Tutorial letter 203/1/2018

Theoretical Computer Science 1

COS1501

Semester 1

School of Computing

Discussion of assignment 03

Dear Student,

By this time you should have received the tutorial matter listed below. These can be downloaded from *my*Unisa.

- COSALLP/301/4/2018 General information regarding the School of Computing including lecturers' information;
- COS1501/101/3/2018 General information about the module and the assignments;
- COS1501/201/1/2018 Solutions to the first assignment, and **examination information**;
- COS1501/202/1/2018 Solutions to the second assignment;
- COS1501/203/1/2018 This tutorial letter;
- MO001/4/2018 Learning units, example exam paper and other important information;
- COS1501/102/3/2018 Solutions to self-assessment questions in assignments 02 and 03, example assignments & solutions, example examination paper & solutions, and extra questions & solutions.

Everything of the best with the exam!
Regards
COS1501 team

Discussion of assignment 03 semester 1.

Suppose U = {1, 2, 3, 4, 5, a, b, c} is a universal set with the subset A = {a, b, c, 1, 2, 3, 4}. Answer questions 1 and 2 by using the given sets U and A.

Question 1

Which one of the following relations on A is NOT functional?

1. {(1, 3), (b, 3), (1, 4), (b, 2), (c, 2)}
2. {(a, c), (b, c), (c, b), (1, 3), (2, 3), (3, a)}
3. {(a, a), (c, c), (2, 2), (3, 3), (4, 4)}
4. {(a, c), (b, c), (1, 3), (3, 3)}

Answer: Alternative 1

Discussion

First we look at the definition of functionality:

Suppose R $\subseteq$ B × C is a binary relation from a set B to a set C. We may call R functional if the elements of B that appear as first co-ordinates of ordered pairs in R do not appear in more than one ordered pair of R.

We consider the relations provided in the different alternatives:

1. Let L = {(**1**, 3), (**b**, 3), (**1**, 4), (**b**, 2), (c, 2)} (say). L is not functional since the elements 1 and b appear more than once as first co-ordinates in ordered pairs of L.

2. Let M = {(a, c), (b, c), (c, b), (1, 3), (2, 3), (3, a)} (say).
   M is a relation on A = {a, b, c, 1, 2, 3, 4}: {a, b, c, 1, 2, 3} = dom(M) $\subseteq$ A and ran(M) = {a, b, c, 3} $\subseteq$ A. Each first co-ordinate appears only once as first co-ordinate thus M is functional.

3. Let N = {(a, a), (c, c), (2, 2), (3, 3), (4, 4)} (say).
   N is a relation on A = {a, b, c, 1, 2, 3, 4}: {a, c, 2, 3, 4} = dom(N) $\subseteq$ A and ran(N) = {a, c, 2, 3, 4} $\subseteq$ A. Each first co-ordinate appears only once as first co-ordinate thus N is functional.

4. Let S = {(a, c), (b, c), (1, 3), (3, 3)} (say).
   S is a relation on A = {a, b, c, 1, 2, 3, 4}: {a, b, 1, 3} = dom(S) $\subseteq$ A and ran(S) = {c, 3} $\subseteq$ A. Each first co-ordinate appears only once as first co-ordinate thus S is functional.

From the arguments provided we can deduce that alternative 1 should be selected.

Refer to study guide, p 98.

Question 2

Which one of the following alternatives represents a **surjective function** from **U** to **A**?

1. {(1, 4), (2, b), (3, 3), (4, 3), (5, a), (a, c), (b, 1), (c, b)}
2. {(a, 1), (b, 2), (c, a), (1, 4), (2, b), (3, 3), (4, c)}
3. {(1, a), (2, c), (3, b), (4, 1), (a, c), (b, 2), (c, 3)}
4. {(1, a), (2, b), (3, 4), (4, 3), (5, c), (a, a), (b, 1), (c, 2)}

Answer: Alternative 4

Discussion

We look at the definitions of a function and of surjectivity:

Suppose R $\subseteq$ B × C is a binary relation from a set B to a set C. We may call R a **function** from B to C if every element of B appears exactly once as the first co-ordinate of an ordered pair in R (i.e. f is functional), and the domain of R is exactly the set B, ie dom(R) = B.

