Increased; And; Grows; Added
Subtraction (-)	Less than; Less
Multiplication (×)	Times; Per
Equals to (=)	Result

Solution

We are told that the variable y indicates pollution and the variable x indicates elapsed time.

The first line says "the pollution level in the city centre at 06:00 is 25 parts per million"

So from this alone we can say:

$$y = 25$$

Then it continues to say "**it grows** linearly by **30** parts **per** million **every hour**"

From this statement, we know we need to add (+) something to our first equation. We also know that we need to multiply (x) the number 30 with something [grows = plus; per = multiply].

Notice that the last part is talking about elapsed time x ("every hour"). Which gives you what you must multiply 30 with.

So, from this we finally get:

$$y = 25 + 30x$$

Question 11

A tutor at a college is required to do teaching and research for at most 40 hours per week. His research work pays R30 per hour and teaching pays R70 per hour. The tutor needs at least R2000 per week for his own studies.

Let x be the number of hours he spent for research per week.

Let y be the number of hours he spent for teaching per week.

Derive the system of inequalities that shows the number of hours he can spend on research and teaching.

Tip: The best way to derive a system of linear inequalities from word statements is to remember the following:

Descriptive words	Inequality
At least Minimum Not smaller than	$\geq$
At most Maximum Not more than	$\leq$
Greater than More than Above	$>$
Less than Smaller than Below	$<$

Solution

They ask us here to determine a "system" of inequalities. Meaning we have to derive more than one inequality.

The first sentence tell us that the tutor is "required to do teaching **and** research for **at most** 40 hours per week"

From this statement we already know that x represents teaching hours per week and y represent research hours per week. Remember that the descriptive word "and" means addition (+).
We also know that "at most" is represented by the inequality **"≤"**.

So from this we get,

$$\boldsymbol{x + y \ \leq 40}$$

The second sentence tells us that his research pays 30 **per** hour **and** teaching pays R70 **per** hour. It also tells us that he needs **at least** R2000 per week for his own studies.

Remember that the descriptive word "per" means multiplication (x).
For research we have $\boldsymbol{30 \times x \ hours\ of\ research\ per\ week = 30x}$
For teaching we have $\boldsymbol{70 \times y\ hours\ of\ teaching\ per\ week = 70y}$

And to complete our inequality, we take into account the last statement where "at least" is represented by "≥".

So from this we get,

$$\boldsymbol{30x + 70y \geq 2000}$$

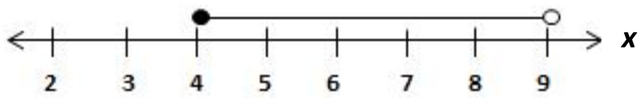
Finally, our system of inequalities is:

$$\boldsymbol{x + y \leq 40}$$

$30x + 70y \geq 2000$

<u>Question 12</u>

Consider the following diagram:



Derive an inequality that represents the above.

Tip:

An open circle indicates strictly less than “<” or strictly greater than “>”.
A closed or shaded circle indicates less than or equal to “≤” or greater than or equal to”≥”.

<u>Solution</u>

We are only dealing with x here meaning that the above diagram indicates that x lies between two values.

Let’s consider the shaded circle. The line moves from the circle to the right, it includes numbers that are greater than or equal to 4.
Let’s consider the open circle. The line moves from the circle towards the left, it includes numbers that are less than 9.

Putting this together, we get:

$$4 \leq x < 9$$

<u>Question 13</u>

Draw the graphical representation of the following inequalities
$x - y > 2$
$x + 2y \geq -2$

Tip: There are few easy steps to follow when you need to draw a graph.
Step 1: Rewrite the inequality to make y the subject of the formula
Step 2: Find the y-intercept by making x = 0. This is the point (0; y)
Step 3: Find the x-intercept by making y = 0. This is the point (x; 0)
Step 4: Plot your points on the graph
Step 5: Colour or use lines to indicate the region of all (x; y) points that satisfy the inequality
Then the overlapping part of both inequalities is the solution.

