Tutorial letter 203/2/2018

Theoretical Computer Science 1

COS1501

Semester 2

School of Computing

Discussion of assignment 03

Dear Student,

By this time you should have received the tutorial matter listed below. These can be downloaded from *my*Unisa.

- COSALLP/301/4/2018 General information regarding the School if Computing including lecturers' information;
- COS1501/101/3/2018 General information about the module and the assignments;
- COS1501/201/2/2018 Solutions to the first assignment, and **examination information**;
- COS1501/202/2/2018 Solutions to the second assignment;
- COS1501/203/2/2018 This tutorial letter;
- MO001/4/2018 Learning units, sample exam paper and other important information;
- COS1501/102/3/2018 Solutions to self-assessment questions in assignments 02 and 03, example examination paper & solutions, and discussion class questions & solutions & example assignment questions & solutions.

Everything of the best with the exam!
Regards
COS1501 team

Discussion of assignment 03 semester 2.

Suppose U = {2, 4, 6, a, b, c, {b, c}} is a universal set with the following subsets:
A = {6, b, c, {b, c}} and B = {2, 6, b, c}.

Answer questions 1 and 2 by using the given sets.

Question 1
Which one of the following relations from **A** to **B** is functional?
1. {(2, 6), (6, b), (b, c), (c, {b, c}}
2. {(6, 6), (c, c), (b, 2), ({b, c}, 6)}
3. {(6, 2), (b, 6), (c, b), (6, {b, c})}
4. {(b, 2), (b, 6), (b, b), (b, c)}

Answer: Alternative 2

Discussion
First we look at the definition for functionality:
Suppose R $\subseteq$ B × C is a binary relation from a set B to a set C. We may call R functional if the elements of B that appear as first co-ordinates of ordered pairs in R do not appear in more than one ordered pair of R.

We consider the relations provided in the different alternatives:

1. Let L = {(2, 6), (6, b), (b, c), (c, {b, c}} (say). L is not functional since 2 $\notin$ A. Therefore (2, 6) cannot be in L, since L is not defined from A to B.

2. Let M = {(6, 6), (c, c), (b, 2), ({b, c}, 6)} (say). M is functional because each first co-ordinate is in A, and each first co-ordinate is only used once.

3. Let N = {(6, 2), (b, 6), (c, b), (6, {b, c})} (say). N is a not functional because the element 6 appears twice as first co-ordinate in the relation.

4. Let S = {(b, 2), (b, 6), (b, b), (b, c)} (say). N is a not functional because b appears as first co-ordinate in every ordered pair in the relation.

From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, p 98.

Question 2

Let F = {(6, {b, c}), (b, 2), (c, c), ({b, c}, 6)} be a function from **A** to **U**. Which one of the following alternatives is true for function **F**?

1. F is injective and surjective.
2. F is surjective, but not injective.
3. F is neither injective nor surjective.
4. F is injective, but not surjective.

Answer: Alternative 4

Discussion

First we look at the definition of a function:

Suppose R $\subseteq$ B × C is a binary relation from a set B to a set C. We may call R a function from B to C if every element of B appears exactly once as the first co-ordinate of an ordered pair in R (i.e. f is functional), and the domain of R is exactly the set B, i.e. dom(R) = B.

We also look at the definition of an injective and surjective function.

Injective function:

A function f: B $\rightarrow$ C is injective iff f has the property that whenever $f(a_1) = f(a_2)$ then $a_1 = a_2$ (or whenever $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$) .

Surjective function:

Given a function f: B $\rightarrow$ C we say that f: B $\rightarrow$ C is surjective iff the range of f is equal to the codomain of f, ie f(B) = C.

Each first co-ordinate is paired with only one unique second co-ordinate, therefore F is an injective function. But F is not surjective. Although ran(A) = {2, 6, c, {b, c})} $\subseteq$ U, it is not true that ran(A) = U.

From the arguments provided we can deduce that alternative 4 should be selected.

Refer to study guide, pp 105 - 107.

Question 3

Which one of the following relations is a bijective function on the set B = {t | (−4 < t < 4, t $\in \mathbb{Z}$}?

