Solutions for Assignment 7, Semester 1, 2018

Chapters 3 & 4

Question 1

A paraboloid can be parametrized as

$$\begin{aligned} x(u,\, v) &= au\cos v \\ y(u,\, v) &= au\sin v \\ z(u,\, v) &= u^2 \end{aligned}$$

(a) Find the line element for the surface.

(b) What is the metric tensor and the dual metric tensor?

(c) Determine the values of all the Christoffel coefficients of the surface.

(d) What is the value of the component $R^1_{\ 212}$ of the Riemann curvature tensor?

(e) What is the Ricci tensor for the surface?

(f) What is the curvature scalar R for the surface?

(g) What is the Gaussian curvature of the surface?

(h) Is the surface Euclidean? Explain your answer.

(i) Suppose that the surface is filled with non-interacting particles, or dust. Use the two dimensional version of the energy-momentum tensor for dust and Einstein's field equation to find an expression for the Einstein constant κ for this surface.

Solution

Part A

In Cartesian coordinates, the line element is given by

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2 .$$

We have

$$\begin{aligned} x(u,\, v) &= au\cos v \\ y(u,\, v) &= au\sin v \\ z(u,\, v) &= u^2 \end{aligned}$$

so that

$$\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ &= \frac{\partial}{\partial u}(au\cos v)\, du + \frac{\partial}{\partial v}(au\cos v)\, dv \\ &= a\cos v\, du - au\sin v\, dv \end{aligned}$$

Similarly, we get

$$\begin{aligned} dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \\ &= a\sin v\, du + au\cos v\, dv \end{aligned}$$

$$\begin{aligned} dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \\ &= 2u\, du \end{aligned}$$

Substituting this into the Cartesian line element and simplifying gives

$$\begin{aligned} (dl)^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= (a\cos v\, du - au\sin v\, dv)^2 + (a\sin v\, du + au\cos v\, dv)^2 + (2u\, du)^2 \\ &= a^2\cos^2 v\, du^2 - 2a^2u\cos v\sin v\, dudv + a^2u^2\sin^2 v\, dv^2 + a^2\sin^2 v\, du^2 + a^2u\cos v\sin v\, dudv + a^2u^2 \mathrm{c} \\ &= a^2\left(\cos^2 v + \sin^2 v\right) du^2 + \left(2a^2u\cos v\sin v\, dudv - 2a^2u\cos v\sin v\, dudv\right) + a^2u^2\left(\sin^2 v + \cos^2 v\right) d \end{aligned}$$

$$
\begin{aligned}
&= a^2\,du^2 + a^2u^2dv^2 + 4u^2\,du^2 \\
&= \left(a^2 + 4u^2\right)\,du^2 + a^2u^2dv^2
\end{aligned}
$$

Part B

We know that the line element has the form

$$dl^2 = \sum_{i,j=1}^{n} g_{ij}dx^i dx^j$$

If we choose $x^1 = u$ and $x^2 = v$, this reduces to (for this case)

$$
\begin{aligned}
dl^2 &= \sum_{i,j=1}^{2} g_{ij}dx^i dx^j \\
&= g_{11}dx^1dx^1 + 2g_{12}dx^1dx^2 + g_{22}dx^2dx^2 \\
&= g_{11}\left(du\right)^2 + 2g_{12}dudv + g_{22}\left(dv\right)^2
\end{aligned}
$$

Above we used the fact that the metric tensor is symmetric $g_{ij} = g_{ji}$. Comparing this to the line element calculated in part A allows us to identify

$$g_{11} = a^2 + 4u^2, \quad g_{12} = 0, \quad g_{22} = a^2u^2$$

so that the metric tensor for the surface is

$$[g_{ij}] = \begin{pmatrix} a^2 + 4u^2 & 0 \\ 0 & a^2u^2 \end{pmatrix}$$

We know that we must have

$$\sum_k g^{ik}g_{kj} = \delta^i_j$$

so that the dual metric $[g^{ij}]$ is just the matrix inverse of $[g_{ij}]$. We find

$$\left[g^{ij}\right] = \begin{pmatrix} \frac{1}{a^2+4u^2} & 0 \\ 0 & \frac{1}{a^2u^2} \end{pmatrix}$$

