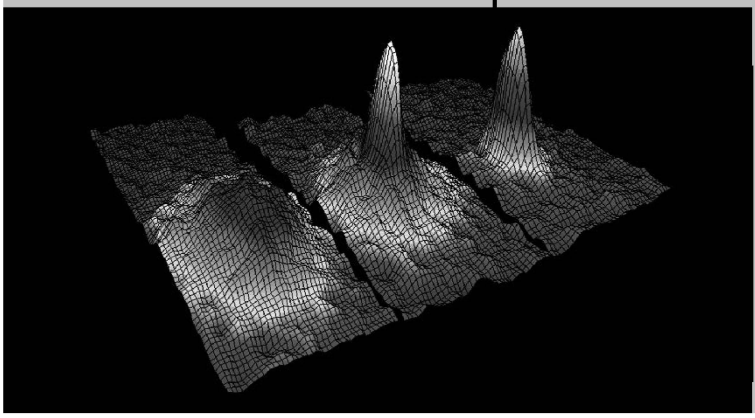
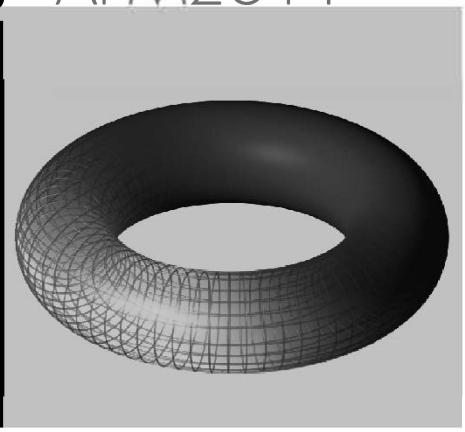
Differential Equations

ONLY STUDY GUIDE FOR

APM2611





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PREFACE

When Newton invented calculus in the 1660s, he also invented differential equations (DEs) and used them to describe physical processes, as he did in his famous second law of motion, $F = m\frac{d^2x}{dt^2}$. Now we know that a great number of natural phenomena can be described by such DEs and solving them gives us insight into the behavior of the universe.

Many industrial procedures can also be described by DEs.

The main application of the theory of DEs, then, is this: if we can describe some phenomenon – either from nature or industry – as a DE, solving it allows us to predict the future.

That is why we study DEs and techniques for solving them.

The prescribed book for this module is:

> Zill, Dennis G. and Wright, Warren S.
> *Differential Equations with Boundary-Value Problems*
> (eighth edition), Brooks/Cole, 2013

The 6^{th} edition is also perfectly suitable, although the page numbering may differ slightly from that of the 8^{th} edition. We will refer to the textbook as **Z & W** in these notes.

Since the textbook is very suitable for self-study, in these notes we will give you a few pointers to guide you in your reading. The exception is in Chapter 4, where we present in these notes another method for solving DEs, namely the **Operator Method**. (Note that this method is optional, and it is mentioned here just for you to know about its existence.)

You need to study the following sections in **Z & W**:

Chapter 1 §1.1 – §1.3
Chapter 2 §2.1 – §2.4
Chapter 3 §3.1
Chapter 4 §4.1 – §4.7 AND notes in this study guide
Chapter 5 §5.1
Chapter 6 §6.1 – §6.2
Chapter 7 §7.1 – §7.4
Chapter 11 §11.1 – §11.3
Chapter 12 §12.1 – §12.3

See Chapter 0 for the specific outcomes that you need to achieve in order to pass this course. All questions in the assignments and exam test the points mentioned in that chapter. The following sections are for additional reading:

Chapter 11 §11.4;
Chapter 12 §12.2, §12.4 and §12.5.

Don’t forget to read the review at the end of each chapter in Z & W, as it serves as a useful summary of the contents of the chapter.

Z & W also contains a large number of problems at the end of each section. You should try to do as many of these as possible. This is the only way you can be sure you have mastered the techniques and ideas in that particular section. The answers to odd-numbered exercises are given at the end of the prescribed book.

You need to learn all definitions and statements of theorems in the study sections. You must be able to write them down and apply them. However, no proofs of theorems will be required in the exam.

Chapter 0

Specific Outcomes and Assessment Criteria

0.1 Classify and recognise the basic types of DEs and PDEs

Assessment Criteria:

1.1 Differential equations are classified according to whether they are ordinary differential equations (DEs) or partial differential equations (PDEs).

1.2 Differential equations are classified according to whether they are homogeneous or non- homogeneous and linear or non-linear.