***Hint**: First list the elements of set B.*

1. {(-3, -3), (-1, -1), (1, 1), (3, 3)}
2. {(0, -3), (0, -2), (0, -1), (0, 0), (0, 1), (0, 2), (0, 3)}
3. {(-3, 0), (-2, -1), (-1, -1), (0, 0), (1, -3), (2, -3), (3, 3)}
4. {(-3, 0), (-2, -3), (-1, 3), (0, -1), (1, -2), (2, 2), (3, 1)}

Answer: Alternative 4

Discussion

Let us first look at the definition of a bijective function. A function is bijective iff it is both a surjective and an injective function. A function f: A → B is injective iff it has the property that *whenever f(a1) = f(a2) then a1 = a2. Alternatively, a function f: A → B is injective iff it has the property that whenever a1 ≠ a2 then f(a1) ≠ f(a2). Given a function f: A → B, we say that f: A → B is surjective iff the range of f is equal to the codomain of f, ie f[A] = B.*

The set B is defined as B = {t | (−4 < t < 4, t ∈ ℤ}, ie B = {-3, -2, −1, 0, 1, 2, 3}. Now let us look at the different alternatives:

1. Let L = {(-3, -3), (-1, -1), (1, 1), (3, 3)} (say). Although L is injective, (ie each first co-ordinate is paired with only one unique second co-ordinate), L is not a function since dom(L) ≠ B. Furthermore, L is not surjective because ran(L) = {-3, -1, 1, 3} ≠ B. Thus L is not a bijective function.

2. Let N = {(0, -3), (0, -2), (0, -1), (0, 0), (0, 1), (0, 2), (0, 3)} (say). N is not a function because 0 appears as first co-ordinate in every ordered pair, thus dom(N) ≠ B.

3. Let S = {(-3, 0), (-2, -1), (-1, -1), (0, 0), (1, -3), (2, -3), (3, 3)} (say). There isn't a one-to-one relationship between the first and second co-ordinates, because first co-ordinates -2 and -1 both have -1 as second co-ordinate, and the first co-ordinates 1 and 2 both have -3 as second co-ordinates. Therefore S is not injective. Furthermore, S is not surjective because ran(S) = {-3, -1, 0, 3} ≠ B. Thus S is not a bijective function.

4. Let M = {(-3, 0), (-2, -3), (-1, 3), (0, -1), (1, -2), (2, 2), (3, 1)} (say). It is clear that M is surjective and injective. M is injective because each first co-ordinate is paired with only one unique second co-ordinate. M is surjective, because the range of M is exactly equal to its codomain, which is B.

From the discussion we can deduce that alternative 4 provides the only bijective function on set B.

Refer to study guide, pp 105-107, 112. Also refer to the diagram on p 108 for clarification.

Let f be a function on $\mathbb{Z}^+$ (the set of positive integers) defined by

$(x, y) \in f$ iff $y = 2x^2 - 7$ ($f \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$)

and g be a function from $\mathbb{Z}^+$ to $\mathbb{Q}$ (the set of rational numbers) defined by

$(x, y) \in g$ iff $y = \frac{3}{5}x + 5$ ($g \subseteq \mathbb{Z}^+ \times \mathbb{Q}$).

Answer questions 4 to 7 by using the given functions f and g.

Question 4

Which one of the following is NOT an ordered pair in g?

1. $(3, 6\frac{4}{5})$
2. $(4, 7\frac{1}{5})$
3. $(5, 8)$
4. $(6, 8\frac{3}{5})$

Answer: Alternative 2

Discussion

The first co-ordinates of ordered pairs in g are elements of Z^+ and the second co-ordinates are elements of Q.

We consider the ordered pairs provided in the different alternatives:

1. $(x, y) \in g$ iff $y = \frac{3}{5}x + 5$. Is $(3, 6\frac{4}{5}) \in g$?

   Let x = 3 then

   $y = \frac{3}{5}x + 5$

   $= \frac{3}{5}.(3) + 5$

   $= \frac{9}{5} + 5$

   $= 1\frac{4}{5} + 5$

   $= 6\frac{4}{5}$

   Thus $(3, 6\frac{4}{5}) \in g$.

2. Is $(4, 7\frac{1}{5}) \in g$?

   Let x = 4 then

   $y = \frac{3}{5}x + 5$

   $= \frac{3}{5}.(4) + 5$

   $= 7\frac{2}{5}$

   Thus $(4, 7\frac{2}{5}) \in g$ but $(4, 7\frac{1}{5}) \notin g$.

3. Is $(5, 8) \in g$?
   Let x = 5 then
   $y = \frac{3}{5}x + 5$
   $= \frac{3}{5} \cdot (5) + 5$
   $= \frac{15}{5} + 5$
   $= 8$

   Thus $(5, 8) \in g$.