Solution

Let's follow the steps. Starting with rewriting the inequalities to obtain y on the left-hand side:

For the first inequality it is

$$\boldsymbol{x - y > 2}$$
$$\boldsymbol{-y > 2 - x}$$
$$\boldsymbol{y < -2 + x}$$

The sign changed from > to < because we divided by a negative number.

For the second inequality it is

$$\boldsymbol{x + 2y \geq -2}$$
$$\boldsymbol{2y \geq -2 - x}$$
$$\boldsymbol{y \geq -1 - \frac{1}{2}x}$$

The y- intercepts are:

Inequality 1: $\boldsymbol{y < -2 + x} \rightarrow \boldsymbol{y < -2 + 0} \rightarrow \boldsymbol{y < -2}$

$\boldsymbol{(0; -2)}$

Inequality 2: $\boldsymbol{y \geq -1 - \frac{1}{2}x} \rightarrow \boldsymbol{y \geq -1 - \frac{1}{2}(0)} \rightarrow \boldsymbol{y \geq -1}$

$\boldsymbol{(0; -1)}$

The x –intercepts are:

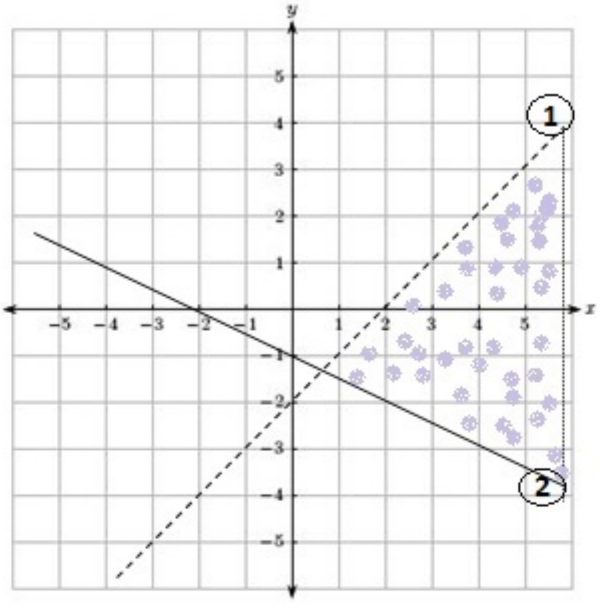
Inequality 1: $\boldsymbol{y < -2 + x} \rightarrow \boldsymbol{0 < -2 + x} \rightarrow \boldsymbol{x > 2}$

$\boldsymbol{(2; 0)}$

Inequality 2: $\boldsymbol{y \geq -1 - \frac{1}{2}x} \rightarrow \boldsymbol{0 \geq -1 - \frac{1}{2}x} \rightarrow \boldsymbol{x \geq -2}$

$\boldsymbol{(-2; 0)}$

Now to graph the inequalities



Question 14

The quadratic function

$$\boldsymbol{y - 6 = 3[(x - 2)(x - 1)]}$$

Can be rewritten in the form $y = ax^2 + bx + c$.
Identify the constants a, b and c.

Tip: After multiplying out your equation**,** "a" will be the number before x^2, "b" will be the number before x and "c" will be the stand alone number. Remember to also include the sign before the number (positive or negative).

Solution

$$\boldsymbol{y - 6 = 3[(x - 2)(x - 1)]}$$
$$\boldsymbol{y - 6 = 3[x^2 - x - 2x + 2]}$$
$$\boldsymbol{y - 6 = 3[x^2 - 3x + 2]}$$
$$\boldsymbol{y - 6 = 3x^2 - 9x + 6}$$
$$\boldsymbol{y = 3x^2 - 9x + 12}$$

From this we see that:
$\boldsymbol{a = +3 = 3}$
$\boldsymbol{b = -9}$
$\boldsymbol{c = +12 = 12}$

Question 15

Find the vertex of the following function

$$\boldsymbol{y = x^2 + 14x + 40}$$

Tip: The vertex is also known as the turning point. The value of x at the vertex is calculated by using the formula $\boldsymbol{x_m = \frac{-b}{2a}}$.
The value of y at the vertex is found by substituting the value of x into the quadratic function.