1.3 DEs and PDEs are classified according to their order.

1.4 Ordinary DEs include:

  (a) Linear first-order DEs;
  (b) Linear homogeneous higher-order DEs with constant coefficients;
  (c) Linear non-homogeneous higher-order DEs;
  (d) Bernoulli DEs;
  (e) Cauchy-Euler DEs.

1.5 The heat equation is recognized as a linear first-order PDE among partial DEs.

1.6 The concepts of boundary values and initial values are used.

0.2 Solve specific types of differential equations

Assessment Criteria:

2.1 A given function is tested to see whether it is a solution of a given DE and its boundary conditions.

2.2 First-order linear DEs is solved using an integrating factor - it is fundamental for all other methods.

2.3 A given DE is converted into a simpler form using the method of substitution and solved before the answer is converted back into the original format.

2.4 Linear homogeneous higher-order (especially second-order) DEs are solved using a specific method.

2.5 Linear non-homogeneous higher-order (especially second-order) DEs are solved using the characteristic equation method and particular solutions.

2.6 Methods for obtaining particular solutions such as the method of undetermined coefficients and the method of variation of parameters are used. The operator method may also be used (this is optional).

2.7 Bernoulli equations are solved using the substitution method.

2.8 Cauchy-Euler equations are solved using the substitution method.

0.3 Perform basic operations on infinite series

Assessment Criteria:

3.1 Direct computations of addition, subtraction, scalar multiplication and multiplication of infinite series are performed correctly.

3.2 Computation of the integrals and derivatives of functions are written as infinite series.

3.3 A given infinite series is shown as the solution of a given DE.

3.4 The solution of a given DE is written as an infinite series and the coefficients of the infinite series are computed given the boundary conditions.

0.4 Use the Fourier and Laplace transforms

Assessment Criteria:

4.1 Laplace and Fourier transforms of functions are calculated from the definition in simple cases.

4.2 The definition of odd and even functions is given and the Fourier transform for them is calculated.

4.3 Elementary operations on the Laplace transform are performed:

- (a) translation and scalar multiplication;
- (b) differentiation and integration;
- (c) the relationship between these operations on the function and complementary operations on the transform of the function.

4.4 Elementary operations on the Fourier transform are performed:

- (a) translation and scalar multiplication;
- (b) differentiation and integration;
- (c) the relationship between these operations on the function and complementary operations on the transform of the function.

4.5 Laplace and Fourier transforms of more complicated functions are calculated using the above described operations.

4.6 A given DE is converted into an algebraic equation using the Laplace transforms, it is solved and the inverse Laplace transform is applied to obtain a solution of the given DE.

0.5 Use the heat equation, Fourier transforms and method of separation of variables to solve problems

Assessment Criteria:

5.1 The heat equation (a linear PDE) is defined and its motivation is given.

5.2 The general form of the solution of the heat equation is calculated as a Fourier series.

5.3 The heat equation is solved by transformation into a separable PDE.

5.4 The coefficients of the Fourier series are calculated using the boundary values.

5.5 The solution is applied to modelling the dissipation and distribution of heat in various objects.

0.6 Use DEs to model practical situations and interpret the solutions

Assessment Criteria:

6.1 DEs that model the mixing of substances in a tank are set up and solved; the solution of the DE is interpreted in terms of the concentration of the substance at any given time.

6.2 DEs that model the motion of objects (spring-mass systems) are set up, solved and interpreted.

Chapter 1

Introduction to Differential Equations

Study the following sections thoroughly:

1.1 Definitions and Terminology

This section introduces you to some of the basic definitions and terminology used in the study of differential equations (DEs). You must familiarize yourself with these concepts and be able to

- classify a DE according to *type, order* and *linearity.*
- determine whether a given function is a solution of a given DE by substituting the function and its derivatives into the DE.

1.2 Initial-Value Problems

Make sure that you know the statement of Theorem 1.2.1 and are able to apply it. The existence and uniqueness of solutions for DEs is very important. Determining that there is a unique solution for any given DE is not the same as actually solving it, but often a DE has no solution, so trying to find one would be futile.
This is why the theorem is important.

1.3 Differential Equations as Mathematical Models

Read through this section for the sake of interest. It contains examples showing how DEs arise as mathematical models or real-life systems. As you progress with this module you will be able to solve more and more 'application' problems of this type.

