Study Guide

Quantitative Modelling 1
DSC1520

Semesters 1 and 2

Department of Decision Sciences

This document contains guidelines to be used together with the prescribed textbook

*Essential Mathematics for Economics and Business*

by Teresa Bradley
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Chapter 1: Introduction

These notes are intended to guide you through the textbook. Important parts will sometimes be highlighted or explained in another way than in the book.

1.1 The textbook

The book prescribed for this module is

*Essential Mathematics for Economics and Business* by Theresa Bradley.

When we feel that the notes in the textbook can be explained in another way or if there is something interesting that can enhance your understanding of a concept, we will provide additional notes. It is, however, of utmost importance that you work attentively through the textbook, using these notes only as a guideline.

The textbook uses pounds (£) and sometimes euros (€) as currency, which is of course not what we use in South Africa. To keep it simple, we convert pounds and euros directly to rand (R), that is £1 = R1 and €1 = R1, without taking exchange rates into account.

In South Africa we also use the decimal comma and not the decimal point as in the textbook. Even though you will not be penalised for using the decimal point, these notes and all tutorial letters will contain the decimal comma.

At the beginning of each chapter and some sections in the textbook, objectives are given. These objectives are mostly the same as the assessment criteria of the study units. They differ only where parts of the material in the textbook are excluded and do not need to be studied.

At the end of each chapter, a summary of important concepts is provided.

> From Chapter 2 in these notes, the chapter and section numbers correspond with those in the textbook.

1.2 Knowledge assumed to be in place

The mathematical preliminaries that are discussed in Chapter 1 of the textbook are regarded as pre-knowledge, since you should have done it at school. Please work through this chapter if you find that you need to catch up with any of the following topics:

- arithmetic operations (addition, subtraction, multiplication, division),
- the order in which such operations should be carried out,
fractions (addition, subtraction, multiplication, division),
> solving equations (including factorisation),
> converting currencies,
> solving simple inequalities and writing down intervals representing inequalities,
> calculating percentages,
> evaluating and transforming formulae (changing the subject of a formula).

We furthermore assume that you are proficient with the use of a calculator.

1.3 Software

At the end of each chapter in the textbook the use of Microsoft Excel for the topic under consideration is shown. You are welcome to work through these notes, but since it is quite cumbersome to plot functions in Excel, we rather introduce you to the mathematical package Maxima. This is free software that can be downloaded from the internet. See the notes for installation and the use of Maxima under Additional Resources on myUnisa.

Maxima is easy to use. For example, to plot a function you simply select the Plot function, enter a mathematical expression, choose begin and end values on the axes and when you enter, a graph of the function is shown.

When you graph functions regularly, you may develop an intuitive sense of the properties of certain functions. It can also help you to check your answers to assignment questions.

1.4 Activities

Do the activities at the end of each section by hand (with pen on paper) before you look at the solutions. This will give you an indication of how well you understand the preceding study material and provide the opportunity to exercise your mathematical writing skills.

1.5 Errata

We have identified small errors in the fourth edition of the textbook. These are listed below. If you find more such errors (also in other editions), please bring it to our attention.

> Page 77, line 1: ... the corresponding change in $x$, given as ...
> Page 110, line 6 of the solution: ... (4) adding equations (1) and (3)
> Page 128: The consumer surplus is calculated incorrectly. At $A$ on the graph, $P = 100$. Therefore $CS = 0.5(90)(100 - 55) = 2025$. (Ignore the 110 inserted in Figures 3.11 and 3.12.)
> Page 132: Questions 4 and 5 are the same. Ignore either one of them.
> Page 167: In Table 4.8, the last value of $x$ should be 12.
> Page 186: Rule 4 that is mentioned in the note before the progress exercise, is on page 191.
> Page 288: The equation in (ii) should be $AC = 3Q^2 - 18Q + 34$.
> Page 357: Number 1, six lines from the bottom should be $y = e^{f(x)} \rightarrow \frac{dy}{dx} = f'(x) \times e^{f(x)}$.
> Page 456, line 1: ... sketch the supply function and then shade the producer surplus ...
> Page 483, Tables 9.3 and 9.4: The column heading Selling price should be Revenue.
STUDY UNIT I

LINEAR FUNCTIONS
Chapter 2: The straight line and applications

2.1 The equation of a straight line

The straight line is the simplest mathematical function, defined by

\[ f(x) = mx + c \quad \text{or} \quad y = mx + c, \]

where \( m \) is the slope of the line and \( c \) is the intercept on the vertical axis.

You should be able to plot linear functions – it was part of the school syllabus.

The rest of the study unit relies heavily on your ability to plot and interpret such linear functions and you may expect assignment and/or exam questions on it. We therefore advise you to work through this section to refresh your memory and maybe learn something you didn’t know!

In the textbook different methods of plotting a straight line and finding the equation of a straight line are discussed in detail.

You need to be able to determine the equation of a straight line when given

- the slope and the \( y \) intercept,
- the slope and a point on the line, or
- two points on the line,

and graph a linear function by using

- its slope and intercept, or
- two points on the line.

Activity

Apply what you have learnt and answer the following questions, writing it down with pen on paper.

Solutions are available on page 52.

1. For the straight line given by \( y = -4x + 3 \), find the slope and both horizontal and vertical intercepts.

2. Plot the straight line given by \( 2y + 6x = 12 \).
2.2 Mathematical Modelling (Excluded)

In a subject like Decision Sciences, mathematical models are created and solved for a wide range of practical problems. These are then used to optimise some quantity, for example to minimise cost or to maximise revenue and/or profit.

This paragraph gives the background of where mathematical modelling fits into the bigger picture, how such models are constructed and it discusses economic models in general.

This paragraph is not for examination purposes, but it may put the notion of modelling as such into context for you. We therefore encourage you to read it.

2.3 Applications: demand, supply, cost, revenue

2.3.1 Demand and supply

For this module we consider the simplest models of demand and supply where the demand for and the supply of a product depend on price only. All other possible variables are considered to be constant.

(a) Demand

The term demand represents the quantity of a product or service (denoted by \( Q \) in the textbook) that consumers would buy at a certain price \( (P) \).

The demand for a product is negatively related to the price that is asked for the product, that is

\[
\text{the quantity demanded will decrease when the price increases.}
\]

Consumers will buy less of a product if the price of it gets higher, while they will buy more of a product at a lower price.

Consider Figure 2.16(a) (page 60 of the textbook).

The demand function \( Q = 200 - 2P \) is graphed. Here \( Q \) is the dependent variable (on the vertical axis).

We see that the slope of the demand line is negative \((-2\)) When the price is zero, \( Q = 200 \) items are demanded. On the other hand, when the price is R100, the quantity demanded is zero. This means that the price is too high and nobody will buy the product.

In Figure 2.16(b) the same demand function is graphed with \( P \) as the dependent variable, that is \( P = 100 - 0,5Q \). The slope is again negative \((-0,5\)) showing that higher demand corresponds with lower prices.

To draw the linear demand functions as in Figure 2.16, we can either find two points on the line, or use the slope and intercept.

For example, in Figure 2.16(a), two points on the line are found. By setting \( P = 0 \) we find \( Q = 200 - 2(0) = 200 \) (the point \((0; 200)\)) and by setting \( Q = 0 \) we find \( 0 = 200 - 2P \) or \( P = 100 \) (the point \((100; 0)\)).

The general form of the demand function is

\[
P = a - bQ,
\]
where \( a \) is the vertical intercept and \( b \) is the slope of the line. This is illustrated in Figure 2.17 on page 61 of the textbook.

Applying the same method as above gives the two coordinates \((Q; P)\) where this line cuts the axes as \((0; a)\) and \((\frac{a}{b}; 0)\).

Work through the study material on the demand function and Worked example 2.6.

(b) **Supply**

The term *supply* represents the quantity of a product or service (also denoted by \( Q \)) that is made available in the market, depending on the price \((P)\) of the product or service. There is a *positive relationship* between the supply of a product and the price of it, that is

\[ \text{the quantity supplied increases when the price increases.} \]

When the price of a good is high, suppliers want to sell more in order to make more profit, thus increasing supply.

As was the case with the demand function, the supply function can be graphed with either supply \((Q)\) or price \((P)\) as dependent variable. Figures 2.21 and 2.22 in the textbook show such graphs.

For the case where \( Q \) is the dependent variable (on the \( y \) axis) the equation of the supply line is \( Q = -10 + 2P \) (Figure 2.21) and for \( P \) as dependent variable, it is \( P = 5 + 0.5Q \) (Figure 2.22).

Note that the slope of the line in each of the graphs is positive, indicating a positive relationship between price and supply.

The general form of the supply function is

\[ P = c + dQ, \]

where \( c \) is the vertical intercept on the graph and \( d \) is the slope of the line. This is illustrated in Figure 2.20 on page 65 of the textbook.

Work through the study material on the supply function and Worked examples 2.7 and 2.8.

You should now be able to

- graph linear demand and supply functions with quantity/price as dependent variable, using either the slope and intercept, or the intercepts on the vertical and horizontal axes;
- transform linear demand/supply functions from having quantity as dependent variable to having price as dependent variable, and vice versa.

### 2.3.2 Cost

Work through the study material on cost in Paragraph 2.3.2 and Worked example 2.9.

The *linear* total cost function \( TC = FC + VC \) can be graphed as before by using either the slope and the vertical intercept (or another point on the line), or any two points on the line (for instance the coordinates on the axes).
For example, in the given total cost function

\[ TC = 20 + 4Q, \]

with \( Q \) the number of units produced, the slope is 4 and the intercept on the vertical axis is 20 (when \( Q = 0 \)).

We could graph this function by using the vertical intercept \((0; 20)\) and another point on the line. If \( Q = 5 \), for example, \( TC = 20 + 4 \times 5 = 40 \), giving the coordinate \((5; 40)\).

Alternatively, we could use the vertical intercept and the slope. Here, the vertical intercept is at \( C = 20 \) and for a slope of +4, we move 4 units up for every unit we move to the right. If we therefore move 5 units to the right from \((0; 20)\), we have to move 20 units up, reaching the point \((5; 40)\).

The graph is shown in Figure 2.1.

![Figure 2.1: The graph of the cost function \( TC = 20 + 4Q \).](image)

### 2.3.3 Revenue

In retail stores, prices for products usually are fixed. The amount of money that a storekeeper will receive from selling a certain product is called the total revenue from the product.

Total revenue is simply the price per unit of a product \( P \) multiplied by the number of units demanded \( Q \), that is

\[ TR = P \times Q. \]

So, if a product is sold for R10 per unit, total revenue is given by

\[ TR = 10Q. \]

To graph this function, we compare it to the standard form of the straight line, namely \( y = mx + c \) and find that the slope \( m = 10 \) and the intercept \( c = 0 \), meaning that the line goes through the origin \((0; 0)\). The graph is shown in Figure 2.2.

Work through the study material on revenue in Paragraph 2.3.3 and Worked example 2.10a.
2.3.4 Profit

*BusinessDictionary.com* defines profit as

> The surplus remaining after total costs are deducted from total revenue, and the basis on which tax is computed and dividend is paid. It is the best known measure of success in an enterprise. ([www.businessdictionary.com/definition/profit.html#ixzz3as2wc1F7](http://www.businessdictionary.com/definition/profit.html#ixzz3as2wc1F7))

Furthermore, *Investopedia* explains profit as follows:

> Profit is the money a business makes after accounting for all the expenses. Regardless of whether the business is a couple of kids running a lemonade stand or a publicly traded multinational company, consistently earning profit is every company’s goal. ([www.investopedia.com/terms/p/profit.asp](http://www.investopedia.com/terms/p/profit.asp))

The profit function is given by

\[
\pi = \text{total revenue} - \text{total cost} = TR - TC = TR - (FC + VC).
\]

[Note that profit is denoted by \(\pi\) to distinguish it from \(P\) that is reserved to denote price.]