4. Is $(6, 8\frac{3}{5}) \in g$?
   Let x = 6 then
   $y = \frac{3}{5}x + 5$
   $= \frac{3}{5} \cdot (6) + 5$
   $= \frac{18}{5} + 5$
   $= 8\frac{3}{5}$

5.

Thus $(6, 8\frac{3}{5}) \in g$.
From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, pp 98-99.

Question 5
Which one of the following alternatives represents the range of g (ie ran(g))?

1. $\{y \mid \frac{5}{3}(y - 5) \in \mathbb{Q}\}$
2. $\{y \mid \frac{3}{5}x + 5 \in \mathbb{Q}\}$
3. $\{y \mid \text{for some } y \in \mathbb{Q},\ y = \frac{3}{5}x + 5 \in \mathbb{Q}\}$
4. $\mathbb{Q}$

Answer: Alternative 1

By the definition of the range of a function

$\text{ran}(g) = \{y \mid \text{for some } x \in \mathbb{Z}^+, (x, y) \in g\}$
$= \{y \mid \text{for some } x \in \mathbb{Z}^+,\ y = \frac{3}{5}x + 5\}$
$= \{y \mid \text{for some } \mathbf{x} \in \mathbb{Z}^+,\ \mathbf{x} = \frac{5}{3}(y - 5\}$
$= \{y \mid \frac{5}{3}(y - 5) \in \mathbb{Z}^+\}$

Ordered pairs ($\frac{5}{3}(y - 5)$, y) are elements of g. By the definition of g it is the case that ran(g) $\subseteq \mathbb{Q}$. It is important to make sure that the first co-ordinates are paired with second co-ordinates are elements of $\mathbb{Q}$ since g is a function from $\mathbb{Z}^+$ to $\mathbb{Q}$.
From the arguments provided, alternative 1 is the correct alternative.

Refer to study guide, pp 98, 104-105

Question 6

The relation f is NOT surjective. Which of one of the following values for y provides a counterexample that can be used to prove that f is not surjective?

1. y = 43
2. y = 25
3. y = 12
4. y = 1

Answer: Alternative 3

The function f is NOT surjective thus ran(f) ≠ $\mathbb{Z}^+$ ($\mathbb{Z}^+$ is the codomain). A counterexample provides a value y $\in \mathbb{Z}^+$ for which there is ***no*** *element x $\in \mathbb{Z}^+$ ie x = $\sqrt{\frac{y+7}{2}}$ $\in \mathbb{Z}^+$ such that y = $2x^2 - 7 \in \mathbb{Z}^+$.*

We consider the elements provided in the different alternatives:

1. Let y = 43 then

   $x = \sqrt{\frac{43+7}{2}}$

   $= \sqrt{\frac{50}{2}}$

   $= \sqrt{25}$

   $= 5 \in \mathbb{Z}^+$ thus 43 $\in$ ran(f).

2. Let y = 25 then

   $x = \sqrt{\frac{25+7}{2}}$

   $= \sqrt{\frac{32}{2}}$

   $= \sqrt{16}$

   $= 4 \in \mathbb{Z}^+$ thus 25 $\in$ ran(f)

3. Let y = 12 then

   $x = \sqrt{\frac{12+7}{2}}$

$= \sqrt{\frac{19}{2}} \notin \mathbb{Z}^+$

Thus ($\sqrt{\frac{19}{2}}$, 12) $\notin$ f.

This means that 12 $\notin$ ran(f) since no x $\in \mathbb{Z}^+$ exists such that (x, 12) $\in$ f. Thus y = 12 can be used in a counterexample to prove that f is not surjective, ie ran(f) $\neq \mathbb{Z}^+$.

4. Let y = 1 then

$x = \sqrt{\frac{1+7}{2}}$

$= \sqrt{\frac{8}{2}}$

$= \sqrt{4}$

= 2 $\in \mathbb{Z}^+$ thus 1 $\in$ ran(f).

.

From the arguments provided we can deduce that alternative 3 should be selected.

Refer to study guide, pp 89, 104-106.

Question 7

Which one of the following alternatives represents the image of x under g $\circ$ f (ie g $\circ$ f(x))?