Chapter 2

First-Order Differential Equations

This chapter is concerned with methods for solving first-order DEs. Since the method of solution depends on the kind of DE, it is important for you to

- be able to classify a first-order DE. In this module we deal only with the following 5 types: *separable, homogeneous, exact, linear* and *Bernoulli.*
- know and be able to apply the appropriate method of solution.

Once you have worked through this chapter you should be able to determine whether a given DE has one or more of the above classifications.

Study the following sections thoroughly:

2.1 Solution Curves Without a Solution

2.2 Separable Equations

2.3 Linear Equations

2.4 Exact Equations

Chapter 3

Modelling with First-Order Differential Equations

This chapter is concerned with solving some of the more commonly occurring linear and non-linear first-order DEs that arise in applications. In particular, be sure to understand the problem of the concentration of a substance in a tank under various conditions (§3.1, example 5, p 87 in Z&W 8th edition).
Study the following section thoroughly:

3.1 Linear Models

Chapter 4

Higher-Order Differential Equations

This chapter is concerned with *higher*-order linear DEs. The first two sections discuss the underlying theory of linear DEs in general. Thereafter, methods of solution for linear DEs with *constant coefficients* are developed. (Methods of solution for linear DEs with *non-constant coefficients* are discussed in Chapter 6.) The third section deals with the general solution of *homogeneous* linear DEs and the remaining sections (including the extra section, §4.8 of this guide), all discuss methods of solving *non-homogeneous* linear DEs. Study the following sections thoroughly:

4.1 Preliminary Theory

Pay an attention to the concepts of **linear independence of functions** (Definition 4.1.1 in Z&W 8th edition) and the **reduction of order** of an ODE as shown by the example in the next section.

4.2 Reduction of Order

Recall that a set of functions $f_1(x), f_2(x), \cdots, f_n(x)$ defined on an interval I is said to be **linearly dependent** if one of the functions in the set can be expressed as a linear combination of one or more of the other functions in the set. If none of the functions in the set can be expressed as a linear combination of any other functions of the set, then the set is said to be **linearly independent**.

In other words, linear independence means that we have for every $x \in I$,

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0$$

Example *4.2.1.* Knowing that $y_1(x) = x^{-1}$ is a solution to the equation

$$2x^2 y'' + xy' - 3y = 0$$

on $I = (0, +\infty)$, find its general solution.

As mentioned above, reduction of order requires that a solution should be given, and without this known solution, it is impossible to carry out reduction of order. Using this first solution y_1, we are looking for a second solution y_2 so that the set $\{y_1,\ y_2\}$ is **linearly independent** on I. This means that their quotient y_2/y_1 is not a constant on I. That is, $y_2(x)/y_1(x) = u(x)$, where $u(x)$ is a variable function depending on x. Hence, the second solution will have the form

$$y_2(x) = u(x)y_1(x) \tag{4.1}$$

where $u(x)$ is to be determined. For that, we substitute the guess into the differential equation and get a new differential equation that can be solved for $u(x)$. Hence, we have

$$y_2(x) = x^{-1}u, \quad y_2'(x) = -x^{-2}u + x^{-1}u', \quad y_2''(x) = 2x^{-3}u - 2x^{-2}u' + x^{-1}u''.$$

Substituting these into the differential equation gives

$$2x^2(2x^{-3}u - 2x^{-2}u' + x^{-1}u'') + x(-x^{-2}u + x^{-1}u') - 3(x^{-1}u) = 0.$$

Rearranging and simplifying yields

$$2xu'' + (-4+1)u' + (4x^{-1} - x^{-1} - 3x^{-1})u = 0$$

equivalently

$$2xu'' - 3u' = 0$$

Note that upon simplifying the only terms remaining are those involving u and its derivatives, but the term involving u drops out. Generally, if you have done all of your work correctly this should always happen. In some special cases, such as the repeated roots case, the first derivative term of u will also drop out. So, in order for (4.3) to be a solution, u must satisfy

$$2xu'' - 3u' = 0 \tag{4.2}$$

This appears to be a problem. In order to find a solution to a second-order non-constant coefficient DE we need to solve a different second-order non-constant coefficient DE.

However, this is not the problem that it appears to be. Because the term involving the u drops out we can actually solve (4.2), and we can do it with the knowledge that we already have at this point. We will solve this by making the following change of variable:

$$w = u' \Longrightarrow w' = u''.$$

With this change of variable, (4.2) becomes

$$2xw' - 3w = 0$$

and this is a linear, first-order DE that we can solve. This also explains the name of this method. We have managed to reduce a second-order DE down to a first-order DE.