When \(TR = TC\), no profit is made (\(\pi = 0\)) and we say the company *breaks even*; when \(TR > TC\) (i.e. \(\pi > 0\)), the company *makes a profit* and when \(TR < TC\) (i.e. \(\pi < 0\)), the company *makes a loss.*

Work through the study material on profit in Paragraph 2.3.4 and *Worked example 2.10b.*

You should now be able to

- model problems in the business world in terms of linear functions (demand, supply, cost, revenue and profit) and
- describe, plot, manipulate and interpret such linear functions.
Activity

Apply what you have learnt and answer the following questions, writing it down with pen on paper. Solutions are available on page 52.

1. Suppose the demand function for joyrides at a merry-go-round is given by $Q = 64 - 4P$, where $Q$ is the number of rides per hour and $P$ is the price per ride in rand.
   
   (a) Find the demand when $P = 0$ and the price when $Q = 0$ and write these results as coordinates.
   
   (b) Use these coordinates to plot the demand function with $Q$ on the vertical axis.
   
   (c) What is the change in demand ($Q$) if price ($P$) increases by one unit?
   
   (d) Transform the demand function to have price ($P$) as dependent variable.

2. The demand and supply functions for baby marrows are $Q = 210 - 3.5P$ and $P = 0,25Q + 22,5$, respectively, with $P$ the price in rand and $Q$ is the quantity in boxes.

   (a) Transform the supply function to have $Q$ as the dependent variable.

   (b) Graph the demand and supply functions on the same diagram with $Q$ as dependent variable, using the intercepts on the vertical and horizontal axes.

   (c) Find the point where the demand and supply functions intersect. What does this point tell us?

3. A firm produces calculators for a certain shop. Their fixed cost is R1 000 and each calculator costs R15 to produce. The calculators are sold for R35 each.

   (a) Write down the equation for total weekly cost.

   (b) Graph the total cost function for $Q = 0$ to 100.

   (c) What is the total cost to produce 25 calculators?

   (d) How many calculators are produced if total costs amount to R7 000?

   (e) Find the total revenue function.

   (f) Graph the $TR$ function for $Q = 0$ to 100.

   (g) How many calculators are sold when $TR = 1 750$?

   (h) Does the firm’s total revenue exceed total costs when 80 calculators are produced? What does this mean?

   (i) Find the firm’s profit when 80 calculators are produced.
2.4 More mathematics on the straight line

In this section, methods to determine the slope and equation of a straight line are discussed. These are the following:

2.4.1 Calculating the slope of a line, given two points on the line.
2.4.2 Finding the equation of a line, given the slope and any point on the line.
2.4.3 Finding the equation of a line, given two points.

Work through Sections 2.4.1 to 2.4.3, and Worked examples 2.11 – 2.13.

Activity

Solutions are available on page 55.

1. A supplier supplies 50 T-shirts when the price is R60 per T-shirt and 90 T-shirts when the price is R110 per T-shirt.
   
   (a) Determine the equation of the supply function as a function of $Q$.
   
   (b) How many additional T-shirts are supplied for each successive R1 increase in price?
   
   (c) How many T-shirts are supplied when the price is R85?
   
   (d) What is the price when 120 T-shirts are supplied?

2.5 Translations of linear functions (Excluded)

2.6 Elasticity of linear demand and supply functions

As we have seen before, one can find helpful definitions and explanations of concepts like elasticity from the internet, by simply entering it in a search engine like Google. For example, the following definition from www.sparknotes.com makes the concept elasticity quite clear:

Elasticity refers to the degree of responsiveness in supply or demand in relation to changes in price. If a curve is more elastic, then small changes in price will cause large changes in quantity consumed. If a curve is less elastic, then it will take large changes in price to effect a change in quantity consumed.

On YouTube (www.youtube.com) many videos of lecturers explaining mathematical/economical concepts can be found by googling something like price elasticity of demand youtube. If you have problems understanding some concept, you may benefit from such videos.

Work through the study material on elasticity (pages 83 – 90) and Worked example 2.19.
You should now be able to

- calculate point elasticity of demand;
- calculate arc price elasticity of demand over an interval on the demand function;
- calculate the price elasticity of supply;
- interpret and explain what the resulting coefficient of elasticity in each of these cases means in practice.

Note: The notation $|\varepsilon|$ stands for the absolute value of $\varepsilon$. This is the size of $\varepsilon$, without considering the sign. The formal definition for absolute value is

$$|\varepsilon| = \begin{cases} 
\varepsilon & \text{if } \varepsilon \geq 0, \\
-\varepsilon & \text{if } \varepsilon < 0.
\end{cases}$$

**Activity**

Solutions are available on page 55.

1. Suppose the demand function for a certain kind of calculator is $Q = 250 - 5P$, with $Q$ the number of calculators demanded at price $P$.
   
   (a) Find the expression for point elasticity of demand in terms of $P$ only.
   
   (b) Calculate the point elasticity of demand at prices $P = 20$ and $P = 30$ and explain each of the results.
   
   (c) Calculate the arc elasticity of demand if the price increases from R25 to R35.

2. Suppose the supply function for a certain product is given by $P = 90 + 0.05Q$, with $P$ and $Q$ the price and quantity, respectively.
   
   (a) Find the formula for price elasticity of supply in terms of $P$.
   
   (b) Determine the price elasticity of supply when the price is R70.
   
   (c) Calculate the arc elasticity of supply when the price increases from R40 to R60.
STUDY UNIT II

SIMULTANEOUS LINEAR FUNCTIONS
Chapter 3: Simultaneous equations

To calculate economical values like *equilibrium price/quantity, break-even points, consumer surplus*, etc it is necessary to find the point where two (or more) equations intersect.

### 3.1 Solving simultaneous linear equations

In the textbook, the method of elimination is mostly applied. The method of substitution that we will use in the examples throughout this chapter, is discussed on page 106.

Let us, for instance, solve the simultaneous equations of *Worked Example 3.2* algebraically by using substitution.

The equations are

\[
\begin{align*}
2x + 3y &= 12.5 \\
-x + 2y &= 6
\end{align*}
\]  \hspace{1cm} (1) \hspace{1cm} (2)

From equation (2) we find that

\[x = 2y - 6.\] \hspace{1cm} (3)

When we replace the \(x\) in equation (1) with \(2y - 6\), we find

\begin{align*}
2(2y - 6) + 3y &= 12.5 \\
4y - 12 + 3y &= 12.5 \\
7y &= 24.5 \\
y &= 3.5.
\end{align*}

When we now substitute \(y = 3.5\) into equation (3), we get \(x = 2(3.5) - 6 = 1\).

The solution to these simultaneous equations is therefore \(x = 1\) and \(y = 3.5\).

Let us also solve the equations of *Worked example 3.3* by using substitution.

These equations are

\[
\begin{align*}
2x + 3y &= 0.75 \\
5x + 2y &= 6
\end{align*}
\]  \hspace{1cm} (1) \hspace{1cm} (2)

19
From equation (1) we find
\[ 2x = -3y + 0.75 \]
\[ x = -1.5y + 0.375. \]  
(3)
Substituting this into equation (2) gives
\[ 5(-1.5y + 0.375) + 2y = 6 \]
\[ -7.5y + 1.875 + 2y = 6 \]
\[ -5.5y = 6 - 1.875 \]
\[ y = \frac{4.125}{-5.5} = -0.75. \]

When we substitute \( y = -0.75 \) into equation (3), we find
\[ x = -1.5(-0.75) + 0.375 = 1.5. \]

The solution to this system of simultaneous equations is the point where the lines representing the equations intersect, that is the point \( (x; y) = (1.5; -0.75) \)

Work through *Worked examples* 3.1 – 3.5 to also get acquainted with the method of elimination.

Note: It is important to be able to tell when a set of simultaneous equations has a unique solution, no solution or infinitely many solutions.

**Three equations in three variables**

In the case of three equations containing three variables (unknowns), the same methods can be applied. To solve the set of three simultaneous equations in *Worked example* 3.6, we apply a combination of elimination and substitution.

The equations are
\[ 2x + y - z = 4 \]  
(1)
\[ x + y - z = 3 \]  
(2)
\[ 2x + 2y + z = 12 \]  
(3)

When we subtract equation (2) from equation (1), we eliminate both \( y \) and \( z \) and get \( x = 1 \).

When we substitute \( x = 1 \) into equations (2) and (3) we find
\[ 1 + y - z = 2 \]  
giving \[ y - z = 2 \]  
(4)
\[ 2 + 2y + z = 12 \]  
giving \[ 2y + z = 10 \]  
(5)

Adding equations (4) and (5) eliminates \( z \), and we find
\[ 3y = 12 \]  
or \[ y = 4. \]

We now substitute \( y = 4 \) into equation (4) and find
\[ 4 - z = 2 \]  
or \[ z = 2. \]

The solution to this system of simultaneous equations is the point where all three equations pass through, that is where \( x = 1, y = 4 \) and \( z = 2 \) in the three dimensional space.
Activity

Solutions are available on page 56.

1. Solve the following system of simultaneous equations:

\[ \begin{align*}
4x - 3y + 1 &= 13 \\
0.5x + y - 3 &= 4
\end{align*} \]

2. Find the solution to the following simultaneous equations:

\[ \begin{align*}
P &= \frac{18 - 10Q}{5} \\
2 &= \frac{3Q + 5P}{2}
\end{align*} \]

3. Solve the following simultaneous equations:

\[ \frac{5}{2}q - 3p = \frac{7}{2} \quad \text{and} \quad 3p = 3(q - 3). \]

3.2 Equilibrium and break-even

When we google *equilibrium*, the general definition is given as "a state in which opposing forces or influences are balanced."

### 3.2.1 Equilibrium in the goods and labour markets

*Market equilibrium* occurs when the demand and supply of a product (or good\(^1\)) are balanced, that is when the number of units that consumers demand (\(Q_d\)) is equal to the number of units that producers supply (\(Q_s\)). Equilibrium also occurs when the price that consumers are willing to pay (\(P_d\)) for a good is equal to the price that the producer is willing to accept (\(P_s\)).

At market equilibrium, the following conditions hold:

\[ Q_d = Q_s \quad \text{and} \quad P_d = P_s. \]

Work through the study material on market equilibrium in Section 3.2.1 and *Worked example 3.7.*

Note that *consumer demand* is the number of units of a product that consumers (people) buy in stores and *producer supply* is the number of units that a producer produces and makes available to be sold in retail stores.

*Labour market equilibrium* occurs when the labour that firms demand (\(L_d\)) is equal to the labour that workers supply (\(L_s\)). Also, when the wage (or salary) that a firm is willing to pay (\(w_s\)) is equal to the wage that workers are willing to accept (\(w_d\)), the labour market is in equilibrium. At labour market equilibrium, the following conditions hold:

\[ L_d = L_s \quad \text{and} \quad w_d = w_s. \]

Work through the study material on labour market equilibrium and *Worked example 3.8.*

\(^1\)Goods are products that are purchased for consumption by the average consumer.
3.2.2 Price controls and government intervention in markets

Price ceilings
When government believes that the equilibrium price is too high for consumers to pay, they may establish a price ceiling which is below market equilibrium. This is also known as maximum price control, where a price is not allowed to go over the maximum (or ceiling) price.

Let us now work through Worked example 3.9.

