Tutorial letter 101/3/2018

DIFFERENTIAL EQUATIONS

APM2611

Semesters 1 & 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains important information about your module.

BARCODE

UNISA university of south africa

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1 INTRODUCTION

Dear Student

Welcome to the APM2611 module in the Department of Mathematical Sciences at Unisa. We trust that you will find this module both interesting and rewarding.

Some of this tutorial matter may not be available when you register. Tutorial matter that is not available when you register will be posted to you as soon as possible, but is also available on *my*Unisa.

1.1 *my*Unisa

You must be registered on *my*Unisa (`http://my.unisa.ac.za`) to be able to submit assignments online, gain access to the library functions and various learning resources, download study material, "chat" to your lecturers and fellow students about your studies and the challenges you encounter, and participate in online discussion forums. *my*Unisa provides additional opportunities to take part in activities and discussions of relevance to your module topics, assignments, marks and examinations.

1.2 Tutorial matter

A tutorial letter is our way of communicating with you about teaching, learning and assessment. You will receive a number of tutorial letters during the course of the module. This particular tutorial letter contains important information about the scheme of work, resources and assignments for this module as well as the admission requirements for the examination. We urge you to read this and subsequent tutorial letters carefully and to keep it at hand when working through the study material, preparing and submitting the assignments, preparing for the examination and addressing queries that you may have about the course (course content, textbook, worked examples and exercises, theorems and their applications in your assignments, tutorial and textbook problems, etc.) to your APM2611 lecturers.

2 PURPOSE AND OUTCOMES FOR THE MODULE

2.1 Purpose

This module will be useful to students interested in developing those skills in modeling physical problems using differential equations and then solving them. Indeed, these techniques can be used in the natural, economic, social and mathematical sciences and have been central to our understanding of the world since Newton developed the idea of a differential equation nearly 350 years ago. Students who successfully complete this module will have a knowledge of those basic techniques required to recognise and solve certain types of well-known and commonly appearing differential equations. Also, you will be able to use differential equations to model, explain and predict the behaviour of certain physical processes.

2.2 Outcomes

2.2.1 Classify and recognise the basic types of differential equations
(Prescribed textbook, chapter 1).

2.2.2 Solve specific types of differential equations
(Prescribed textbook, chapter 2 and chapter 4).

2.2.3 Use differential equations to model practical situations
(Prescribed textbook, chapter 3).

2.2.4 Perform basic operations on infinite series
(Prescribed textbook, chapter 6).

2.2.5 Use Fourier Series and Laplace Transform
(Prescribed textbook, chapter 7 and chapter 11).

2.2.6 Solve a partial differential equation using separation of variables
(Prescribed textbook, chapter 12).

3 LECTURER(S) AND CONTACT DETAILS

3.1 Lecturer(s)

The contact details for the lecturer responsible for this module is

Postal address: The APM2611 Lecturers
Department of Mathematical Sciences
Private Bag X6
Florida
1709
South Africa

For 2018 the lecturer is:
Dr Emile Franc D. Goufo .
dgoufef@unisa.ac.za, Tel: +27 11 670 9159, Fax: +27 11 670 9171.
Office: GJ Gerwel C6-38, Science Campus, Florida

Additional contact details for the module lecturers will be provided in a subsequent tutorial letter.

All queries that are not of a purely administrative nature but are about the content of this module should be directed to your lecturer(s). Tutorial letter 301 will provide additional contact details for your lecturer. Please have your study material with you when you contact your lecturer by telephone. If you are unable to reach us, leave a message with the departmental secretary. Provide your name, the time of the telephone call and contact details. If you have problems with questions that you are unable to solve, please send your own attempts so that the lecturers can determine where the fault lies.

Please note: Letters to lecturers may not be enclosed with or inserted into assignments.

3.2 Department

The contact details for the Department of Mathematical Sciences are:

Departmental Secretary: (011) 670 9147 (SA) +27 11 670 9147 (International)

3.3 University

If you need to contact the University about matters not related to the content of this module, please consult the publication *Study @ Unisa* that you received with your study material. This booklet contains information on how to contact the University (e.g. to whom you can write for different queries, important telephone and fax numbers, addresses and details of the times certain facilities are open). Always have your student number at hand when you contact the University.

4 RESOURCES

4.1 Prescribed books

Prescribed books can be obtained from the University's official booksellers. If you have difficulty locating your book(s) at these booksellers, please contact the Prescribed Books Section at (012) 429 4152 or e-mail vospresc@unisa.ac.za.

Your prescribed textbook for this module is:

Title:	Differential Equations with Boundary-Value Problems
Authors:	Dennis G. Zill and Warren S. Wright
Edition:	International Edition - 8th edition
Publishers:	Cengage Learning (Brooks/Cole)
ISBN:	9781133492467

Please buy the textbook as soon as possible since you have to study from it directly – you cannot do this module without the prescribed textbook.

4.2 Recommended books

The book "Elementary Differential Equations with Boundary Value Problems" by William F. Trench and the corresponding solutions manual are available for free at the following web sites:

`http://digitalcommons.trinity.edu/mono/9/`
`http://digitalcommons.trinity.edu/mono/10/`

4.3 Electronic reserves (e-Reserves)

There are no e-Reserves for this module.

4.4 Library services and resources information

For brief information go to:
`http://www.unisa.ac.za/brochures/studies`

For more detailed information, go to the Unisa website: http://www.unisa.ac.za/, click on Library. For research support and services of Personal Librarians, go to:
http://www.unisa.ac.za/Default.asp?Cmd=ViewContent&ContentID=7102

The Library has compiled numerous library guides:

- find recommended reading in the print collection and e-reserves
  - http://libguides.unisa.ac.za/request/undergrad
- request material
  - http://libguides.unisa.ac.za/request/request
- postgraduate information services
  - http://libguides.unisa.ac.za/request/postgrad
- finding , obtaining and using library resources and tools to assist in doing research
  - http://libguides.unisa.ac.za/Research_Skills
- how to contact the Library/find us on social media/frequently asked questions
  - http://libguides.unisa.ac.za/ask

5 STUDENT SUPPORT SERVICES

For information on the various student support services available at Unisa (e.g. student counseling, tutorial classes, language support), please consult the publication *Study @ Unisa* that you received with your study material.

6 STUDY PLAN

The following table provides an outline of the outcomes and ideal dates of completion, and other study activities.

	Semester 1	Semester 2
Outcome 2.2.1 – 2.2.3 to be achieved by	1 March 2018	15 August 2018
Outcome 2.2.4 – 2.2.6 to be achieved by	4 April 2018	19 September 2018
Work through previous exam paper by Revision	25 April 2018	10 October 2018

See the brochure *Study @ Unisa* for general time management and planning skills.

7 PRACTICAL WORK AND WORK INTEGRATED LEARNING

There are no practicals for this module.

8 ASSESSMENT

8.1 Assessment criteria

8.2 Assessment plan

A final mark of at least 50% is required to pass the module. If a student does not pass the module then a final mark of at least 40% is required to permit the student access to the supplementary examination. The final mark is composed as follows:

Year mark			Final mark	
Assignment 01:	50%	$\longrightarrow$	Year mark:	20%
Assignment 02:	50%		Exam mark:	80%

Please note: if you fail the examination with **less than 40%**, the year mark will **not** be used, i.e. the exam counts 100% towards your final mark.

8.3 Assignment numbers

8.3.1 General assignment numbers

The assignments for this module are Assignment 01, Assignment 02, etc.

8.3.2 Unique assignment numbers

Please note that each assignment has a unique assignment number which must be written on the cover of your assignment.

8.3.3 Assignment due dates

The dates for the submission of the assignments are listed in the relevant ADDEMDUM here below.

8.4 Submission of assignments

You may submit written assignments either by post or electronically via *my*Unisa. Assignments may **not** be submitted by fax or e-mail.

For detailed information on assignments, please refer to the *Study @ Unisa* brochure which you received with your study package.

Please make a copy of your assignment before you submit!

To submit an assignment via *my*Unisa:

- Go to *my*Unisa.
- Log in with your student number and password.
- Select the module.
- Click on "Assignments" in the menu on the left-hand side of the screen.
- Click on the assignment number you wish to submit.
- Follow the instructions.

