Tutorial Letter 202/2/2015

Quantitative Modelling
DSC1520

Semester 2

Department of Decision Sciences

Important Information:
This tutorial letter contains the solutions for Assignment 02.
Dear Student

By now you should have worked through chapters 1 to 3 of the textbook and completed your first and second assignments. As the assignments contain questions from old examination papers, you are in fact already preparing for the examination. Do as many examples as possible – the more examples you work through, the better you will be able to recognise a problem and solve it.

Remember, help is just a phone call or an e-mail away. Please contact me if you need any help with the third assignment. My contact details are as follows:

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ASSIGNMENT 2: SOLUTIONS

Question 1

To determine the point of intersection of two lines we need to determine a point \((x; y)\) so that the \(x\) and \(y\) values satisfy the equations of both lines. Thus, we need to solve the two equations simultaneously. There are different methods you may use to solve the set of equations.

(a) **Elimination method:**

**Step 1:** Eliminate one variable by adding or subtracting one equation or multiple of an equation from another equation.

The equations of the two lines are

\[
y + 2x = 3 \quad (1)
\]

and

\[
y - x = 2. \quad (2)
\]

We see that subtracting equation (2) from equation (1) will eliminate \(y\).

Now, equation (1) minus equation (2) gives

\[
\begin{align*}
y + 2x &= 3 \\
-(y - x) &= 2
\end{align*}
\]

\[
3x = 1
\]

\[
x = \frac{1}{3}
\]

**Step 2:** Substitute the value \(x\) into any one of the equations solve for \(y\). Substituting the value of \(x = \frac{1}{3}\) into equation (2), for instance, gives

\[
y + 2 \left(\frac{1}{3}\right) = 3
\]

or

\[
y = 2 \frac{1}{3}
\]

The two lines intersect in the point \((x; y) = \left(\frac{1}{3}; \frac{2}{3}\right)\).

(b) **Substitution method:**

**Step 1:** Change one of the equations so that a variable is the subject of the equation.

Say we consider \(y\) in equation (1):

From

\[
y + 2x = 3, \quad (1)
\]

it follows that

\[
y = 3 - 2x. \quad (3)
\]

**Step 2:** Now, substitute the value of \(y\) (equation (3)) into equation (2) and solve for \(x\), that is

\[
(3 - 2x) - x = 2,
\]
from which it follows that

$$x = \frac{1}{3}.$$

**Step 3:** Calculate the value of $y$. Substitute $x = \frac{1}{3}$ into equation (1) or equation (2) and let’s say we choose equation (2), then

$$y - \frac{1}{3} = 2$$

$$y = 2\frac{1}{3}.$$

The two lines intersect in the point $(x; y) = (\frac{1}{3}; 2\frac{1}{3})$.

**Question 2**

We need to solve the following system of equations:

$$x + 2y - z = 5 \quad (1)$$

$$2x - y + z = 2 \quad (2)$$

$$y + z = 2 \quad (3)$$

Add equations (1) and (2):

$$\begin{align*}
    x + 2y - z &= 5 \\
    2x - y + z &= 2 \\
    \hline
    3x + y &= 7 \quad (4)
\end{align*}$$

Subtract equation (3) from equation (2):

$$\begin{align*}
    2x - y + z &= 2 \\
    y + z &= 2 \\
    \hline
    2x - 2y &= 0 \quad (5)
\end{align*}$$

Make $y$ the subject of equation (4):

$$y = 7 - 3x \quad (6)$$

Substitute equation (6) into equation (5):

$$\begin{align*}
    2x - 2(7 - 3x) &= 0 \\
    2x - 14 + 6x &= 0 \\
    8x &= 14 \\
    x &= \frac{7}{4}
\end{align*}$$

Substitute $x = 1,75$ into equation (6) and solve for $y$:

$$\begin{align*}
    y &= 7 - 3 \left(\frac{7}{4}\right) \\
    &= \frac{28-21}{4} \\
    &= \frac{7}{4} \\
    &= 1,75.
\end{align*}$$
Substitute \( y = 1.75 \) into equation (3) and solve for \( z \):

\[
1.75 + z = 2
\]

\[
z = 0.25
\]

Therefore, the solution is \( x = 1.75; \ y = 1.75 \) and \( z = 0.25 \).

**Question 3**

Equilibrium is the price and quantity where the demand and supply functions are equal. This is the point where the lines of the demand and supply function intersect.

We therefore need to determine the value of \( P \) and \( Q \) for which \( P_d = P_s \) or \( Q_d = Q_s \). It is given that \( Q_d = 400 - \frac{1}{2}P \) and \( Q_s = 5 + \frac{1}{8}P \). At equilibrium

\[
400 - \frac{1}{2}P = 5 + \frac{1}{8}P
\]

\[
-\frac{1}{2}P - \frac{1}{8}P = 5 - 400
\]

\[
-\frac{5}{8}P = -395
\]

\[
P = -\frac{8}{5} \times -395
\]

\[
P = 632.
\]

To calculate the quantity at equilibrium, we substitute the value of \( P \) into the demand or supply function and calculate \( Q \). If we use the demand function, we find

\[
Q = 400 - 316
\]

\[
Q = 84.
\]

The equilibrium price is equal to R632 and the quantity is 84.