A function R from B to C is **surjective** iff ran(R) = C.

We have U = {1, 2, 3, 4, 5, a, b, c} and A = {a, b, c, 1, 2, 3, 4}.

We consider the relations provided in the different alternatives:

1. Let L = {(1, 4), (2, b), (3, 3), (4, 3), (5, a), (a, c), (b, 1), (c, b)} (say). It is the case that 2 ∈ A but 2 is not a second co-ordinate in any ordered pair of L thus ran(L) ≠ A. Thus L is not surjective.

2. Let M = {(a, 1), (b, 2), (c, a), (1, 4), (2, b), (3, 3), (4, c)} (say). It is the case that 5 ∈ U but 5 is not a first co-ordinate in any element of M. We have dom(M) = {1, 2, 3, 4, a, b, c} ≠ U. Thus M is not a function from U to A and thus cannot be a surjective function.

3. Let N = {(1, a), (2, c), (3, b), (4, 1), (a, c), (b, 2), (c, 3)} (say). For the same reason as provided in alternative 2, N is not a function. Thus N is not a surjective function.

4. Let S = {(1, a), (2, b), (3, 4), (4, 3), (5, c), (a, a), (b, 1), (c, 2)} (say). S is a function since it is functional and dom(S) = U, and furthermore, ran(S) = A thus S is a surjective function.

From the arguments provided we can deduce that alternative 4 should be selected.

Refer to study guide, p 105.

Question 3

Let G and L be relations on A = {1, 2, 3, 4} with

G = {(1, 2), (2, 3), (4, 3)} and L = {(2, 2), (1, 3), (3, 4)}.

Which one of the following alternatives represents the relation L ○ G = G; L?

1. {(2, 3), (3, 3)}
2. {(1, 2), (2, 4), (4, 4)}
3. {(1, 2), (2, 1), (3, 3), (4, 4)}
4. {(2, 4), (4, 4)}

Answer: Alternative 2

Discussion
We first look at a definition of a composition relation:
Given relation P from A to B and R from B to C, the composition of P followed by R (R ○ P or P; R) is the relation from A to C defined by

R ○ P = P; R = {(a, c) | there is some b ∈ B such that (a, b) ∈ P and (b, c) ∈ R}.

G = {(1, 2), (2, 3), (4, 3)} and L = {(2, 2), (1, 3), (3, 4)} are defined on A = {1, 2, 3, 4}.

To determine G; L we start with the pair (1, **2**) of G, and then we look for a pair in L that has as first co-ordinate an **2**, and then see where it takes us.

Link (1, **2**) of G with (**2**, 2) of L, then (1, 2) ∈ G; L,
link (2, **3**) of G with (**3**, 4) of L, then (2, 4) ∈ G; L, and
link (4, **3**) of G with (**3**, 4) of L, then (4, 4) ∈ G; L.

No other pairs can be linked, so G; L = {(1, 2), (2, 4), (4, 4)}. Thus alternative 2 should be selected.

Refer to study guide, pp 79, 108, 109.

Let g be a function from $\mathbb{Z}^+$ (the set of positive integers) to $\mathbb{Q}$ (the set of rational numbers) defined by

$(x, y) \in g$ iff $y = 4x - \frac{3}{7}$ ($g \subseteq \mathbb{Z}^+ \times \mathbb{Q}$) and

let f be a function on $\mathbb{Z}^+$ defined by

$(x, y) \in f$ iff $y = 5x^2 + 2x - 3$ ($f \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$).

Answer questions 4 to 7 by using the given functions g and f.
Hint: Draw graphs of the f and g before answering the questions. Keep in mind that $g \subseteq \mathbb{Z}^+ \times \mathbb{Q}$ and $f \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$.

Question 4
Consider the function f on $\mathbb{Z}^+$. For which values of x is it the case that $5x^2 + 2x - 3 > 0$?
Hint: Solve $5x^2 + 2x - 3 > 0$ and keep in mind that $x \in \mathbb{Z}^+$.