8.5 The assignments

Please make sure that you submit the correct assignments for the 1st semester, 2nd semester or year module for which you have registered. For each assignment there is a **fixed closing date**, the date at which the assignment must reach the University. When appropriate, solutions for each assignment will be dispatched, as Tutorial Letter 201 (solutions to Assignment 01) and Tutorial Letter 202 (solutions to Assignment 02) etc., a few days after the closing date. They will also be made available on *my*Unisa. Late assignments **will not** be marked!

Note that Assignment 01 is the compulsory assignment for admission to the examination and must reach us by the due date.

8.6 Other assessment methods

There are no other assessment methods for this module.

8.7 The examination

During the relevant semester, the Examination Section will provide you with information regarding the examination in general, examination venues, examination dates and examination times. For general information and requirements as far as examinations are concerned, see the brochure *Study @ Unisa* which you received with your study material.

Registered for . . .	Examination period	Supplementary examination period
1st semester module	May/June 2018	October/November 2018
2nd semester module	October/November 2018	May/June 2019
Year module	January/February 2019	May/June 2019

9 FREQUENTLY ASKED QUESTIONS

The *Study @ Unisa* brochure contains an A–Z guide of the most relevant study information.

10 IN CLOSING

We hope that you will enjoy APM2611 and we wish you all the best in your studies at Unisa!

ADDENDUM A: ASSIGNMENTS – FIRST SEMESTER

ASSIGNMENT 01
Due date: Wednesday, 7 March 2018
UNIQUE ASSIGNMENT NUMBER: 835495

ONLY FOR SEMESTER 1

First order separable, linear, Bernoulli, exact and homogeneous equations. Higher order homogeneous DE's. Solving non-homogeneous DE's using the undetermined coefficients, variation of parameters and operator methods.

Answer all the questions. Show all your workings.

If you choose to submit via *my*Unisa, note that only PDF files will be accepted.

Note that only some questions will be marked. However, it is highly recommended to attempt all questions since the questions you leave undone might be the chosen ones, giving you **ZeRo** mark. The questions which will be marked **will not be announced in advance**.

Question 1

Solve the following differential equations:

(1.1) $x^2\frac{dy}{dx} + 2xy = 5y^3$, where $x > 0$.

(1.2) $\frac{dy}{dx} = \frac{xy-y^2}{x^2-y^2}$

(1.3) $(6xy - y^3)dx + (4y + 3x^2 - 3xy^2)dy = 0$.

(1.4) $x\frac{dy}{dx} + 4y = x^3 - x$, where $x > 0$.

(1.5) $x^2\frac{dy}{dx} = y^2 + 2y + 1$, where $x > 0$.

(1.6) $\dfrac{dy}{dx} - y = xy^5$. ()

Question 2

Consider the DE

$$y'' - y = e^{2x}.$$

Using the **method of variation of parameters**,

(2.1) Find a solution for the homogeneous part of the DE.

(2.2) Find a particular solution.

(2.3) Write down the general solution for the DE. ()

Question 3

Solve the following differential equation and determine the interval of validity of the solution:

$$\frac{dy}{dx} = 6y^2x, \qquad y(1) = \frac{1}{25}.$$

Question 4

Using the method of variation of parameters find the general solution of the differential equation

$$x^2y'' + xy' - y = x,$$

given that $y_1 = x$ and $y_2 = 1/x$ are solutions of the corresponding homogeneous equation.

Question 5

Radium decomposes at a rate proportional to the quantity of radium present. Suppose that it is found that in 30 years approximately 1.5% of a certain quantity of radium has decomposed. Determine approximately how long it will take for one-half of the original amount of radium to decompose.

– End of assignment –

ASSIGNMENT 02
Due date: Wednesday, 11 April 2018
UNIQUE ASSIGNMENT NUMBER: 840402

ONLY FOR SEMESTER 1

Series solutions, Laplace transforms and Fourier series, solving PDE's by separation of variables.

Answer all the questions. Show all your workings.

If you choose to submit via *my*Unisa, note that only PDF files will be accepted.

Note that only some questions will be marked. However, it is highly recommended to attempt all questions since the questions you leave undone might be the chosen ones, giving you **ZeRo** mark. The questions which will be marked **will not be announced in advance**.

Question 1

Use the power series method to solve the initial value problem:

$$(x^2+1)y'' - 6xy' + 12y = 0; \qquad y(0) = 1, y'(0) = 1.$$

Question 2

Calculate the Laplace transform of the following functions:

(2.1) $t \sin 2t$.

(2.2) $(2t+1)\mathcal{U}(t-1)$.

(2.3) $t \int_0^t \tau e^{-\tau} d\tau$.

Question 3

Calculate the Laplace transform of the following function from first principles:

$$f(t) = \begin{cases} \cos t & 0 \le t < \pi \\ 0 & t \ge \pi \end{cases}$$

Question 4

Calculate the following inverse Laplace transforms:

(4.1) $\mathcal{L}^{-1}\left\{\dfrac{2s+1}{s^2+4s+13}\right\}$ ()

(4.2) $\mathcal{L}^{-1}\left\{\dfrac{e^{-2s}}{s-2}\right\}$ ()

Question 5

Solve the following initial value problem by using Laplace transforms:

$$x''(t) + 5x'(t) + 6x(t) = e^{-2t}, \qquad x(0) = 1, \quad x'(0) = -2.$$

Question 6

Compute the Fourier series for

$$x + \pi \qquad \text{on} \qquad (-\pi, \pi).$$

Question 7

Use separation of variables to find a product solution of the following partial differential equation (other than $u(x, y) = 0$):

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u$$

– End of assignment –

ADDENDUM B: ASSIGNMENTS – SECOND SEMESTER

ASSIGNMENT 01
Due date: Wednesday, 22 August 2018
UNIQUE ASSIGNMENT NUMBER: 751360

ONLY FOR SEMESTER 2

First order separable, linear, Bernoulli, exact and homogeneous equations. Higher order homogeneous DE's. Solving non-homogeneous DE's using the undetermined coefficients, variation of parameters and operator methods.

Answer all the questions. Show all your workings.

If you choose to submit via *my*Unisa, note that only PDF files will be accepted.

Note that only some questions will be marked. However, it is highly recommended to attempt all questions since the questions you leave undone might be the chosen ones, giving you **ZeRo** mark. The questions which will be marked **will not be announced in advance**.

Question 1

Solve the following differential equations:

(1.1) $2(y+3)dx - xydy = 0.$

(1.2) $(x^2 - xy + y^2)dx - xydy = 0.$

(1.3) $3x(xy-2)dx + (x^3 + 2y)dy = 0.$ ()

(1.4) $\dfrac{dy}{dx} = -\dfrac{3x+4y-1}{3x+4y+2}.$ ()

(1.5) $\dfrac{dy}{dx} + 2xy = 3x.$ ()

(1.6) $\dfrac{dy}{dx} + 2xy + xy^4 = 0.$ ()

Question 2

(2.1) Solve the following boundary value problem:

$$y'' - 2y' - 3y = 0, \qquad y(0) = 4, \quad y'(0) = 0$$

(2.2) Use the method of undetermined coefficients to solve the following differential equation:

$$y'' - 4y' + 3y = 2\cos x + 4\sin x.$$

Question 3

Solve the following differential equation and determine the interval of validity of the solution:

$$\frac{dy}{dx} = \frac{xy^3}{\sqrt{1+x^2}}, \qquad y(0) = -1.$$

Question 4

Using the method of variation of parameters find the general solution of the differential equation

$$x^2y'' - 3xy' + 3y = 12x^4,$$

given that $y_1 = x$ and $y_2 = x^3$ are solutions of the corresponding homogeneous equation.

Question 5

A tank contains 300 litres of fluid in which 20 grams of salt is dissolved. Brine containing 1 gram of salt per litre is then pumped into the tank at a rate of 4 litres per minute; the well-mixed solution is then pumped out at the same rate. Find the number $N(t)$ of grams of salt in the tank at time t.

– End of assignment –

ASSIGNMENT 02
Due date: Wednesday, 26 September 2018
UNIQUE ASSIGNMENT NUMBER: 892122

ONLY FOR SEMESTER 2

Series solutions, Laplace transforms and Fourier series, solving PDE's by separation of variables.

Answer all the questions. Show all your workings.