Solution

$$\boldsymbol{y = x^2 + 14x + 40}$$

From our quadratic function, we already know that,
a = 1
b = 14
c = 40

Therefore,

$$x_m = \frac{-b}{2a} = \frac{-14}{2\times1} = \frac{-14}{2} = -7$$

Substituting this back into the quadratic function,

$$y = x^2 + 14x + 40$$

$$y = (-7)^2 + 14(-7) + 40$$

$$y = 49 - 98 + 40 = -9$$

Finally, the vertex is at the point $(-7; -9)$

Question 16

Simplify the expression

$$log_7 49^{-1}$$

Tip: A log is just another way to write an exponent! We can use three easy steps to solve such problems.

Step 1: Set the log equal to x
Step 2: Use the definition of a log to write the equation in exponential form
i.e. $if\ y = a^x\ with\ a > 0\ and\ a \neq 1, then\ log_a y = x$
Step 3: Solve for x

Solution

First we notice that any expression written as a^{-1}, can also be written as $\frac{1}{a}$

So, $log_7 49^{-1} = log_7 \frac{1}{49}$

Now we follow the steps.

Let's set our log to x,

$$log_7 \frac{1}{49} = x$$

Then use the definition to write it as an exponent,

$$7^x = \frac{1}{49}$$

And now we solve for x,

The best way to solve for x is to think about how the equation on the right hand side can be written in a similar form to the left hand side. For example, we know that:

$$\mathbf{49 = 7^2}$$

So,

$$\mathbf{7^x = \frac{1}{49} = \frac{1}{7^2}}$$

Which already know can be written as:

$$\mathbf{7^x = 7^{-2}}$$

And finally,

$$\boldsymbol{x = -2}$$

Question 17

Find the equation of the line that passes through the points (-3; 8) and (4,-2)

Tip: The basic form a linear equation is $\boldsymbol{y = ax + b}$

Where **a** is the **slope of the line** and **b** is **the intercept on the y – axis.**
To find the slope a line, we use the formula $\boldsymbol{a = \frac{y_2 - y_1}{x_2 - x_1}}$
To find b, since the line passes through both given points, either point can be used to find b by substituting the x and y value of the selected point into our line after having found a.

Solution

$$\boldsymbol{Let\ (x_1; y_1) = (-3; 8)\ \ and\ (x_2; y_2) = (4; -2)}$$

Let's start with finding a,

$$\boldsymbol{a = \frac{-2 - 8}{4 - (-3)}}$$

$$\boldsymbol{a = \frac{-10}{7}}$$

This reduces our general expression to:

$$\boldsymbol{y = -\frac{10}{7}x + b}$$

To find b, let us use the point $\boldsymbol{(x_2; y_2) = (4; -2)}$, remember you can use either point and you will still get to the same answer. So,

$$\boldsymbol{y = -\frac{10}{7}x + b}$$

$$\boldsymbol{y_2 = -\frac{10}{7}x_2 + b}$$

$$-2 = -\frac{10}{7}(4) + b$$
$$-2 = -\frac{10\times4}{7} + b$$
$$-2 = -\frac{40}{7} + b$$
$$\frac{26}{7} = b$$

The expression for the line passing through the points (-3; 8) and (4;-2) is therefore

$$y = -\frac{10}{7}x + \frac{26}{7}$$

Question 18

It is assumed that consumption C depends on income I and this relationship takes the form of the linear function

$$C = aI + b$$

When I is R500, C is R550. When I is R1000, C is R800. Find the linear function that is parallel to the above.

Tip: It makes it easier to solve such equations if we consider the characteristics of a straight line

Specific cases of a straight line	Characteristics and descriptions
Two lines that are parallel	Both lines will have the same slope
A straight line that passes through the origin	There is no constant term (b-value) in the equation
A line that is parallel to the x-axis.	Expression is y = b where b is the intercept on the y-axis
Two lines that intersect at the origin	One line has a positive slope while the other has a negative slope – both with b= 0
A line parallel to the y-axis.	Expression is x = c where c is the intercept on the x-axis.