1. $x < 5, x \in \mathbb{Z}^+$
2. $\frac{3}{5} < x < 1, x \in \mathbb{Z}^+$
3. $x \geq 1, x \in \mathbb{Z}^+$
4. $x < 1, x \in \mathbb{Z}^+$

Answer: Alternative 3

We solve for x:
$5x^2 + 2x - 3 > 0$
ie $(5x - 3)(x + 1) > 0$
ie $[(5x - 3) > 0 \text{ and } (x + 1) > 0]$ OR $[(5x - 3) < 0 \text{ and } (x + 1) < 0]$
ie $[x > \frac{3}{5} \text{ and } x > -1]$ OR $[x < \frac{3}{5} \text{ and } x < -1]$
ie $x > \frac{3}{5}$ OR $x < -1$ (which is impossible)

It is not possible for x to be less than –1 since f is defined on $\mathbb{Z}^+$. It is also stated that $x > \frac{3}{5}$, but x cannot be a rational number less than 1 since f is defined on $\mathbb{Z}^+$ so we have that $x \geq 1, x \in \mathbb{Z}^+$ (If $x > \frac{3}{5}$, then it is also true that $x \geq 1$). This is the correct statement provided in alternative 3.

Question 5

Which one of the following is an ordered pair belonging to f?

1. (–1, 0)
2. (2, 21)
3. (1, 5)
4. (3, 44)

Answer: Alternative 2

Discussion

The first and second co-ordinates of elements (x, y) of f are elements of $\mathbb{Z}^+$*.*

We consider the ordered pairs provided in the different alternatives:

1. $(x, y) \in f$ iff $y = 5x^2 + 2x - 3$. Is $(-1, 0) \in f$?

   No, f is defined on $\mathbb{Z}^+$ but neither –1 nor 0 are elements of $\mathbb{Z}^+$.

   Thus $(-1, 0) \notin f$.

2. Is $(2, 21) \in f$?

   Let x = 2 then

   $y = 5(2)^2 + 2(2) - 3$

   $= 20 + 4 - 3$

   $= 21$

   Thus $(2, 21) \in f$.

3. Is $(1, 5) \in f$?

   Let x = 1 then

   $y = 5(1)^2 + 2(1) - 3$

   $= 4$

   $(1, 4) \in f$ but $(1, 5) \notin f$.

4. Is $(3, 44) \in f$?

   Let x = 3 then

   $y = 5(3)^2 + 2(3) - 3$

   $= 48$

   $(3, 48) \in f$ but $(3, 44) \notin f$.

From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, pp 71, 72.

Question 6

Which one of the following alternatives represents the **image of x under g ○ f** (ie g ○ f(x)))?

1. $20x^2 + 8x - 12\frac{3}{7}$
2. $80x^2 + 4\frac{4}{7}x - \frac{180}{49}$
3. $20x^2 + 8x + 3\frac{3}{7}$
4. $80x^2 + 4\frac{4}{7}x - 3$

Answer: Alternative 1

Discussion

Given the functions f: A → B and g: B → C the composite function

g ○ f: A → C is defined by g ○ f(x) = g(f(x)).

g ○ f(x)

= g(f(x))

= $4(f(x)) - \frac{3}{7}$

= $4(5x^2 + 2x - 3) - \frac{3}{7}$

= $20x^2 + 8x - 12 - \frac{3}{7}$

= $20x^2 + 8x - 12\frac{3}{7}$

Note that we define the **function g ○ f** by g ○ f: $\mathbb{Z}^+ \to \mathbb{Q}$ defined by g ○ f(x) = $20x^2 + 8x - 12\frac{3}{7}$.

From the above we can deduce that alternative 1 should be selected.

Refer to study guide, p 110.

Question 7

Which one of the following statements regarding the function g is TRUE?

(Remember, g $\subseteq \mathbb{Z}^+ \times \mathbb{Q}$.)

1. g can be presented as a **straight line** graph.
2. g is injective.
3. g is surjective.
4. g is bijective.