The demand and supply functions are the same as in Worked example 3.8. These are the following (also showing the transformed equations with \( Q \) as subject):

\[
\begin{align*}
P_d &= 100 - 0.5Q_d & \text{or} & \quad Q_d &= 200 - 2P_d \\
P_s &= 10 + 0.5Q_s & \text{or} & \quad Q_s &= 2P_s - 20
\end{align*}
\]

The equilibrium price and quantity are found to be

\[
Q_e = 55 \quad \text{and} \quad P_e = 90.
\]

(a) A price ceiling of R40 is introduced. To analyse the situation, we calculate the numbers demanded and supplied at the price \( P = 40 \):

\[
\begin{align*}
Q_d &= 200 - 2(40) = 120 \\
Q_s &= 2(40) - 20 = 60
\end{align*}
\]

Since demand is higher than supply, there is a shortage of 120 - 60 = 60 units in the market. (Producers will only make 60 units available to be sold in retail stores at the lower price.)

(b) This shortage opens an opportunity for someone to buy the 60 units at the lowered price of R40 and sell it on the black market for more than the equilibrium price. (There is a demand.)

The price that consumers are willing to pay if these 60 units are made available, is

\[
P_d = 100 - 0.5) = 70.
\]

The potential profit that the black marketeer can make, is

\[
\pi = TR - TC = \text{number sold} \times \text{price asked} - \text{number bought} \times \text{cost per unit}
\]

\[
= 60 \times 70 - 60 \times 40
\]

\[
= 1\,800.
\]

See Figure 3.7 in the textbook for a graphical representation of this situation.

Price floors
When governments believe that the equilibrium price is too low for producers to receive, a minimum price, called a price floor can be set to protect the producers. Such a price operates above market equilibrium.

Work through Worked example 3.10.
Activity

Solutions are available on page 57.

Do the following questions from Progress exercise 3.2 (p. 117), providing your interpretation of the values obtained.

1. Question 2
2. Question 7

3.2.3 Market equilibrium for substitute and complementary goods (Excluded)

3.2.4 Taxes, subsidies and their distribution

The important aspects highlighted in this section are the following:

1. When a fixed tax per unit is imposed on a product, the producer will receive the price $P$ minus tax $t$, that is $P - t$. The supply function $P = a + dQ$ will then become $P - t = a + dQ$.

2. When a product is subsidised by $s$ per unit, the producer will receive the price $P$ plus the subsidy, that is $P + s$. In this case the supply function becomes $P + s = a + bQ$.

In both these cases the equilibrium shifts, affecting the equilibrium consumer price.

Work through Worked examples 3.12 and 3.13

Activity

Do Question 7 of Progress exercise 3.3 in the textbook (page 127).
Solutions are provided at the end of the textbook on page 582.

3.2.5 Break-even analysis

At break-even the revenue received is equal to the costs associated with the processes that are involved to receive the revenue. This means that at break-even,

$$TR = TC \quad \text{or} \quad TR = FC - VC,$$

with $FC$ the fixed cost(s) and $VC$ the variable cost.

Work through Worked example 3.14.

Activity

Solutions are available on page 59.

1. Do Question 8 from Progress exercise 3.3 (p. 126). Provide your interpretation of the values obtained.
   
   (a) Calculate the equilibrium price and quantity.

   (b) Calculate the value of total revenue and total cost at break-even.
3.3 Consumer and producer surplus

Consumer surplus

*Consumer surplus* is defined as the difference between what consumers are willing and able to spend on a product and what they actually spend (at market price).

The market price of a product is often lower than what consumers are willing to pay. In terms of what we have learnt so far,

- the market price is given by the equilibrium price and
- the demand function gives the price that consumers are willing to pay.

Let us consider the demand function $P = 100 - 0.5Q$ and the supply function $P = 10 + 0.5Q$ for a certain product. The price and quantity at equilibrium are found to be $P_0 = 55$ and $Q_0 = 90$, respectively. (See Section 3.2.)

The demand function is graphed in Figure 3.1(a) with the shaded area under the demand line representing the amount that consumers are willing to spend on the product.

At the market price of R55 per unit, 90 units of the product are sold and the amount that consumers spend is $P_0 \times Q_0 = 55 \times 90 = R4,950$. This is represented by the shaded area of the rectangle in Figure 3.1(b).

As defined earlier, the consumer surplus is the amount that consumers *would be willing to spend over and above expenditure at market price*. This is represented by the area under the demand line from $Q = 0$ to $Q_0$ and above the line representing the market price. This area is shown in Figure 3.1(c).

![Figure 3.1: Calculating consumer surplus](image)

The consumer surplus for our problem is calculated as follows:

$$CS = \text{amount consumers are willing to spend} - \text{amount actually spent}$$

$$= \text{area under demand line from 0 to } Q_0 - P_0 \times Q_0$$

$$= \text{area of triangle } P_0E_0A$$

$$= 0.5 \times 90 \times (100 - 55)$$

$$= R2,025.$$

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**Producer surplus**

When considering the situation from the producer’s point of view, the market price might be higher than the minimum price at which producers are willing to produce.

For the supply function \( P = 10 + 0.5Q \) as before, the producer sells \( Q_0 = 90 \) units of the product at market price of \( P_0 = \text{R55} \) per unit. The resulting revenue is \( 90 \times 55 = \text{R}4950 \), which is the area represented by the shaded area in Figure 3.2(a).

However, the supply line represents prices that are acceptable to the producer. The shaded area in Figure 3.2(b) therefore shows the revenue acceptable to the producer. This is the area under the supply line, between \( Q = 0 \) and \( Q_0 = 90 \).

Producer surplus is therefore given by the revenue at market price, minus the revenue the producer would be willing to accept. This is shown in Figure 3.2(c).

![Figure 3.2: Calculating producer surplus](attachment:image.png)

Producer surplus for our supply function is calculated as follows:

\[
PS = P_0 \times Q_0 - \text{area under supply line to the left of } Q_0 \\
= 0.5 \times 90 \times (55 - 10) \\
= \text{R2025}.
\]

The *total surplus* at market price is simply the sum of the consumer and producer surplus at market price. An example is graphically shown in Figure 3.14 in the textbook (page 131).

A lecture on consumer and producer surplus is available on the Kahn Academy’s website at [www.khanacademy.org/economics-finance-domain/microeconomics/consumer-producer-surplus](http://www.khanacademy.org/economics-finance-domain/microeconomics/consumer-producer-surplus).

Watch this if you need more insight into this topic.

Work through the notes in the textbook and *Worked example 3.15*.

### 3.3.1 Activity

Solutions are available on page 59.

1. Do Question 2 of Progress Exercise 3.4
Chapter 9: Linear algebra and applications

Only Section 9.1 of this chapter is included as study material for this module.

9.1 Linear programming

Linear programming (LP) is a method where a problem is modelled by representing the constraints that exist in the situation as linear functions. The objective function of the model, such as maximising profit or minimising cost, also consists of a linear relationship between the variables of the problem. When such a model is solved, values for the variables are found that optimise the objective function.

The important aspects of an LP model are the constraints and the objective function. Also the way variables are chosen to represent elements of the problem have an impact on the success of such a model and should be done cautiously.

Consider the following problem situation:

A manufacturer of leather articles produces boots and jackets. The manufacturing process consists of two activities, namely Making which involves cutting and stitching and Finishing.

There are 800 labour hours available per month for making the articles and 1200 hours for finishing them. It takes four hours to make a pair of boots and three hours to finish it. It takes two hours to make a jacket and four hours to finish it. They sell a pair of boots for R900 and a jacket for R1200.

Formulate an LP model for this problem with the objective to maximise monthly revenue.

When we read this problem statement, we realise that the manufacturer needs to determine how many pairs of boots and how many jackets they should manufacture to maximise revenue. The decision variables are therefore chosen to represent this objective, namely

- $x$ is the number of pairs of boots to manufacture; and
- $y$ is the number of jackets to manufacture.

To give structure to the given information, we set up the following table:

<table>
<thead>
<tr>
<th></th>
<th>Boots (x)</th>
<th>Jackets (y)</th>
<th>Hours available</th>
</tr>
</thead>
<tbody>
<tr>
<td>Making</td>
<td>4</td>
<td>2</td>
<td>800</td>
</tr>
<tr>
<td>Finishing</td>
<td>3</td>
<td>4</td>
<td>1200</td>
</tr>
<tr>
<td>Price</td>
<td>900</td>
<td>1200</td>
<td></td>
</tr>
</tbody>
</table>
The revenue from selling boots is \( R = 900x \) and the revenue from selling jackets is \( R = 1200y \). The objective function is to maximise total revenue, namely

\[
TR = 900x + 1200y.
\]

The constraints of the problem are represented by the following linear inequalities:

\[
\begin{align*}
4x + 2y & \leq 800 \quad \text{(only 800 hours available for making)} \\
3x + 4y & \leq 1200 \quad \text{(only 1200 hours available for finishing)} \\
x & \geq 0 \quad \text{(not possible to produce a negative number)} \\
y & \geq 0 \quad \text{(not possible to produce a negative number)}
\end{align*}
\]

Since there are only two variables, we can graph the inequalities on the same axes, determine the feasible area and solve the optimisation problem.

To graph an inequality, we first treat it as an equation. For example, to graph \( 4x + 2y \leq 800 \), we consider the equation \( 4x + 2y = 800 \).

By setting \( x = 0 \), we find the \( y \) intercept to be \( y = 400 \) (from \( 4(0) + 2y = 800 \)).

If we set \( y = 0 \), we find the \( x \) intercept to be \( x = 200 \) (from \( 4x + 2(0) = 800 \)).

We can now draw the line through the coordinates \( (0; 400) \) and \( (200; 0) \).

To determine which side of the line represents the inequality, we take the origin \( (0; 0) \) and see whether it satisfies the inequality. Substituting \( x = 0 \) and \( y = 0 \) into the inequality, we find \( 4(0) + 2(0) = 0 \) which is less than 800 and therefore true. The origin therefore falls inside the feasible area of this inequality and we can shade this area as in Figure 9.1(a).

The same procedure can be followed to draw the function \( 3x + 4y = 1200 \) and find the feasible area as shown in Figure 9.1(b).

The non-negative constraints \( x \geq 0 \) and \( y \geq 0 \) restricts us to the quadrant where \( x \) and \( y \) are always positive.

In Figure 9.1(c) all the inequalities are drawn on the same axes. The area where the feasible areas overlap (the checkered area) is the feasible area of the model. In this area all the constraints are satisfied.
Now, to find the point in this feasible area where the total revenue function $TR = 900x + 1200y$ is a maximum, we can either calculate the value at each corner point and choose the highest number, or we can draw the isorevenue lines to find the optimum. These procedures are both described in the textbook.

The feasible area and isolines are shown in Figure 9.2 where it is clear that revenue is a maximum at the point $(80; 240)$. (Find this point by solving the simultaneous equations $4x + 2y = 800$ and $3x + 4y = 1200$.)

![Figure 9.2: Feasible area and isorevenue lines](image)

Work through the notes in Section 9.1 and *Worked examples* 9.1 and 9.2.

**Activity**

Do the following problems of Progress exercise 9.1 (page 487). Check your answers against the solutions provided at the back of the textbook (page 642).

1. Question 2
2. Question 6
3. Question 10
STUDY UNIT III

NONLINEAR FUNCTIONS
Chapter 4: Nonlinear functions and applications

4.1 Quadratic, cubic and other polynomial functions

A polynomial is a function of the form
\[ f(x) = a^n x^n + a^{n-1} x^{n-1} + \ldots + a^2 x^2 + a^1 x^1 + a^0. \]
The degree of a polynomial is the highest power of \( x \) in its expression.

In the following table polynomials of degrees 0, 1, 2 and 3 are shown with the function names, general forms and examples.