Part C

The Christoffel coefficients are defined by

$$\Gamma^h_{\ ij} = \sum_k \frac{1}{2} g^{hk} \left(g_{ki,j} + g_{jk,i} - g_{ij,k} \right)$$

$$\Gamma^1_{\ 11} = \frac{1}{2} g^{11} \left(g_{11,1} + g_{11,1} - g_{11,1} \right) + \frac{1}{2} g^{12} \left(g_{21,1} + g_{12,1} - g_{11,2} \right)$$

All the g_{ik} and g^{ik} where $i \neq k$ will be zero, so their derivatives will also be zero. Remembering this will reduce the calculations a lot. So we have

$$\begin{aligned}
\Gamma^1_{\ 11} &= \frac{1}{2} g^{11} \left(g_{11,1} + g_{11,1} - g_{11,1} \right) + \frac{1}{2} (0) \left((0) + (0) - g_{11,2} \right) \\
&= \frac{1}{2} g^{11} g_{11,1} \\
&= \frac{1}{2} \left(\frac{1}{a^2 + 4u^2} \right) \frac{\partial}{\partial u} \left(a^2 + 4u^2 \right) \\
&= \frac{4u}{a^2 + 4u^2}
\end{aligned}$$

Using the symmetric property of the Christoffel coefficients $\Gamma^h_{\ ij} = \Gamma^h_{\ ji}$ will also cut down on calculations

$$\begin{aligned}
\Gamma^1_{\ 12} = \Gamma^1_{\ 21} &= \frac{1}{2} g^{11} \left(g_{11,2} + g_{21,1} - g_{12,1} \right) + \frac{1}{2} g^{12} \left(g_{21,2} + g_{22,1} - g_{12,2} \right) \\
&= \frac{1}{2} g^{11} g_{11,2} \\
&= \frac{1}{2} \left(\frac{1}{a^2 + 4u^2} \right) \frac{\partial}{\partial v} \left(a^2 + 4u^2 \right) \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^1_{\ 22} &= \frac{1}{2} g^{11} \left(g_{12,2} + g_{21,2} - g_{22,1} \right) + \frac{1}{2} g^{12} \left(g_{22,2} + g_{22,2} - g_{22,2} \right) \\
&= -\frac{1}{2} g^{11} g_{22,1} \\
&= -\frac{1}{2} \left(\frac{1}{a^2 + 4u^2} \right) \frac{\partial}{\partial u} \left(a^2 u^2 \right) \\
&= \frac{-a^2 u}{a^2 + 4u^2}
\end{aligned}$$

$$\begin{aligned}
\Gamma^2_{\ 11} &= \frac{1}{2}g^{21}\left(g_{11,1}+g_{11,1}-g_{11,1}\right)+\frac{1}{2}g^{22}\left(g_{21,1}+g_{12,1}-g_{11,2}\right)\\
&= -\frac{1}{2}g^{22}g_{11,2}\\
&= -\frac{1}{2}\left(\frac{1}{a^2u^2}\right)\frac{\partial}{\partial v}\left(a^2+4u^2\right)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^2_{\ 12}=\Gamma^2_{\ 21} &= \frac{1}{2}g^{21}\left(g_{12,1}+g_{11,2}-g_{21,1}\right)+\frac{1}{2}g^{22}\left(g_{22,1}+g_{12,2}-g_{21,2}\right)\\
&= \frac{1}{2}g^{22}g_{22,1}\\
&= \frac{1}{2}\left(\frac{1}{a^2u^2}\right)\frac{\partial}{\partial u}\left(a^2u^2\right)\\
&= \frac{1}{u}
\end{aligned}$$

$$\begin{aligned}
\Gamma^2_{\ 22} &= \frac{1}{2}g^{21}\left(g_{12,2}+g_{21,2}-g_{22,1}\right)+\frac{1}{2}g^{22}\left(g_{22,2}+g_{22,2}-g_{22,2}\right)\\
&= \frac{1}{2}g^{22}g_{22,2}\\
&= \frac{1}{2}\left(\frac{1}{a^2u^2}\right)\frac{\partial}{\partial v}\left(a^2u^2\right)\\
&= 0
\end{aligned}$$

In summary, the only non-zero Christoffel coefficients that we have are $\Gamma^1_{\ 11}=4u/\left(a^2+4u^2\right)$, $\Gamma^1_{\ 22}=-a^2u/\left(a^2+4u^2\right)$ and $\Gamma^2_{\ 12}=\Gamma^2_{\ 21}=1/u$.

