Tutorial Letter 201/2/2015

NUMERICAL METHODS 1

COS2633

Semester 2

Department of Mathematical Sciences

This tutorial letter contains solutions for assignment 01

BAR CODE

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Introduction

By this time you should have received the following tutorial matter. If you have not received all these tutorial matter, please contact the Department of Dispatch at the telephone number given in the inventory you received upon registration.
Tutorial letters:

- COS2633/101/3/2015 General information about the module and the assignments
- COS2633/102/3/2015 Background material
- COS2633/201/2/2015 This letter: Solutions to assignment 1

1 Discussion

1.1 The marking

The marking of this assignment was automated and performed at the assignment division of UNISA. The result published to students is therefore not, in any case, under lecturers control. All the questions were marked.

Next we present a model solution on which the marking was based.

Question 1

In this question, we are looking at the fixed point method to compute the k^{th} root of a given positive number, N. As stated in the problem statement, we need to solve the nonlinear equation $x^k = N$ iteratively. This problem can be seen as solving the equation $f(x) = 0$, where $f(x) = x^k - N$.

(1.1)

This question involves determining the suitability of the following fixed point algorithms for computing the required root.

(a) $g_1(x) = N + x - x^k$

(b) $g_2(x) = 1 + x - \frac{x^k}{N}$

(c) Newton's method (that is, $g_3(x) = x - \frac{f(x)}{f'(x)}$*)*

For each of the functions, we want to determine whether the corresponding fixed point iteration scheme $x_{n+1} = g_i(x_n)$ is (locally) convergent to $\sqrt[3]{N}$. From the theory of fixed point method, [1, pp.56-60], the function g gives a fixed point problem that is equivalent to the equation $f(x) = 0$ if $g(x)$ is defined by

(1) $$g(x) = x - f(x).$$

The fixed point algorithms defined in (a) and (b) above are of the form (1), hence options (1) and (2) of this question are correct. Newton's iterative method to determine the root

r of the equation $f(x) = 0$ is given by $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, thus the algorithm in option (3) of this question is correct, since $f(x) = x^k - N$ and $f'(x) = kx^{k-1}$. Finally, option (4) of this question is also correct since the condition for application of the algorithm in option (3) to find the root of a negative number is to start the iteration with a complex number. ***Thus the correct option for this question is option (5)****.*

(1.2)

This question involves studying the behaviour, convergence and the errors of the fixed point iteration $x_{n+1} = g_1(x_n)$. For a particular case where $N = 7$ and $k = 3$, this scheme approximates $\sqrt[3]{7}$ as follows:

$$x = g_1(x) \iff x = 7 + x - x^3 \iff 0 = 7 - x^3.$$

We have the following convergence condition: If $g_1(x)$ and $g_1'(x)$ are continuous on an interval about a root r of the equation $x = g_1(x)$, and if $|g_1'(x)| < 1$ for all x in the interval, then

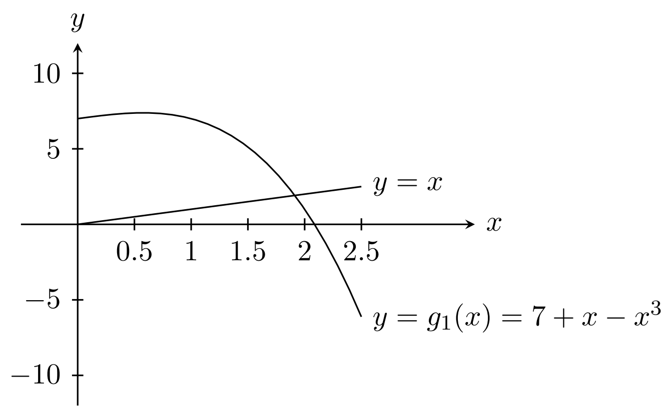
$$x_{n+1} = g_1(x_n), \; n = 0, 1, 2, 3, \ldots$$

will converge to the root r, provided that x_0 is chosen in the interval. Note that this is a sufficient condition only. So, to be able to provide an appropriate answer to this question, we need to do the following:

- *calculate the derivative of the given function (g_1' in this case)*
- *establish the continuity of the function and its derivative (g_1 and g_1' in this case)*
- *find an interval about $\sqrt[3]{7}$ where $|g_1'(x)| < 1$ for all x in the interval.*

Here,

(2) $$r = \sqrt[3]{7} \quad \text{and} \quad g_1'(x) = 1 - 3x^2.$$



Clearly, g_1 and g_1' are continuous functions. Let us now consider the graphs of $y = g_1(x)$ and $y = x$ in Figure 1 (the figure above). The fixed points of the given scheme are the intersections of the line $y = x$ with the curve $y = g_1(x)$. From equations (1) and (2), it follows that $|g_1'(x)| < 1$ for $0 < x < 0.816496580927726$.
From the above and the stated convergence condition we conclude the following: For $x = g_1(x)$, convergence is not guaranteed because the interval of convergence does not contain $r = \sqrt[3]{7} \approx 1.912931182$. ***Thus the correct option for this question is option (3)****.*

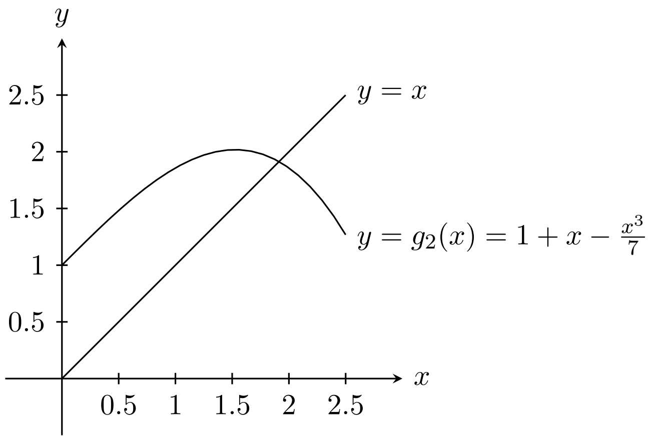
(1.3)

Following the same procedure as above, we have $x_{n+1} = g_2(x_n)$. For a particular case where $N = 7$ and $k = 3$, this scheme approximates $\sqrt[3]{7}$ as follows:

$$x = g_2(x) \iff x = 1 + x - \frac{x^3}{7} \iff 0 = 7 - x^3.$$

The convergence criterion given above for g_1 also applies to g_2. Furthermore,

(3) $$r = \sqrt[3]{7} \quad \text{and} \quad g_2'(x) = 1 - \frac{3}{7}x^2.$$



Clearly, g_2 and g_2' are continuous functions. Let us now consider the graphs of $y = g_2(x)$ and $y = x$ in Figure 2 (the figure above). The fixed points of the given scheme are the intersections of the line $y = x$ with the curve $y = g_2(x)$. From equations (1) and (3), it follows that $|g_2'(x)| < 1$ for $0 < x < 2.160246899469287$.
From the above and the stated convergence condition we conclude the following: For $x = g_2(x)$, convergence is guaranteed if $x_0 \in (0, 2.16)$. ***Thus the correct option for this question is option (3)***.

(1.4)

The procedure to follow in this question is similar to that of the question above. ***Thus the correct option for this question is option (5)***.

(1.5)

For this question, note that the error is the difference between the exact value and the approximate value. Also note that none of Options (1), (2), (3) and (4) involves the exact value of the root of f, and hence none of them is correct. ***Thus the correct option for this question is option (5)***.