We see clearly that the order of the ODE has been reduced from two to one. Then we have a fairly simple first-order ODE in w, and there are many methods to solve it. Using the integrating factor method and putting the equation into the form $w' - \frac{3}{2x}w = 0, \quad x \neq 0$, gives integrating factor

$$e^{\int \frac{-3}{2x} dx} = e^{\frac{-3}{2} \ln |x|} = x^{-\frac{3}{2}}.$$

This yields $\frac{d}{dx}[x^{-\frac{3}{2}}w] = 0$ equivalent to $x^{-\frac{3}{2}}w = k_1$. Then, the solution to this DE is

$$w = k_1 x^{\frac{3}{2}},$$

where k_1 is a real constant. Because we are after a solution to (4.2), we can now find this. Using our change of variable $w = u'$ gives

$$u(x) = \int w(x)dx = \int k_1 x^{\frac{3}{2}} dx = \frac{2}{5} k_1 x^{\frac{5}{2}} + k_2,$$

where k_2 is a real constant. This is the most general possible $u(x)$ that we can use to get a second solution. Just as we did in the repeated roots section, we can choose the constants to be anything we want; so we choose them to clear out all the extraneous constants. In this case we can use

$$k_1 = \frac{5}{2}, \qquad k_2 = 0.$$

Using these gives the following expression for $u(x)$ and for the second solution:

$$u(x) = x^{\frac{3}{2}} \quad \Longrightarrow \quad y_2(x) = x^{-1}(x^{\frac{5}{2}}) = x^{\frac{3}{2}}.$$

Finally, the general solution is

$$y(x) = c_1 x^{-1} + c_2 x^{\frac{3}{2}},$$

where $c_1, \quad c_2$ are real constants.

Note that if we had been given initial conditions we could then differentiate, apply the initial conditions and solve for the constants $c_1, \quad c_2$.

In the next example, we have the opportunity to follow the same steps as above.

Example *4.2.2.* Find the general solution to

$$x^2y'' + 2xy' - 2y = 0$$

on $I = (0, +\infty)$, knowing that $y_1(x) = x$ is one of its solutions.

Hint: The second solution will have the form

$$y_2(x) = xu(x) \tag{4.3}$$

Substituting into the DE and integrating will yield

$$u(x) = x^{-3}$$

and the general solution will be

$$y(x) = c_1x + \frac{c_2}{x^2}.$$

4.3 Homogeneous Linear Equations with Constant Coefficients

4.4 Undetermined Coefficients - Superposition Approach

There are two approaches to the method of undetermined coefficients - the superposition approach, which we discuss in this section, and the annihilator approach, which is discussed in §4.5.
Unless an approach is specified you may use either one if you are asked to use the method of undetermined coefficients. If you are asked to use the superposition approach, then you must use the method described in this section.
NOTE: The method of undetermined coefficients is **limited** to non-homogeneous linear DEs

- that have constant coefficients;
- where the right-hand side of the DE (the non-homogeneous part) is a polynomial function, an exponential function, $\sin \beta x, \cos \beta x$ or sums and products of these functions.

4.5 Undetermined Coefficients - Annihilator Approach

Please read the comments contained in §4.4 of the study guide. Note that the same limitations apply to the annihilator approach.
If you are asked to use the annihilator approach, you must use the method described in this section. Do not confuse the 'operator method', which is described in §4.8 of this guide, with the annihilator approach. If you are asked to use the operator method, then you must use the method described in §4.8.

4.6 Variation of Parameters

Although the method of variation of parameters can in some cases be more tedious than the method of undetermined coefficients, it has some advantages:

- The method is not restricted to sums and products of polynomial functions, exponential functions, sin βx and cos βx.
- It is also applicable to DEs with *non-constant* coefficients.

4.7 Cauchy-Euler Equation

4.8 The Operator Method

You will notice that this method is not mentioned in your prescribed book. However, it is good for all of you to know about its existence. For this reason, the operator method is mentioned in this section just for information purposes.

This method is very useful for finding particular solutions of non-homogeneous DEs. Ultimately, the method boils down to three things: ten rules, infinite series and the calculation of partial fractions. The ten rules allow one to quickly compute particular solutions of DEs without much trouble. These must be applied, sometimes in combination, to reduce a DE to simpler and simpler forms until we are left with a particular solution.

We start by discussing the concept of an **operator,** and then show how to use it to solve DEs.

