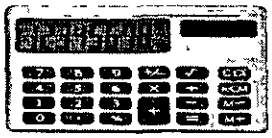
UNIVERSITY EXAMINATIONS



UNIVERSITEITSEKSAMENS

UNISA university of south africa

APM1514

May/June 2016

MATHEMATICAL MODELLING

Duration 2 Hours

100 Marks

EXAMINERS

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Use of a non-programmable pocket calculator is permissible

Closed book examination

This examination question paper remains the property of the University of South Africa and may not be removed from the examination venue

This paper consists of 4 pages

Answer all questions

[TURN OVER]

QUESTION 1

1 (a) A difference equation is given by

$$a_{n+1} = a_n^2$$

(i) Find all the equilibrium points of the system

(ii) Find the solution to the system, when the initial value is $a_0 = 3$

(b) Solve the initial value problem

$$\frac{dy}{dt} = (10 - y), \qquad y(2) = 0 \tag{15}$$

2 Find the solution to the following

$$\frac{dy}{dt} = 2y - 1, \qquad y(2) = 0 \tag{10}$$

[25]

QUESTION 2

1 (a) Assume that a Malthusian population decreases by 10 percent each year. How long does it take for the population to decrease to half of its initial size?

(b) Assume that a radioactive substance decays from 6 g to 1 g in 500 years

(i) How long does it take until only 5% of an original quantity N_0 is present?

(ii) If today there is 1 kg of the substance present, how much was there 2000 years ago?

(10)

2 A flu infection is spreading in a school of 2000 pupils. Let $X(t)$ denote the number of pupils who have the flu after t days. Assume that the epidemic model for the spread of the flu infection is modelled by the following two differential equations

$$\text{(A)} \quad \frac{dX}{dt} = a(2000)$$

$$\text{(B)} \quad \frac{dX}{dt} = a(2000 - X)$$

where a is a positive constant.

(a) Draw the phase line of the model

(b) Explain how the models differ from each other. Which model is more realistic? Justify your answer!

(15)

[25]

[TURN OVER]

QUESTION 3

1 A cake with the initial temperature 20°C is placed in an oven, the temperature of which is kept at 190°C Assume that the cake heats up in such a way that the rate of change of the temperature $T(t)$ of the cake is directly proportional to the difference between the temperature of the oven and the temperature of the cake After 30 minutes the temperature of the cake is 100°C

(a) How long does it take for the cake to reach the temperature 120°C?

(b) What is the initial rate of change of the temperature of the cake?

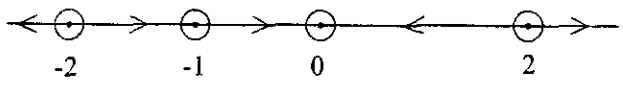
(c) How long does it take until the rate of change of the temperature of the cake has dropped to 0 5°C per minute?

(10)

2 Consider the model

$$\frac{dy}{dt} = F(y)$$

where the phase line of the model looks as follows



(a) Draw a rough sketch of the function F (8)

(b) Draw a rough sketch of the solution curve $y(t)$ if the initial value is $y(0) = 1$ (7)

[25]

QUESTION 4

1 The following differential equation is suggested to model the heating up of an object subjected to a radiation source

$$\frac{dT}{dt} = k(T - \alpha)^2$$

where k is a positive constant, $T(t)$ is the temperature of the object, and α is the temperature of the radiation source

(a) In this model, describe how the temperature of the object behaves, in each of the following three cases Justify your answers in each case

(i) If $T > \alpha$,

(ii) if $T = \alpha$,

(iii) if $T < \alpha$

(b) According to this model, what happens if $T(0) = 0$?

(c) Is this a special case of the logistic model? Justify your answer (10)

2 Draw the phase diagram of the system

$$\frac{dx}{dt} = xy,$$

$$\frac{dy}{dt} = x - 1$$

You should include all four quadrants of the xy–plane For full marks, all the following must be given in your answer The isoclines and all the equilibrium points, and the signs of dx/dt and dy/dt in different parts of the xy–plane In addition, the following must be correctly and clearly annotated in your phase diagram The coordinate axes, all the isoclines all the equilibrium points, the allowed directions of motion (both vertical and horizontal) in all the regions into which the isoclines divide the xy–plane, direction of motion along isoclines, where applicable, examples of allowed trajectories in all regions, and examples of trajectories crossing from a region to another, whenever such a crossing is possible

Specify also which equilibrium points of the system are stable and which are unstable Justify your answers!

You do NOT have to list the possible outcomes of the system, beyond the stability of the equilibrium points (15)

[25]

TOTAL. 100