1. $\frac{6}{5}x^2 + \frac{4}{5}$
2. $\frac{18}{25}x^2 + 12x + 43$
3. $\frac{9}{25}x^2 - 8$
4. $4x^2 - 21$

Answer: Alternative 1

Discussion
Given the functions g: A $\rightarrow$ B and f: B $\rightarrow$ C the composite function

g $\circ$ f: A $\rightarrow$ C is defined by g$\circ$f(x) = g(f(x)).

g $\circ$ f(x)

= g(f(x))

$= \frac{3}{5}(f(x)) + 5$

$= \frac{3}{5}(2x^2 - 7) + 5$

$= \frac{6}{5}x^2 - \frac{21}{5} + 5$

$= \frac{6}{5}x^2 + \frac{4}{5}$

From the above we can deduce that alternative **1** should be selected.

Refer to study guide, p 109 - 111.

Let A = {□, ◊, ☼, ⌂}. Consider the following table for the binary operation *: A × A → A:

*	□	◊	☼	⌂
□	⌂	☼	◊	⌂
◊	☼	⌂	□	◊
☼	◊	◊	⌂	☼
⌂	⌂	◊	☼	☼

Answer questions 8 and 9 by referring to the table of *.

Question 8

The binary operation * does not satisfy associativity. Which one of the following alternatives can be used in the calculations of a counterexample to prove that * does **NOT** satisfy associativity?

1. Determine (☼ * □) * ◊ and ☼ * (□ * ◊).

2. Determine (□ * ⌂) * ⌂ and □ * (⌂ * ⌂).

3. Determine (□ * ☼) * □ and □ * (☼ * □).

4. Determine (□ * ⌂) * □ and □ * (⌂ * □).

Answer: Alternative 2

Discussion

Definition of associativity for a binary operation:

A binary operation ◊: X × X → X is associative iff (y ◊ x) ◊ z = y ◊ (x ◊ z) for all x, y, z ∈ X. To prove that ◊ is not associative a counterexample can be used to show that for some x, y, z ∈ X it is the case that (y ◊ x) ◊ z ≠ y ◊ (x ◊ z).

We look at the different alternatives:

1. From the table (☼ * □) * ◊ = ◊ * ◊ = ⌂ and ☼ * (□ * ◊) = ☼ * ☼ = ⌂. This is not a counterexample since both the expressions are equal to ⌂.

2. We determine (□ * ⌂) * ⌂ = ⌂ * ⌂ = ☼ and □ * (⌂ * ⌂) = □ * ☼ = ◊. Clearly (□ * ⌂) * ⌂ ≠ □ * (⌂ * ⌂). Thus this counterexample proves that * is not associative.

3. From the table (□ * ☼) * □ = ◊ * □ = ☼ and □ * (☼ * □) = □ * ◊ = ☼. This is not a counterexample since both the expressions are equal to ☼.

4. From the table (□ * ⌂) * □ = ⌂ * □ = ⌂ and □ * (⌂ * □) = □ * ⌂ = ⌂. This is not a counterexample since both the expressions are equal to ⌂.

From the above we can deduce that alternative 2 should be selected.

Refer to study guide, p 119.

Question 9

Which of the following is true regarding an identity element for operation * ?

1. ◊ is the identity element.
2. ⌂ is the identity element.
3. * does not have an identity element
4. ☼ is the identity element.

Discussion:

Definition of an identity element of a binary operation:

An element e of X is an identity element in respect of the binary operation ◊ : $X \times X \rightarrow X$ iff e ◊ x = x ◊ e = x for all x $\in$ X. To prove that e is not the identity element, a counterexample can be used to show that for some x $\in$ X it is the case that e ◊ x ≠ x ◊ e.

Answer: Alternative 3

We consider the different alternatives:

1. ◊ is not the identity element. We provide a counterexample: From the table ◊ * ☼ = □ ≠ ☼ and ☼ * ◊ = ◊ ≠ ☼. Thus this alternative provides a counterexample that proves that is ◊ not the identity element. It is also true that ◊ * ☼ ≠ ☼ * ◊.

2. ⌂ is not the identity element. We provide a counterexample: From the table ⌂ * □ = ⌂ ≠ □ and □ * ⌂ = ⌂ ≠ □. Thus this alternative provides a counterexample that proves that is ⌂ not the identity element.

3. In a similar fashion as in alternatives 1, 2 and 4 we can also prove that □ is not an identity element, which means none of the elements of A is an identity element. If an operation has an identity element, the row to the right of the identity element must be the same as the top row of the table. Similarly the column below the identity element must be the same as the leftmost column of the table.