Part D

The Riemann Curvature tensor is defined by

$$R^l_{\ ijk}\equiv\frac{\partial\Gamma^l_{\ ik}}{\partial x^j}-\frac{\partial\Gamma^l_{\ ij}}{\partial x^k}+\sum_m\Gamma^m_{ik}\Gamma^l_{\ mj}-\sum_m\Gamma^m_{ij}\Gamma^l_{\ mk}$$

since we are dealing with a two dimensional surface, the only independent entry will be $R^1{}_{212}$, so it will be sufficient to only calculate this. We have

$$
\begin{aligned}
R^1{}_{212} &= \frac{\partial \Gamma^1{}_{22}}{\partial x^1} - \frac{\partial \Gamma^1{}_{21}}{\partial x^2} + \sum_m \Gamma^m{}_{22}\Gamma^1{}_{m1} - \sum_m \Gamma^m{}_{21}\Gamma^1{}_{m2} \\
&= \frac{\partial \Gamma^1{}_{22}}{\partial u} - \frac{\partial \Gamma^1{}_{21}}{\partial v} + \Gamma^1{}_{22}\Gamma^1{}_{11} + \Gamma^2{}_{22}\Gamma^1{}_{21} - \Gamma^1{}_{21}\Gamma^1{}_{12} - \Gamma^2{}_{21}\Gamma^1{}_{22} \\
&= \frac{\partial \Gamma^1{}_{22}}{\partial u} + \Gamma^1{}_{22}\Gamma^1{}_{11} - \Gamma^2{}_{21}\Gamma^1{}_{22} \\
&= \frac{\partial}{\partial u}\left(\frac{-a^2 u}{a^2+4u^2}\right) + \left(\frac{-a^2 u}{a^2+4u^2}\right)\left(\frac{4u}{a^2+4u^2}\right) - \left(\frac{1}{u}\right)\left(\frac{-a^2 u}{a^2+4u^2}\right) \\
&= \frac{-a^2}{a^2+4u^2} + \frac{8a^2u^2}{\left(a^2+4u^2\right)^2} - \frac{4a^2u^2}{\left(a^2+4u^2\right)^2} + \frac{a^2 u}{u\left(a^2+4u^2\right)} \\
&= \frac{-a^2 u\left(a^2+4u^2\right) + 8a^2u^3 - 4a^2u^3 + a^2 u\left(a^2+4u^2\right)}{u\left(a^2+4u^2\right)^2} \\
&= \frac{-a^4 u - 4a^2u^3 + 4a^2u^3 + a^4 u + 4a^2u^3}{u\left(a^2+4u^2\right)^2} \\
&= \frac{4a^2u^2}{\left(a^2+4u^2\right)^2}
\end{aligned}
$$

For the Riemann curvature tensor we have

$$R^1{}_{212} = R^2{}_{121} = \frac{4a^2u^2}{\left(a^2+4u^2\right)^2}$$

$$R^1{}_{221} = R^2{}_{112} = \frac{-4a^2u^2}{\left(a^2+4u^2\right)^2}$$

With all other entries equal to zero.

Part E

The Ricci tensor is defined by

$$R_{ij} \equiv \sum_k R^k{}_{ijk}$$

Using the fact that the Ricci tensor is symmetric we find the 4 entries of the Ricci tensor

$$
\begin{aligned}
R_{11} &= R^1{}_{111} + R^2{}_{112} \\
&= \frac{-4a^2u^2}{\left(a^2+4u^2\right)^2}
\end{aligned}
$$

$$\begin{aligned} R_{12} = R_{21} &= R^1{}_{121} + R^2{}_{122} \\ &= 0 \end{aligned}$$

$$\begin{aligned} R_{22} &= R^1{}_{221} + R^2{}_{222} \\ &= \frac{-4a^2u^2}{\left(a^2+4u^2\right)^2} \end{aligned}$$

Part F

The Ricci scalar is defined by

$$R \equiv \sum_{i,j} g^{ij} R_{ij}$$

So we have for the paraboloid

$$\begin{aligned} R &= g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22} \\ &= g^{11}R_{11} + g^{22}R_{22} \\ &= \left(\frac{1}{a^2+4u^2}\right)\left(\frac{4a^2u^2}{\left(a^2+4u^2\right)^2}\right) + \left(\frac{1}{a^2u^2}\right)\left(\frac{4a^2u^2}{\left(a^2+4u^2\right)^2}\right) \\ &= \frac{4a^2u^2}{\left(a^2+4u^2\right)^3} + \frac{4\left(a^2+4u^2\right)}{\left(a^2+4u^2\right)^3} \\ &= \frac{4a^2u^2+4a^2+16u^4}{\left(a^2+4u^2\right)^3} \\ &= \frac{4\left(4u^4+a^2\left(u^2+1\right)\right)}{\left(a^2+4u^2\right)^3} \end{aligned}$$