<table>
<thead>
<tr>
<th>Degree</th>
<th>Function name</th>
<th>General form</th>
<th>Example(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Constant</td>
<td>( a, b, c, \ldots )</td>
<td>( 2 = 2x^0, 45 = 45x^0 )</td>
</tr>
<tr>
<td>1</td>
<td>Linear function</td>
<td>( y = ax + b )</td>
<td>( y = 2x^1 + x^0 = 2x + 1 )</td>
</tr>
<tr>
<td>2</td>
<td>Quadratic function</td>
<td>( y = ax^2 + bx + c )</td>
<td>( f(x) = 3x^2 + x + 3 )</td>
</tr>
<tr>
<td>3</td>
<td>Cubic function</td>
<td>( y = ax^3 + bx^2 + cx + d )</td>
<td>( TC = 0,5Q^3 - 5Q^2 + 1,5Q + 25 )</td>
</tr>
</tbody>
</table>

For this module, the nonlinear polynomials mentioned above, namely quadratic functions (degree 2) and cubic functions (degree 3) are discussed in detail.

The graphs of these functions have distinct features that we will discuss here. In Figures 4.1(a) and 4.1(b) the quadratic function \( f(x) = 3x^2 + x + 3 \) and the cubic function \( TC = 0,5Q^3 - 5Q^2 + 1,5Q + 25 \) are shown\(^1\).

4.1.1 Solving a quadratic equation

An equation \( y = ax^2 + bx + c \) is called a quadratic equation. Here, \( a \), \( b \) and \( c \) are constant values. Also, \( a \) and \( b \) are called the coefficients of \( x^2 \) and \( x \), respectively.

Solving a quadratic function implies that the values of \( x \) where the graph of the function cuts the \( x \) axis, must be found. These values are called the roots of the quadratic function.

To find the roots of the quadratic equation, we set \( y = 0 \) to find \( ax^2 + bx + c = 0 \) and solve this equation in standard form.

\(^{1}\)These graphs were generated by Maxima. We encourage you to use this software to visualise nonlinear functions.
The equation $x^2 = 3 - 2x$, for example, is written in standard form as either $x^2 + 2x - 3 = 0$ or $-x^2 - 2x + 3 = 0$. These equations are the same, since we can simply multiply either one of them on both sides of the equal sign by $-1$, to find the other.

There are two methods to solve quadratic equations, namely by using factorisation\(^2\) or by using the formula as given on page 149.

1. By using factorisation, we solve the equation as follows:

\[
x^2 + 2x - 3 = 0
\]

\[
(x + 3)(x - 1) = 0.
\]

(factorise)

When either $x + 3 = 0$ or $x - 1 = 0$, the left-hand side of the equation is zero, which makes the statement true. The roots of the equation are therefore at $x = -3$ and $x = 1$.

2. It is not always possible to easily factorise a quadratic equation and we need another method to solve such equations. In such cases we use the ‘minus $b$’ formula as given on page 149 of the textbook, namely

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\]

Using this formula, the roots of $x^2 + 2x - 3 = 0$, with $a = 1$, $b = 2$ and $c = -3$, are found as

\[
x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-3)}}{2(1)}
\]

\[
= \frac{2 \pm \sqrt{4+12}}{2}
\]

\[
= \frac{2 \pm \sqrt{16}}{2}
\]

\[
x = \frac{2 + \sqrt{16}}{2} = 3
\]

or

\[
x = \frac{2 - \sqrt{16}}{2} = -3
\]

Note: One may multiply by $-1$ when working with equations, but not when working with functions. The functions $f(x) = x^2 + 2x - 3$ and $f(x) = -x^2 - 2x + 3$ are obviously not the same, as can be seen from their graphs in Figure 4.2.

Work through the study material in Sections 4.1.1 and 4.1.2 and Worked examples 4.1 and 4.2.

Note the different types of solutions/roots that are possible for quadratic equations.

\(^2\)We assume that you are familiar with factorisation. If you need to refresh your memory on this, we suggest you consult the internet. For example, visit the website www.purplemath.com/modules/solvquad.htm.
Figure 4.2: Quadratic function multiplied by \(-1\) (using Maxima)

Activity

Do the following problems in *Progress exercise* 4.1 (page 152 of textbook). Worked solutions on page 59.
1. Question 3
2. Question 8
3. Question 16
4. Question 20
5. Question 21

4.1.2 Properties and graphs of quadratic functions (parabolas)

In the textbook, Excel is used to graph quadratic functions and their properties are deduced from the graphs. Here, we show how to graph quadratic functions by using their properties.

The properties that follow, are given in terms of a quadratic function in *standard form*, namely

\[ f(x) = ax^2 + bx + c. \]

(a) **The general form of the graph**

From Figure 4.2(a) (above) we see that when the coefficient of \(x^2\) is positive, the turning point of the graph is at the bottom, and the “legs” of the graph point upward.

On the other hand, as can be seen from Figure 4.2(b), when the coefficient of \(x^2\) is negative, the graph’s turning point is at the top and the “legs” point down.

When the coefficient of \(x^2\) is positive \((> 0)\) the function has at a minimum turning point, and when the coefficient of \(x^2\) is negative \((< 0)\) the graph has a maximum turning point.

(b) **The \(y\) axis intercept**

The \(y\) intercept of a quadratic function \(f\) is the point where it cuts the \(y\) axis, that is where \(x = 0\). The vertical intercept of \(f\) is therefore \(f(0) = a(0)^2 + b(0) + c = c\).
The y intercept of \( f(x) = ax^2 + bx + c \) is given by \( c \).

(c) The x axis intercept(s)

The points where a quadratic function cuts the x axis, are found where \( y = 0 \). We therefore need to set \( f(x) = ax^2 + bx + c = 0 \) to find the quadratic equation \( ax^2 + bx + c = 0 \) and solve for \( x \).

We find these roots by either factorisation or by using the ‘minus b’ formula. (See above.)

The kind of roots of a function can be determined by looking at the discriminant \( (b^2 - 4ac) \) that we find under the square root in the ‘minus b’ formula. This quantity is known as the discriminant since it discriminates between different types of intercepts.

- When \( b^2 - 4ac > 0 \), we can calculate \( \sqrt{b^2 - 4ac} \) and two different roots are found.
- When \( b^2 - 4ac = 0 \), the ‘minus b’ formula becomes \( -\frac{b}{2a} \), giving a single root. The function touches the x axis in this turning point.
- When \( b^2 - 4ac < 0 \), we cannot evaluate the square root. The square root of a negative number is not a real number\(^3\). In this case the function doesn’t touch the x axis and has no x intercepts.

(d) The turning point

An important aspect of a quadratic function that is necessary to graph the function, is the coordinates of the turning point. The \( x \) value\(^4\) of the turning point is called the vertex of the graph and it is given by

\[
x_m = -\frac{b}{2a}.
\]

The \( y \) value at this point is \( f(x_m) \).

The coordinates of the turning point of a quadratic function is \( \left( \frac{-b}{2a}; f \left( \frac{-b}{2a} \right) \right) \).

Example 1

Consider the quadratic function (in standard form)

\[
f(x) = -x^2 - 2x + 3.
\]

(a) We see that \( a = -1 < 0 \) and we know the parabola will have a maximum turning point.

(b) The y intercept is 3. (If we set \( x = 0 \), we find \( y = 3 \).)

(c) Since the discriminant is \( b^2 - 4ac = (-2)^2 - 4(-1)(3) = 16 > 0 \), the equation \(-x^2 - 2x + 3 = 0\) has two distinct roots, namely \( x = -3 \) and \( x = -1 \) as shown before.

(d) The vertex of \( f \) is at \( x = -\frac{b}{2a} = \frac{-(-2)}{2(-1)} = -1 \). At \( x = -1 \) we find \( f(-1) = -(-1)^2 - 2(-1) + 3 = 4 \), so the turning point is at \( (-1; 4) \).

The graph of \( f \) is shown in Figure 4.3. The points that were calculated above are indicated as black dots.

\(^3\)It is an imaginary number, which falls outside the scope of this module.

\(^4\)When we come to differentiation later on, we will see how this point is derived.
Activity

For each of the following quadratic functions, determine the intercepts on the axes and the turning point. Indicate the shape of each by means of a rough graph.

Solutions are available on page 61.

1. \( f(x) = 2x^2 - x - 3 \)
2. \( f(x) = 4x^2 - 16x + 16 \)
3. \( f(x) = -3x^2 + 3x - 2 \)

4.1.3 Quadratic functions in economics

Work through Worked examples 4.7, 4.8 and 4.9.

Do not use Excel as shown in the textbook. Either draw the graphs by using the technique discussed in the previous section or use Maxima.

Example

The ACE company decides to enter the market with a new microwave oven, the ACE2015. The production costs entail fixed costs of R120 000 per month and a unit cost of R400 per oven produced. A market survey established that at a wholesale price of R600 per unit, demand will be 1 000 units per month, but if the price is increased to R1 000 per unit, there will be no demand. Assuming that the demand function is linear, they conclude that demand is

\[ Q = 2500 - 2.5P, \]

where \( P \) is price in rand and \( Q \) is the number of units demanded.

At what value should the wholesale price be set and how many units must be produced if the profit is to be maximised?
The total cost function is given by fixed costs plus variable cost, that is

\[ TC = 120\,000 + 400Q. \]

From the demand function we find the total cost function in terms of price to be

\[ TC = 120\,000 + 400(2500 - 2.5P) = 1120\,000 - 1000P. \]

We know that total revenue is given by the number of units sold times the price, that is

\[ TR = P \times Q = P(2500 - 2.5P) = 2500P - 2.5P^2. \]

The profit function is therefore

\[
\pi = TR - TC \\
= 2500P - 2.5P^2 - (1120000 - 1000P) \\
= -2.5P^2 + 3500P - 1120000.
\]

This is a quadratic function with \( a = -2.5, \ b = 3500 \) and \( c = 1120000 \).

Since the coefficient of \( P^2 < 0 \), the profit function has a maximum turning point.

The discriminant \( b^2 - 4ac = (3500)^2 - 4(-2.5)(1120000) = 23450000 > 0 \). The function therefore has two distinct roots.

The vertex is at

\[
P_m = \frac{-b}{2a} = \frac{-3500}{2 \times -2.5} = 700,
\]

which means that profit is a maximum when \( P = \text{R700} \). The maximum profit at this price is

\[
\pi_m = -2.5(700)^2 + 3500(700) - 1120000 = 105000
\]

that is R105000.

**Activity**

Solutions are available on page 62.

1. Consider the example above. ACE considers incorporating two robots into the production line. If they do so, the saving on labour cost can reduce the unit cost of the ACE2015 to R300. However, the capital expenditure incurred will raise their fixed costs to R200000. How will incorporating the robots affect the optimal price and profitability?

2. Clancy’s Chariots, a car-rental firm, has a fleet of 100 identical vehicles. The fixed daily costs amount to R150 per car, while each car used incurs an additional cost of R50 per day. Experience shows that if the rent is set at R200 per day, all cars are rented out, whereas for each increase of R20 the number of cars rented out drops by ten. Determine the optimal price, the number of cars rented out per day at that price and the maximum profit.

[Hint: Determine the total cost and demand functions from the information given and then obtain an expression for profit.]
3. For the following questions from Progress exercise 4.3, use the method described above to calculate intercepts, vertices and turning points and to draw graphs of quadratic functions. Use Maxima to confirm that your graphs are correct. Check your answers against the solutions at the back of the textbook.

(a) Question 1
(b) Question 4
(c) Questions 6 and 7

4.1.4 Cubic functions

A cubic function in general notation is

\[ f(x) = ax^3 + bx^2 + cx + d, \]

where \(a, b, c\) and \(d\) are constants.

Let us work through Worked examples 4.10a and 4.10b together.