Answer: Alternative 2

We consider the statements provided in the different alternatives:

1. g is not defined on the set of real numbers thus g **cannot** be depicted as a straight **line** graph. Only positive integers can be present in the domain of g, ie dom(g) = $\mathbb{Z}^+$. It is the case that

ordered pairs such as $(1, 3\frac{4}{7}), (2, 7\frac{4}{7}), (3, 11\frac{4}{7}),\ldots$ belong to g and these pairs can be presented as **dots** in a graph.

2. We prove that g is indeed injective:

   Assume g(u) = g(v) then
   $4u - \frac{3}{7} = 4v - \frac{3}{7}$
   ie 4u = 4v
   ie u = v

3. The function g is NOT surjective since ran(g) ≠ $\mathbb{Q}$. A counterexample provides a value $y \in \mathbb{Q}$ for which there is **no** element x such that $x = (y + \frac{3}{7})/4$ and $x \in \mathbb{Z}^+$.
   (If $y = 4x - \frac{3}{7}$ then $4x = y + \frac{3}{7}$, ie $x = (y + \frac{3}{7})/4$.)

   Let $y = \frac{4}{7}$ then we determine whether a positive integer x can live together with this y in an ordered pair: $(x, \frac{4}{7})$.

   $x = (y + \frac{3}{7})/4$
   $= (\frac{4}{7} + \frac{3}{7})/4$
   $= (\frac{7}{7})/4$
   = ¼ which is not a positive integer.

   We see that for $y = \frac{4}{7}$ there is **no** positive integer x such that $x = (y + \frac{3}{7})/4$.

   Thus $y = \frac{4}{7} \notin$ ran(g), therefore ran(g) ≠ $\mathbb{Q}$.
   Since ran(g) ≠ $\mathbb{Q}$ we may conclude that g is not surjective.

4. In alternative 3 we proved that g is not surjective thus g is not bijective since it is not injective **and** surjective.

From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, pp 98, 105, 106, 112.

Let A = {□, ◊, ☼, ⌂} and let # be a binary operation from A × A to A presented by the following table:

#	□	◊	☼	⌂
□	□	◊	☼	⌂
◊	◊	□	◊	□
☼	☼	◊	☼	⌂
⌂	⌂	□	⌂	⌂

Answer questions 8 and 9 by referring to the table for #.

Question 8

Which one of the following statements pertaining to the binary operation # is TRUE?

1. ☼ is the identity element for **#.**
2. From the table it can be observed that # is commutative.
3. # is associative.
4. (⌂ # ◊) # ☼ = ⌂ # (◊ # ☼)

Answer: Alternative 2

We consider the statements provided in the different alternatives:

1. *Definition of an identity element of a binary operation:*
   An element e of X is an identity element in respect of the binary operation
   ** : X × X → X iff e * **x** = **x** * e = **x** for all x ∈ X. (Note that the output is **x**.)*

   Is it possible to identify an element *e* in A such that *e* # x = x # *e* = x for all x ∈ A?

   Yes, e = □ is such an element of A:

	e # x	= x # e	= x
x = □:	□ # □	= □ # □	= □,
x = ◊:	□ # ◊	= ◊ # □	= ◊,
x = ☼:	□ # ☼	= ☼ # □	= ☼, and
x = ⌂:	□ # ⌂	= ⌂ # □	= ⌂.

   So □ acts as an identity element for #.

   The highlighted row and column in the table confirm that □ is the identity element for #. For an explanatory example, refer to study guide, p 121.

To **prove** that ☼ is **not** an identity element for #, we provide a counterexample:

Let e = ☼ and **x** = □. Is it the case that e # **x** = **x** # e = **x**?
e # **x** = **x** # e becomes
☼ # □ = □ # ☼ = ☼, but if ☼ was the identity element, the output should have been
x = □ which is not the case. Thus ☼ is **not** the identity element.

2. From the table it can be observed that # is commutative since there is symmetry about the diagonal from the top left to the bottom right corners of the table. In a formal proof it can be shown that # is indeed commutative by proving that x # y = y # x for all x, y ∈ A.