Part G

The Gaussian curvature of a two dimensional surface is given by

$$K = \frac{R_{1212}}{g}$$

where $g = \det\left[g_{ij}\right]$ (see Exercise 3.16 p105).

The determinant of a diagonal matrix is just the product of its diagonal entries so that

$$g = \prod_i g_{ii}$$

$$
\begin{aligned}
&= \left(a^2+4u^2\right)\left(a^2u^2\right) \\
&= a^4u^2+4a^2u^4 \\
&= a^2u^2\left(a^2+4u^2\right)
\end{aligned}
$$

R_{1212} is the element of the Riemann curvature tensor with an index lowered, i.e.

$$
\begin{aligned}
R_{1212} &= \sum_i g_{i1}R^i{}_{212} \\
&= g_{11}R^1{}_{212}+g_{21}R^2{}_{212} \\
&= \left(a^2+4u^2\right)\left(\frac{4a^2u^2}{\left(a^2+4u^2\right)^2}\right) \\
&= \frac{4a^2u^2}{a^2+4u^2}
\end{aligned}
$$

So we have for the Gaussian curvature

$$
\begin{aligned}
K &= \frac{R_{1212}}{g} \\
&= \left(\frac{4a^2u^2}{a^2+4u^2}\right)\left(\frac{1}{a^2u^2\left(a^2+4u^2\right)}\right) \\
&= \frac{4}{\left(a^2+4u^2\right)^2}
\end{aligned}
$$

Part H

No, the paraboloid is not Euclidean (flat). The necessary and sufficient condition for a surface to be flat is that the Riemann curvature tensor (all its components) should vanish (be equal to zero) at all points on the surface. This is only true at the point where $u = 0$, and not for all values of u and v.

Part I

Einstein's field equation for two dimensions is

$$
R_{ij}-\frac{1}{2}Rg_{ij}=-\kappa T_{ij}
$$

where i and j can take the values of 1 or 2, as with the rest of the calculations regarding the surface above. The only non-zero component of the energy-momentum tensor $[T^{ij}]$ for dust is $T^{11} = \rho c^2$.

$[T^{ij}]$ is related to $[T_{ij}]$ by

$$T_{ij} = \sum_{m,n} g_{im} g_{jn} T^{mn}$$

Clearly, the only non-zero component of $[T_{ij}]$ will be T_{11}. We find

$$\begin{aligned} T_{11} &= \sum_{m,n} g_{1m} g_{1n} T^{mn} \\ &= g_{11}g_{11}T^{11} + g_{11}g_{12}T^{12} + g_{12}g_{11}T^{21} + g_{12}g_{12}T^{22} \\ &= g_{11}g_{11}T^{11} \\ &= \rho c^2 \left(a^2 + 4u^2\right)^2 \end{aligned}$$

Now all the quantities in the Einstein field equation are known. We substitute and solve for κ

$$\begin{aligned} R_{11} - \frac{1}{2} R g_{11} &= -\kappa T_{11} \\ \frac{4a^2u^2}{(a^2+4u^2)^2} - \frac{1}{2}\left(\frac{4\left(4u^4 + a^2\left(u^2+1\right)\right)}{\left(a^2+4u^2\right)^3}\right)\left(a^2+4u^2\right) &= -\kappa\rho c^2 \left(a^2+4u^2\right)^2 \\ \frac{2a^2u^2 - 2a^2 - 8u^4}{\left(a^2+4u^2\right)^2} &= -\kappa\rho c^2 \left(a^2+4u^2\right)^2 \\ \kappa &= \frac{2a^2\left(1-u^2\right) + 8u^4}{\rho c^2 \left(a^2+4u^2\right)^4} \end{aligned}$$