If you choose to submit via *my*Unisa, note that only PDF files will be accepted.

Note that only some questions will be marked. However, it is highly recommended to attempt all questions since the questions you leave undone might be the chosen ones, giving you **ZeRo** mark. The questions which will be marked **will not be announced in advance**.

Question 1

Use the power series method to solve the initial value problem:

$$(1-x^2)y'' - 6xy' - 4y = 0; \qquad y(0) = 1, \quad y'(0) = 2.$$

Question 2

Calculate the Laplace transform of the following functions:

(2.1) $t^2 \sin t$. ()

(2.2) $t^2\,\mathcal{U}(t-2)$. ()

(2.3) $\displaystyle\int_0^t \frac{1-e^{-u}}{u}\,du$. ()

Question 3

Calculate the Laplace transform of the following function from first principles:

$$f(t) = \begin{cases} t & 0 \le t < 4 \\ 5 & t \ge 4 \end{cases}$$

Question 4: **Marks**

Calculate the following inverse Laplace transforms:

(4.1) $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$ ()

(4.2) $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\}$ ()

Question 5: **Marks**

Solve the following initial value problem by using Laplace transforms:

$$y''(t) + 9y = \cos 3t, \qquad y(0) = 2, \quad y'(0) = 5.$$

Question 6: **Marks**

Compute the Fourier series for the function

$$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$$

on $[-1, 1]$.

Question 7: **Marks**

Using the method of separation of variables, solve the following boundary value problem:

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}; \qquad u(0, y) = 8e^{-3y}.$$

– End of assignment –

ADDENDUM C: USEFUL COMPUTER SOFTWARE

It is possible to check the correctness of your calculations by hand. If you are interested in software that may help to check your results please consult the following resources. **Note however that the software will not be available at exam time, so it is recommended to be proficient at checking your own results by hand**.

Maxima:
`http://maxima.sourceforge.net/`
`http://maxima.sourceforge.net/docs/intromax/intromax.html`
`http://maxima.sourceforge.net/docs/manual/en/maxima_44.html`

Maxima is also available for Android devices:
`https://sites.google.com/site/maximaonandroid/`

See addendum D for a brief introduction to Maxima for differential equations.

Wolfram Alpha:
`http://www.wolframalpha.com/`
`http://www.wolframalpha.com/examples/DifferentialEquations.html`

Please note that the use of software **is not required** for this module.

ADDENDUM D: DIFFERENTIAL EQUATIONS USING MAXIMA

A complete guide to Maxima is beyond the scope of this module. Here we list only the most essential features. Please consult `http://maxima.sourceforge.net/` for documentation on Maxima.

Please note that the use of software **is not required** for this module.

D.1 The `contrib_ode` package

First we load the package `contrib_ode`. Type only the line following `(%i1)` in the white boxes, i.e. `load(contrib_ode);`

```
(%i1) load(contrib_ode);
```

```
(%o1)          /usr/maxima/5.29/share/contrib/diffequations/contrib_ode.mac
```

The output `(%o1)` may be different, but there should be no error messages. Note the semicolon `;` after every command.

D.2 Solving differential equations

Consider the differential equation (initial value problem)

$$y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x), \qquad y(0) = 0, y'(0) = 1.$$

The derivative y' is written as `diff(y,x)` in Maxima, and the second derivative is written as `diff(y,x,2)`. However, we must first state that y is the dependent variable:

```
(%i2) depends(y,x);
```

```
(%o2)                                [y(x)]
```

Now we input the differential equation. The number e is written in maxima as `%e`:

```
(%i3) contrib_ode(diff(y,x,2) - 2*diff(y,x) + 2*y
                          = %e^(2*x)*(cos(x) - 3*sin(x)), y, x);
```

```
                                                  2 x              2 x
           x                                    %e    sin(x) - 7 %e    cos(x)
(%o3) [y = %e  (%k1 sin(x) + %k2 cos(x)) - ------------------------------]
                                                             5
```

Type carefully to reproduce the input `(%i3)` correctly. Note that x^y is written as `x^y`, and that multiplication has to be written explicitly, i.e. xy is written as `x*y`. The constants of integration are `%k1` and `%k2`. We solve for them from the initial conditions $y(0) = 0$ and $y'(0) = 1$:

```
(%i4) append(subst([y=0,x=0], %o3),
             subst([diff(y,x)=1,x=0], diff(%o3,x)));
```

```
                          7                   13
(%o4)          [0 = %k2 + -, 1 = %k2 + %k1 + --]
                          5                   5
```

Here `append` is used to combine the two equations into a system of equations (in Maxima: a list of equations). The operation `subst` performs the substitutions. The order is important! We substitute *first* $y(x) = 0$ and *then* $x = 0$ into the equation. The second `subst` operation implements $y'(0) = 1$. Since `%o3` is the equation for y, we simply differentiate both sides of the equation i.e. `diff(%o3,x)`. Then we again substitute (in order!) $y'(x) = 1$ and $x = 0$. Now we can solve for the constants of integration:

```
(%i5) solve(%, [%k1, %k2]);
```

```
                             1         7
(%o5)             [[%k1 = - -, %k2 = - -]]
                             5         5
```

Here `%` means the last result, i.e. `%o4`. To obtain the final solution, we substitute the constants of integration into the equation for y:

```
(%i6) subst(%, %o3);
```

```
                                          2 x              2 x
               x     sin(x)    7 cos(x)     %e    sin(x) - 7 %e    cos(x)
(%o6)   [y = %e  (- ------ - --------) - -----------------------------]
                       5         5                     5
```

In other words, the solution is

$$y = -\frac{e^x}{5}(\sin(x) + 7\cos(x)) - \frac{e^{2x}}{5}(\sin(x) - 7\cos(x)).$$

D.3 Partial fraction decomposition

Suppose we wish to find the partial fraction decomposition of

$$\frac{5s+2}{s^2+3s+2}$$

This can be achieved as follows.

```
(%i7) partfrac((5*s+2)/(s^2+3*s+2), s);
```

```
                        8       3
(%o7)                 ----- - -----
                      s + 2   s + 1
```

The `s` here (in `partfrac(..., s)`) is the variable over which the fractions are split. For example, in

$$\frac{as+2}{s^2+3s+2}$$

we have two variables (a and s) but we want the partial fraction decomposition over s:

```
(%i8) partfrac((a*s+2)/(s^2+3*s+2), s);
```

```
                                 2 a - 2   2 - a
(%o8)                            ------- + -----
                                  s + 2    s + 1
```

D.4 Laplace transform

Consider again the differential equation (initial value problem)

$$y'' - 2y' + 2y = e^{2t}(\cos t - 3\sin t), \qquad y(0) = 0, y'(0) = 1.$$

Let's remove the dependency of y on x:

```
(%i9) remove(y, dependency);
```

```
(%o9)                                  done
```

Now we take the Laplace transform on both sides of the differential equation

```
(%i10) laplace(diff(y(t),t,2) - 2*diff(y(t),t) + 2*y(t)
                    = %e^(2*t)*(cos(t) - 3*sin(t)), t, s);
```

```
                      !
         d            !
(%o10) - -- (y(t))!      - 2 (s laplace(y(t), t, s) - y(0))
         dt           !
                      !t = 0
          2
       + s  laplace(y(t), t, s) + 2 laplace(y(t), t, s) - y(0) s
                      s - 5
                 = ------------
                     2
                    s  - 4 s + 5
```

Substituting in the initial values

```
(%i11) subst([y(0)=0, diff(y(t),t)=1],%);
```

```
        2
(%o11) s  laplace(y(t), t, s) - 2 s laplace(y(t), t, s)

                                                 s - 5
             + 2 laplace(y(t), t, s) - 1 = ------------
                                             2
                                            s  - 4 s + 5
```

yields an algebraic equation for $Y(s)$ (denoted here by `laplace(y(t), t, s)`). So we solve for $Y(s)$

```
(%i12) solve(%, laplace(y(t),t,s));
```

```
                                                    2
                                                   s  - 3 s
(%o12)       [laplace(y(t), t, s) = -----------------------------]
                                         4      3       2
                                        s  - 6 s  + 15 s  - 18 s + 10
```

Once again we have a list of (one) solutions. Applying the inverse laplace transform will yield the solution in t:

```
(%i13) ilt(first(%), s, t);
```

```
                   2 t  7 cos(t)   sin(t)      t    sin(t)   7 cos(t)
(%o13)   y(t) = %e    (-------- - ------) + %e  (- ------ - --------)
                           5         5                5         5
```

In other words, the solution is

$$y = \frac{e^{2t}}{5}(7\cos(t) - \sin(t)) - \frac{e^{t}}{5}(\sin(t) + 7\cos(t)).$$

Compare this result to our first example above.