Worked example 4.10a

To plot \( f(x) = x^3 \) and \( f(x) = -x^3 \) in Maxima, we simply enter the functions as \(x^3\) and \(-x^3\) to get Figures 4.4(a) and 4.4(b).

![Graphs of f(x) = x^3 and f(x) = -x^3](image)

Figure 4.4: Graphs of \( f(x) = x^3 \) and \( f(x) = -x^3 \)

It is clear that neither of these functions has any turning points. The roots of \( f(x) = y = x^3 \) are found by setting \( y = 0 \), that is \( x^3 = 0 \) which gives \( x = 0 \) as the only root. The same applies for \( f(x) = -x^3 \).

Worked example 4.10b

Instead of calculating function values for a range of values for \( x \), we enter the functions into Maxima as \(0.5*x^3-5*x^2+8.5*x+27\) and \(-0.5*x^3-5*x^2+8.5*x+27\), set the range of \( x \) values to show the graphs properly and enter to get the graphs in Figures 4.5(a) and 4.5(b).

To find the roots of polynomials like cubic functions in Maxima, select Allroots under the Equations tab, enter the function as shown above and press enter.

For \( f(x) = 0.5x^3 - 5x^2 + 8.5x + 27 \) it gives one root at \( x = -1.54820 \approx -1.5 \).
Figure 4.5: Graphs for Worked example 4.10b

For \( f(x) = -0.5x^3 - 5x^2 + 8.5x + 27 \) it gives the following three roots: \( x = 2.81984 \approx 2.8, \ x = 1.72621 \approx -1.7 \) and \( x = 11.09363 \approx -11.1 \). [Note that \( \approx \) stands for approximately equal to.]

You will find the tools to determine the turning points of cubic functions when we get to differentiation in Study unit 4.

Work through Worked example 4.11. Use Maxima to draw the graphs.

Cubic functions with different numbers of roots and turning points

The graph of a cubic function can have no turning points and a single root, as shown in Figure 4.6(a). It can also have two turning points with one root (Figure 4.6(b)), two roots (Figure 4.6(c)) or three roots (Figure 4.6(d)).
4.2 Exponential functions

4.2.1 Definition and graphs of exponential functions

Exponential functions have a variable \((x)\) as the power (or index) of a constant, that is

\[ f(x) = a^x. \]

The following are graphs of typical exponential functions:

![Exponential function graphs](image)

**Figure 4.7: Exponential functions**

**Activity**

Do the following questions in Progress exercise 4.5. The solutions are on page 589 of the textbook.

1. Question 4(a)
2. Question 9(b)
3. Question 13
4. Question 20
5. Question 24

4.2.2 Solving equations that contain exponentials

**Activity**

Do the following questions in Progress exercise 4.6 (the answers are on page 594 of the textbook):

1. Question 1
2. Question 5
3. Question 16
4. Question 23
4.2.3 Applications of exponential functions

Work through *Worked examples* 4.15 and 4.16

Activity

Use Maxima to graph the functions of the following questions in *Progress exercise* 4.7 and 4.8 (solutions are available on page 595 – 597 of the textbook):

1. Question 2 (*Progress exercise* 4.7)
2. Question 6 (*Progress exercise* 4.7)
3. Question 2 (*Progress exercise* 4.8)

[Note: In Maxima, enter the value e as %e.]

4.3 Logarithmic functions

4.3.1 How to find the log of a number

Work through the study material on logs (Sections 4.3.1 – 4.3.4) and *Worked examples* (4.17 – 4.22).

Activity

Do the following questions in *Progress exercise* 4.10 (the answers are at the back of the textbook, page 598):

1. Question 2
2. Question 8
3. Question 10 (Use Maxima for (b))

4.3.2 Graphs and properties of logarithmic functions

In the following figure, \( \log(x) \) and \( \ln(x) \) — the functions in *Worked example* 4.19 — are graphed on the same diagram by using Maxima:
[Note: Maxima uses \( \log(x) \) for \( \ln(x) \). We therefore enter \( \log_{10}(x) \) as \( \log(x)/\ln(10) \) – see Rule 4 in Table 4.16 (page 191).]

This is the same graph as shown on page 190 of the textbook. Take note of the properties of log functions deduced from it.

**Activity**

Do the following questions in *Progress exercise* 4.11 (the answers are at the back of the textbook):

1. Question 2
2. Question 12(b)
3. Question 15
4. Question 21
5. Question 24(b)

### 4.4 Hyperbolic functions

Work through sections 4.4.1 and 4.4.2 and *Worked examples* 4.23 – 4.24.

The graph required for *Worked example* 4.24 as drawn by using Maxima is as follows:

![Graph for Worked example 4.24](image)

**Activity**

Do the following questions in *Progress exercise* 4.13 (the answers are at the back of the textbook):

1. Question 2
2. Question 5
3. Question 7
STUDY UNIT IV

DIFFERENTIATION
Chapter 6: Differentiation and applications

6.1 Slope of a curve and differentiation

6.1.1 The slope of a curve

The study material in Sections 6.1.1 and 6.1.2 gives background on differentiation from first principles.

Work through sections 6.1.1 and 6.1.2 and Worked example 6.1.

Important concepts and terminology to take note of are the following:

▷ The slope of a line tangent to a curve (simply called a tangent) varies.

▷ The formula used to find the equation of the tangent at a point \((x_1; y_1)\) on a curve is

\[
y - y_1 = m(x - x_1),
\]

where \(m\) is the slope of the tangent.

▷ The limit when the length of cords connecting points on a curve tends to zero, is given by

\[
\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx},
\]

and is called the derivative of \(y\) with respect to \(x\).

Note: We mainly use the following notations for differentiation in these notes: \(\frac{dy}{dx}\), \(\frac{d}{dx} f(x)\) and \(f'(x)\).

6.1.3 The derivative

The process of finding the derivative of \(y\) with respect to \(x\), denoted by \(\frac{dy}{dx}\), is called differentiation.

Work through the study material in sections 6.1.3 to 6.1.6 and Worked examples 6.2, 6.3 and 6.4.

The power rule is the basic rule to determine the derivative of \(x^n\), where \(n\) may be any real number. The rule is as follows:

\[
\text{If } y = x^n \text{ then } \frac{dy}{dx} = nx^{n-1}.
\]

For example, if \(y = x^5\), then \(\frac{dy}{dx} = 5x^4\).
Important rules for differentiation using the power rule are the following:

\(\text{\textbullet} \) The derivative of a mathematical term multiplied (or divided) by a constant is the derivative of the term multiplied (or divided) by the constant.

For any term of the form \(kx^n\), the derivative is \(k \frac{dx^n}{dx} = k \times nx^{n-1}\). The constant \(k\) can be positive, negative or a fraction.

For example, \(\frac{d}{dx}10x^3 = 10 \times \frac{d}{dx}x^3 = 10 \times 3x^2 = 30x^2\).

\(\text{\textbullet} \) The derivative of a constant term is zero.

Remember that anything to the power zero is equal to one. A constant like 5 can be written as \(5 \times 1\) which is also equal to \(5 \times x^0\). The derivative of 5 is therefore \(\frac{d}{dx}5 = 5 \times 0 \times x^{0-1} = 0\) since anything multiplied by zero is zero.

\(\text{\textbullet} \) The derivative of a polynomial \(y = f(x) = ax^n + bx^{n-1} + \cdots + cx^2 + dx^1 + e\) with \(a, b, c, d\) and \(e\) constants, is the sum of the derivatives of the terms.

For example, \(\frac{d}{dx}(x^4 + 2x^3 - 5x^2 + 3x + 100) = 4x^3 + 6x^2 - 10x + 3\).

After working through the material and doing the examples, you should have a good understanding of finding the derivative of single mathematical terms in the form \(y = x^n\) and polynomials. If this is not the case, please go back and work through these parts again.

It is very important that you are able to differentiate functions by applying the power rule.

Activity

Do at least the following questions in Progress exercise 6.1: (The answers are at the back of the textbook and the solutions are written out on page 64.)

1. Question 3(c)
2. Question 3(e)
3. Question 5(b)
4. Question 6(b) – differentiate the result.
5. Question 8(c)

6.2 Applications of differentiation

This section focuses on marginal and average functions.

Marginal functions are found by differentiating functions like total revenue \((TR)\) and total cost \((TC)\). They give the rate at which \(TR\) or \(TC\) changes per unit increase in \(Q\) at any point.

On the other hand, average functions are found by dividing the \(TR\) or \(TC\) by the number of units, \(Q\). It gives the average revenue or cost throughout an interval.

Work through the study material in Sections 6.2.1 and 6.2.2 and Worked examples 6.6 – 6.10.
The following variables are regularly used in the textbook:

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>TR</td>
<td>Total revenue</td>
</tr>
<tr>
<td>TC</td>
<td>Total cost</td>
</tr>
<tr>
<td>FC</td>
<td>Fixed cost</td>
</tr>
<tr>
<td>VC</td>
<td>Variable cost</td>
</tr>
<tr>
<td>MR</td>
<td>Marginal revenue</td>
</tr>
<tr>
<td>MC</td>
<td>Marginal cost</td>
</tr>
<tr>
<td>MVC</td>
<td>Marginal variable cost</td>
</tr>
<tr>
<td>AR</td>
<td>Average revenue</td>
</tr>
<tr>
<td>AC</td>
<td>Average cost</td>
</tr>
<tr>
<td>AFC</td>
<td>Average fixed cost</td>
</tr>
<tr>
<td>AVC</td>
<td>Average variable cost</td>
</tr>
</tbody>
</table>

**Activity**

Do the following questions in *Progress exercise* 6.3 (answers are at the back of the textbook). Use Maxima to plot the graphs.

1. Question 2
2. Question 3
3. Question 6
4. Question 9

**6.3 Optimisation for functions of one variable**

The optimum of a function is always at a turning point – where the function is either a maximum or a minimum. Turning points occur where the slope of a function is equal to zero.

Work through the study material of Sections 6.3.1 – 6.3.3 and *Worked examples* 6.16 – 6.19.

**Activity**

Do the following questions in *Progress exercise* 6.5. Find the answers at the back of the textbook.

1. Question 1
2. Question 6

Do the following questions in *Progress exercise* 6.6.

1. Question 8
2. Question 16

Do the following questions in *Progress exercise* 6.7.

1. Question 2
2. Question 7

Do the following questions in *Progress exercise* 6.8.

1. Question 3
2. Question 6
6.4 Economic applications of optimisation

Work through the study material of this section. Worked examples 6.21 – 6.25 are very important. Work through them with pen on paper.

Activity

Do the following questions in Progress exercise 6.9:
1. Question 3 (Solution on page 65 of this document.)
2. Question 7 (Solution on page 618 of textbook.)

6.5 Curvature and other applications (Excluded)

6.6 Further differentiation and applications

In Section 6.6.1, the differentiation of special functions like exponential and logarithmic functions is discussed. Table 6.19 provides the rules.

Work through Worked example 6.33.

Activity

Do the following questions in Progress exercise 6.12: (Solutions on page 620 of textbook.)
1. Question 3
2. Question 12
3. Question 13
4. Question 17

Chain rule

The chain rule for differentiation (Section 6.6.2) is used for a function of a function. Consider for example the function

\[ y = (2x + 3)^2. \]

This is a function of a function with the inner function \( u(x) = 2x + 3 \) and \( y = (u(x))^2 \).
We therefore denote the function \( 2x + 3 \) by \( u \), so we have \( y = u^2 \) which is easy to differentiate in terms of \( u \), namely

\[
\frac{dy}{du} = 2u.
\]
If we differentiate $u = 2x + 3$ with regard to $x$, we find
\[
\frac{du}{dx} = \frac{d}{dx}(2x + 3) = 2.
\]
When we put these results together, we find
\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 2u \times 2 = 4(2x + 3).
\]

Work through material on page 336 and *Worked example* 6.34 with pen in hand.