Question 2

In a given frame of reference, with coordinates (x^0, x^1, x^2, x^3), the components of a symmetric second order tensor $[A^{\mu\nu}]$ are given by $A^{03} = -2$, $A^{13} = 1$ and $A^{02} = -1$, with all unspecified components being zero. Find the value of the component A'^{03} of the tensor $[A'^{\mu\nu}]$ in a frame obtained by the coordinate transformation

$$\begin{aligned} x'^0 &= 4x^0 - 13x^1 + 4x^2 - 15x^3 \\ x'^1 &= -13x^0 + 14x^1 - 3x^2 + 12x^3 \\ x'^2 &= x^0 - 4x^1 + 11x^2 - 5x^3 \\ x'^3 &= 8x^0 + 14x^1 - 3x^2 + 17x^3 \end{aligned}$$

Solution

We know $[A^{\mu\nu}]$ is a second order contravariant tensor, so its components transform according to the rule

$$A'^{\mu\nu} = \sum_{\alpha,\beta=0}^{4} \frac{\partial x'^{\mu}}{\partial x^{\alpha}} \frac{\partial x'^{\nu}}{\partial x^{\beta}} A^{\alpha\beta} .$$

Writing this out completely for A'^{03} gives

$$\begin{aligned} A'^{03} &= \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^3}{\partial x^0}A^{00} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^3}{\partial x^0}A^{10} + \frac{\partial x'^0}{\partial x^2}\frac{\partial x'^3}{\partial x^0}A^{20} + \frac{\partial x'^0}{\partial x^3}\frac{\partial x'^3}{\partial x^0}A^{30} \\ &\quad + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^3}{\partial x^1}A^{01} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^3}{\partial x^1}A^{11} + \frac{\partial x'^0}{\partial x^2}\frac{\partial x'^3}{\partial x^1}A^{21} + \frac{\partial x'^0}{\partial x^3}\frac{\partial x'^3}{\partial x^1}A^{31} \\ &\quad + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^3}{\partial x^2}A^{02} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^3}{\partial x^2}A^{12} + \frac{\partial x'^0}{\partial x^2}\frac{\partial x'^3}{\partial x^2}A^{22} + \frac{\partial x'^0}{\partial x^3}\frac{\partial x'^3}{\partial x^2}A^{32} \\ &\quad + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^3}{\partial x^3}A^{03} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^3}{\partial x^3}A^{13} + \frac{\partial x'^0}{\partial x^2}\frac{\partial x'^3}{\partial x^3}A^{23} + \frac{\partial x'^0}{\partial x^3}\frac{\partial x'^3}{\partial x^3}A^{33} \end{aligned}$$

We use the given components and the fact that $[A^{\mu\nu}]$ is symmetric to determine that all the components of $[A^{\mu\nu}]$ is equal to zero, except for $A^{03} = A^{30} = -2$, $A^{13} = A^{31} = 1$ and $A^{02} = A^{20} = -1$. Substituting all the components that are equal to zero reduces the above equation to

$$\begin{aligned} A'^{03} &= \frac{\partial x'^0}{\partial x^2}\frac{\partial x'^3}{\partial x^0}A^{20} + \frac{\partial x'^0}{\partial x^3}\frac{\partial x'^3}{\partial x^0}A^{30} + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^3}{\partial x^2}A^{02} + \frac{\partial x'^0}{\partial x^3}\frac{\partial x'^3}{\partial x^1}A^{31} + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^3}{\partial x^3}A^{03} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^3}{\partial x^3}A^{13} \\ &= \left(\frac{\partial x'^0}{\partial x^2}\frac{\partial x'^3}{\partial x^0} + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^3}{\partial x^2}\right)A^{20} + \left(\frac{\partial x'^0}{\partial x^3}\frac{\partial x'^3}{\partial x^0} + \frac{\partial x'^0}{\partial x^0}\frac{\partial x'^3}{\partial x^3}\right)A^{30} + \left(\frac{\partial x'^0}{\partial x^3}\frac{\partial x'^3}{\partial x^1} + \frac{\partial x'^0}{\partial x^1}\frac{\partial x'^3}{\partial x^3}\right)A^{13} \quad (1) \end{aligned}$$