For piecewise defined functions we need the unit step function $\mathscr{U}(x)$. In Maxima, the unit step function is written `unit_step(x)`.

D.5 Fourier transform

First we load the package `fourie`.

```
(%i14) load(fourie);
```

```
(%o14)      /usr/pkg/share/maxima/5.32.1/share/calculus/fourie.mac
```

Now we can calculate the coefficients for the Fourier transform. Consider the Fourier transform of $f(x) = 2x$ on $(-1, 1)$. Thus we type `fourier(`$f(x)$`,`x`,`p`)` for the interval $(-p, p)$ (and in our case $p = 1$).

```
(%i15) fourier(2*x,x,1);
```

```
(%t15)                                  a  = 0
                                         0

(%t16)                                  a  = 0
                                         n

                              2 sin(%pi n)   2 cos(%pi n)
(%t17)                b  = 2 (------------ - ------------)
                       n              2  2         %pi n
```

```
                                %pi  n

(%o17)                      [%t15, %t16, %t17]
```

The variable `%t15` is a temporary variable introduced during the Fourier transform. Obviously b_n can be simplified, we use `foursimp` to do so.

```
(%i18) foursimp(%);
```

```
(%t18)                             a  = 0
                                    0

(%t19)                             a  = 0
                                    n

                                            n
                                      4 (- 1)
(%t20)                         b  = - --------
                                n      %pi n

(%o20)                      [%t18, %t19, %t20]
```

It is clear that $f(x) = 2x$ is an odd function, so we can instead find the coefficients of the Fourier sine series using `foursin`.

```
(%i21) foursin(2*x,x,1);
```

```
                           2 sin(%pi n)   2 cos(%pi n)
(%t21)             b  = 2 (------------ - ------------)
                    n            2  2         %pi n
                              %pi  n

(%o21)                            [%t21]
```

```
(%i22) foursimp(%);
```

```
                                            n
                                      4 (- 1)
(%t22)                         b  = - --------
                                n      %pi n

(%o22)                            [%t22]
```

For an even function we can compute the coefficients of the Fourier cosine series using `fourcos` in a similar way.

D.6 Some assignment type questions with solutions

D.6.1 Assignment 1 type

Question 8: **Marks**

Solve the following differential equation by separating the variables:

$$\frac{dy}{dx} = xy^3.$$

Rewriting the equation as $\frac{dy}{y^3} = xdx$, *we get*

$$\int \frac{dy}{y^3} = \int xdx \quad \text{or} \quad \int y^{-3}dy = \int xdx,$$

giving

$$\frac{1}{-2}y^{-2} = \frac{1}{2}x^2 + c_1.$$

Hence, the solution of the differential equation is given by the family

$$y^{-2} + x^2 = +c_2,$$

where $c_2 = -2c_1$ *is the constant of integration.*
Give an interval of validity for your solution subject to the initial condition

(8.1) $y(0) = 0,$ ()

The solution becomes $y^2 = \pm\sqrt{\frac{1}{c_2 - x^2}}$ *and when* $x = 0, y = 0$, *so* $0 = \pm\sqrt{\frac{1}{c_2 - 0}}$, *meaning* $\frac{1}{c_2} = 0$. *there is no such constant. Therefore, the interval of validity is the empty set* ϕ

(8.2) $y(0) = 1.$ ()

Now, when $x = 0, y = 1$, *so* $1^2 = \pm\sqrt{\frac{1}{c_2 - 0^2}}$, *yielding* $c_2 = 1$. *The solution becomes* $y^2 = \pm\sqrt{\frac{1}{1-x^2}}$. *This solution is valid for* $1 - x^2 \neq 0$ *giving* $x \neq 1$ *or* $x \neq -1$. *Therefore, the interval of validity is* $\mathbb{R} - \{-1, 1\} = (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$.

Question 9: **Marks**

Solve the following differential equations:

(9.1) $x\frac{dy}{dx} + 2y = x^{-3}.$ ()

This equation is linear in $y(x)$. *Dividing by* x *yields the standard form*

$$\frac{dy}{dx} + \frac{2}{x}y = x^{-4}, \quad x \neq 0.$$

The integrating factor is

$$\mu(x) = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2 = x^2.$$

Multiplying the equation by the integrating factor $\mu(x)$ *yields*

$$\frac{d(x^2y)}{dx} = x^{-2}$$

Integrating both sides with respect to x *yields*

$$x^2y = -\frac{1}{x} + c, \quad x \neq 0,$$

where c *is the constant of integration. Finally, we solve for* y *to obtain*

$$y = \frac{c}{x^2} - \frac{1}{x^3}.$$

(9.2) $(1/y)dx - (3y + x/y^2)dy = 0.$ ()

Identifying $M(x,y) = 1/y$ *and* $N(x,y) = -3y - x/y^2$ *we find*

$$\frac{\partial M}{\partial y} = -1/y^2, \qquad \frac{\partial N}{\partial x} = -1/y^2 = \frac{\partial M}{\partial y}$$

This is an exact differential equation and from Theorem 2.4.1 (in the prescribed book), there is a function $f(x,y)$ *such that*

(1)
$$\frac{\partial f}{\partial x} = 1/y, \qquad \frac{\partial f}{\partial y} = -3y - x/y^2.$$

Hence, from the first of these equations, we have

$$f(x;y) = \int (1/y)dx + g(y) = x/y + g(y)$$

where we treat y *as constant and* $g(y)$ *is the "constant" of integration. It follows that*

$$-x/y^2 + g'(y) = \frac{\partial f}{\partial y} = -3y - x/y^2$$

so that $g'(y) = -3y$ *yielding* $g(y) = -(3/2)y^2$*. The implicit solution is* $f(x,y) = c$*, i.e.*

$$\frac{x}{y} - \frac{3}{2}y^2 = c,$$

where c *is the constant of integration.*

Alternative*: We can find the function* $f(x,y)$ *from the seconde equation of* (**??**) *as follows:*

$$f(x;y) = \int (-3y - x/y^2)dy + h(x) = -(3/2)y^2 + x/y + h(x)$$

where we treat x *as constant and* $h(x)$ *is the "constant" of integration. It follows that*

$$1/y + h'(x) = \frac{\partial f}{\partial x} = 1/y$$

so that $h'(x) = 0$ *yielding* $h(x) = c_1$ *= constant. The implicit solution is* $f(x,y) = c$*, i.e.*

$$-(3/2)y^2 + x/y + c_1 = c \quad \text{equivalent to} \quad \frac{x}{y} - \frac{3}{2}y^2 = k$$

where $k = c - c_1$ *is the constant of integration.*

(9.3) $\dfrac{dy}{dx} = \dfrac{x^2 - y^2}{3xy}$. ()

The equation is equivalent to $(x^2 - y^2)dx - 3xydy = 0$ *This equation is homogeneous of degree 2 since*

$$M(tx, ty) = (tx)^2 - (ty)^2 = t^2(x^2 - y^2) = t^2M(x, y)$$

and

$$N(tx, ty) = -3(tx)(ty) = t^2(-3xy) = t^2N(x, y).$$

Thus, we use the substitution $y = ux$ *so that* $dy = xdu + udx$. *The equation becomes*

$$(x^2 - u^2x^2)dx - 3ux^2(xdu + udx) = 0$$

$$x^2(1 - 4u^2)dx - 3ux^3du = 0$$

This equation is separable and separating variables

$$\frac{1}{x}dx = \frac{3u}{1 - 4u^2}du, \quad 1 - 4u^2 \neq 0, \quad x \neq 0.$$