3. *Definition of an associative binary operation:*
*The binary operation * : $X \times X \rightarrow X$ is associative iff (x * y) * z = x * (y * z) for all x, y, z ∈ X.*

is **not** an associative binary operation. A counterexample is provided in alternative 4.

4. We determine (⌂ # ◊) # ☼ and ⌂ # (◊ # ☼) then compare the results:
(⌂ # ◊) # ☼
= □ **#** ☼
= ☼

⌂ # (◊ # ☼)
= ⌂ # ◊
= □

We see that [(⌂ # ◊) # ☼] ≠ [⌂ # (◊ # ☼)] since ☼ ≠ □.
This is a counterexample which proves that # is not associative.

From the above we can deduce that alternative 2 should be selected.

Refer to study guide, pp 116-122.

Question 9
can be written in list notation. Which one of the following ordered pairs is an element of the list notation set representing #?

1. ((□, ◊), ⌂)
2. ((⌂, ☼), ◊)
3. ((☼, ◊), ◊)
4. ((⌂, ◊), ◊)

Answer: Alternative 3

We consider the ordered pairs provided in the different alternatives:

1. From the table □ # ◊ = ◊
   thus ((□, ◊), ◊) ∈ # but ((□, ◊), ⌂) ∉ #.

2. ⌂ # ☼ = ⌂
   thus ((⌂, ☼), ⌂) ∈ # but ((⌂, ☼), ◊) ∉ #.

3. From the table ☼ # ◊ = ◊
   thus ((☼, ◊), ◊) ∈ #.

4. From the table ⌂ # ◊ = □
   thus ((⌂, ◊), □) ∈ # but ((⌂, ◊), ◊) ∉ #.

From the arguments provided we can deduce that alternative 3 should be selected.

Refer to study guide, pp 118, 119.

Question 10

Perform the following matrix multiplication operation:

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 4 & 1 \\ -1 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -3 & 0 & 0 \\ -1 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer: Alternative 3

Discussion
A 3 $\times$ 3 matrix multiplied by a 3 $\times$ 3 matrix gives a 3 $\times$ 3 matrix.

We determine a_{ij}:
$a_{11} = 2(-3) + 1(-1) + 0(1) = -7$
$a_{12} = 2(0) + 1(0) + 0(1) = 0$
$a_{13} = 2(0) + 1(-1) + 0(1) = -1$
$a_{21} = 3(-3) + 4(-1) + 1(1) = -12$
$a_{22} = 3(0) + 4(0) + 1(1) = 1$
$a_{23} = 3(0) + 4(-1) + 1(1) = -3$
$a_{31} = -1(-3) + 0(-1) + -1(1) = 2$
$a_{32} = -1(0) + 0(0) + -1(1) = -1$
$a_{33} = -1(0) + 0(-1) + -1(1) = -1$

Thus the answer to the multiplication of the given matrices is

$$\begin{bmatrix} -7 & 0 & -1 \\ -12 & 1 & -3 \\ 2 & -1 & -1 \end{bmatrix}$$

From the above we can deduce that alternative 3 should be selected.

Refer to study guide, pp 131, 132.

Question 11

Consider the truth table for the connective '↔' with two simple declarative statements p and q.

p	**q**	**p ↔ q**
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**T**

Which one of the given alternatives represents '↔' as a binary operation on the set of truth values {T, F}? (b) Does the operation '↔' have an identity element?

Answer: Alternative 4

(a) Discussion

We see that in the table for the connective '↔' with two simple declarative statements p and q, when p and q are true or when p and q are false that p ↔ q is true. In the table below we see that T ↔ T = T and F ↔ F = T. When p or q is false, then p ↔ q is false. In the table below we see that T ↔ F = F and F ↔ T = F. (We can write ↔ in list notation: { ((T, T), T), ((T, F), F), ((F, T), F), ((F, F), T) }.)

So we can present the binary operation ↔ in the following table:

↔	**T**	**F**
T	T	F
F	F	T

(b) The binary operation '↔' has an identity element, namely **T**:
We have T ↔ **T** = T and **T** ↔ T = T; **T** ↔ F = F and F ↔ **T** = F.