**Activity**

Do the following questions in *Progress exercise* 6.13: (Solutions on page 620 of textbook.)

1. Question 2
2. Question 3
3. Question 9
4. Question 17

**Product rule**

The product rule for differentiation (Section 6.6.3) is used when we have a product of two functions.

Consider the function
\[
f(x) = (x^2 + 2x + 1)(3x + 2).
\]
Here, $u(x) = x^2 + 2x + 1$ with $\frac{du}{dx} = 2x + 2$ and $v(x) = 3x + 2$, with $\frac{dv}{dx} = 3$. The derivative of $f$ according to the product rule is
\[
f'(x) = v \frac{du}{dx} + u \frac{dv}{dx} = (3x + 2)(2x + 2) + (x^2 + 2x + 1) \times 3
\]
\[= 6x^2 + 10x + 4 + 3x^2 + 6x + 3
\]
\[= 9x^2 + 16x + 7.
\]

Another example of a function containing the product of two functions is
\[
f(x) = x^2 e^{2x+1}.
\]
Here, $u(x) = x^2$ with $\frac{du}{dx} = 2x$ and $v(x) = e^{2x+1}$, with $\frac{dv}{dx} = e^{2x+1} \cdot 2$ and the derivative of $f$ is
\[
f'(x) = v \frac{du}{dx} + u \frac{dv}{dx} = e^{2x+1} \cdot 2x + x^2 \cdot 2e^{2x+1}
\]
\[= 2xe^{2x+1} + 2x^2 e^{2x+1}
\]
\[= 2xe^{2x+1}(1 + x).
\]

Work through the study material and *Worked example* 6.35.
Activity

Do the following questions in *Progress exercise* 6.14: (Solutions on page 621 of textbook.)
1. Question 5
2. Question 7
3. Question 16

**Quotient rule**

The quotient rule for differentiation (Section 6.6.4) is used when we have a *quotient* of two functions.

Consider the function

\[ f(x) = \frac{x^2 + 2x + 1}{3x + 2}. \]

Here, \( u(x) = x^2 + 2x + 1 \) with \( \frac{du}{dx} = 2x + 2 \) and \( v(x) = 3x + 2 \), with \( \frac{dv}{dx} = 3 \). Now, according to the quotient rule,

\[
 f'(x) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2} 
 = \frac{(3x + 2)(2x + 2) - (x^2 + 2x + 1)(3)}{(3x + 2)^2} 
 = \frac{6x^2 + 10x + 4 - (3x^2 + 6x + 3)}{(3x + 2)^2} 
 = \frac{3x^2 + 4x + 1}{(3x + 2)^2}. 
\]

Note: The quotient of two functions can always be transformed so it is the product of two functions. For example, the function \( f(x) = \frac{2x^2 + 3}{3x + 2} \) can be written as

\[ f(x) = (2x + 2)(3x + 2)^{-1}. \]

Applying the product rule with \( u = 2x + 2 \) and \( v = (3x + 2)^{-1} \) results in

\[
 f'(x) = u \frac{dv}{dx} + v \frac{du}{dx} 
 = (3x + 2)^{-1}(2) + (2x + 2) \cdot (-3(3x + 2)^{-2}) 
 = \frac{2}{3x + 2} \frac{3(2x + 2)}{(3x + 2)^2} 
 = 2(3x + 2) - 3(2x + 2) 
 = \frac{-2}{(3x + 2)^2}. 
\]

Work through the study material and *Worked example* 6.36.
Activity

Do the following questions in Progress exercise 6.15: (Solutions on page 621 of textbook.)
1. Question 1
2. Question 3
3. Question 5

The applications in Worked examples 6.37 and 3.38 are very important. Work through them.

Activity

Do the following questions in Progress exercise 6.16: (Solutions on page 622 of textbook.)
1. Question 8
2. Question 10

6.7 Elasticity and the derivative

In Section 2.6 price elasticity of demand (with linear demand functions) was defined as

$$\varepsilon_d = \frac{\Delta Q}{\Delta P} \cdot \frac{P}{Q} = -\frac{1}{b} \cdot \frac{P}{Q}.$$  

For a non-linear demand function, price elasticity of demand is given by

$$\varepsilon_d = \frac{dQ}{dP} \cdot \frac{P}{Q}.$$  

Work through the study material and Worked examples 6.39 and 3.40.

[Note: Sections 6.7.2 and 6.7.3 are excluded for this module.]

Activity

Do the following questions in Progress exercise 6.17: (Solutions on page 624 of textbook.)
1. Question 1
2. Question 2
3. Question 3
STUDY UNIT V

INTEGRATION
Chapter 8: Integration and applications

8.1 Integration as the reverse of differentiation

The study material in this section shows that integration is simply the reverse of differentiation. Read through this section, taking special note of the constant that should be added when integrating.

8.2 The power rule for integration

The power rule of integration is as follows:

\[ \int x^n \, dx = \frac{x^{n+1}}{n+1} + c. \]

As an example, consider the function

\[ f(x) = x^3 + 3x^2 + x + 1. \]

When we apply the power rule of integration, we find

\[ \int f(x) \, dx = \frac{x^{3+1}}{3 + 1} + 3 \frac{x^{2+1}}{2 + 1} + \frac{x^{1+1}}{1 + 1} + \frac{x^{0+1}}{0 + 1} + c \]
\[ = \frac{x^4}{4} + x^3 + \frac{x^2}{2} + x + c. \]

Work through the study material and Worked examples 8.1, 8.2 and 8.3.

Activity

Do the following questions in Progress exercise 8.1: (Solutions on page 631 of textbook.)

1. Question 1
2. Question 6
3. Question 12
8.3 Integration of the exponential function

You will often encounter problems where the exponential function needs to be integrated. It is therefore important that you work through this section in detail.

We know that \( \frac{d}{dx} e^x = e^x \). So, the solution to \( \int e^x \, dx \) is \( e^x \) plus some constant.

As an example, consider the function \( f(x) = 3e^x \). Integrating \( f \) gives \( \int f(x) \, dx = 3e^x + c \).

Do Worked example 8.4 with pen in hand.

8.4 Integration by algebraic substitution

The method described in this section is the reverse of the chain rule for differentiation.

As an example, consider the function

\[
f(x) = \frac{10}{2x-1} \, dx.
\]

To integrate \( f \), we set \( u = 2x - 1 \) and differentiate to get \( \frac{du}{dx} = 2 \) or \( du = 2 \, dx \) which gives \( dx = \frac{du}{2} \). Now,

\[
\int f(x) \, dx = \int \frac{10}{u} \, dx = \int \frac{10 \, du}{u} = 5 \int \frac{1}{u} \, du = 5 \ln u + c \quad \text{(since } \frac{d}{dx} \ln x = \frac{1}{x}\text{)}
\]

\[
= 5 \ln(2x - 1) + c.
\]

Work through the study material and Worked examples 8.5, 8.6 and 8.7.

Activity

Do the following questions in Progress exercise 8.2: (Solutions on page 632 of textbook.)

1. Question 5
2. Question 13
3. Question 17
4. Question 28
8.5 The definite integral and the area under a curve

Work through the explanation of how to find the approximate area under a curve. Note that

\[ \int_a^b f(x) \, dx = F(b) - F(a). \]

Although the textbook uses the notation where the variable is included when stating the limits, we show it without the variable as in many books on calculus.

Activity

Do the following questions in Progress exercise 8.3: (Solutions on page 633 of textbook.)
1. Question 3
2. Question 6
3. Question 11
4. Question 15
5. Question 19

8.6 Consumer and producer surplus

Work through the notes on consumer and producer surplus and Worked examples 8.12, 8.13 and 8.14.

Activity

Do the following questions in Progress exercise 8.4: (Solutions on page 635 of textbook.)
1. Question 4
2. Question 8
3. Question 17
4. Question 20
Appendix A: Solutions to activities

A.1 Section 2.1 (The equation of a straight line)

1 Slope $= -4$; horizontal intercept $x = 0.75$ (when $y = 0$); vertical intercept $y = 3$ (when $x = 0$).

2 In standard form $y = -3x + 6$.

The slope of the line is $-3$ and the intercept on the vertical axis is 6.

Using this information, the line is plotted as follows:

A.2 Section 2.3 (Applications: demand, supply, cost, revenue)

1a When $P = 0$, then $Q = 64$ and when $Q = 0$, then $P = \frac{64}{4} = 16$.

These give the coordinates $(0; 64)$ and $(16; 0)$.

1b The demand function is graphed in figure A.1.

1c When price $P$ increases by R1, the demand ($Q$) will decrease by 4 rides per hour (the slope is $-4$).

1d Writing the demand function $Q = 64 - 4P$ with $P$ on the left-hand side gives $4P = 64 - Q$ which results in $P = 16 - \frac{1}{4}Q = 16 - 0.25Q$ when simplified.

2a Writing the supply function $P = 0.25Q + 22.5$ with $Q$ as the subject, we find $-0.25Q = 22.5 - P$ which results in $Q = -90 + 4P$.

2b To graph the demand and supply functions, we find the coordinates on the axes for each. For the demand function $Q = 210 - 3.5P$, when $P = 0$, $Q = 210$ and when $Q = 0$, $P = 60$, giving the coordinates $(0; 210)$.
Figure A.1: \( Q = 64 - 4P \)

For the supply function \( P = 0.25Q + 22.5 \), when \( P = 0 \), \( Q = -90 \) and when \( Q = 0 \), \( P = 22.5 \), giving the coordinates \((0; -90)\) and \((22.5; 0)\). The functions are graphed on the same axes as follows:

Since negative prices and quantities do not make sense, only the quadrant where both \( P \) and \( Q \) are positive are shown in demand and supply graphs. You may, however, use the coordinate on the negative axes to draw the line.

2c The point of intersection can either be read off the graph or it can be found by solving the functions simultaneously [see Paragraph 3.1.]. This point is at \((40; 70)\) which says that when price \( P = 40 \), demand and supply are equal with \( Q = 70 \). (When demand and supply are equal, we say the market is in equilibrium.)

3a From the given information we find that \( FC = 1000 \) and \( VC = 15 \). The total weekly cost function is therefore given by

\[ TC = FC + VC = 1000 + 15Q, \]

where \( Q \) is the number of calculators produced per week. In function notation, we write this as

\[ TC(Q) = 1000 + 15Q. \]

3b From the total cost function, we see that the slope is 15 and the intercept on the \( TC \) axis is 1000. So, when nothing is produced, the cost is R1 000, and when 100 calculators are produced, \( TC = 1000 + 15 \times 100 = 2500 \).

The graph is shown in Figure A.2.

3c The total cost of producing 25 calculators is \( TC(25) = 1000 + 15(25) = 1375 \).
To determine the number of calculators produced when $TC = 7000$, we solve for $Q$ in the equation $7000 = 1000 + 15Q$ and find that $15Q = 6000$ which gives $Q = 400$.

Total revenue is given by the number of calculators sold/demanded ($Q$) multiplied by the price per calculator. Therefore, $TR = 35Q$.

To graph the $TR$ function for $Q = 0$ to 100, we need to find two points on the line. When no calculators are sold, $TR = 0$, that is the coordinate $(0; 0)$. When 100 calculators are sold, $TR = 3500$ giving the coordinate $(100; 3500)$.

The graph of $TR$ is shown in Figure A.3.

To find the number of calculators sold when $TR = 1750$, we set $1750 = 35Q$ from which it follows that $Q = 50$. Therefore, 50 calculators are sold.