The functions we use in this course take a number as input and then output a number, such as

$$f : x \longmapsto x^2 + x + 3$$

$$f(1) = 5.$$

An **operator** is similar, except that it takes a **function** as input and then outputs a **function**. There are many examples of operators, but you are already very familiar with one: the **differential operator** D, which takes a (differentiable) function and then outputs its derivative:

$$\begin{aligned} D(\sin x) &= \cos x \\ D(e^r x) &= re^x \\ D(y) &= y'. \end{aligned}$$

From the differential operator D we can make endless combinations, such as

$$D^2 - 3D \qquad D^4 + 4D^2 + 3 \qquad \cdots$$

In fact, every DE can be written in terms of such operators. Take the DE

$$y'' + 3y' - 4y = \sin 2x.$$

We can write this as

$$(D^2 + 3D - 4)y = \sin 2x. \tag{4.4}$$

The great English engineer Oliver Heaviside, who thought up this scheme for solving DEs, suggested that we work with such differential operators **as if** they were simply numbers. Of course they are not numbers, but they behave similarly in many cases. So in equation (4.4), solving for y, we write

$$y = \frac{1}{D^2 + 3D - 4} \sin 2x \tag{4.5}$$

(just as if $D^2 + 3D-4$ is a number).

4.8.1 Partial Fractions

Now our task is to work out what operator $\frac{1}{D^2+3D-4}$ might be. First, let us simplify it using partial fractions: factorizing $D^2 + 3D-4 = (D + 4)(D-1)$, we solve for α and β in the equation

$$\frac{1}{D^2 + 3D-4} = \frac{\alpha}{D+4} + \frac{\beta}{D-1}$$
$$1 = \alpha(D-1) + \beta(D + 4)$$

Substituting $D = 1$, we get $\beta = 1/5$ and for $D = -4$, we get $\alpha = -1/5$, so

$$\frac{1}{D^2 + 3D-4} = -\frac{1}{5}\frac{1}{D-1} + \frac{1}{5}\frac{1}{D+4}$$

We put this back in equation (4.5) to get

$$y = -\frac{1}{5}\frac{1}{D-1} \sin 2x + \frac{1}{5}\frac{1}{D+4} \sin 2x. \tag{4.6}$$

Now we solve

$$y = \frac{1}{D-1} \sin 2x$$
$$(D-1)y = \sin 2x$$
$$y' - y = \sin 2x$$

This is just a simple first-order linear DE, which we solve in the usual way: the integrating factor is $e^{\int -1dx} = e^{-x}$ and so

$$\frac{d}{dx}[e^{-x}y] = e^{-x} \sin 2x$$

$$y = \frac{1}{5}[-e^{-x} \sin 2x - 2e^{-x} \cos 2x].$$

Similarly a solution of $y = \frac{1}{D+4} \sin 2x$ is $y' + 4y = \sin 2x$ giving

$$y = \frac{1}{20}[4e^{4x} \sin 2x - 2e^{4x} \cos 2x].$$

So a solution of equation (4.6) is

$$y = -\frac{1}{25}[-e^{-x}\sin 2x - 2e^{-x}\cos 2x] + \frac{1}{100}[4e^{4x}\sin 2x - 2e^{4x}\cos 2x].$$

There are many ways of generalising this problem. If you are interested, you are welcome to consult references such as the ones below for more details:

- The D operator, Solving DEs using the D operator <http://www.codecogs.com/library/maths/calculus/differential/the-d-operator.php>.
- Find a solution to a linear system using the D operator and method of elimination <http://math.stackexchange.com/questions/864114/find-a-solution-to-a-linear-system-using-the-d-operator-and-method-of-eliminatio>

Chapter 5

Modelling with Higher-Order Differential Equations

This chapter is concerned primarily with one application: the motion of a mass attached to a spring. In the last section of the chapter certain electric circuits are discussed, since the mathematics involved is identical to that of a vibrating spring-mass system.
Study the following section thoroughly:

5.1 Linear Models: Initial-Value Problems

Chapter 6

Series Solutions of Differential Equations

This chapter is concerned with the solution of linear DEs of order two or higher with *non-constant* coefficients. The solutions of these equations cannot usually be expressed in terms of the elementary functions constructed from powers of x, cosines, sines, logarithms and exponentials. Solutions are written as infinite series (Taylor series).
One of the very important uses of these methods is that they show how to **approximate** the solution of a DE, allowing us to compute solutions even though the full, exact solution cannot easily be described. Indeed, it is by using infinite series that computers and calculators evaluate trigonometric, exponential and logarithmic functions.
Study the following sections thoroughly:

6.1 Review of Power Series; Power Series Solutions

When using the power series expansion to solve DEs, we will have to sum two or more series expansions. In order for us to do this, two things are necessary:

1. The sums must have the same power of x.
2. The sums must start with the same value of n.

For example, let us write the following sum of two series as a single series:

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} n a_n x^{n-1} \tag{6.1}$$

We deal with item (1) first: in the second term we must change x^{n-1} to x^n. We substitute $k = n{-}1$ to get

$$\sum_{k=1}^{\infty} (k+1)a_{k+1}x^k = \sum_{n=1}^{\infty} (n+1)a_{n+1}x^n.$$

For item (2), note that the first term can be written as

$$a_0 + \sum_{n=1}^{\infty} a_n x^n.$$

Thus equation (6.1) equals

$$a_0 + \sum_{n=1}^{\infty} a_n x^n + \sum_{n=1}^{\infty} (n+1)a_{n+1}x^n$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n + (n+1)a_{n+1})x^n.$$

6.2 Solutions About Singular Points

Chapter 7

The Laplace Transform

This chapter is concerned with the definitions and properties of an integral known as the Laplace transform and its use in solving certain types of linear higher-order DEs with constant coefficients. In the exam, we will provide you with a table of Laplace transforms similar to the one found at the back of the book.
Study the following sections thoroughly:

7.1 The Definition of the Laplace Transform

VERY IMPORTANT:

$$\mathcal{L}\left(f(t)\cdot g(t)\right)\neq\mathcal{L}(f(t))\mathcal{L}(g(t))$$

You have to use convolution:

$$\mathcal{L}f(t)*g(t)=\mathcal{L}(f(t))\mathcal{L}(g(t))$$

For example, $\mathcal{L}(\sin^2 t)\neq\mathcal{L}(\sin t)\mathcal{L}(\sin t)$.

7.2 Inverse Transforms and Transforms of Derivatives

VERY IMPORTANT:

$$\mathcal{L}^{-1}\left(F(s)\cdot G(s)\right) \neq \mathcal{L}^{-1}(F(s))\mathcal{L}^{-1}(G(s))$$

You have to use convolution:

$$\mathcal{L}^{-1}\left(F(s)\cdot G(s)\right) = \mathcal{L}^{-1}(F(s))*\mathcal{L}^{-1}(G(s))$$

For example,

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)(s+2)(s+4)}\right) \neq \mathcal{L}^{-1}\left(\frac{1}{s-1}\right)\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)\mathcal{L}^{-1}\left(\frac{1}{s+4}\right)$$

7.3 Operational Properties I & II

Chapter 8

This chapter is optional, but it is essential for those of you who want to continue with any specialized Mathematical Sciences programme.

Chapter 9

This chapter is optional, but it is essential for those of you who want to continue with any specialized Mathematical Sciences programme.

Chapter 10

This chapter is optional, but it is essential for those of you who want to continue with any specialized Mathematical Sciences programme.

Chapter 11

Orthogonal Functions and Fourier Series

Study the following sections thoroughly:

11.1 Orthogonal Functions

11.2 Fourier Series

We have already used the fact that a function can be written as an infinite sum of polynomials - this is the Power Series method that you studied in Chapter 6. There are many other ways of writing a given function as an infinite sum of other, simpler functions. One such method was invented by the great French mathematician and Egyptologist Joseph Fourier in the 1820s. It shows us how to write a periodic function as a sum of sine and cosine functions. Using this technique, we are able to solve many difficult partial differential equations, including those describing some of the most fundamental physical problems - the dissipation of heat and the motion of waves - successfully. We work with periodic functions. Remember that a function: $\mathbb{R} \longrightarrow \mathbb{R}$ is **periodic with period p** if

$$f(x+p) = f(x) \quad \text{for all} \quad x \in \mathbb{R}. \tag{11.1}$$

In other words, f repeats itself after an interval of length p. For example, $f(x) = \sin x$ has period 2π.
Most often, the functions are only defined on some interval. The way to interpret such functions is as periodic, with period the length of the interval. Therefore, it is possible to extend such functions to another interval. For example, suppose the following function is of period 1:

$$f(x) = x^2 \quad \text{for} \quad x \in [1,2).$$

It follows that $f(0.5) = f(1.5)$ from equation (11.1).
Here is our definition of a Fourier series (compare this with Definition 11.2.1, which is on page 427 in the 8th edition of Z&W).