Solution

We know that when I = 500, C = 550 and when I =1000, C = 800. Giving us the following points

$$(I_1; C_1) = (500; 550)$$
$$(I_2; C_2) = (1000; 800)$$

We know that any line parallel to our equation will have the same slope, so let us find the slope a.

$$a = \frac{C_2 - C_1}{I_2 - I_1} = \frac{800 - 550}{1000 - 500} = \frac{250}{500} = \frac{1}{2}$$

So finally, our linear function will be parallel to any other linear equation of the form

$$C = \frac{1}{2}I + b$$

Question 19

Consider the following table

Item	2010		2012	
	Price	Quantity	Price	Quantity
A	260	200	520	380
B	250	650	265	350

Calculate the Laspeyres price index for 2012 using 2010 as the base year.
Tip:
We know that the Laspeyres price index is calculated as

$$P_l(n) = \frac{\sum p_n q_0}{\sum p_0 q_0} \times 100$$

Where,
p_n *is the price in the nth year*
p_0 *is the price in the base year*
q_0 *is the quantity in the base year*

Solution
It is easier to answer this question by constructing a table

Item	2010		2012			
	p_0	q_0	p_n	q_n	$p_0 \times q_0$	$p_n \times q_0$
A	260	200	520	380	260 x 200 = 52000	520 x 200 = 104000
B	250	650	265	350	250 x 650 = 162500	265 x 650 = 172250
Total					$\sum p_0 q_0 = 214500$	$\sum p_n q_0 = 276250$

Therefore, our Laspeyres price index for 2012 is calculated as

$$P_l(2012) = \frac{\sum p_{2012} q_{2010}}{\sum p_{2010} q_{2010}} \times 100 = \frac{276250}{214500} \times 100 = 128.79$$

Question 20

At a certain point in time the South African rand (R) is converted to the American dollar ($) at a rate of

$$\mathbf{1.00\$ = \mathit{R}6.50}$$

If an article costs $15 in the USA, what is the number of articles that can be bought for R2 535?

Tip: To answer such questions, always convert the total amount you have to the currency of the goods. The goods in this case are "articles" which cost $15, so we must convert our total rands of R2 535 to dollars so that we can answer the question.

Solution

Let us first convert our total of R2 535 to dollars.

To do so, the first thing we must do is determine the exchange rate in dollars.

This can be interpreted as a ratio:

$$\boldsymbol{rand : dollar}$$
$$\mathbf{6.50 : 1}$$
$$\mathbf{\frac{6.50}{6.50} : \frac{1}{6.50}}$$
$$\mathbf{1 : \frac{1}{6.50}}$$

Hence **1** rand is equal to $\mathbf{\frac{1}{6.50}}$ dollars. This is our exchange rate.
If we multiply both sides of our ratio by 2 535 we get,

$$\mathbf{1 \times 2535 : \frac{1}{6.50} \times 2535}$$
$$\mathbf{2535 : 390}$$

This means out of 2535 rands we have 390 dollars to spend on articles.
We already know that one article costs **$15** and now we also know that we have **$390** to spend.

Therefore, the number of articles we can afford is equal to:

$$\mathbf{\frac{390}{15} = 26\ \boldsymbol{articles}}$$

Question 21

You go into a foreign exchange bureau to buy US dollars for your holiday. You exchange £200 and receive $343. When you get home you discover that you have lost the receipt. You know that the exchange bureau charges a fixed £4 fee on all transactions. Calculate the exchange rate used for your money.

Tip: The easiest way to tackle this kind of exchange rate question is to use a ratio.

Solution

£ : $

We exchanged £200 but the question tells us that the bureau charges us £4 for a transaction. So in total we gave the bureau £200 + £4 = £204

Therefore,

$$\textbf{£200} + \textbf{£4} : \textbf{\$343}$$

$$\textbf{£204} : \textbf{\$343}$$

$$\frac{\textbf{£204}}{\textbf{204}} : \frac{\textbf{\$343}}{\textbf{204}}$$

$$\textbf{£1} : \textbf{\$1.681}$$

The exchange rate is therefore:

$$\textbf{£1} = \textbf{\$1.681}$$

Question 22

Differentiate the function

$$f(x) = -\frac{x^6}{2} + x^3 - \frac{4}{x} - 3x + 5$$

Tip: Transform $\frac{4}{x}$ to $4x^{-1}$ before differentiating.