From the above we can deduce that alternative 4 should be selected.

Refer to study guide, pp 136 – 146.

Question 12

Let p, q and r be simple declarative statements. Which alternative provides the truth values for the biconditional '↔' of the compound statement provided in the given table?

Hint: Determine the truth values of $p \rightarrow r$, $q \vee r$, $(p \rightarrow r) \wedge (q \vee r)$, $q \rightarrow p$, $\neg(q \rightarrow p)$ *and* $\neg(q \rightarrow p) \wedge r$ *in separate columns before determining the truth values of*

$[(p \rightarrow r) \wedge (q \vee r)] \leftrightarrow [\neg(q \rightarrow p) \wedge r]$. **p**	**q**	**r**	$[(p \rightarrow r) \wedge (q \vee r)] \leftrightarrow [\neg(q \rightarrow p) \wedge r]$
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

1.

↔
F
T
F
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T

2.

↔
F
T
F
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T
F
T

3.

↔
T
F
F
F
T
F
F
F

4.

↔
F
F
T
F
T
F
F
T

Answer: Alternative 1

Discussion
We apply the **definitions** of the logical connectives which are discussed and also provided in **truth tables** in the study guide. Refer to these to determine the truth values of the statement provided in the question.

We look at the truth values highlighted in one row of the table below:
With reference to the statement $[(p \to r) \wedge (q \vee r)] \leftrightarrow [\neg(q \to p) \wedge r]$ we first want to determine the truth value of $p \to r$. If **p** is T and **r** is F it means that the hypothesis **p** is true and the conclusion is **r** false, thus the conditional statement 'if p then r' ie $p \to r$ is false. Next determine the truth value of $q \vee r$. If **q** is F and **r** is F it means that the disjunction of q or r is false. Now $(p \to r)$ and $(q \vee r)$ are false, so the conjunction $(p \to r) \wedge (q \vee r)$ is false. And so we can go on to determine the truth values of all the expressions. (Refer to the study guide for the tables of all the logical connectives.)

We compile a truth table for the given expression:

p	**q**	**r**	**$p \to r$**	**$q \vee r$**	**$(p \to r) \wedge (q \vee r)$**	**$\leftrightarrow$**	**$q \to p$**	**$\neg(q \to p)$**	**$\neg(q \to p) \wedge r$**
T	T	T	T	T	T	**F**	T	F	F
T	T	F	F	T	F	**T**	T	F	F
T	F	T	T	T	T	**F**	T	F	F
T	F	F	F	F	F	**T**	T	F	F
F	T	T	T	T	T	**T**	F	T	T
F	T	F	T	T	T	**F**	F	T	F
F	F	T	T	T	T	**F**	T	F	F
F	F	F	T	F	F	**T**	T	F	F

(Considering the final column containing Ts and Fs, it is clear that the expression is neither a tautology nor a contradiction.)

From the above table we can deduce that alternative 1 should be selected.

Refer to study guide, pp 136 – 146.

Question 13

Consider the following quantified statement:

$$\forall x \in \mathbb{Z}, [(x^2 \geq 0) \vee (x^2 + 2x - 8 > 0)]$$

Which one of the alternatives provides a true statement regarding the given statement or its negation?

1. The negation $\exists x \in \mathbb{Z}, [(x^2 < 0) \vee (x^2 + 2x - 8 \leq 0)]$ is not true.
2. $x = -3$ would be a counterexample to prove that the negation is not true.
3. $x = -6$ would be a counterexample to prove that the statement is not true.
4. The negation $\exists x \in \mathbb{Z}, [(x^2 < 0) \wedge (x^2 + 2x - 8 \leq 0)]$ is true.