**Question 4**

The linear equation for the total cost function is

\[
TC(x) = 335x + 1250,
\]

with \( x \) the number of lessons per day.

The total revenue function is given as

\[
TR(x) = 9000 + 25x.
\]

At break-even, cost is equal to revenue, that is

\[
TC(x) = TR(x)
\]

\[
335x + 1250 = 9000 + 25x.
\]

Now we need to solve for \( x \)

\[
335x + 1250 = 9000 + 25x
\]

\[
335x - 25x = 9000 - 1250
\]

\[
310x = 7750
\]

\[
x = \frac{7750}{310}
\]

\[
x = 25.
\]

To break even, 25 lessons must be provided.
**Question 5**

(a) Let \((x_1; y_1) = (1; 20)\) and \((x_2; y_2) = (5; 60)\).

To determine the equation of a line \(y = mx + c\), we need to determine the slope \(m\) and \(y\)-intercept \(c\) of the line.

The slope \(m\) is

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{60 - 20}{5 - 1} = \frac{40}{4} = 10,
\]
giving

\[y = 10x + c.\]

To determine the \(y\)-intercept \(c\) of the line we use one of the given points say \((1; 20)\), to find

\[20 = 10 + c\]

which results in

\[c = 10.\]

The equation of the line is

\[y = 10x + 10.\]

(b) To draw a line we need two points:

Let us choose \(x = 0\), then

\[y = 10(0) + 10 = 10.\]

Therefore, one point is \(= (0; 10)\).

Now we choose \(y = 0\), then

\[0 = 10x + 10\]

or \(x = -1\). Therefore, a second point is \(= (-1; 0)\).

Note that you may use any \(x\) or \(y\) values to calculate the two points. Normally \(x = 0\) and \(y = 0\) are used to simplify the calculations.

Next we plot the two calculated points of the line and draw the line:
Question 6
We need to find the consumer surplus for the demand function

\[ P = 90 - 5Q, \]

when the market price \( P = 20 \).

From the textbook page 128, we know that consumer surplus is calculated as

\[ CS = \text{Amount willing to pay} - \text{Amount actually paid}. \]

This is determined by calculating the area of the triangle \( P_0E_0a \) in the following graph, which is equal to

\[ CS = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times Q_0(a - P_0) \]

with

- \( P_0 \) the market price,
- \( Q_0 \) the number of units demanded at price \( P_0 \), and
- \( a \) the \( y \)-intercept of the demand function \( P = a - bQ \).

In general we can summarise the steps of determining the consumer surplus as follows:

1. Calculate \( Q_0 \) if \( P_0 \) is given.
2. Draw a rough graph of the demand function.
3. Read the value of \( a \) from the demand function – that is the \( y \)-intercept of the demand function.
4. Calculate the area of \( CS = \frac{1}{2} \times Q_0(a - P_0) \).

**Step 1:**
First we need to determine \( Q \) from the demand function

\[
P = 90 - 5Q
\]

if \( P = 20 \). That is

\[
20 = 90 - 5Q
\]
giving

\[
Q = 14.
\]

**Step 2:**
Next we draw the demand function by using the point \((14; 20)\) found before and \((0; a) = (0; 90)\).

The value of \( a \) is found by substituting \( Q = 0 \) into the demand function, that is

\[
P = 90 - b0 = 90.
\]

**Step 3:**
The consumer surplus is the area of the shaded triangle in the sketch, that is

\[
CS = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times 14 \times (90 - 20) = 490.
\]

The consumer surplus is equal to 490 if the price \( P = 20 \).

**Question 7**

**Step 1:**
To graph a linear inequality we first change the inequality sign (\( \geq \) or \( \leq \) or \( > \) or \( < \)) to an equal sign (=) and draw the graph of the line. We need two points to draw a line and for the sake of simplicity, we choose the \( x \)-axis intercept \((x_1; 0)\) and \( y \)-axis intercept \((0; x_2)\). Calculate these points, plot them and draw the line. See the table below for the calculations.
Step 2:
Determine the feasible region for each inequality by substituting a point on either side of this line into the equation of the inequality. The inequality region is the area where the selected point makes the inequality true. The calculations are shown in the last column of the table:

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Step 1 x-axis intercept Point (0 ; (x_2))</th>
<th>x-axis intercept Point ((x_1) ; 0) if (x_2 = 0)</th>
<th>Step 2 Inequality region</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1) -x_1 + x_2 \leq 3)</td>
<td>(0 + x_2 = 3) (x_2 = 3) Point: (0; 3)</td>
<td>(-x_1 + 0 = 3) (x_1 = -3) Point: (-3; 0)</td>
<td>Select points (0; 0) below the line. (0 + 0 \leq 3 \rightarrow) True Area below the line is true.</td>
</tr>
<tr>
<td>((2) x_1 + x_2 \geq 5)</td>
<td>(0 + x_2 = 5) (x_2 = 5) Point (0; 5)</td>
<td>(x_1 \leq 3)</td>
<td>Select points (0; 0) below the line. (0 + 0 \geq 5 \rightarrow) False Area below the line is false.</td>
</tr>
<tr>
<td>((3) x_1 \leq 3)</td>
<td></td>
<td>(x_1 \leq 3)</td>
<td>Select points (0; 0) to the left of the line. (0 \leq 3 \rightarrow) True Area to the left the line is true.</td>
</tr>
<tr>
<td>(x_1, x_2 \geq 0)</td>
<td></td>
<td></td>
<td>Area above the (x_1)-axis and to the right of the (x_2)-axis is true.</td>
</tr>
</tbody>
</table>

Step 3:
The feasible region (grey shaded area in the figure below) is where all the inequalities are true simultaneously.
Question 8

Step 1:
The corner points of the feasible region are the points A, B, C.

Point A:
The point where line (2) cuts the \( y \)-axis.
Point A is the point (0; 10).

Point B:
This is the point where lines (1) and (2) intersect, that is where
\[
4x + 2y = 20 \quad \text{and} \quad 2x + 6y = 30
\]
intersect. Writing these with \( y \) as the subject, gives
\[
y = 10 - 2x \quad \text{and} \quad y = 5 - \frac{1}{3}x.
\]
From this follows that
\[
10 - 2x = 5 - \frac{1}{3}x \\
-2x + \frac{1}{3}x = 5 - 10 \\
-\frac{5}{3}x = -5 \\
x = 3.
\]
Substituting the value of \( x = 3 \) into equation (2) gives
\[
4(3) + 2y = 20 \\
y = \frac{20-12}{4} \\
= 4.
\]
Point B is the point (3; 4).
Point C is the point where lines (3) and (1) intersect.

\[ y = 2 \quad \text{and} \quad 2x + 6y = 30 \]

Substituting the value of \( y = 2 \) into equation (1) gives

\[ 2x + 6(2) = 30 \]
\[ x = \frac{18}{2} \]
\[ = 9. \]

Point C is the point \((2; 9)\).

<table>
<thead>
<tr>
<th>Corner points of feasible region</th>
<th>Value of ( Z = 18x + 12y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: ( x = 0; \ y = 10 )</td>
<td>( Z = 18(0) + 12(10) = 120 )</td>
</tr>
<tr>
<td>B: ( x = 3; \ y = 4 )</td>
<td>( Z = 18(3) + 12(4) = 102 ) ← Minimum</td>
</tr>
<tr>
<td>C: ( x = 9; \ y = 2 )</td>
<td>( Z = 18(9) + 12(2) = 186 )</td>
</tr>
</tbody>
</table>

Minimum of \( Z \) is at point B where \( x = 3, y = 4 \) with \( Z = 102 \).

**Question 9**

To help us with the formulation, we summarise the information given in the following table:

<table>
<thead>
<tr>
<th>Items with restrictions</th>
<th>Containers of Zombie mixed ((x))</th>
<th>Containers of Skyjack mixed ((y))</th>
<th>Capacity/ requirements</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vodka</td>
<td>3 litres</td>
<td>5 litres</td>
<td>1500 litres</td>
</tr>
<tr>
<td>Vermouth</td>
<td>6 litres</td>
<td>3 litres</td>
<td>1500 litres</td>
</tr>
<tr>
<td>Ginger ale</td>
<td>1 litre</td>
<td>2 litres</td>
<td>400 litres</td>
</tr>
<tr>
<td>Number of containers</td>
<td></td>
<td></td>
<td>Never negative</td>
</tr>
</tbody>
</table>

The variables are defined as
\[ x = \text{number of containers of Zombie mixed} \]
\[ y = \text{containers of Skyjack mixed}. \]

Using the table, the constraints are as follows:

\[ 3x + 5y \leq 1500 \]
\[ 6x + 3y \leq 1500 \]
\[ x + 2y \geq 400 \]
\[ x, y \geq 0. \]
Question 10

Let the price of the T-shirt before the mark-up be \( x \).

The current price after the mark-up of 20\% is given as R36.

This means the current price of the T-shirt is equal to the original price of the T-shirt + 20\% mark-up on original price, that is

\[
36 = x + (20\% \text{ of } x) \\
= x + 0.2x \\
= 1.2x.
\]

which results in

\[
x = \frac{36}{1.2} = 30.
\]

The price of the T-shirt before the mark-up is R30.00.