Solution

The function f(x) can first be expressed as $f(x) = -\frac{x^6}{2} + x^3 - 4x^{-1} - 3x + 5$

Since $\frac{4}{x} = 4x^{-1}$

Differentiating gives

$f^1(x) = -6\frac{x^5}{2} + 3x^2 - (-1)4x^{-2} - 3$

Hence

$f^1(x) = -3x^5 + 3x^2 + \frac{4}{x^2} - 3$

Question 23

Suppose that the cost in rand for a daily production of x kilograms of copper is given by the function

$$C(x) = -\frac{1}{20}x^2 + 150x + 2200.$$

Calculate the marginal cost to produce the 1001^{th} kilogram of copper.

Tip: you have to substitute 1000kg into the marginal cost function in order to determine the marginal cost for the 1001^{th} kg produced.

Solution

Differentiating the cost function

$$C(x) = -\frac{1}{20}x^2 + 150x + 2200$$

Gives the marginal cost function

$$C^1(x) = -(2)\frac{1}{20}x + 150$$

$$C^1(x) = -\frac{1}{10}x + 150$$

Substituting 1000kg into the marginal cost function gives the marginal cost for the 1001^{th} kg produced as follows

$$C^1(1000) = -\frac{1}{10}(1000) + 150$$

$$C^1(1000) = 50$$

Hence the cost of producing the 1001^{th} kilogram of copper is R50.

Question 24

The profit function to produce and sell x chairs is given by

$$p(x) = -x^2 + 100x - 20$$

Calculate the number of chairs that have to be produced and sold in order to maximize the profit.

Tip: Equate the marginal profit function to zero and solve for x.

Solution

Firstly calculate the marginal profit function by differentiating the profit function

$$p(x) = -x^2 + 100x - 20 \text{ to obtain}$$

$$p^1(x) = -2x + 100$$

The number of units that must be produced and sold to maximize the profit is obtained by equating the marginal profit to zero and solving for x.

Hence $$P^1(x) = 0$$

$$-2x + 100 = 0 \quad \text{solving gives}$$

$$2x = 100$$

$$x = 50$$

Question 25

Lindi invested R10 000 at 10% per annum **simple interest** for the first 5 years and thereafter at 15% per annum **simple interest** for the next 3 years. Calculate the value of the investment at the end of 8 years.

Tip: Use the formula S = P(1+RT) to find the accumulated sum after 5 years. This amount will then be used as the principal for the next 3 years.

Solution

We will use the formula S = P (1+RT) to find the accumulated sum after 5 years. This amount will then be used as the principal for the next 3 years.

P = R10 000

R = 10% = 0.10

T = 5 years

S = P (1+RT)

S = 10000(1+ 0.10(5))

S = 10000(1.5)

S = 15 000

This amount is used as the principal for the second part of the investment to find the new accumulated amount as follows

P = R15 000

R = 15% = 0.15

T = 3 years

S = P (1+RT)

S = 15000(1+ 0.15(3))

S = 15000(1.45)

S = 21 750

Question 26

Peter receives R16 375 when taking out a loan from a lender with the simple discount rate of 14.5%. The loan has to be repaid in fifteen months' time. Calculate the amount that he as to repay.

Tip: Note that the time is given in months. You have to convert it to years by dividing by 12. Secondly, you can use the discount formula P = S (1 - dT) to solve for S.

Solution

In this question, we have to convert the time to years.

P = R16 375

d = 14.5% = 0.145

T = $\frac{15}{12}$ = 1.25 years

P = S (1 - dT)

Substituting into the formula gives

16 375 = S (1 – 0.145(1.25))

16 375 = S (1 – 0.18125)

16 375 = S (0.81875)

S = $\frac{16\ 375}{0.81875}$

S = 20 000

Question 27

You invest R12 000 at 15% simple interest per year. The investment amounts to R20 100. Calculate the period of the investment in months.