Answer: Alternative 2

Discussion

Firstly we will derive the negation of the given statement step by step:

$\neg[\forall x \in \mathbb{Z}, [(x^2 \geq 0) \vee (x^2 + 2x - 8 > 0)]]$ (always write down this step)

$\equiv \exists x \in \mathbb{Z}, \neg[(x^2 \geq 0) \vee (x^2 + 2x - 8 > 0)]$

$\equiv \exists x \in \mathbb{Z}, \neg(x^2 \geq 0) \wedge \neg(x^2 + 2x - 8 > 0)$ (de Morgan's law)

$\equiv \exists x \in \mathbb{Z}, (x^2 \ngeq 0) \wedge (x^2 + 2x - 8 \ngtr 0)$

$\equiv \exists x \in \mathbb{Z}, (x^2 < 0) \wedge (x^2 + 2x - 8 \leq 0)$

Let us look at the statement first:

$$\forall x \in \mathbb{Z}, [(x^2 \geq 0) \vee (x^2 + 2x - 8 > 0)]$$

This statement states that for all integers x, it is true that $(x^2 \geq 0)$ **or** $(x^2 + 2x - 8 > 0)$. This means that at least $(x^2 \geq 0)$ **or** $(x^2 + 2x - 8 > 0)$ must be true for the statement to be true. Both may also be true. (Refer to the truth table for the disjunction '∨'.)

We substitute a few integers for x in both ($x^2 \geq 0$) and ($x^2 + 2x - 8 > 0$) and see where that takes us:

x	$x^2 \geq 0$	$x^2 + 2x - 8 > 0$
-1	$(-1)^2 \geq 0$ which is true	$(-1)^2 + 2(-1) - 8 > 0$ which is not true
-2	$(-2)^2 \geq 0$ which is true	$(-2)^2 + 2(-2) - 8 > 0$ which is not true
3	$3^2 \geq 0$ which is true	$(3)^2 + 2(3) - 8 > 0$ which is true
4	$4^2 \geq 0$ which is true	$(4)^2 + 2(4) - 8 > 0$ which is true
	etc…	etc…

We know that $x^2 \geq 0$ is true for all integers x. Thus ($x^2 \geq 0$) $\vee$ ($x^2 + 2x - 8 > 0$) is true for all integers x.

What about the negation statement?

$$\exists x \in \mathbb{Z},\ (x^2 < 0) \wedge (x^2 + 2x - 8 \leq 0)$$

The negation statement states that there exists an integer x such that **both** ($x^2 < 0$) **and** ($x^2 + 2x - 8 \leq 0$) are true. But we know that there is no integer x such that $x^2 < 0$, thus the negation statement is not true. Any integer could play a role in a counterexample to prove that the negation statement is not true, thus $x = -3$ can play this role.

From the discussion above, alternative 2 should be selected.

Refer to study guide, pp 152-158.

Question 14

Consider the following proposition:

For any predicates P(x) and Q(x) over a domain D, the negation of the statement

$$\exists x \in D,\ P(x) \wedge Q(x)$$

is the statement

$$\forall x \in D,\ P(x) \rightarrow \neg Q(x).$$

We can use this truth to write the negation of the following statement:

“There exist integers a and d such that a and d are negative and a/d = 1 + d/a.”

Which one of the alternatives provides the negation of this statement?

1. There exist integers a and d such that a and d are positive and a/d = 1 + d/a.
2. For all integers a and d, if a and d are positive then a/d ≠ 1 + d/a.

3. For all integers a and d, if a and d are negative then $a/d \neq 1 + d/a$.
4. For all integers a and d, a and d are positive and $a/d \neq 1 + d/a$.

Answer: Alternative 3

Discussion

We provide the following substitutions:

Let D be a set such that a, d $\in$ D (a and d are the only elements of D); and

let P(x) be the predicate 'a and d are negative'; and

let Q(x) be the predicate 'a/d = 1 + d/a'.

The statement 'There exist integers a and d such that a and d are negative and a/d = 1 + d/a.' can now be written as

$\exists x \in D, P(x) \wedge Q(x)$. We are required to write down the negation of this statement:

In the question statement it is given that

$\neg [\exists x \in D, P(x) \wedge Q(x)]$

$\equiv \forall x \in D, P(x) \rightarrow \neg Q(x)$. By using our initial substitutions we can write this statement as:

For all integers a and d belonging to D, if a and d are negative then it is not the case that a/d = 1 + d/a.

i.e. For all integers a and d, if a and d are negative then $a/d \neq 1 + d/a$.

