UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

APM1514

October/November 2017

MATHEMATICAL MODELLING

Duration 2 Hours

100 Marks

EXAMINERS :

FIRST	DR AS KUBEKA	
SECOND	DR B HUNTLEY	DR JM MANALE

Use of a non-programmable pocket calculator is permissible.

Closed book examination

This examination question paper remains the property of the University of South Africa and may not be removed from the examination venue

This paper consists of 4 pages

Answer all questions

[TURN OVER]

QUESTION 1

1 1 Consider the following proportionally growing population The birth rate is 0 1 births per population member per year There are no deaths but, at the end of each year 50 % of all the population members who were present at the beginning of the year move out

(a) Write down the difference equation to model the population Use P_n to denote the number of population members present at the end of year n

(b) For which initial population values P_0 does the population stay constant? Justify your answer

(c) Assume that we change the birth rate from 0 1 to c births per population member per year What should the value of c be, to ensure that the population size will stay constant if the initial population size is $P_0 = 10\ 000$? Justify your answer (10)

1 2 Consider the population model given by the differential equation

$$\frac{dP}{dt} = -2P^2 + 4P - 2$$

(a) Which of the following claims are TRUE, and which are FALSE? In each case, justify your answer

(i) This population grows according to the logistic model

(ii) If the population starts at $P_0 = 0$, it will forever stay at zero

(iii) The limit population size is 2

(iv) The population here will grow without bound

(b) Draw rough sketches of the solution curves starting from the following two initial points

(i) $P_0 = 0\ 5$,

(ii) $P_0 = 3$ (15)

[25]

QUESTION 2

Assume that paraffin and diesel are mixed together in a tank with a capacity of 1000 litres Pure diesel is pumped into the tank at a rate of 20 litres per day while pure paraffin is pumped into the tank at the rate of 60 litres per day The mixture in the tank is always kept well mixed together, and is pumped out at a rate of 80 litres per day Let X_t denote the amount of diesel in the tank on day t, measured in litres

(a) Write down the differential equation for X_t

(b) If $X_0 = 500$, solve the differential equation for X_t in (a) above, to determine the initial rate of change of the amount of diesel in the tanks

(c) Prove that for any initial value X_0, as t increases, X_t will converge towards the value 250 litres

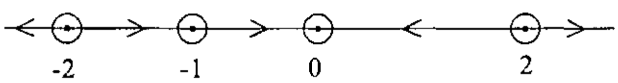
[25]

QUESTION 3

3 1 Consider the model

$$\frac{dy}{dt} = F(y)$$

where the phase line of the model is given as follows



(a) Draw a rough sketch of the function F (5)

(b) Draw a rough sketch of the solution curve $y(t)$, if the initial value is $y(0) = 1$ (5)

3 2 An autonomous differential equation can be written as

$$\frac{dx}{dt} = F(x)$$

where $F(x)$ is a function of x An example could be $F(x) = x$

(a) Give an example of a function $F(x)$ for which the differential equation has no equilibrium points

(b) Give an example of a function $F(x)$ for which no solutions to the differential equations will grow without bound

(c) Give an example of a function $F(x)$ for which the differential equation has one unstable equilibrium point at $x = 0$ and one stable equilibrium point at $x = 1$, with no other equilibrium points

(d) Give an example of a function $F(x)$ for which the rate of change of $x(t)$, the solution to the differential equation, is equal to 10 when $x = 1$ and equal to 5 when $x = 2$

(15)

[25]

QUESTION 4

Draw the phase diagram of the system

$$\frac{dx}{dt} = 3 - 2y,$$

$$\frac{dy}{dt} = xy$$

You should include all four quadrants of the xy–plane For full marks, all the following must be given in your answer The isoclines and all the equilibrium points, and the signs of dx/dt and dy/dt in different parts of the xy–plane In addition, the following must be correctly and clearly annotated in your phase diagram The coordinate axes, all the isoclines, all the equilibrium points, the allowed directions of motion (both vertical and horizontal) in all the regions into which the isoclines divide the xy–plane, direction of motion along isoclines, where applicable, examples of allowed trajectories in all regions, and examples of trajectories crossing from a region to another whenever such a crossing is possible

Specify also which equilibrium points of the system are stable and which are unstable Justify your answers

[25]

TOTAL: 100 Marks