Tip: Use the future value formula S = P (1 + RT) to find the period of investment, T.

Solution

Use the future value formula S = P (1 + RT), to find the period of investment, T as follows

S = 20 100

P = 12 000

R = 15% = 0.15

S = P (1 + RT)

20 100 = 12 000 (1 + 0.15T) hence

20 100 = 12 000 + 1800T

20 100 – 12 000 = 1800T

8 100 = 1800T

$T = \frac{8\,100}{1800}$

T = 4.5 years.

In order to get the time in months, we will multiply 4.5 by 12 to get 54 months.

Question 28

You invest R15 000 at 16.5% per year, compounded monthly. The investment amounts to R40 097.17. Calculate the period of the investment in months.

Tip: Use the formula $S = P\ (1 + R)^T$, to find the investment period. You will need to use logarithms in order to find T.

Solution

Future value, S = 40 097.17

Present value, P = 15 000

Effective Interest rate per month $= \frac{16.5\%}{12} = \frac{0.165}{12}$ = 0.01375 (it is the effective interest rate paid every month since it is compounded monthly)

Substituting into the formula $S = P\ (1 + R)^T$ gives

$40097.17 = 15\,000(1 + 0.01375)^T$

$40097.17 = 15\,000(1.01375)^T$

$\frac{40097.17}{15\ 000} = (1.01375)^T$

$2.67314 = (1.01375)^T$

Taking logs of both sides in order to solve for T gives

Log (2.67314) = T Log (1.01375)

$T = \left[\frac{log 2.67314}{\log 1.01375}\right]$

T = 72 months

Question 29

Johan will need R35 000 for his daughter's first year's university fees starting in 4 years' time. How much money should he invest now, to provide for the first year of study if the interest rate is calculated at 18% per year, compounded annually?

Tip: Use the formula $S = P(1 + R)^T$ to find the present value, P.

Solutions

Future value, S = 35 000

Time, T = 4 years

Effective Interest rate per year 18% = 0.18 (it is the effective interest rate paid every year since it is compounded annually)

Substituting into the formula $S = P(1 + R)^T$ gives

$35\ 000 = P(1 + 0.18)^4$

$35\ 000 = P(1.18)^4$

$P = \frac{35\ 000}{1.18^4}$

P = 18 052.61

We could have easily solved the question using the recommended calculator as follows

2ndF CA

2ndF P/Y 1 ENT

ON/C

+/- 35 000 FV

18 I/Y

4 2ndF xP/Y N

COMP PV

PV = 18 052.61

Question 30

Thabo starts a small business and takes out a loan of R50 000. He agrees to repay R20 000 at the end of 2 years, and then follow up with another payment of R25 000 one year later. After 5 years he expands his business and takes out an additional loan of R75 000, and agrees to repay R55 000 one year later and the rest of his debt at the end of the third year. If the interest rate is 16% per year compounded annually, calculate the final payment.

Tip: The key thing in this question is to draw a timeline of this scenario. Secondly, find the accumulated amounts of both the debts and payments at the time the last payment is made. Finally, equate total payments to total obligations and solve for the unknown.

Solution

The future value of the payments at the end of 8 years must be equal to the future value of both loans at the end of 8 years.

Hence, at time T = 8, the future values are calculated using $S = P(1 + R)^T$ as follows:

$20\,000(1 + 0.16)^6 + 25\,000(1 + 0.16)^5 + 55\,000(1 + 0.16)^2 + X = 50\,000(1 + 0.16)^8 + 75\,000(1 + 0.16)^3$

$20\,000(1.16)^6 + 25\,000(1.16)^5 + 55\,000(1.16)^2 + X = 50\,000(1.16)^8 + 75\,000(1.16)^3$

48 727.93 + 52 508.54 + 74008.00 + X = 163 920.74 + 117 067.20

175 244.47 + X = 280 987.94

X = 280 987.94 – 175 244.47

X = 105 743.47

Hence the final payment is R105 743.47

Question 31

John pays R2 500 at the end of each quarter into a sinking fund for 9 years. Interest is 18% per year compounded quarterly. Calculate the amount John will receive at the end of 9 years.

