Tutorial letter 202/2/2018

Mathematical Modelling

APM1514

Semester 2

Department of Mathematical Sciences

Ths tutorial letter contains solutions for assignment 02.

BARCODE

Second Semester 2014 (1)

Assignment 2 Solutions APM1514

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1. <u>Data given:</u>

$t_1 = 2000$

$P_0 = 100\,000$

$t_2 = 2002$

$P_1 = 60000$

a) $100\ 000 = P_0\, e^{K(2000)}$ (1)

$60\ 000 = P_0\, e^{K(2002)}$ (2)

$(1)/(2) \Rightarrow \dfrac{100\,000}{60\,000} = e^{K(2000) - K(2002)} = e^{-2K}$

$\therefore \ln\left(\dfrac{10}{6}\right) = -2K$

$\therefore K = -\dfrac{1}{2}\ln\left(\dfrac{5}{3}\right) = -\dfrac{1}{2}(0.512) = -0.26$

b) i) If we take $t=2000$ as our initial time frame, then we get

$10\,000 = 100\,000\, e^{-0.26 \cdot t}$

$\dfrac{1}{10} = e^{-0.26 \cdot t}$

$\therefore t = \dfrac{1}{-0.26} \cdot \ln\left(\dfrac{1}{10}\right) = -\dfrac{1}{0.26} \cdot 2.30$

$= 9$

$\therefore t = 2000 + 9 = 2009$

ii) If we take $t = 2002$ as an initial time frame we get

$$10\ 000 = 60\ 000\, e^{-0.26 \cdot t}$$

$$\frac{1}{6} = e^{-0.26 \cdot t}$$

$$\ln\left(\frac{1}{6}\right) = -0.26 \cdot t$$

$$\therefore \quad t = -\frac{1}{0.26} \cdot \ln\left(\frac{1}{6}\right) = -\frac{1}{0.26} \cdot (-1.79)$$

$$= 7$$

$$\therefore \quad t = 2002 + 7 = 2009$$

c) taking $t = 2000$, as our initial time frame, then we have

$$100\ 000 = e^{-0.26(2000)} \qquad \text{— (1)}$$

$$P(2010) = e^{-0.26(2010)} \qquad \text{— (2)}$$

$(1)/(2)$ gives:

$$\frac{100\ 000}{P(2010)} = \frac{e^{-0.26(2000)}}{e^{-(0.26)(2010)}}$$

$$= e^{-0.26(2000-2010)}$$

$$= e^{0.26(10)}$$

Therefore: $P(2010) = 100\ 000\, e^{-0.26(10)}$

NOTE: We can just straight use the general equation where, $P_0 = 100\ 000$, $t = 2010 - 2000 = 10$, and $k = -0.26$.

It then follows that

$P(2010) \simeq 7427 \simeq 7000$ rounded to the nearest thousand.

ii) OR We can take $t = 2002$ as our initial time frame provided we know the value of K and we do in this case:

$$P(2010) = 60\,000 e^{-0.26(8)}$$

$$= 7495 \approx 7000$$ rounded to the nearest thousand

d) $P(t_0) = P(0) e^{-K t_0}$, where $t_0 = 0$.

$$= 7000 e^{-K t_0}.$$

Assignment 2 Solutions

2. <u>Data</u>

$k = 0.001$
$N_0 = 15$ grams
$N_t = 3$ grams

a) We know that: $N_t = N_0 e^{-kt}$, from which we have

$$3g = 15g\, e^{-(0.001)t}$$

$$\therefore \frac{3}{15} = e^{-(0.001)t}$$

$$\therefore -(0{,}001)t = \ln\left(\frac{3}{15}\right)$$

$$\therefore t = -\frac{1}{0{,}001} \ln\left(\frac{3}{15}\right) = -\frac{1}{0{,}001} \cdot \ln(0{,}2)$$

$$= -\frac{1}{0.001} \cdot (-1.609)$$

$$\therefore t = 1609 \text{ yrs}$$

b) Today: $5\%\, N_0 = \frac{5}{100} N_0$, $t_{today} = ?$

Thus: $$\frac{5}{100} N_0 = N_0 e^{-k t_{today}}$$

$$-k\, t_{today} = \ln\left(\frac{5}{100}\right)$$

$$\therefore t_{today} = -\frac{1}{k} \ln\left(\frac{5}{100}\right) = 2996.$$

But: initialy: $N_t = N_0 e^{-k t_{in}}$ — (1)

today: $\frac{5}{100} N_t = N_0 e^{-k t_{today}}$ — (2)

then taking ratios: we have

$$(1)/(2): \quad \frac{N_t}{\frac{5}{100} N_t} = \frac{N_0 e^{-k t_{in}}}{N_0 e^{-k t_{today}}}$$