4. ☼ is not the identity element. We provide a counterexample: From the table ☼ * □ = ◊ ≠ □ and □ * ☼ = ◊ ≠ □. Thus this alternative provides a counterexample that proves that is ☼ not the identity element.

From the above arguments we can deduce that alternative 3 should be selected.

Refer to study guide, pp 119, 120.

Question 10

Perform the following matrix multiplication operation:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & -3 \end{bmatrix} . \begin{bmatrix} -4 & 0 \\ -2 & -1 \\ 0 & 1 \end{bmatrix}$$

Which one of the following alternatives represents the correct answer to the above operation?

1. The operation is not possible.

2. $\begin{bmatrix} -8 & -16 \\ 1 & -3 \end{bmatrix}$

3. $\begin{bmatrix} -4 & 0 \\ -4 & 0 \\ 0 & -3 \end{bmatrix}$

4. $\begin{bmatrix} -8 & 1 \\ -16 & -3 \end{bmatrix}$

Answer: Alternative 4

Discussion

*A **2** $\times$ 3 matrix multiplied by a 3 $\times$ **2** matrix gives a **2** $\times$ **2** matrix.*

We determine a_{ij}:

a_{11} = 1(-4) + 2(-2) + 3(0) = -8

a_{12} = 1(0) + 2(-1) + 3(1) = 1

a_{21} = 4(-4) + 0(-2) + -3(0) = -16

a_{22} = 4(0) + 0(-1) + -3(1) = -3

Thus the answer to the multiplication of the matrices is

$\begin{bmatrix} -8 & 1 \\ -16 & -3 \end{bmatrix}$.

From the above we can deduce that alternative 4 should be selected.

Refer to study guide, p 131, 132.

Question 11

Let p and q be simple declarative statements. Which one of the following statements is **not** a logical equivalence – in other words which one of the expressions is **false**? (*Hint:* Use truth tables to get to a conclusion)

1. $(\neg p \vee q) \equiv ((\neg(\neg p)) \rightarrow q)$
2. $(\neg q \rightarrow (p \wedge q)) \equiv ((\neg q \vee p) \wedge \neg q)$
3. $((q \wedge p) \rightarrow (\neg p \vee \neg q)) \equiv (p \rightarrow \neg q\ )$
4. $(p \vee (q \rightarrow p)) \equiv (p\ \ \vee \neg q)$

Answer: Alternative 2

Discussion:

Remember that if such an equivalence is true, it means that the expression is a tautology, which can easily be determined using truth tables. The ≡ equivalence symbol has the same meaning as the ↔ symbol. We also show how the LHS and RHS of the expression can be simplified using the important logical equivalences in the study guide p. 147, and then a conclusion can be made.

We consider the different alternatives:

1. Is $(\neg p \vee q)) \equiv ((\neg(\neg p)) \rightarrow q)$?

p	q	¬p	¬(¬p)	(¬p ∨ q)	((¬(¬p)) → q)	(¬p ∨ q)) ↔ ((¬(¬p)) → q)
T	T	F	T	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	F	T	T	T

From the last column it is clear that the expression is a tautology, in other words

$(\neg p \vee q)) \equiv ((\neg(\neg p)) \rightarrow q)$.

We also show how to simplify the LHS and RHS by using logical equivalences:

LHS: $(\neg p \vee q)$

≡ $p \rightarrow q$ (*Comment: This is proven on p 147 of the study guide by using truth tables. You should always remember this, because you will be using it very often in proofs.)

RHS: $((\neg(\neg p)) \rightarrow q)$

≡ $p \rightarrow q$ (double negation rule)

It is clear that LHS ≡ RHS.

2. Is (¬q → (p ∧ q)) ≡ ((¬q ∨ p) ∧ ¬q)?

p	q	¬q	p ∧ q	¬q ∨ p	(¬q → (p ∧ q)	((¬q ∨ p) ∧ ¬q)	(¬q → (p ∧ q)) ↔ ((¬q ∨ p) ∧ ¬q)
T	T	F	T	T	T	F	F
T	F	T	F	T	F	T	F
F	T	F	F	F	T	F	F
F	F	T	F	T	F	T	F

From the last column it is clear that the expression is not a tautology, but a contradiction.