Tip: use the future value annuity formula.

Solution

Payment, R = 2 500

Effective Interest rate per quarter = $\frac{18\%}{4} = \frac{0.18}{4}$ = 0.045 (it is the effective interest rate paid every quarter since it is compounded quarterly)

Number of payments, n = 36, (9 x 4 quarters in a year)

We will use the Future value annuity formula to find the future value, S as follows:

$S = R\left[\frac{(1+i)^n - 1}{i}\right]$

$S = 2\,500\left[\frac{(1+0.045)^{36} - 1}{0.045}\right]$

S = 215 409.91

We could have easily solved the question using the recommended calculator as follows

2ndF CA

2ndF P/Y 4 ENT

ON/C

+/- 2 500 PMT

18 I/Y

9 2ndF xP/Y N

COMP FV

FV = 215 409.91

Question 32

Nandi wants to invest R3 500 *at the end of each second quarter* for 7 years. The investment will earn 11.5% per year, compounded half yearly. Calculate how much she will accumulate at the end of the 7 years.

Tip: use the future value annuity formula. Note that paying an amount at the end of each second quarter is the same as paying it half yearly.

Solution

Payment, R = 3 500

Effective Interest rate per half year = $\frac{11.5\%}{2} = \frac{0.115}{2}$ = 0.0575 (it is the effective interest rate paid every half year since it is compounded half yearly)

Number of payments, n = 14,   (7 x 2 half years in a year)

We will use the Future value annuity formula to find the future value, S as follows:

$S = R\left[\frac{(1+i)^n - 1}{i}\right]$

$S = 3\ 500\left[\frac{(1+0.0575)^{14} - 1}{0.0575}\right]$

S = 72 275.62

We could have easily solved the question using the recommended calculator as follows

2ndF CA

2ndF P/Y 2 ENT

ON/C

+/- 3 500 PMT

11.5 I/Y

7 2ndF xP/Y N

COMP FV

FV = 72 275.62

Question 33

Anthony wants to invest the same amount of money at the end of each quarter for five years. The investment will earn 9% interest per year, compounded quarterly. Calculate the amount of each quarterly payment if he wants to accumulate R50 000 at the end of 5 years.

Tip: Use the future value annuity formula to calculate the quarterly payment.

Solution

Future value, FV = 50 000

Effective Interest rate per quarter = $\frac{9\%}{4} = \frac{0.09}{4}$ = 0.0225 (it is the effective interest rate paid every quarter since it is compounded quarterly)

Number of payments, n = 20,   (5 x 4 quarters in a year)

We will use the Future value annuity formula to find the payment, R as follows:

$S = R\left[\frac{(1+i)^n - 1}{i}\right]$

$50\ 000 = R\left[\frac{(1+0.0225)^{20} - 1}{0.0225}\right]$

$50\ 000 = 24.9115R$

$R = \frac{50\ 000}{24.9115}$

$R = 2\ 007.11$

Question 34

Andile takes out a five year loan to buy her dream car. The loan is financed at 15% per year compounded monthly and her monthly payments are R4 150. In addition to the loan, she makes a deposit of R30 000 on the car. Calculate the price of the car.

Tip: use the present value annuity formula to find the present value of the payments. Then add R30 000 to get the price of the car.

Solution

Payment, PMT = 4 150

Effective Interest rate per month = $\frac{15\%}{12} = \frac{0.15}{12}$ = 0.0125 (it is the effective interest rate paid every month since it is compounded monthly)

Number of payments, n = 60,   (5 x 12 months in a year)

We will use the present value annuity formula to find the present value, P as follows:

$$P = R\left[\frac{(1+i)^n - 1}{i(1+i)^n}\right]$$

$$P = 4\ 150\left[\frac{(1+0.0125)^{60} - 1}{0.0125(1+0.0125)^{60}}\right]$$

$$P = 4\ 150\left[\frac{(1.0125)^{60} - 1}{0.0125(1.0125)^{60}}\right]$$

P = 174 443.56

Hence the cost of the car is R204 443.56 (174 443.56 + 30 000 deposits)