When 80 calculators are sold,

$$TC(80) = 1000 + 15(80) = 2200 \quad \text{and} \quad TR(80) = 35(80) = 2800.$$ 

Since $TR > TC$ we conclude that revenue exceeds costs, so a profit is made.
3i The profit when \( Q = 80 \) is given by

\[
\pi(80) = TR(80) - TC(80) = 2800 - 2200 = 600,
\]

that is R600.

### A.3 Section 2.4 (More mathematics on the straight line)

1a We have two distinct points on the supply line, namely \((Q_1; P_1) = (50; 60)\) and \((Q_2; P_2) = (90; 110)\). The equation of the supply function is in the form \( P = c + dQ \).

We first find the slope \( d \), that is

\[
d = \frac{P_2 - P_1}{Q_2 - Q_1} = \frac{110 - 60}{90 - 50} = \frac{50}{40} = 1.25.
\]

Now, to find the intercept \( c \) on the \( P \) axis, we substitute one of the given points, say \((50; 60)\), into the equation \( P = c + 1.25Q \), that is \( 60 = c + 1.25(50) \) from which it follows that \( c = -2.5 \). The supply function is therefore

\[
P = -2.5 + 1.25Q.
\]

1b The number of T-shirts that will be supplied additionally for each R1 increase in price is given by the slope of the supply function with \( Q \) as subject. We therefore need to transform the equation from \( P = -2.5 + 1.25Q \) to have \( Q \) at the left, that is

\[
1.25Q = P + 2.5 \quad \text{or} \quad Q = 0.8P + 2.
\]

Therefore, for each R1 increase in price, 0.8 T-shirts are supplied additionally.

1c When the price is R85, then \( Q(85) = 0.8(85) + 2 = 70 \) T-shirts are supplied.

1d When 120 T-shirts are supplied, the price is \( P(120) = -2.5 + 1.25(120) = R147.50 \) per T-shirt.

### A.4 Section 2.6 (Elasticity of linear demand and supply functions)

1a We need the demand function in the form \( P = a - bQ \) to apply the derived formulas. Therefore, \( Q = 250 - 5P \) is manipulated to get \( P = 50 - 0.2Q \). From this we see that \( a = 50 \) and \( b = 0.2 \).

The point elasticity of demand is given by

\[
\varepsilon_d = -\frac{1}{b} \times \frac{P}{Q} = -\frac{1}{0.2} \frac{P}{250 - 5P} = -\frac{P}{50 - P} = \frac{P}{P - 50}.
\]

Note that this can be written down directly by using Equation 2.14, namely \( \varepsilon_d = \frac{P}{P - a} \).
1b When $P = 20$, $\varepsilon_d = \frac{20}{20-30} = -\frac{2}{5}$. Since $|\varepsilon_d| = \frac{2}{5} < 1$, we say that demand is inelastic.\footnote{The notation $|\varepsilon_d|$ is used for the absolute value of $\varepsilon_d$, ignoring the sign.} This means that a change in price will not have a significant effect on demand. In fact, if price is increased by 1%, demand will decrease by 0.667%.

When $P = 30$, $\varepsilon_d = \frac{30}{30-30} = -\frac{3}{2}$. Since $|\varepsilon_d| = \frac{3}{2} > 1$, we say that demand is elastic. This means that a change in price will have a significant effect on demand. In fact, if price is increased by 1%, demand will decrease by 1.333%.

1c We need the values of demand at these prices, i.e. $Q(25) = 250 - 125 = 125$ and $Q(35) = 250 - 5 \times 35 = 75$.

The arc elasticity of demand is therefore

$$\varepsilon_d = \frac{1}{b} \cdot \frac{P_1 + P_2}{Q_1 + Q_2} = -5 \cdot \frac{25 + 35}{125 + 75} = -1.5.$$ 

Since $|\varepsilon_d| > 1$ demand is on average elastic over the interval $25 \leq P \leq 35$.

2a If $P = c + dQ$, then $dQ = P - c$. Therefore, point elasticity of supply becomes

$$\varepsilon_s = \frac{1}{d} \cdot \frac{P}{Q} = \frac{P}{dQ} = \frac{P}{P - c}.$$ 

2b When $P = 70$, $\varepsilon_s = \frac{70}{70-90} = -3.5$. Since $|\varepsilon_s| > 1$, supply is elastic at price R70. When the price increases by 1%, supply will increase by 3.5%.

2c Arc elasticity of supply is given by $\varepsilon_s = \frac{1}{d} \cdot \frac{P_1 + P_2}{Q_1 + Q_2}$. From the supply function $P = 90 + 0.05Q$ we find that $Q = 20P - 1800$. So, at $P = 40$, $Q = -1000$ and when $P = 60$, $Q = -600$. Therefore,

$$\varepsilon_s = \frac{1}{0.05} \cdot \frac{40 + 60}{-1000 - 600} = 20 \cdot \frac{100}{-1600} = -1.25.$$ 

Since $|\varepsilon_s| > 1$, supply is elastic over the given price interval.

### A.5 Section 3.1 (Solving simultaneous linear equations)

1 First, we simplify the equations to find

$$4x - 3y = 12 \quad \quad (1)$$

$$0.5x + y = 7 \quad \quad (2)$$

From (2) we find that $y = 7 - 0.5x$. When we substitute this into (1) we find

$$4x - 3(7 - 0.5x) = 12$$

$$4x - 21 + 1.5x = 12$$

$$5.5x = 33$$

$$x = 6$$

and $y = 7 - 0.5(6) = 4$.\footnote{The notation $|\varepsilon_d|$ is used for the absolute value of $\varepsilon_d$, ignoring the sign.}
2 Again, we first simplify and get

\[ 5P + 10Q = 18 \]  \hspace{1cm} (1)
\[ 5P + 3Q = 4 \]  \hspace{1cm} (2)

When we substitute the first equation as it is given \( P = \frac{18 - 10Q}{5} \) into (2), we find

\[
\begin{align*}
5 \left( \frac{18 - 10Q}{5} \right) + 3Q &= 4 \\
18 - 10Q + 3Q &= 4 \\
-7Q &= -14 \\
Q &= 2.
\end{align*}
\]

Substituting this value into (1) gives

\[ 5P + 10 \times 2 = 18 \]  \hspace{1cm} or  \hspace{1cm} \[ P = \frac{-2}{5} = -0.4. \]

3 Simplifying the equations gives

\[
\begin{align*}
5q - 6p &= 7 & (1) \\
q - p &= 3 & (2)
\end{align*}
\]

From (2) we find \( q = p + 3 \). Substituting this into (1) gives \( 5(p + 3) - 6p = 7 \) which results in \( p = 8 \).

Substituting this into (2) gives \( q = 11 \).

A.6 Section 3.2 (Equilibrium and break-even)

1 (PE 3.2 Q2)

The demand and supply functions with both \( P \) and \( Q \) as subject, are as follows:

\[
\begin{align*}
P_d &= 800 - 2Q_d & \text{or} & & Q_d &= 400 - 0.5P_d \\
P_s &= -40 + 8Q_s & \text{or} & & Q_s &= 5 + 0.125P_s
\end{align*}
\]

(a) To find the equilibrium, we set \( P_d = P_s \), that is

\[
\begin{align*}
800 - 2Q &= -40 + 8Q \\
-2Q - 8Q &= -800 - 40 \\
-10Q &= -840 \\
Q &= 84.
\end{align*}
\]

Therefore, at equilibrium \( Q = 84 \) rings are supplied/demanded. When we substitute this into \( P_d = 800 - 2Q \), we find the equilibrium price to be \( P = 800 - 2(84) = R632 \).

(b) The level of excess supply when \( P = 720 \):

\[
\begin{align*}
Q_s - Q_d &= (5 + 0.125(720)) - (400 - 0.5(720)) \\
&= 95 - 40 \\
&= 55.
\end{align*}
\]

If the price per ring is raised to R720, the producer will supply 55 more rings.
(c) The level of excess demand when \( P = 560 \):

\[
Q_d - Q_s = (400 - 0.5(560)) - (5 + 0.125(560)) \\
= 120 - 75 \\
= 45.
\]

If the price per ring is lowered to R560, customers will demand 45 more rings.

2 (PE 3.2 Q7)
The demand and supply functions with both \( w \) and \( L \) as subject, are as follows:

\[
w_d = 70 - 4L_d \quad \text{or} \quad L_d = 17.5 - 0.25w_d \\
w_s = 10 + 2L_s \quad \text{or} \quad L_s = -5 + 0.5w_s
\]

(a) At equilibrium, \( w_d = w_s \), that is

\[
70 - 4L_d = 10 + 2L_s \\
-4L - 2L = 10 - 70 \\
-6L = -60 \\
L = 10.
\]

Therefore, at equilibrium, 10 labourers are working. When we substitute \( L = 10 \) into \( w_d = 70 - 4L \), we find the equilibrium wages to be \( w = 70 - 4(10) = R30 \).

(b) The excess demand for labour when \( w = 20 \):

\[
L_d - L_s = (17.5 - 0.25(20)) - (-5 + 0.5(20)) \\
= 12.5 - 5 \\
= 7.5.
\]

If the wages are lowered to R20, then \( 7.5 \approx 8 \) more workers will be needed to work.

(c) The excess supply for labour when \( w = 40 \):

\[
L_s - L_d = (-5 + 0.5(40)) - (17.5 - 0.25(40)) \\
= 15 - 7.5 \\
= 7.5.
\]

If wages are raised to R40, then 8 more workers will be willing to work.

A.7 Section 3.2.5 (Break-even analysis)

1a (PE 3.3 Q8)
It is given that the price per unit \( P = 30 \), fixed costs \( FC = 200 \) and variable cost \( VC = 5 \) per unit.

From the available information we find the formulæ for total revenue and total cost to be

\[
TR = P \times Q = 30Q \quad \text{and} \quad TC = FC + VC = 200 + 5Q.
\]
At the break-even point,

\[ TR = TC \]

\[ 30Q = 200 + 5Q \]

\[ 25Q = 200 \]

\[ Q = 8. \]

The firm should produce and sell 8 units to break even.

1b At break-even, \( TR = 30(8) = 240 \) and \( TC = 200 + 5(8) = 240 \) which are equal as we would expect.

### A.8 Section 3.3 (Consumer and producer surplus)

1 (PE 3.4 Q2)

(a) Demand and supply functions are given as \( P_d = 58 - 0.2Q \) and \( P_s = 4 + 0.1Q \)

At equilibrium, \( 58 - 0.2Q = 4 + 0.1Q \) which results in \( 0.3Q = 54 \) or \( Q = 180 \). Substituting this into either function gives \( P_0 = 22 \).

This means that at equilibrium 180 seats on the bus will be sold for R22 each. See the graph in the solution at the back of the textbook.

(b) (i) At equilibrium consumers spend \( P_0 \times Q_0 = 22 \times 180 = R3\,960 \).

(ii) Consumers are willing to pay \( P_0 \times Q_0 + 0.5 \times 22 \times (58 - 22) = 3,960 + 3,240 = R7\,200 \).

(iii) Consumer surplus \( CS = R3\,240 \).

(c) (i) At equilibrium, the bus company receives R3\,960.

(ii) The bus company is willing to accept \( 180 \times 4 + 0.5 \times 180 \times (22 - 4) = 720 + 1,620 = 2,340 \).

(iii) \( PS = 3,960 - 2,340 = 1,620 \).

### A.9 Section 4.1.1 (Solving a quadratic equation)

1 (PE 4.1 Q3)

We use factorisation to solve this quadratic equation:

\[-Q^2 + 6Q - 5 = 0\]

\[Q^2 - 6Q + 5 = 0\]

\[(Q - 5)(Q - 1) = 0.\]

This gives \( Q - 5 = 0 \) or \( Q - 1 = 0 \) which results in \( Q = 5 \) or \( Q = 1 \).