Integrating both sides yields

$$\ln|x| = -(3/8)\ln|1 - 4u^2| + c, \quad 1 - 4u^2 \neq 0, \quad x \neq 0,$$

where c is the constant of integration. Multiplying by 8, using properties of the logarithm function and exponentiating gives the equation

$$|x|^8 = e^{8c}|1 - 4u^2|^{-3}$$

Substituiting $u = y/x$ *gives*

$$|x|^8 = \frac{e^{8c}}{|1 - 4(\frac{y}{x})^2|^3}$$

which is the implicit solution. This needs the restriction $1 - 4(\frac{y}{x})^2 \neq 0$ *equivalent to* $4y^2 \neq x^2$

Next, we show that $y^2 = x^2/4$ *is a singular solution for the equation*

$$\frac{dy}{dx} = \frac{x^2 - y^2}{3xy}.$$

Implicit differentiation provides $\frac{d}{dx}y^2 = \frac{d}{dx}x^2/4$, *in other words* $2y\frac{dy}{dx} = x/2$. *It follows that, when* $y^2 = x^2/4$,

$$\frac{dy}{dx} = \frac{x}{4y}.$$

On the other hand, when $y^2 = x^2/4$,

$$\frac{x^2 - y^2}{3xy} = \frac{x^2 - x^2/4}{3xy} = \frac{x}{4y} = \frac{dy}{dx}.$$

(9.4) $\dfrac{dy}{dx} + \dfrac{y}{x-2} = 5(x-2)y^{1/2}$. ()

Notice that $y = 0$ satisfies the equation. Now assume $y \neq= 0$. This equation is a Bernoulli equation with $n = 1/2$. To make the equation linear, we set $u = y^{1-1/2} = y^{1/2}$ so that

$$\frac{du}{dx} = \frac{1}{2}y^{-1/2}\frac{dy}{dx} \quad \Longrightarrow \quad \frac{dy}{dx} = 2y^{1/2}\frac{du}{dx}.$$

Hence,

$$2y^{1/2}\frac{du}{dx} + \frac{y}{x-2} = 5(x-2)y^{1/2}$$

which simplifies to

$$\frac{du}{dx} + \frac{y^{1/2}}{2(x-2)} = \frac{5}{2}(x-2)$$

$$\frac{du}{dx} + \frac{1}{2(x-2)}u = \frac{5}{2}(x-2)$$

Thus we have a linear equation with integrating factor

$$\mu(x) = e^{\int \frac{1}{2(x-2)}dx} = e^{(1/2)\ln|x-2|} = |x-2|^{1/2}$$

and multiplying by the integrating factor yields

$$\frac{d}{dx}(u(x-2)^{1/2}) = \frac{5}{2}(x-2)(x-2)^{1/2}$$

$$\frac{d}{dx}(u(x-2)^{1/2}) = \frac{5}{2}(x-2)^{3/2}$$

Integrating both sides of the equation leads to

$$u(x-2)^{1/2} = \frac{5}{2}\frac{1}{\frac{3}{2}+1}(x-2)^{\frac{3}{2}+1} + c,$$

giving

$$u = (x-2)^2 + c(x-2)^{-1/2}.$$

Then,

$$y^{1/2} = u = (x-2)^2 + c(x-2)^{-1/2}$$

so that

$$y = \left[(x-2)^2 + c(x-2)^{-1/2}\right]^2,$$

with c is the constant of integration.

Question 10: **Marks**

The air in a room, of size 12m by 8m by 8m, is 3% carbon monoxide. Starting at time $t = 0$, fresh air containing no carbon monoxide is blown into the room at a rate of $100\text{m}^3/\text{min}$. If air in the room flows out through a vent at the same rate, when will the air in the room be 0.01% carbon monoxide?

In this exercise, it is easy to think in terms of the volume of the CO (= carbon monoxide)as part of the gas in the room. There are many ways of solving the problem, but note that we are told nothing related to gas density or mass. Let $v(t)$ be the volume of CO in the room (measured in m^3) at the time t (measured in minutes). Then $v(0) = \frac{3}{100} \times 768 = 23.04m^3$ because the room has volume $8 \times 8 \times 12 = 768m^3$ and is 3% CO. From the basic principle of mathematical modeling, we say $\frac{dv}{dt}$ = (input) - (output) giving

$$\frac{dv}{dt} = 0 - 100 \times \frac{v(t)}{768}$$

where the "input=0" is due to the fact that no CO enters the room while the "output" is given by the fact that the volume of CO leaving the room is $100m^3/min$ times the fraction of the gas which is CO (given by $\frac{v(t)}{768}$). (We assume the gases are well-mixed!) hence,

$$\frac{dv}{dt} = -\frac{100}{768}v(t)$$

which has solution

$$v(t) = v(0)e^{-\frac{100}{768}t} = 23.04e^{-\frac{100}{768}t}.$$

We want to find the time when

$$\frac{0.01}{100} \times 768 = 23.04 \times e^{-\frac{100}{768}t}$$

Dividing by 23.04 and using the properties of the logarithm function yield

$$t = -\frac{768}{100} \times \ln\left(\frac{0.0001 \times 768}{23.04}\right) \simeq 43.8$$

This time is about $t = 43.8$ minutes.

Alternative: *let r be the percentage of room air changed in one minute then*

$$r = \frac{100}{768} \times 100 \simeq 13.02\% = 0.1302$$

let p be the percentage volume of CO in the room at any time t.

$$\frac{dp(t)}{dt} = -rp(t)$$

with the solution

$$p = p(0)e^{-rt} = 3e^{-rt}$$

We want to find the time when $p = 0.01$ giving $0.01 = p(0)e^{-rt} = 3e^{-rt}$. Hence,

$$t = \frac{\ln 300}{r} = \frac{\ln 300}{0.1302} = 43.8 \text{ minutes}$$

Question 11: **Marks**

(11.1) Find a general solution for the following differential equation: ()

$$y^{(4)} - 6y''' + 17y'' - 28y' + 20y = 0$$

given that

$$x^4 - 6x^3 + 17x^2 - 28x + 20 = (x-2)^2(x^2 - 2x + 5).$$

This is a homogeneous linear equation with constant coefficients. As shown in Chapter 4, the equation requires the calculation of the roots of a quartic polynomial given as

$$m^4 - 6m^3 + 17m^2 - 28m + 20 = 0$$

Since

$$m^4 - 6m^3 + 17m^2 - 28m + 20 = (m-2)^2(m^2 - 2m + 5)$$

then the roots of

$$m^4 - 6m^3 + 17m^2 - 28m + 20 = 0$$

are

$$m_1 = m_2 = 2, \quad m_3 = 1 + \sqrt{-4} = 1 + 2i, \quad m_4 = 1 - \sqrt{-4} = 1 - 2i.$$

Thus we find the general solution

$$y(x) = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^x \cos 2x + c_4 e^x \sin 2x.$$

(11.2) Find a general solution for the differential equation ()

$$y'' - 5y' + 6y = e^{3x} - x^2$$

We first find the complimentary solution. The auxiliary equation is

$$m^2 - 5m + 6 = 0$$

which has the following two roots

$$m_1 = 2, \quad m_2 = 3$$

Thus the complementary solution is

$$y_c = c_1 e^{2x} + c_2 e^{3x}$$

(a) using the method of undetermined coefficients. ()

The right hand side of the equation has the form

$$Ae^{3x} + Bxe^{3x} + C + Dx + Ex^2$$

However, note that Ae^{3x} is already included in the expression of y_c, (which is $c_2 e^{3x}$). So it is ignored and then, the particular solution takes the form

$$y_p = Bxe^{3x} + C + Dx + Ex^2.$$

The derivatives are

$$y_p' = D + 2xE + Be^{3x} + 3xBe^{3x}$$

$$y_p'' = 2E + 6Be^{3x} + 9xBe^{3x}$$

Inserting the derivatives into the equation yields

$$\begin{aligned} 1 &: \quad 2E - 5D + 6C = 0 \\ x &: \quad 10E + 6D = 0 \\ x^2 &: \quad 6E = -1 \\ e^{3x} &: \quad 6B - 5B = 1 \\ xe^{3x} &: \quad 9B - 15B + 6B = 0 \end{aligned}$$

It follows that $B = 1$, $C = -19/108$, $D = -5/18$ *and* $E = -1/6$. *Thus the particular solution is*

$$y_p = xe^{3x} - \frac{19}{108} - \frac{5}{18}x - \frac{1}{6}x^2$$

and the general solution is

$$y(x) = y_c + y_p = c_1 e^{2x} + c_2 e^{3x} + xe^{3x} - \frac{19}{108} - \frac{5}{18}x - \frac{1}{6}x^2$$