Table 1: Newton's method with starting point $x_0 = 1.5$

i	$x_0 = 1.5$
1	10.5371308363960701
2	6.6093246475799132
3	5.4243003271404149
4	5.2921791692299189
5	5.2904937969182884
6	5.2904935225588332

Question 2

To answer this question, we need to do the following:

- *Derive an appropriate mathematical expression of a word problem (for (2.1)).*
- *Apply Newton's method with different starting points [-1, -0.5 and 0.5] (for (2.2)and (2.5)).*
- *Apply the secant method with starting points* $[1.5, -1]$, $[1.5, 7]$, $[1, 3]$, *and* $[3, 5]$ *for the following number of iterations* $[2, 4, 6]$ *(for (2.3)).*
- *Apply Regula Falsi method with starting point $x_0 = 1$ and $x_1 = 2$ as well as $x_0 = -1$ and $x_1 = -2$ for different number of iterations* $[2, 5, 7]$ *(for (2.4) and (2.5)).*

Let the two numbers be denoted by x and y, with $y > x$. Then the word problem gives the two equations

$$y - x = 3 \quad \text{and} \quad x^2 - \sqrt{x+3} - 25.11 = 0.$$

We then find the zeros of the function $x^2 - \sqrt{x+3} - 25.11 = 0$, using Newton, Regula Falsi and the secant methods.

(a) *Applying Newton's method with the corresponding starting points gives the following results in table 1. Newton's method involves calculating the derivative of the function whose zeros are to be determined. The first derivative of the above function in our problem is a rational expression in x. Hence starting the Newton's method with $x_0 = -3$ may lead to a division by zero (singular point). So to remedy this situation we choose our initial starting point to be away from -3. (see Table 1)*

(b) *Applying secant method with the corresponding starting points gives the following results in Table 2.*

(c) *Applying Regula Falsi method with the corresponding starting points ($x_0 = 1$ and $x_1 = 2$) turns out not to be necessary since these points have images of the same sign.*

(2.1)

As mentioned above, if the two numbers are x and y with $x < y$, then $y - x = 3$; then we have $Y = y + \sqrt{y}$ and $X = x + x^2$; the question says

$$(y - x)(Y - X) = -66.33, \iff 3\left(y + \sqrt{y} - x - x^2\right) = -66.33, \iff x^2 - \sqrt{x+3} - 25.11 = 0.$$

Hence the correct option is option (4).

Table 2: Secant method with various starting points

i	$[x_0, x_1] = [1.5, 7]$	$[x_0, x_1] = [1, 3]$	$[x_0, x_1] = [3, 5]$	$[x_0, x_1] = [1.5, -1]$
1	4.505909855546	7.916088876257	5.376213474921	116.537909287939
2	5.171590149859	4.727704923755	5.288039646619	−0.778928684284
3	5.300369060668	5.171530002805	5.290473368007	−0.554223155837
4	5.290378893004	5.297417797645	5.290493527337	−16.432372299423
5	5.290493413320	5.290413085185	5.290493522559	−2.097201522649
6	5.290493522560	5.290493468797	−	−3.261962902632
7	−	5.290493522559	−	−5.595677745833

(2.2)

The first few iterations of Newton's method with starting point $x_0 = 1.5$ are given in Table 1 and from this table it is obvious that options (2), (3) and (4) are not correct. ***The correct option is option (1)***.

(2.3)

The first few iterations of the secant method with various pairs of starting points are given in Table 2 and this table gives us the information we need. ***Hence the correct option is option (4)***.

(2.4)

Note that we are applying the Regula Falsi method on $f(x) = x^2 - \sqrt{x+3} - 25.11$, with starting points $x_0 = 1$ and $x_1 = 2$. A simple substitution shows that $f(x_0) < 0$ and $f(x_1) < 0$, which means they have the same sign, and therefore there is no point moving forward. ***Hence the correct option is option (5)***.

(2.5)

Performing a few iterations in each case leads to an approximation which is less than -3, and $\sqrt{x+3}$ will give the square root of a negative number, which is a complex number. ***Hence the correct option is option (4)***.

Below (in Table 3) is the summary for the answers of all the questions:

Table 3: Summary of Correct Answers

Question:	(1.1)	(1.2)	(1.3)	(1.4)	(1.5)	(2.1)	(2.2)	(2.3)	(2.4)	(2.5)
Answer	(5)	(3)	(5)	(5)	(5)	(4)	(1)	(4)	(5)	(4)

REFERENCES

[1] Richard L. Burden and Douglas J. Faires. *Numerical Analysis*. ninth edition. BROOKS/COLE, CENGAGE learning 2011.