Definition 11.2.1. The **Fourier series** of a function f defined on an interval (a, b) is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi}{L} x + b_n \sin \frac{2n\pi}{L} x \right)$$

where $L = b - a$, the period of the function and

$$a_0 = \frac{2}{L} \int_a^b f(x) dx$$
$$a_n = \frac{2}{L} \int_a^b f(x) \cos \frac{2n\pi}{L} x dx$$
$$b_n = \frac{2}{L} \int_a^b f(x) \sin \frac{2n\pi}{L} x dx$$

Note that the book's definition is exactly this one, where $b = p$, $a = -p$ and $L = 2p$. The slightly more general definition given above will help us explain the Fourier Sine and Cosine series discussed in the next section. The a_n and b_n are called the Fourier coefficients of f.
Very important: **to compute the integrals for the Fourier coefficients, you must be able to use integration by parts very well.** If necessary, practise some of these integrals from first-year calculus.

11.3 Fourier Sine and Cosine series

Two important classes of functions are the **even functions** and the **odd functions**. A function f is said to be **even** if $f(x) = f(-x)$. So f is symmetric around the y-axis. For example, as $\cos(x) = \cos(-x)$, cos is even.
A function f is said to be **odd** if $-f(x) = f(-x)$. For example, as $-\sin(x) = \sin(-x)$, sin is odd. Note that for f odd, $-f(0) = f(-0) = f(0)$, so $f(0) = 0$. The labels even

and odd come from the fact that if $n \in \mathbb{N}$ is even, the function $f(x) = x^n$ is even; if n is odd, f is odd.
As far as Fourier series are concerned, if f is even, it can't have any sine terms in it, because these are odd. Likewise, if f is odd, it can't have any cosine terms in it, because these are even.
So if f is an even function, its Fourier series must be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2n\pi}{L} x \right).$$

If f is an odd function, its Fourier series must be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(b_n \sin \frac{2n\pi}{L} x \right).$$

These are called the **Fourier sine series** and **Fourier cosine series** respectively.
How do we compute them? Here, suppose we are given a function f defined on $[0,\ p)$. To get either the Fourier sine or cosine series, it **must** be either odd or even. For this to happen, we must extend its definition to $(-p,\ 0)$.
If we set $f(x) = f(-x)$ for $x \in (-p,\ 0)$, we have made f even. If we compute the Fourier series of this new function, all the coefficients $b_n = 0$ and from Definition 11.2.1, its Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{p} x \right).$$

This is called the Fourier cosine series of f. Note that as we have defined f on $(-p,\ p)$, the period is $2p$.
Similarly, if we set $f(x) = -f(-x)$ for $x \in (-p,\ 0)$, we have made f odd. If we compute the Fourier series of this new function, all the coefficients $a_n = 0$ and from Definition 11.2.1, its Fourier series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(b_n \sin \frac{n\pi}{p} x \right).$$

This is called the Fourier sine series of f. Note that as we have defined f on $(-p,\ p)$, the period is $2p$.
Let's summarise the situation. For a given function f defined on an interval $(0,\ p)$ we can define three series: the Fourier series, the Fourier sine series and the Fourier cosine series.
To obtain each of the three, we extend f from $(0,\ p)$ to a periodic function on all of $\mathbb{R}$ by setting f on $(-p,\ 0)$ in one of three ways:

1. $f(x) = f(x + p)$ for $x \in (-p,\ 0)$ (**Fourier series**)

2. $f(x) = f(-x)$ for $x \in (-p, 0)$ (**Fourier cosine series**)

3. $f(x) = -f(-x)$ for $x \in (-p, 0)$ (**Fourier sine series**)

For actually calculating the sine and cosine series, we use Definition 11.3.1 of Z&W. As before, we have written it in a slightly different form.

Definition 11.3.1. Let f be a function defined on $(0,\ p)$.

1. The **cosine series** of f is given by

$$
\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{p} x \\
a_0 &= \frac{2}{p} \int_0^p f(x) dx \\
a_n &= \frac{2}{p} \int_0^p f(x) \cos \frac{n\pi}{p} x dx
\end{aligned}
$$

2. The **sine series** of f is given by

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{p} x \\
b_n &= \frac{2}{p} \int_0^p f(x) \sin \frac{n\pi}{p} x dx
\end{aligned}
$$

Let us do an example. Suppose $f(x) = x,\ 0 < x < \pi$. We will calculate the Fourier series, the Fourier cosine series and the Fourier sine series of f.