(b) using variation of parameters. ()

We assume from the form of the complimentary solution that $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$ *where* $y_1(x) = e^{2x}$ *and* $y_2(x) = e^{3x}$. *The equation* $y'' - 5y' + 6y = e^{3x} - x^2$ *is already written in standard form (i.e. the leading coefficient should be 1.) The Wronskian is given by*

(2) $$W(e^{2x}, e^{3x}) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = e^{5x}$$

and

(3) $$W_1 = \begin{vmatrix} 0 & e^{3x} \\ e^{3x} - x^2 & 3e^{3x} \end{vmatrix} = x^2 e^{3x} - e^{6x}$$

(4) $$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & e^{3x} - x^2 \end{vmatrix} = e^{5x} - x^2 e^{2x}$$

We find

$$u_1' = \frac{W_1}{W} = \frac{x^2 e^{3x} - e^{6x}}{e^{5x}} = x^2 e^{-2x} - e^x$$

$$u_2' = \frac{W_2}{W} = \frac{e^{5x} - x^2 e^{2x}}{e^{5x}} = 1 - x^2 e^{-3x}$$

Integrating by parts yields

$$u_1 = -\frac{1}{2}\left(x^2 + x + \frac{1}{2}\right)e^{-2x} - e^x$$

$$u_2 = x + \frac{1}{3}\left(x^2 + \frac{2}{3}x + \frac{2}{9}\right)e^{-3x}$$

so that

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$$

$$y_p = \left(-\frac{1}{2}\left(x^2 + x + \frac{1}{2}\right)e^{-2x} - e^x\right)e^{2x} + \left(x + \frac{1}{3}\left(x^2 + \frac{2}{3}x + \frac{2}{9}\right)e^{-3x}\right)e^{3x}$$

$$y_p = xe^{3x} - \frac{19}{108} - \frac{5}{18}x - \frac{1}{6}x^2$$

The general solution is

$$y(x) = y_c + y_p = c_1e^{2x} + c_2e^{3x} + xe^{3x} - \frac{19}{108} - \frac{5}{18}x - \frac{1}{6}x^2$$

(c) using the D-operator method. ()

The equation can be rewritten in terms of the D-operator

$$(D^2 - 5D + 6)y = e^{3x} - x^2$$

equivalent to

(5)
$$y = \frac{1}{D^2 - 5D + 6}(e^{3x} - x^2)$$

Factorising

$$D^2 - 5D + 6 = (D-2)(D-3)$$

We solve for α and β in the equation

$$\frac{1}{D^2 - 5D + 6} = \frac{\alpha}{D-2} + \frac{\beta}{D-3}.$$

This gives $1 = \alpha(D-3) + \beta(D-2)$ Substituting $D = 3$, we get $\beta = 1$ and for $D = 2$, we get $\alpha = -1$. Hence

$$\frac{1}{D^2 - 5D + 6} = -\frac{1}{D-2} + \frac{1}{D-3}.$$

*We put this back in equation (**??**) to get*

$$y = -\frac{1}{D-2}(e^{3x} - x^2) + \frac{1}{D-3}(e^{3x} - x^2)$$

Now we solve

$$y = -\frac{1}{D-2}(e^{3x} - x^2)$$

$(D-2)y = -e^{3x} + x^2$ equivalent to $y' - 2y = -e^{3x} + x^2$ This is just a simple first-order linear DE, which we solve in the usual way: the integrating factor is $e^{\int -2dx} = e^{-2x}$ and so

$$\frac{d}{dx}[e^{-2x}y] = e^{-2x}(-e^{3x} + x^2) = x^2e^{-2x} - e^x$$

Integrating by parts the right hand side yields

$$y = -\frac{1}{2}\left(x^2 + x + \frac{1}{2}\right) - e^{3x}$$

Similarly a solution of $y = \frac{1}{D-3}(e^{3x} - x^2)$ yields

$$y' - 3y = e^{3x} - x^2$$

$$\frac{d}{dx}[e^{-3x}y] = e^{-3x}(-e^{3x} + x^2) = 1 - x^2e^{-3x}$$

$$y = xe^{3x} + \frac{1}{3}\left(x^2 + \frac{2}{3}x + \frac{2}{9}\right)$$

Then, adding them gives

$$y_p = -\frac{1}{2}\left(x^2 + x + \frac{1}{2}\right) - e^{3x} + xe^{3x} + \frac{1}{3}\left(x^2 + \frac{2}{3}x + \frac{2}{9}\right)$$

The general solution is

$$y(x) = y_c + y_p = c_1e^{2x} + c_2e^{3x} + xe^{3x} - \frac{19}{108} - \frac{5}{18}x - \frac{1}{6}x^2$$

Question 12: **Marks**

Solve the initial value problem

$$y'' - 2y' = 8x, \qquad y(0) = 1,\ y'(0) = 2$$

using the method of variation of parameters.
For the complimentary solution, the auxiliary equation is

$$m^2 - 2m = 0$$

with the solutions $m = 0$ and $m = 2$ and yielding the the complementary solution

$$y_c = c_1 + c_2e^{2x}$$

Hence, from the form of the complimentary solution we take $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$ where $y_1(x) = 1$ and $y_2(x) = e^{2x}$. The equation $y'' - 2y' = 8x$ is already written in standard form (i.e. the leading coefficient should be 1.) The Wronskian is given by

(6) $$W(1, e^{2x}) = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x}$$

and

$$W_1 = \begin{vmatrix} 0 & e^{2x} \\ 8x & 2e^{2x} \end{vmatrix} = -8xe^{2x} \tag{7}$$

$$W_2 = \begin{vmatrix} 1 & 0 \\ 0 & 8x \end{vmatrix} = 8x \tag{8}$$

We find

$$u_1' = \frac{W_1}{W} = \frac{-8xe^{2x}}{2e^{2x}} = -4x$$
$$u_2' = \frac{W_2}{W} = \frac{8x}{2e^{2x}} = 4xe^{-2x}$$

Integrating by parts yields

$$u_1 = -2x^2$$
$$u_2 = -\left(2x+1\right)e^{-2x}$$

so that

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) = -2x^2 - \left(2x+1\right)e^{-2x}e^{2x} = -2x^2 - 2x - 1$$

The general solution is

$$y(x) = y_c + y_p = c_1 + c_2e^{2x} - 2x^2 - 2x - 1$$

The initial value $y(0) = 1$ *yields* $1 = c_1 + c_2 - 1$. *The initial condition* $y'(0) = 2$ *yields*

$$2 = 2c_2e^{2\cdot 0} - 4\cdot 0 - 2 = 2c_2 - 2$$

Thus $c_2 = 2$ *and* $c_1 = 0$. *The solution is*

$$y = 2e^{2x} - 2x^2 - 2x - 1.$$

***Alternative**:*
The general solution is equivalent to

$$y(x) = c + c_2e^{2x} - 2x^2 - 2x \quad \text{where} \quad c = c_1 - 1$$

The initial value $y(0) = 1$ *yields* $1 = c + c_2$. *The initial condition* $y'(0) = 2$ *yields*

$$2 = 2c_2e^{2\cdot 0} - 4\cdot 0 - 2 = 2c_2 - 2$$

Thus $c_2 = 2$ *and* $c = 1 - c_2 = -1$. *The solution is*

$$y = -1 + 2e^{2x} - 2x^2 - 2x.$$

D.6.2 Assignment 2 type

Question 13: **Marks**

Use the power series method to solve the initial value problem:

$$y'' + 2xy' + x^2y = 2, \qquad y(0) = 0,\, y'(0) = 1.$$

Let $y = \sum_{j=0}^{\infty} a_j x^j$*. The initial value problem becomes*

$$\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} + 2x\sum_{j=1}^{\infty} j a_j x^{j-1} + x^2 \sum_{j=0}^{\infty} a_j x^j = 2, \qquad a_0 = 0, \quad a_1 = 1.$$

Thus we have

$$\sum_{j=2}^{\infty} j(j-1)a_j x^{j-2} + 2\sum_{j=1}^{\infty} j a_j x^{j} + \sum_{j=0}^{\infty} a_j x^{j+2} = 2, \qquad a_0 = 0, \quad a_1 = 1.$$