2 (PE 4.1 Q8)

Simplifying the equation gives \( Q^2 - 6Q = Q(Q - 6) = 0 \). This results in \( Q = 0 \) or \( Q = 6 \).

3 (PE 4.1 Q16)

The equation is given in factored format, namely \( x(x - 3)(x + 3) = 0 \). The equation holds if \( x = 0 \), \( x = 3 \) or \( x = -3 \).
4 (PE 4.1 Q20)
We first need to simplify the given equation:
\[
Q(2Q - 9) = 4(Q + 3)
\]
\[
2Q^2 - 9Q = 4Q + 12
\]
\[
2Q^2 - 13Q - 12 = 0.
\]
Using the quadratic formula gives
\[
Q = \frac{-(-13) \pm \sqrt{(-13)^2 - 4(2)(-12)}}{2(2)}
\]
\[
= \frac{13 \pm \sqrt{265}}{4}
\]
\[
= 7.32 \text{ or } -0.82.
\]

5 (PE 4.1 Q21)
We are given \( TR = 1800Q - Q^2 - 44375 \).

(a) When \( TR = 0 \), we find \(-Q^2 + 1800Q - 44375 = 0\). Using the quadratic formula, we find
\[
Q = \frac{-1800 \pm \sqrt{(1800)^2 - 4(-1)(-44375)}}{-2}
\]
\[
= \frac{-1800 \pm \sqrt{3062500}}{-2}
\]
\[
= 25 \text{ or } 1775.
\]
\( TR \) is zero when 25 journals are sold and again when 1775 are sold.

(b) When \( TR = 765625 \), we find \(-Q^2 + 1800Q - 44375 = 765625 \) or \(-Q^2 + 1800Q - 810000 = 0\). Using the quadratic formula, we find
\[
Q = \frac{-1800 \pm \sqrt{(1800)^2 - 4(-1)(-810000)}}{-2}
\]
\[
= \frac{-1800 \pm \sqrt{0}}{-2}
\]
\[
= \frac{-1800}{-2}
\]
\[
= 900.
\]
\( TR = 765625 \) when 900 journals are sold.

A.10 Section 4.1.2 (Properties and graphs of quadratic functions)

1 For \( f(x) = 2x^2 - x - 3 \), \( a = 2 \), \( b = -1 \) and \( c = -3 \). Since \( a > 0 \), \( f \) has a minimum.

The vertex is at \( x_m = \frac{-b}{2a} = \frac{-(-1)}{2(2)} = \frac{1}{4} = 0.25 \). The value of \( f \) at the vertex is \( f(0.25) = 2(0.25)^2 - 0.25 - 3 = -3.125 \). The turning point is at \( (0.25; -3.125) \).

The \( y \) intercept is \( c = -3 \).

Since the discriminant \( b^2 - 4ac = (-1)^2 - 4(2)(-3) = 25 > 0 \), \( f \) has two distinct \( x \) intercepts, namely
\[
x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}
\]
\[
= \frac{-(1) - \sqrt{25}}{2 \times 4} = \frac{-(1) + \sqrt{25}}{2 \times 4} = \frac{-4}{4} = -1 = 1.5.
\]

The graph of \( f \) is as follows:

2 For \( f(x) = 4x^2 - 16x + 16 \), \( a = 4 \), \( b = -16 \) and \( c = 16 \). Since \( a > 0 \) \( f \) has a minimum turning point.

The vertex is at \( x_m = \frac{-b}{2a} = \frac{-(-16)}{2 \times 4} = 2 \) and \( f(2) = 4(2)^2 - 16(2) + 16 = 0 \) giving the turning point as \( (2; 0) \).

The \( y \) intercept is \( c = 16 \).

The discriminant \( b^2 - 4ac = (-16)^2 - 4(4)(16) = 0 \). Therefore, the graph touches the \( x \) axis at its minimum point at \( (2; 0) \).

The graph of \( f \) is as follows:

3 For \( f(x) = -3x^2 + 3x - 2 \), \( a = -3 \), \( b = 3 \) and \( c = -2 \). Since \( a < 0 \), \( f \) has a maximum turning point.

The vertex is at \( x_m = \frac{-b}{2a} = \frac{(3)}{2 \times -3} = 0.5 \) and \( f(0.5) = -3(0.5)^2 + 3(0.5) - 2 = -1.25 \) giving the turning point as \( (0.5; -1.25) \).

The \( y \) intercept is at \( c = -2 \).

The discriminant \( b^2 - 4ac = (3)^2 - 4(-3)(-2) = 9 - 24 = -15 < 0 \), therefore the graph doesn’t touch the \( x \) axis.

The graph of \( f \) is as follows:
A.11 Section 4.1.3 (Quadratic functions in economics)

1 If the robots are purchased, the total cost function is

\[ TC = 200000 + 300P. \]

The demand and total revenue functions remain the same as

\[ Q = 2500 - 2.5P \quad \text{and} \quad TR = 2500 - 2.5P^2. \]

The profit function now becomes

\[ \pi = TR - TC \]
\[ = 2500P - 2.5P^2 - (200000 + 300(2500 - 2.5P)) \]
\[ = 2500P - 2.5P^2 - 200000 - 750000 + 750P \]
\[ = -2.5P^2 + 3250P - 950000. \]

Here, \( a = -2.5, \ b = 3250 \) and \( c = -950000. \) The price that maximises profit is

\[ P_m = \frac{-b}{2a} = \frac{-3250}{2(-2.5)} = \text{R650}, \]

which is R50 per unit less than before. The maximum profit at this price is

\[ \pi_m = -2.5(650)^2 + 3250(650) - 950000 = \text{R106250}. \]

We would advise ACE to purchase the robots, since their profit will be 106250 - 105000 = R1250 higher.

2 We know that total cost is fixed costs plus variable cost, that is

\[ TC = 100 \times 150 + 50Q = 15000 + 50Q. \]

To find the demand function, we need two points on the line. It is given that when rent is R200, all the cars are rented out, that is the point \((P_1; Q_1) = (200; 100). \) Also, when the rent goes up by R20, the number rented out drops by 10. Therefore, \((P_2; Q_2) = (220; 90)\) is also on the demand line.

The slope of the demand function is therefore \( b = \frac{90 - 100}{220 - 200} = \frac{-10}{20} = -0.5, \) giving the demand function to be

\[ Q = a - bP = a - 0.5P. \]

To find the value of \( a, \) we use the point \((200; 100)\) to find \( 100 = a - 0.5(200) = a - 100, \)

\[ \text{giving } a = 200. \]

The demand function is therefore

\[ Q = 200 - 0.5P. \]
The total revenue function is given by

\[ TR = PQ = P(200 - 0.5P) = 200P - 0.5P^2 \]

and the profit function is

\[ \pi = TR - TC 
\]
\[ = 200P - 0.5P^2 - (15000 + 50Q) 
\]
\[ = 200P - 0.5P^2 - 15000 - 50(200 - 0.5P) 
\]
\[ = 200P - 0.5P^2 - 15000 - 10000 + 25P 
\]
\[ = -0.5P^2 + 225P - 25000. \]

This is a quadratic function with \( a < 0 \), therefore it has a maximum turning point. The vertex is at

\[ P_m = \frac{-b}{2a} = \frac{-225}{2(-0.5)} = 225. \]

If the rent is R225, \( Q_m = 200 - 0.5(225) = 87.5 \) cars will be rented out per day. Since it is impossible to rent out half a car, we round the number of cars to 88.

The profit when the rent is R225 per car is \( \pi(225) = -0.5(225)^2 + 225(225) - 25000 = 31250 \). This means that if they rent out 88 cars per day for R225, they will make a daily profit of R312,50.

A.12 Section 4.4 (Hyperbolic functions)

1 (PE 4.13 Q2)
Solving the equation gives

\[ \frac{x}{x + 4} = 3 \]
\[ x = 3x + 12 \]
\[ -2x = 12 \]
\[ x = -6. \]

2 (PE 4.13 Q5)
First simplify the equation to find

\[ \frac{Q + 5}{Q - 5} = Q + 1 \]
\[ Q + 5 = (Q_1)(Q - 5) \]
\[ Q + 5 = Q^2 - 4Q - 5 \]
\[ Q^2 - 5Q - 10 = 0. \]

Now use the quadratic formula to solve the equation. That is

\[ Q = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(-10)}}{2} \]
\[ = \frac{5 \pm 8.06}{2} \]
\[ = 6.53 \text{ or } -1.53. \]
3 (PE 4.13 Q7)

(a) The graph of $V$ shows that the value of the car depreciates quickly over the first few years.

(b) Since the value of the car is in thousands of rands, $V = 20$ if the value is R20 000. The value will therefore be R20 000 after

$$20 = 1 + \frac{84}{1 + 2t}$$

$$(20 - 1)(1 + 2t) = 84$$

$$19 + 38t = 84$$

$$38t = 65$$

$$t = \frac{65}{38} = 1.71 \text{ years}.$$  

A.13 Section 6.1.3 (The derivative)

1 (PE 6.1, Q3(c))

$$y = 10 + 5x + x^{-2}$$

$$\frac{dy}{dx} = 0 + 5 - 2x^{-3}$$

$$= 5 - \frac{2}{x^3}.$$ 

2 (PE 6.1 Q3(e))

$$P(Q) = \frac{Q^3}{3} + 700Q - 15Q^2$$

$$P'(Q) = \frac{3Q^2}{3} + 700 - 2 \times 15Q$$

$$= Q^2 + 700 - 30Q.$$ 

3 (PE 6.1 Q5(b))

$$y = \frac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$$

$$\frac{dy}{dx} = -\frac{1}{2}x^{-\frac{3}{2}}$$

$$= -\frac{1}{2x^{\frac{3}{2}}} = -\frac{1}{2\sqrt{x^3}}.$$ 

65
4 (PE 6.1 Q6(b))
Simplifying gives
\[
\frac{4\sqrt{Q}}{q^2} = 4Q^{\frac{1}{2}}Q^{-2} = 4Q^{\frac{1}{2}-2} = 4Q^{-\frac{3}{2}}.
\]
Differentiating the result gives
\[
\frac{d}{dx}4Q^{-\frac{3}{2}} = -6Q^{-\frac{5}{2}} = -\frac{6}{Q^{\frac{5}{2}}}.\]

5 (PE 6.1 Q8(c))
The first derivative of \( P(Q) = 10Q + Q^{0.5} \) is
\[
P'(Q) = 10 + 0.5Q^{-0.5}
\]
and the second derivative is
\[
P''(Q) = -0.25Q^{-1.5}.
\]

A.14 Section 6.4 (Economic applications of optimisation)

1 (PE 6.9 Q 3)
(a) Total revenue is the price per unit times the number of units sold, that is
\[
TR = PQ = (240 - 10Q)Q = 240Q - 10Q^2.
\]
Profit is total revenue minus total cost, that is
\[
\pi = 240Q - 10Q^2 - (120 + 8Q) = -10Q^2 + 232Q - 120.
\]
(b) Profit is a maximum when \( \frac{d \pi}{dQ} = -20Q + 232 = 0 \), that is when \( Q = 11.6 \approx 12 \) units.
Total revenue is a maximum when \( \frac{d}{dQ}TR = 240 - 20Q = 0 \), that is when \( Q = \frac{240}{20} = 12 \) units.
(c) \( MR = \frac{d}{dQ}TR = 240 - 20Q \) and \( MC = \frac{d}{dQ}(120 + 8Q) = 8 \).
At maximum profit (\( Q = 11.6 \)), \( MR = 8 = MC \).
(d) See graphs on page 616 of the textbook. \( MR \) and \( MC \) intersect at the point where profit is a maximum.