We could have easily solve the question using the recommended calculator as follows

2ndF CA

2ndF P/Y 12 ENT

ON/C

+/- 4 150 PMT

15 I/Y

5 2ndF xP/Y N

COMP PV

PV = 174 443.56

Questions 35, 36 and 37 are based on the following information

Sandra buys a house for R480 000. She has to pay a 20% deposit and takes out a loan from the bank for the balance. The loan is amortised over a period of 20 years by means of equal monthly payments. The interest rate is 21% per year, compounded monthly.

Question 35

Calculate Sandra's monthly payment.

Tip: you have to subtract the deposit amount from R480 000 in order to know the value of the loan. Then use the present value annuity formula to find the monthly payment.

Solution

Deposit amount = 20% of 480 000 = 96 000

The present value, PV = R384 000 (R480 000 – R96 000) since he paid a deposit.

Effective Interest rate per month = $\frac{21\%}{12} = \frac{0.21}{12}$ = 0.0175 (it is the effective interest rate paid every month since it is compounded monthly)

Number of payments, n = 240, (20 x 12 months in a year)

We will use the present value annuity formula to find the payment, R as follows:

$P = R\left[\frac{(1+i)^n-1}{i(1+i)^n}\right]$

$384\ 000 = R\left[\frac{(1+0.0175)^{240}-1}{0.0175(1+0.0175)^{240}}\right]$

$384\ 000 = R\left[\frac{(1.0175)^{240}-1}{0.0175(1.0175)^{240}}\right]$

384 000 = 56.2543R

$R = \frac{384\ 000}{56.2543}$

R = 6 826.15

We could have easily solve the question using the recommended calculator as follows

2ndF CA

2ndF P/Y 12 ENT

ON/C

+/- 384 000 PV

21 I/Y

20 2ndF xP/Y N

COMP PMT

PMT = 6 826.15

Question 36

Considering an amortisation schedule, calculate the interest due at the end of the first two months.

Tip: Draw up an amortisation table for two months and determine the interest due.

Solution

The amortization table for the loan is as follows

Month	Outstanding balance at beginning of month	Interest due at the end of the month	payment	Principal repaid
1	384 000	6 720	6 826.15	106.15
2	383 893.85	6 718.14	6 826.15	108.01

Note that the interest due at the end of month one is

384 000(1.0175) – 384 000 = 6 720

The principal repaid = 6 826.15 – 6 720 = 106.15

Principal outstanding at the beginning of year two = 384 000 – 106.15 = 383 893.85

Interest due at the end of the 2nd month = 383 893.85(1.0175) – 383 893.85 = 6 718.14

The principal repaid = 6 826.15 – 6 718.14 = 108.01

Question 37

Calculate the outstanding principal after 10 years.

Tip: find the present value of the last ten years of monthly instalments that have not yet been paid using the present value annuity formula.

Solution

The outstanding balance is calculated by working out the present value of the last 120 (240-120) remaining monthly payments. From the previous calculation, we know that the monthly payment is R6 826.15

We will use the present value annuity formula to find the present value, P as follows:

$$P = R\left[\frac{(1+i)^{n}-1}{i(1+i)^{n}}\right]$$

$$P = R\left[\frac{(1+i)^{n}-1}{i(1+i)^{n}}\right]$$

$$P = 6\ 826.15\left[\frac{(1+0.0175)^{120}-1}{0.0175(1+0.0175)^{120}}\right]$$

$$P = 6\ 826.15\left[\frac{(1.0175)^{120}-1}{0.0175(1.0175)^{120}}\right]$$

P = 6 826.15 x 50.0171

P = 341 424.08

We could have easily solve the question using the recommended calculator as follows

2ndF CA

2ndF P/Y 12 ENT

ON/C

+/- 384 000 PV

21 I/Y

20 2ndF xP/Y N

COMP PMT

PMT = 6 826.15

RCL N

-120 = N

COMP PV

PV = 341 424.08