From the discussion provided, alternative 3 should be selected.

Question 15

Which one of the alternatives is a proof by contrapositive of the statement

"If $x^3 - x + 4$ is not divisible by 4 then x is even."

1. Required to prove: If $x^3 - x + 4$ is not divisible by 4 then x is even.
   Proof:
   Suppose x is odd. Let x = 2k + 1, then we have to prove that $x^3 - x + 4$ is divisible by 4.
   $x^3 - x + 4 = (2k + 1)^3 - (2k + 1) + 4$
   $= (2k + 1)(4k^2 + 4k + 1) - 2k - 1 + 4$
   $= 8k^3 + 8k^2 + 2k + 4k^2 + 4k + 1 - 2k - 1 + 4$
   $= 8k^3 + 12k^2 + 4k + 4$
   $= 4(2k^3 + 3k^2 + k + 1)$, which is divisible by 4. (4 multiplied by any integer is divisible by 4)

2. Required to prove: If $x^3 - x + 4$ is not divisible by 4 then x is even.
   Proof:
   Assume that $x^3 - x + 4$ is not divisible by 4. Then x can be even or odd. We assume that x is odd.

Let $x = 2k + 1$, then
$x^3 – x + 4 = (2k+1)^3 – (2k + 1) + 4$
$= (2k + 1)(4k^2 + 4k +1) – 2k – 1 + 4$
$= 8k^3 + 8k^2 + 2k + 4k^2 + 4k +1 – 2k – 1 + 4$
$= 8k^3 + 12k^2 + 4k + 4$
$= 4(2k^3 + 3k^2 + k + 1)$, which is divisible by 4. (4 multiplied by any integer is divisible by 4)

But this is a contradiction to our original assumption. Therefore x must be even if $x^3 – x + 4$ is not divisible by 4.

3. Required to prove: If $x^3 – x + 4$ is not divisible by 4 then x is even.
Proof:
Let x = 4 be an even element of $\mathbb{Z}$. We can replace x with 4 in the expression $x^3 – x + 4$.
$x^3 – x + 4$
$= (4)^3 – (4) + 4$
$= 64 – 4 + 4$
$= 64$ which is divisible by 4.

4. Required to prove: If $x^3 – x + 4$ is not divisible by 4 then x is even.
Proof:
Assume that x is even, ie $x = 4k$,
then $x^3 – x + 4$
$= (4k)^3 – (4k) + 4$
$= 64k^3 – 4k + 4$
$= 4(16k^3 – k + 1)$, which is divisible by 4.

Answer: Alternative 1

Discussion: It is very important that you know how to apply each of the proof methods discussed in the study guide.

We look at each of the alternatives:

1. The proof provided in this alternative is a proof by contrapositive. Another way to look at this proof method is the following:

The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$. This means that $p \rightarrow q$ is logically equivalent to $\neg q \rightarrow \neg p$.

The contrapositive of the statement “If $x^3 – x + 4$ is not divisible by 4 then x even.” is “If x odd then $x^3 – x + 4$ is divisible by 4.”
This is exactly what is proven in alternative 1. Thus this is the correct alternative.

2. This alternative provides a proof by contradiction. We assume the ‘if’ part of the given statement is true, ie “$x^3 – x + 4$ is not divisible by 4” is true, then we assume the opposite of the ‘then’ part. The ‘then’ part states that “x is even”, so we assume the opposite, ie “x is odd”, and then try to get to a contradiction.

This alternative does not provide the required contrapositive proof.

3. This alternative is not a proof. One cannot substitute values for x in a proof. One example (ie choosing a value for x and substituting it in the expression) does not provide a general proof to show that
"If $x^3 - x + 4$ is not divisible by 4 then x is even".

4. This proof is not valid. The proof shows that "If x is divisible by 4 then $x^3 - x + 4$ is divisible by 4."

Refer to study guide, pp 152 - 163.