Now we simplify the LHS and RHS and see if we get the same result:

LHS: ¬q → (p ∧ q)

≡ ¬(¬q) ∨ (p ∧ q) (* see comment alternative 1)

≡ q ∨ (p ∧ q) (law of double negation)

≡ (q ∨ p) ∧ (q ∨ q) (distributive laws)

≡ (q ∨ p) ∧ q (idempotent laws)

This expression has the same format as the RHS (¬q ∨ p) ∧ ¬q. Clearly, the LHS and RHS are not equivalent. If you are not convinced, draw truth tables for the simplified LHS and the RHS. Alternative 2 is therefore the correct alternative to choose.

3. Is ((q ∧ p) → (¬p ∨ ¬q)) ≡ (p → ¬q)?

p	q	¬p	¬q	(q ∧ p)	(¬p ∨¬q)	(p → ¬q)	(q ∧ p) →(¬p ∨¬q)	((q ∧ p) →(¬p ∨¬q)) ↔ (p → ¬q)
T	T	F	F	T	F	F	F	T
T	F	F	T	F	T	T	T	T
F	T	T	F	F	T	T	T	T
F	F	T	T	F	T	T	T	T

The expression is clearly a tautology according to the last column.
We also simplify the LHS and RHS of the expression:

LHS: (q ∧ p) → (¬p ∨ ¬q)

≡ ¬(q ∧ p) ∨ (¬p ∨ ¬q) (For any declarative statements r and s it is true that r → s ≡ ¬r ∨ s.)

≡ (¬q ∨ ¬p) ∨ (¬p ∨ ¬q) (de Morgan’s law)

≡ (¬p ∨ ¬q) ∨ (¬p ∨ ¬q) (commutative laws)

≡ ¬p ∨ ¬q (idempotent laws)

≡ (p → ¬q), which is exactly the RHS. The given expression in the question is therefore true.

4. (p ∨ (q → p)) ≡ (p ∨ ¬q)

p	q	¬q	q → p	p ∨ (q → p)	(p ∨ ¬q)	(p ∨ (q → p)) ↔ (p ∨ ¬q)
T	T	F	T	T	T	T
T	F	T	T	T	T	T
F	T	F	F	F	F	T
F	F	T	T	T	T	T

From the last column it is clear that the expression is a tautology.
Also note that we can construct our table so that we do not repeat parts of an expression in different columns. The table below accomplishes exactly the same as the one above, with the result in the shaded column:

p	q	¬q	q → p	p ∨ (q → p)	↔	(p ∨ ¬q)
T	T	F	T	T	T	T
T	F	T	T	T	T	T
F	T	F	F	F	T	F
F	F	T	T	T	T	T

Now we simplify the LHS and RHS and see if we get the same result:

LHS: p ∨ (q → p)

≡ p ∨ (¬q ∨ p) (* see comment alternative 1)

≡ p ∨ (p ∨ ¬q) (commutative laws)

≡ (p ∨ p) ∨ ¬q (associative laws)

≡ p ∨ ¬q (idempotent laws)

which is exactly the RHS of the expression. Thus the expression is true.

Refer to study guide, pp 136-149.

Question 12

Consider the expression ((p → q) → r) ↔ (r ∧ (¬q ∨ ¬p)) as well as an incomplete truth table representing the expression.

p	q	r	((p → q) → r) ↔ (r ∧ (¬q ∨ ¬p))
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

Which one of the alternatives represents the correct truth values for the last column?

1.

↔
F
T
T
F
T
T
T
T

2.

↔
T
T
T
T
T
F
T
F

3.

↔
T
T
T
T
T
T
T
T

4.

↔
F
F
F
F
F
F
F
F

Correct alternative: 1

Discussion:
We determine the truth values in all the columns:

p	q	r	¬p	¬q	(p → q)	(¬q ∨ ¬p)	(p → q) → r	↔	r ∧ (¬q ∨ ¬p)
T	T	T	F	F	T	F	T	F	F
T	T	F	F	F	T	F	F	T	F
T	F	T	F	T	F	T	T	T	T
T	F	F	F	T	F	T	T	F	F
F	T	T	T	F	T	T	T	T	T
F	T	F	T	F	T	T	F	T	F
F	F	T	T	T	T	T	T	T	T
F	F	F	T	T	T	T	F	T	F

From the highlighted column it is clear that alternative 1 should be selected.

Refer to study guide, pp 136 – 149.