1. **Fourier series.** The period of f is $L = \pi$. From Definition 11.2.1,

$$\begin{aligned}
a_0 &= \frac{2}{\pi}\int_0^{\pi} x dx \\
&= \frac{1}{\pi}(\pi^2) = \pi \\
a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos 2nx dx \\
&= \frac{2}{\pi}\left[\frac{x}{2n}\sin 2nx\Big|_0^{\pi} - \frac{1}{2n}\int_0^{\pi}\sin 2nx dx\right] \\
&= \frac{1}{2n^2\pi}\cos 2nx\Big|_0^{\pi} = 0 \\
b_n &= \frac{2}{\pi}\int_0^{\pi} x\sin 2nx dx \\
&= \frac{2}{\pi}\left[-\frac{x}{2n}\cos 2nx\Big|_0^{\pi} + \frac{1}{2n}\int_0^{\pi}\cos 2nx dx\right] \\
&= \frac{2}{\pi}\left[-\frac{\pi}{2n}\cos 2n\pi + \frac{1}{4n^2}\sin 2nx\Big|_0^{\pi}\right] = -\frac{1}{n}
\end{aligned}$$

Therefore the Fourier series of f is

$$f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty}\frac{1}{n}\sin 2nx$$

2. **Fourier cosine series.** The period of f is $L = 2\pi$. From Definition 11.3.1,

$$\begin{aligned}
a_0 &= \pi \\
a_n &= \frac{2}{\pi}\int_0^{\pi} x\cos nx dx \\
&= \frac{2}{\pi}\left[\frac{x}{n}\sin nx\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin nx dx\right] \\
&= \frac{2}{n^2\pi}\cos nx\Big|_0^{\pi} \\
&= \frac{2}{n^2\pi}(\cos nx - 1) \\
&= \frac{2}{n^2\pi}((-1)^n - 1)
\end{aligned}$$

Therefore the Fourier series of f is

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi}((-1)^n - 1)\cos nx$$

3. **Fourier sine series.** The period of f is $L = 2\pi$. From Definition 11.3.1,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^{\pi} x \sin nx dx \\ &= \frac{2}{\pi}\left[-\frac{x}{n}\cos nx \Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi} \cos nx dx \right] \\ &= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos n\pi + \frac{1}{n^2}\sin nx \Big|_0^{\pi} \right] \\ &= -\frac{2}{n}(-1)^n \end{aligned}$$

Therefore the Fourier sine series of f is

$$f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{2}{n}\sin nx$$

NOTE: The following equations are very important for calculating the Fourier coefficients:

$$\boxed{\begin{gathered} \cos n\pi = (-1)^n \\ \cos 2n\pi = 1 \\ \sin n\pi = 0 \end{gathered}}$$

One last remark about applications of Fourier series: they can help in producing remarkable formulae for π and calculating many infinite series. In the example above we obtained

$$x = f(x) = \frac{\pi}{2} - \sum_{n=1}^{\infty} \frac{1}{n}\sin 2nx.$$

Setting $x = \frac{\pi}{4}$ and noting that

$$\sin\frac{2n\pi}{4} = \sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ 1 & \text{if} \quad n = 4m+1 \text{ for some } m \in \mathbb{N} \\ -1 & \text{if} \quad n = 4m+3 \text{ for some } m \in \mathbb{N}, \end{cases}$$

we obtain

$$\begin{aligned} \frac{\pi}{4} &= \frac{\pi}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \cdots \\ \frac{\pi}{4} &= \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \cdots \end{aligned}$$

Chapter 12

Boundary-value Problems and PDEs

The section §12.1 contains a brief introduction to partial DEs (PDEs) as well as a method of solution known as **separation of variables**. This will be combined with the Fourier series method to solve the heat equation in §12.3.
Study the following sections thoroughly:

12.1 Separable Partial DEs

After reading this section you should be able to:

1. classify linear second-order PDEs as either hyperbolic, parabolic, or elliptic
2. use the method to solve elementary PDEs

12.2 Classical PDEs and Boundary-Value Problems

Read through this section, paying special attention to the heat equation.

12.3 The Heat Equation

The heat equation (in one dimension) is solved by the method of separation of variables and Fourier series. The problems in this section consist of taking the general solution (equation (16), p 467 in the 8th edition of Z&W), and calculating this for specific boundary conditions. So, once you understand the derivation of the general solution of the heat equation, all that remains is to calculate the integral

$$\int_0^L f(x) \sin \frac{n\pi}{L} x dx$$

for different values of L and $f(x)$.