To convert the first sum to easily comparable powers, we substitute $j-2 \to j$ *so that* $j \to j+2$ *and* $j(j-1) \to (j+2)(j+1)$ *(the lower bound on the sum becomes* $j+2 = 2$ *so that* $j = 0$*):*

$$\sum_{j=0}^{\infty} (j+2)(j+1)a_{j+2} x^{j} + 2\sum_{j=1}^{\infty} j a_j x^{j} + \sum_{j=0}^{\infty} a_j x^{j+2} = 2, \qquad a_0 = 0, \quad a_1 = 1.$$

Separating terms in the sums which are not common yields

$$\underbrace{2a_2 + a_0x^2}_{j=0} + \sum_{j=1}^{\infty} \left((j+2)(j+1)a_{j+2} + 2ja_j\right) x^j + a_j x^{j+2} = 2.$$

The different terms for each j *to be summed are*

$$\begin{array}{lr}
j = 0 & 2a_2 + a_0x^2 \\
j = 1 & (6a_3 + 2a_1)x + a_1x^3 \\
j = 2 & (12a_4 + 4a_2)x^2 + a_2x^4 \\
\vdots & \\
j = n & ((n+2)(n+1)a_{n+2})x^n + a_nx^{n+2} \\
\vdots &
\end{array}$$

Comparing coefficients of x *and it powers we find*

$$\begin{array}{rrr}
a_0 = 0 & & \\
a_1 = 1 & & \\
a_2 = 1 & & \\
x: & 6a_3 + 2a_1 = 0 & a_3 = -1/3 \\
x^2: & a_0 + 12a_4 + 4a_2 = 0 & a_4 = -1/3 \\
\vdots & & \\
 & a_{j-2} + (j+2)(j+1)a_{j+2} + 2ja_j = 0 & a_{j+2} = \dfrac{-2ja_j - a_{j-2}}{(j+2)(j+1)}, \quad j \geq 2 \\
\vdots & &
\end{array}$$

Thus we find $a_4 = \frac{-4a_2 - a_0}{12} = -1/3$*, Similarly* $a_5 = 1/20, \ldots a_6 = -1/18$*. It follows that*

$$y(x) = \sum_{j=0}^{\infty} a_j x^j = x + x^2 - \frac{x^3}{3} - \frac{x^4}{3} + \frac{x^5}{20} + \cdots .$$

Question 14: **Marks**

Consider the function

$$f(t) = \begin{cases} e^{2t} & 0 \le t < 3 \\ 1 & 3 \le t \end{cases} .$$

(14.1) Find the Laplace transform of $f(t)$ by first principles. ()

We find

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \int_0^3 e^{-st} e^{2t}\, dt + \int_3^\infty e^{-st}\, dt \\ &= \int_0^3 e^{(2-s)t}\, dt + \int_3^\infty e^{-st}\, dt \\ &= \frac{1}{2-s}\left[e^{(2-s)t}\right]_0^3 - \frac{1}{s}\left[e^{-st}\right]_3^\infty \\ &= \frac{1}{2-s}\left(e^{3(2-s)} - 1\right) + \frac{1}{s}e^{-3s} \end{aligned}$$

(14.2) Express $f(t)$ in terms of the Heaviside step function and use the table of Laplace transforms to calculate $\mathscr{L}\{f(t)\}$. ()

Since

$$f(t) = (\mathscr{U}(t-0) - \mathscr{U}(t-3))e^{2t} + (\mathscr{U}(t-3) - \mathscr{U}(t-\infty))$$

we have

$$\begin{aligned} \mathscr{L}\{f(t)\} &= \mathscr{L}\left\{\mathscr{U}(t-0)e^{2t}\right\} - \mathscr{L}\left\{\mathscr{U}(t-3)e^{2t}\right\} + \mathscr{L}\{\mathscr{U}(t-3)\} \\ &= \mathscr{L}\left\{e^{2t}\right\} - \mathscr{L}\left\{\mathscr{U}(t-3)e^{2(t-3+3)}\right\} + \mathscr{L}\{\mathscr{U}(t-3)\} \\ &= \mathscr{L}\left\{e^{2t}\right\} - \mathscr{L}\left\{\mathscr{U}(t-3)e^{2(t-3)}e^{6}\right\} + \mathscr{L}\{\mathscr{U}(t-3)\} \\ &= \frac{1}{s-2} - e^6\mathscr{L}\left\{\mathscr{U}(t-3)e^{2(t-3)}\right\} + \frac{e^{-3s}}{s} \\ &= \frac{1}{s-2} - e^6 e^{-3s}\frac{1}{s-2} + \frac{e^{-3s}}{s} \\ &= \frac{1}{2-s}\left(e^{3(2-s)} - 1\right) + \frac{1}{s}e^{-3s} \end{aligned}$$

Question 15: **Marks**

Solve the following initial value problem using Laplace transforms

$$y'' - 5y' + 6y = 26\cos 2t + 6, \qquad y(0) = 1/2,\ y'(0) = 0.$$

$$\begin{aligned}
\mathscr{L}\{y'' - 5y' + 6y\} &= \mathscr{L}\{26\cos 2t + 6\}\\
\mathscr{L}\{y''\} - 5\mathscr{L}\{y'\} + 6\mathscr{L}\{y\} &= 26\mathscr{L}\{\cos 2t\} + 6\mathscr{L}\{1\}\\
s^2\mathscr{L}\{y\} - sy(0) - y'(0) - 5(s\mathscr{L}\{y\} - y(0)) + 6\mathscr{L}\{y\} &= 26\frac{s}{s^2+4} + \frac{6}{s}\\
s^2\mathscr{L}\{y\} - \frac{s}{2} - 5(s\mathscr{L}\{y\} - \frac{1}{2}) + 6\mathscr{L}\{y\} &= \frac{26s}{s^2+4} + \frac{6}{s}\\
(s^2 - 5s + 6)\mathscr{L}\{y\} + \frac{5-s}{2} &= \frac{26s}{s^2+4} + \frac{6}{s}
\end{aligned}$$

so that

$$\mathscr{L}\{y\} = \frac{s^4 - 5s^3 + 68s^2 - 20s + 48}{2s(s^2+4)(s-2)(s-3)}$$

Partial fraction decomposition

$$\frac{s^4 - 5s^3 + 68s^2 - 20s + 48}{2s(s^2+4)(s-2)(s-3)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} + \frac{D}{s-2} + \frac{E}{s-3}$$

yields

$$\begin{aligned}
&48A - 24sD - 16sE + 20As^2 - 10As^3 + 2As^4 + 12Bs^2 - 10Bs^3 + 2Bs^4 - 10Cs^2 + 2Cs^3\\
&+ 8s^2D + 8s^2E - 6s^3D - 4s^3E + 2s^4D + 2s^4E - 40As + 12Cs = s^4 - 5s^3 + 68s^2 - 20s + 48
\end{aligned}$$

$$\begin{aligned}
&(2A + 2B + 2D + 2E)s^4 + (-10A - 10B + 2C - 6D - 4E)s^3 + (20A + 12B - 10C + 8D + 8E)s^2\\
&+ (-40A + 12C - 24D - 16E)s + 48A = s^4 - 5s^3 + 68s^2 - 20s + 48
\end{aligned}$$

and comparing coefficients of powers of s yields

$$\begin{aligned}
s^4 &: 2A + 2B + 2D + 2E = 1 \quad ①\\
s^3 &: -10A - 10B + 2C - 6D - 4E = -5 \quad ②\\
s^2 &: 20A + 12B - 10C + 8D + 8E = 68 \quad ③\\
s &: -40A + 12C - 24D - 16E = -20 \quad ④\\
1 &: 48A = 48 \quad ⑤
\end{aligned}$$

There are number of methods to solve this system of linear equations. We solve it using the substitution method. Equation ⑤ yields $A = 1$. The substitution into equations ① to ④ gives

$$\begin{aligned}
s^4 &: 2B + 2D + 2E = -1 \quad (1p)\\
s^3 &: -10B + 2C - 6D - 4E = 5 \quad (2p)\\
s^2 &: 12B - 10C + 8D + 8E = 48 \quad (3p)\\
s &: 12C - 24D - 16E = 20 \quad (4p)
\end{aligned}$$

