APM1514

October/November 2013

MATHEMATICAL MODELLING

Duration 2 Hours

100 Marks

EXAMINERS :
FIRST DR EM RAPOO
SECOND MR AS KUBEKA

Use of a non-programmable pocket calculator is permissible

Closed book examination.

This examination question paper remains the property of the University of South Africa and may not be removed from the examination venue

This paper consists of 4 pages

You should answer **ALL** the questions

1 (a) Solve the following differential equation

$$\frac{dy}{dt} = 2t - yt$$

(b) Solve the following initial value problem

$$\frac{dp}{dt} = -k\,(p - 10)\,, \quad p(0) = 5$$

(c) Consider the difference equation

$$a_{n+1} = 3a_n + b$$

(i) For which value of b is $a = 1$ an equilibrium point of the difference equation?

(ii) For which value of b can be have $a_0 = 1$, $a_2 = -3$?

(d) Formulate a difference equation to model the amount of money in a bank account in the following case At the end of each month, the bank account pays interest at a rate of 5 % on the amount of money that was in the account during the month An amount of R500 is added to the bank account at the beginning of each month, and one third of the amount available on the bank account is withdrawn at the end of the month, after the interest has been paid **[17]**

2. In the model of radioactive decay, which of the following statements are true and which are false? Justify your answers!

   (a) The time it takes for a quantity of radioactive material to decay from 300g to 200g is the same as the time it takes for the same material to decay from 200g to 100g

   (b) The time it takes for a quantity of radioactive material to decay from 100 kg to 10 kg is ten times longer than the time it takes for the same material to decay from 10 kg to 1 kg

   (c) How much of the material is left after a given length of time T does not depend on the initial amount of material, but only on T and the constant of decay, k

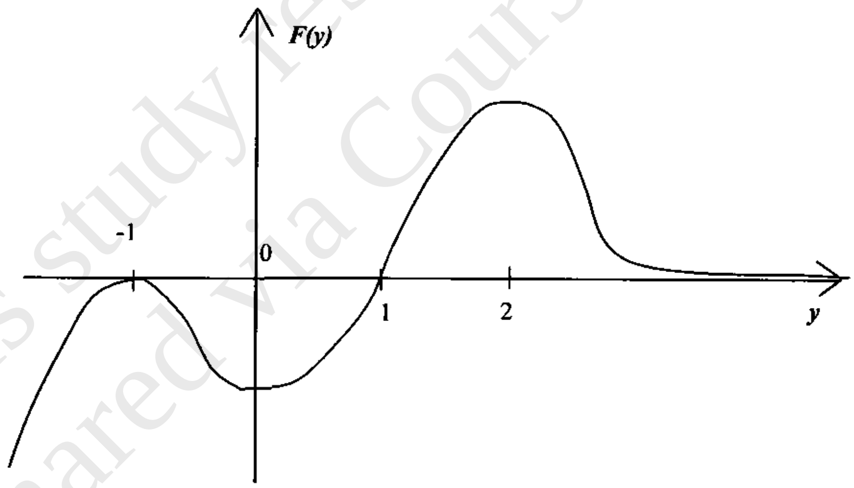
   (d) If the half-life of a certain radioactive material is 50 years, then after 100 years all the material will be gone (The half-life of a material is defined to be the time interval after which only half of the original quantity remains)

   (e) If the half-life of material A is twice as long as the half-life of material B, then it also takes twice as long for material A as it takes for material B to decay until only 10% of the original quantity is left **[13]**

3. Consider the model

$$\frac{dy}{dt} = F(y)$$

where the function F looks as follows



   (a) For which values of y does the solution $y(t)$ decrease?

   (b) Draw a phase line for the model

   (c) Draw a rough sketch of the solution curve $y(t)$ if the initial value is $y(0) = -0.5$, and if it is $y(0) = 1.5$

   (d) Can a solution curve ever increase without bound in this model? Justify your answer! **[14]**

[TURN OVER]

4 A population model is described by the differential equation

$$\frac{dP}{dt} = AP - \frac{B}{P}$$

where $A > 0$ and $B > 0$ are constants. Which of the following claims are true and which are false? Justify your answers!

(a) The bigger the population gets, the faster it grows.

(b) The population will never die out if it starts with an initial population P_0 with $P_0 > 0$.

(c) As P increases, population growth slows down due to overcrowding.

(d) This model behaves more realistically than the logistic model for small population sizes.

(e) This model behaves more realistically than the logistic model for large population sizes. **[16]**

5 An object is dropped into a liquid which is kept at a constant temperature. Assume that the differential equation

$$\frac{dT}{dt} = a\,(B - T)$$

holds when $T(t)$ denotes the temperature of the object at time t, with t measured in minutes.

(a) What is the temperature of the liquid?

(b) Should a be positive or negative for the equation to make sense? Justify your answer!

(c) If $T(0) = T_0$, what is the initial rate of change of the temperature of the object? **[8]**

6 A tank with a volume of 100 litres is used to dissolve a chemical in water. At time $t = 0$ the tank contains none of the chemical. Water containing 0.1 kg of the chemical per litre flows into the tank at the rate 2 litres per minute. The mixture in the tank is stirred thoroughly and the tank is kept full at all times. The mixture is pumped out at the rate of 2 litres per minute. Let $X(t)$ denote the amount of the chemical (in kilograms) in the tank after t minutes.

(a) Derive a differential equation for $X(t)$.

(b) Find the amount of the chemical in the tank after 10 minutes.

(c) Find the rate of change of the chemical in the tank after 10 minutes.

(d) Prove the following result: As $t \to \infty$, the **concentration** of the chemical in the tank converges towards 0.1 kilograms per litre. **[12]**

[TURN OVER]

7 A system of differential equations is given by

$$\begin{cases} \frac{dx}{dt} = x^2 \\ \frac{dy}{dt} = y^3 \end{cases}$$

(a) Find the x- and y-isoclines

(b) Find the equilibrium points

(c) Draw the phase diagram of the system Indicate the directions of motion in all areas of the phase line Sketch possible trajectories in the diagram Show what happens in all four quadrants of the xy-plane

(d) Investigate the stability of all the equilibrium points **[20]**