Question 13

Which one of the alternatives provides the negation of the statement

$$\exists x \in \mathbb{R}, [(\tfrac{x}{2} \geq 0) \wedge (x - 4 < 0)]?$$

1. $\forall x \in \mathbb{R}, [(\tfrac{x}{2} < 0) \wedge (x - 4 \geq 0)]$
2. $\exists x \in \mathbb{R}, [(\tfrac{x}{2} < 0) \vee (x - 4 \geq 0)]$
3. $\forall x \in \mathbb{R}, [(\tfrac{x}{2} < 0) \vee (x \geq 4)]$
4. $\forall x \in \mathbb{R}, [(\tfrac{x}{2} \leq 0) \vee (x > 4)]$

Correct alternative: 3

Discussion:
We will derive the negation of the given statement step by step:

$\neg[\exists x \in \mathbb{R}, [(\tfrac{x}{2} \geq 0) \wedge (x - 4 < 0)]]$ (always write down this step)

$\equiv \forall x \in \mathbb{R}, \neg[(\tfrac{x}{2} \geq 0) \wedge (x - 4 < 0)]$

$\equiv \forall x \in \mathbb{R}, \neg(\tfrac{x}{2} \geq 0) \vee \neg(x - 4 < 0)$ (de Morgan's law)

$\equiv \forall x \in \mathbb{R}, (\tfrac{x}{2} \ngeq 0) \vee (x - 4 \nless 0)$

$\equiv \forall x \in \mathbb{R}, (\tfrac{x}{2} < 0) \vee (x - 4 \geq 0)$

$\equiv \forall x \in \mathbb{R}, (\tfrac{x}{2} < 0) \vee (x \geq 4)$

From the derivation above it is clear that alternative 3 is the correct alternative.

Refer to study guide, pp 152-158.

Question 14

The negation of the statement

$$\forall x \in \mathbb{Z}^+, [(x^2 > 2x) \vee (x - 1 \geq 0)]$$

can be written as:

$$\exists x \in \mathbb{Z}^+, [(x^2 - 2x \leq 0) \wedge (x - 1 < 0)].$$

Which one of the following alternatives is true regarding the statement and its negation?

1. The statement and its negation are false.
2. The statement and its negation are true.
3. The statement is false, but its negation is true.
4. The statement is true, but its negation is false.

Correct alternative: 4

Discussion:

Let us look at the statement first:

$$\forall x \in \mathbb{Z}^+, [(x^2 > 2x) \vee (x - 1 \geq 0)]$$

This statement states, that for all positive integers x, it is true that $(x^2 > 2x)$ **or** $(x - 1 \geq 0)$. This means that at least $(x^2 > 2x)$ **or** $(x - 1 \geq 0)$ must be true for the statement to be true. Both may also be true. (Refer to the truth table for the disjunction '∨'.)

We substitute a few positive integers for x in both $(x^2 > 2x)$ and $(x - 1 \geq 0)$ and see where that takes us:

x	$x^2 > 2x$	$x - 1 \geq 0$
1	$1^2 > 2(1)$ which is false	$1 - 1 \geq 0$ which is true
2	$2^2 > 2(2)$ which is false	$2 - 1 \geq 0$ which is true
3	$3^2 > 2(3)$ which is true	$3 - 1 \geq 0$ which is true
4	$4^2 > 2(4)$ which is true	$4 - 1 \geq 0$ which is true
	etc…	etc…

We can see that for all $x > 4$, both $x^2 > 2x$ and $x - 1 \geq 0$ will always be true. We actually only need one of the statements to be true for $[(x^2 > 2x) \vee (x - 1 \geq 0)]$ to be true, so for x =1 and x = 2 the statement $[(x^2 > 2x) \vee (x - 1 \geq 0)]$ is also true, even though $x^2 > 2x$ is false, because $x - 1 \geq 0$ is true in both cases. We can therefore deduce that the original statement is true.

What about the negation statement?

$$\exists x \in \mathbb{Z}^+, [(x^2 - 2x \leq 0) \wedge (x - 1 < 0)]$$

The negation statement reads that there exists a positive integer, such that **both** $(x^2 - 2x \leq 0)$ **and** $(x - 1 < 0)$ are true. We will substitute some positive values for x in the statements

$(x^2 - 2x \leq 0)$ and $(x - 1 < 0)$ and see whether we can find at least one positive integer for which $[(x^2 - 2x \leq 0) \wedge (x - 1 < 0)]$ is true.

x	$x^2 - 2x \leq 0$	$x - 1 < 0$
1	$1^2 \leq 2(1)$ which is true	$1 - 1 < 0$ which is false
2	$2^2 \leq 2(2)$ which is true	$2 - 1 < 0$ which is false
3	$3^2 \leq 2(3)$ which is false	$3 - 1 < 0$ which is false
4	$4^2 \leq 2(4)$ which is false	$4 - 1 < 0$ which is false
	etc…	etc…

We need not go any further. We can see that $x - 1 < 0$ will always be false for all positive integers, which implies that it is impossible for both $(x^2 - 2x \leq 0)$ and $(x - 1 < 0)$ to be true for any positive integer. The negation of the original statement is therefore false. From the discussion above, alternative 4 is the correct alternative.