Equation (1p) *yields* $B = \frac{-1-2D-2E}{2}$ (1pp)

Equation (4p) *yields* $C = \frac{20+24D+16E}{12}$ (4pp)

Substituting equations (1pp) *and* (4pp) *into equations* (2p) *and* (3p) *yields*

$$
\begin{aligned}
-10\left(\frac{-1-2D-2E}{2}\right) + 2\left(\frac{20+24D+16E}{12}\right) - 6D - 4E = 5 \\
12\left(\frac{-1-2D-2E}{2}\right) - 10\left(\frac{20+24D+16E}{12}\right) + 8D + 8E = 48
\end{aligned}
$$

Reducing this system of equations gives

$$
\begin{aligned}
8D + \frac{26}{3}E = -\frac{10}{3} \quad ⑥ \\
-24D - \frac{52}{3}E = \frac{212}{3} \quad ⑦.
\end{aligned}
$$

Equation ⑥ *yields*

$$D = \frac{-\frac{10}{3} - \frac{26}{3}E}{8} \quad ⑧.$$

Substituting equation ⑧ *into* ⑦ *yields*

$$-24\left(\frac{-\frac{10}{3} - \frac{26}{3}E}{8}\right) - \frac{52}{3}E = \frac{212}{3},$$

which gives

$$E = \left(\frac{212}{3} - 10\right)\frac{3}{26} = 7.$$

Hence, substituting E *into equations* ⑧, (1pp) *and* (4pp) *respectively gives* $D = -8$ $B = 1/2$, $C = -5$. *Finally*

$$A = 1, \quad B = 1/2, \quad C = -5, \quad D = -8, \quad E = 7.$$

$$
\begin{aligned}
y &= \mathscr{L}^{-1}\left\{\frac{1}{s} + \frac{\frac{s}{2} - 5}{s^2+4} + \frac{-8}{s-2} + \frac{7}{s-3}\right\} \\
&= \mathscr{L}^{-1}\left\{\frac{1}{s}\right\} + \frac{1}{2}\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - 5\mathscr{L}^{-1}\left\{\frac{1}{s^2+4}\right\} - 8\mathscr{L}^{-1}\left\{\frac{1}{s-2}\right\} + 7\mathscr{L}^{-1}\left\{\frac{1}{s-3}\right\} \\
&= 1 + \frac{1}{2}\cos 2t - \frac{5}{2}\sin 2t - 8e^{2t} + 7e^{3t}
\end{aligned}
$$

Question 16: **Marks**

Compute the Fourier series for the function

$$f(x) = \begin{cases} x & 0 \le x < \pi \\ -x & \text{otherwise} \end{cases}$$

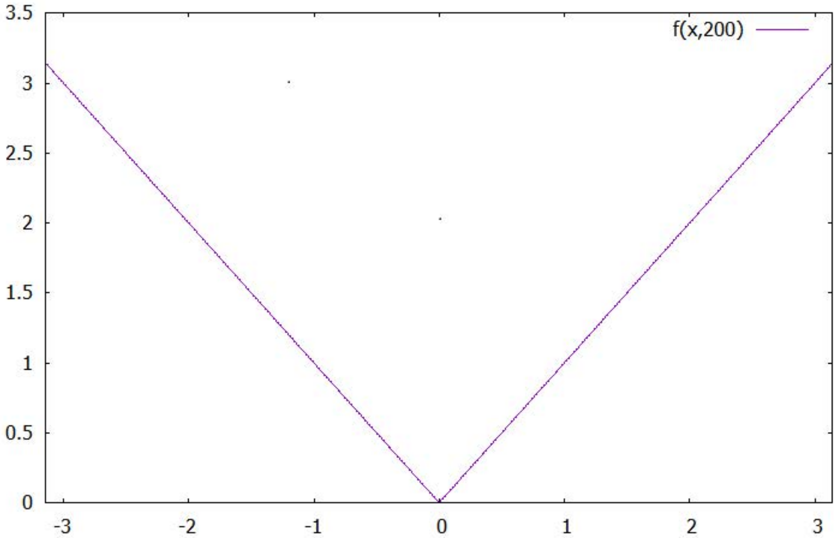
on $(-\pi, \pi)$.
Integrating by parts the coefficients are given by

$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x)dx + \int_0^{\pi} xdx\right) = \pi \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x)\cos nx + \int_0^{\pi} x\cos nx\,dx\right) \\
&= \frac{1}{n\pi}(2\pi\sin\pi n + \int_{-\pi}^{0}\sin nx\,dx - \int_0^{\pi}\sin nx\,dx) \\
&= \frac{1}{n\pi}(2\pi\sin\pi n + \frac{1}{n}\cos\pi n - \frac{1}{n} - \frac{1}{n} + \frac{1}{n}\cos\pi n) \\
&= \frac{1}{\pi n^2}\left(2\cos\pi n + 2\pi n\sin\pi n - 2\right) \\
&= \frac{2}{\pi n^2}\left((-1)^n - 1\right) \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0}(-x)\sin nx + \int_0^{\pi} x\sin nx\,dx\right) \\
&= \frac{-1}{\pi}\int_{-\pi}^{0} x\sin nx + \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx \\
&= \frac{1}{\pi}(-\int_{-\pi}^{0}\frac{1}{n}\cos nx\,dx + \frac{\pi}{n}\cos\pi n + \int_0^{\pi}\frac{1}{n}\cos nx\,dx - \frac{\pi}{n}\cos\pi n) \\
&= \frac{1}{\pi}(-\frac{1}{n}\frac{1}{n}\sin\pi n + \frac{\pi}{n}\cos\pi n + \frac{1}{n}\frac{1}{n}\sin\pi n - \frac{\pi}{n}\cos\pi n) \\
&= 0
\end{aligned}$$

Thus

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left(((-1)^n - 1\right)\cos nx)$$

on $(-\pi, \pi)$.
The following figure shows the truncated series (for $n \le 200$*).*



The figure was generated using GNUplot (`http://gnuplot.info`):

```
set samples 1000
set xrange [-pi:pi]

a(n) =  2.0/(n**2)*((-1)**n-1)/pi

f(x,n) = (n == 0) ? (pi)/2 : a(n)*cos(n*x)  + f(x,n-1)
plot f(x,200)
```

Question 17: **Marks**

Use separation of variables to find a solution of the partial differential equation

$$\frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0,$$

on $x, y \in (0, \infty)$, with boundary value $u(x, 1) = e^x$.
Assume $u(x, y) = X(x)Y(y)$. Then

$$\frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = \frac{dX}{dx} Y + y^2 X \frac{dY}{dy} = 0.$$

The equation separates:

$$\left(\frac{dX}{dx}\right) Y = -\left(y^2 \frac{dY}{dy}\right) X$$

and since we seek a non-zero solution ($X \neq 0$, $Y \neq 0$)

$$\frac{1}{X}\frac{dX}{dx} = -\frac{y^2}{Y}\frac{dY}{dy}.$$

Since the two sides are independent of each other, there must exist a constant k such that

$$\frac{1}{X}\frac{dX}{dx} = k \qquad -\frac{y^2}{Y}\frac{dY}{dy} = k.$$

Integrating these equations yields

$$\ln|X| = kx + c_1 \qquad \ln|Y| = \frac{k}{y} + c_2,$$

where c_1 *and* c_2 *are constants of integration. It follows that*

$$|X| = e^{kx}e^{c_1}, \qquad |Y| = e^{\frac{k}{y}}e^{c_2}$$

so that

$$|u(x,y)| = u(x,y) = e^{c_1+c_2}e^{kx}e^{\frac{k}{y}} = e^{c_1+c_2}e^{kx+\frac{k}{y}}$$

on $x, y \in (0,\infty)$, *with boundary value* $u(x,1) = e^x$. *Thus we find*

$$u(x,y) = Ce^{kx+\frac{k}{y}}$$

where $C = e^{c_1+c_2}$. *Since* $u(x,1) = e^x$ *we have*

$$e^x = u(x,1) = Ce^{kx+k} = Ce^k e^{kx}$$

so that $k = 1$ *and* $C = e^{-1}$. *Thus*

$$u(x,y) = e^{-1}e^{x+\frac{1}{y}} = e^{x+\frac{1}{y}-1}$$