From the given coordinate transformation equations, we determine the derivatives

$$\begin{aligned} \frac{\partial x'^0}{\partial x^0} &= 4,\ \frac{\partial x'^0}{\partial x^1} = -13,\ \frac{\partial x'^0}{\partial x^2} = 4,\ \frac{\partial x'^0}{\partial x^3} = -15 \\ \frac{\partial x'^3}{\partial x^0} &= 8,\ \frac{\partial x'^3}{\partial x^1} = 14,\ \frac{\partial x'^3}{\partial x^2} = -3,\ \frac{\partial x'^3}{\partial x^3} = 17 \end{aligned}$$

$$\begin{aligned} x'^0 &= 4x^0 - 13x^1 + 4x^2 - 15x^3 \\ x'^3 &= 8x^0 + 14x^1 - 3x^2 + 17x^3 \end{aligned}$$

We substitute these and the known components of $[A^{\mu\nu}]$ into (1) to get the value of A'^{03}

$$\begin{aligned} A'^{03} &= [4\times 8+4\times(-3)]\,A^{20}+[(-15)\times 8+4\times 17]\,A^{30}+[(-15)\times 14+(-13)\times 17]\,A^{13} \\ &= 20A^{20}-52A^{30}-431A^{13} \\ &= 20\,(-1)-52\,(-2)-431\,(1) \\ &= -347 \end{aligned}$$

Question 3

Show that if the metric g_{ij} is diagonal, then $\Gamma^i_{kl}=0$ whenever i, k and l are distinct, i.e. whenever $i\neq k\neq l$.

Solution

The Christoffel coefficients are defined by

$$\Gamma^i{}_{kl} = \sum_m \frac{1}{2} g^{im}\left(g_{mk,l}+g_{lm,k}-g_{kl,m}\right)$$

If the metric is diagonal, we have $g^{ij}=0$ whenever $i\neq j$. Therefore, all the terms in the sum above will be zero, except when $i=m$. Thus

$$\Gamma^i{}_{kl} = \frac{1}{2} g^{ii}\left(g_{ik,l}+g_{li,k}-g_{kl,i}\right)$$

Since the metric is diagonal, we also have $g_{ij}=0$ whenever $i\neq j$. If $i\neq k\neq l$, we will have $g_{ik}=g_{li}=g_{kl}=0$, and thus their derivatives will also be zero. Therefore, we have $\Gamma^i_{kl}=0$ if the metric is diagonal and i, k and l are distinct.

Question 4

If $A_{ab}=A_{ba}$ and $B^{ab}=-B^{ba}$ for all a, b, show that

$$\sum_{a,b} A_{ab}B^{ab}=0\,.$$

.

Solution

> **Note:** The easiest way to show that a quantity x is equal to zero is usually to show that $x = -x$.

Using the given symmetric and antisymmetric properties of $[A_{ab}]$ and $\left[B^{ab}\right]$, respectively, we can write

$$\sum_{a,b} A_{ab}B^{ab} = -\sum_{a,b} A_{ba}B^{ba}$$

On both sides of the equation, the indices a and b are just dummy indices. The two indices can be replaced by any other indices without changing the meaning of the expression, since they are just counters to be summed over. In particular, we can replace b with a and a with b, on the RHS so that

$$-\sum_{a,b} A_{ba}B^{ba} = -\sum_{a,b} A_{ab}B^{ab}$$

Using this, we can write

$$\begin{aligned} \sum_{a,b} A_{ab}B^{ab} &= -\sum_{a,b} A_{ba}B^{ba} \qquad \text{(given properties)} \\ &= -\sum_{a,b} A_{ab}B^{ab} \qquad \text{(relabel dummy indices)} \end{aligned}$$

Therefore we have shown that

$$\sum_{a,b} A_{ab}B^{ab} = 0\,.$$

Question 5

The principle of consistency requires that the laws of general relativity should approximate the laws of Newtonian physics in the Newtonian limit. Show that the relativistic momentum $p = \gamma mv$ reduces to the classical momentum when $v \ll c$.

Solution

For the relativistic momentum we have

$$p_{rel} = \gamma m v$$

Using the binomial theorem, for $v \ll c$ we can write

$$\begin{aligned}\gamma &= \left(1 - v^2/c^2\right)^{-1/2} \\ &= 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \frac{5}{16}\frac{v^6}{c^6} + \dots \\ &\approx 1\end{aligned}$$

In the last step we neglected all terms with c in the denominator. Since $v \ll c$, we have $v/c \ll 1$ so that all the terms containing powers of v/c will be very small compared to one.

Therefore, if $v \ll c$, we have $\gamma \approx 1$ and we can write

$$p_{rel} \approx mv = p_{clas}$$

The relativistic momentum therefore reduces to the classical momentum when the speed is much smaller that c.