Refer to study guide, pp 152-158.

Question 15

Which one of the alternatives is a proof by contradiction for the statement

"If $2x^2 - 3x + 7$ is odd, then x is even."

1. Required to prove: If x is odd, then $2x^2 - 3x + 7$ is even.
   Proof: Suppose x is odd. Let $x = 2k + 1$, then we have to prove that $2x^2 - 3x + 7$ is even.
   $2x^2 - 3x + 7 = 2(2k+1)^2 - 3(2k + 1) + 7$
   $= 2(4k^2 + 4k +1) - 6k - 3 + 7$
   $= 8k^2 + 8k + 2 - 6k - 3 + 7$
   $= 8k^2 + 2k + 6$
   $= 2(4k^2 + k + 3)$, which is even (2 multiplied by any integer is even)

2. Assume that $2x^2 - 3x + 7$ is odd. Then x can be even or odd. We assume that x is odd.
   Let $x = 2k + 1$, then
   $2x^2 - 3x + 7 = 2(2k+1)^2 - 3(2k + 1) + 7$
   $= 2(4k^2 + 4k +1) - 6k - 3 + 7$
   $= 8k^2 + 8k + 2 - 6k - 3 + 7$
   $= 8k^2 + 2k + 6$
   $= 2(4k^2 + k + 3)$, which is even (2 multiplied by any integer is even)

   But this is a contradiction to our original assumption. Therefore x must be even if $2x^2 - 3x + 7$ is odd.

3. Let $x = 2$ be an even element of $\mathbb{Z}$. We can replace x with 2 in the expression $2x^2 - 3x + 7$.
   $2x^2 - 3x + 7$

$= 2(2)^2 – 3(2) + 7$
$= 8 – 6 + 7$
$= 9$, which is odd.
We have therefore proven that if $2x^2 – 3x + 7$ is odd, then x is even.

4. Required to prove: if $2x^2 – 3x + 7$ is odd, then x is even.
Proof: Assume that x is even, ie $x = 2k$,
then $2x^2 – 3x + 7$
$= 2(2k)^2 – 3(2k) + 7$
$= 8k^2 – 6k + 7$
$= 2(4k^2 – 3k + 3) + 1$, which is odd. We have therefore proven that if if $2x^2 – 3x + 7$ is odd, then x is even.

Correct alternative: 2

Discussion: It is very important that you know how to apply each of the proof methods discussed in the study guide.

We look at each of the alternatives:

1. The proof provided in this alternative is a proof by contrapositive. Another way to look at this proof method is the following:

The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$. This means that $p \rightarrow q$ is logically equivalent to $\neg q \rightarrow \neg p$.
The contrapositive of the statement '*if $2x^2 – 3x + 7$ is odd, then x is even*' is:
If x is not even, then $2x^2 – 3x + 7$ is not odd,
ie *if x is odd, then $2x^2 – 3x + 7$ is even.*
This is exactly what is proven in alternative 1.

2. This alternative provides a proof by contradiction. We assume the 'if' part of the given statement is true, ie "$2x^2 – 3x + 7$ is odd" is true, then we assume the opposite of the 'then' part. The 'then' part states that "x is even", so we assume the opposite, ie "x is odd", and then try to get to a contradiction. This alternative provides the required proof by contradiction.

3. This alternative is not a proof. One cannot substitute values for x in a proof. One example (ie choosing a value for x and substituting it in the expression) does not provide a general proof to show that
"If $2x^2 – 3x + 7$ is odd, then x is even".

4. This proof is not valid. If p is the statement '*$2x^2 – 3x + 7$ is odd*' and q is the statement '*x is even*', we have to prove that $p \rightarrow q$ is true. The proof in this alternative proves that $q \rightarrow p$ is true.

Refer to study guide, pp 152 - 163.