$$\frac{100}{5} = e^{-k t_{in} + k t_{today}}$$

$$= e^{k(-t_{in} + t_{today})}$$

$$\therefore k(-t_{in} + t_{today}) = \ln\left(\frac{100}{5}\right)$$

$$-t_{in} + t_{today} = \frac{1}{k} \ln\left(\frac{100}{5}\right)$$

$$t_{today} - \frac{1}{k} \ln\left(\frac{100}{5}\right) = t_{in}$$

$$\therefore t_{in} = 2996 - \frac{1}{0.001} \ln\left(\frac{100}{5}\right)$$

$$\therefore t_{in} = 2996 - 2996 = 0$$

(c) because we calculated the time t for the radioactive substance to decay to 5% today. From here we then used the techniques of ratios to calculate t_{in} from which N_0 cancell out.

3. <u>Malthusian model</u>

Country A: $50 \times t$ (doubling time)

Country B: $80 \times t$ (doubling time)

$t = 1950$

$k_A = ?, \; k_B = ?$

$t = 2000 \; (P_A = P_B)$

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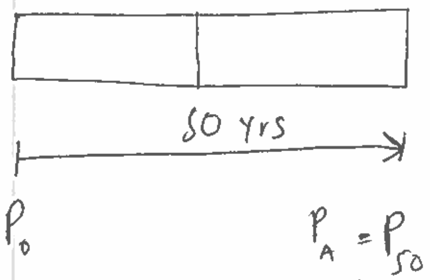
Population A



$(P_A = P_B)$

$$P_A(t) = P_A(0)\, e^{-K_A t}$$

P_A at $t_0 = 1$ (Day!), we have:

$$P_A(t) = P_A(0)\, e^{-K t_0} \quad —(1)$$

and at $t = 50\, t_0$

$$2P_A(t) = P_A(0)\, e^{-K_A 50 t_0} \quad -(2)$$

taking ratios, we have

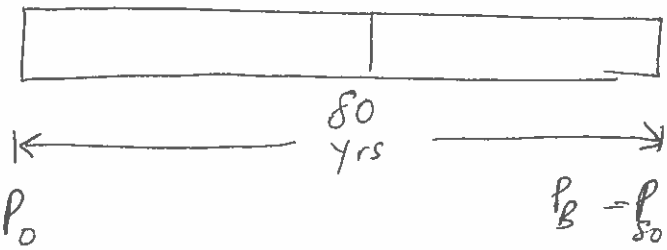
$(1)/(2)$: $\dfrac{1}{2} = e^{-K_A t_0 + K_A 50 t_0}$

$$\frac{1}{2} = e^{-K_A + 50 K_A} \quad (\because t_0 = 1)$$

$$\therefore K_A = \frac{1}{49} \ln\left(\frac{1}{2}\right)$$

$$\therefore K_A = -0{,}0141$$

Population B



$$P_B(t) = P_B(0)\, e^{-K_B t}$$

P_B at $t_0 = 1$ (Day!), we have

$$P_B(t) = P_B(0)\, e^{-K_B t_0} \quad —(1)$$

and at $t = 80\, t_0$, we have

$$2P_B(t) = P_B(0)\, e^{-K_B 80 t_0} \quad —(2)$$

taking ratios, we have

$(1)/(2)$: $\dfrac{P_B(t)}{2P_B(t)} = \dfrac{P_B(0)\, e^{-K_B t_0}}{P_B(0)\, e^{-K_B 80 t_0}}$

$$\frac{1}{2} = \frac{e^{-K_B t_0}}{e^{-K_B 80 t_0}}$$

$$= e^{-K_B t_0 + K_B 80 t_0}$$

$$= e^{-K_B + K_B 80} \quad (\because t_0 = 1)$$

$$\therefore 79 K_B = \ln\left(\frac{1}{2}\right)$$

$$\therefore K_B = \frac{1}{79} \ln\left(\frac{1}{2}\right)$$

$$\therefore K_B = -0.009$$

Now we can calculate the ratio of $P_B(t)$ and $P_A(t)$ at any time t:

$$\frac{P_B(t)}{P_A(t)} = \frac{P_B(0)e^{k_B t}}{P_A(0)e^{k_A t}}$$

but in 2000 $P_B(0) = P_A(0)$

$$\therefore \frac{P_B(t)}{P_A(t)} = \frac{e^{k_B t}}{e^{k_B t}} = e^{k_B t - k_A t} \; e^{(k_B - k_A)t},$$

$$= e^{(-0,009-(-0.0141))t}$$

$$= e^{(-0.009+0.0141)t}$$

$$\approx e^{0.0052t}$$

In particular in year 1950 ($t=-50$) the ratio was $e^{(0.0052)\cdot(-50)} \approx 0.8$, and in year 2050 ($t=+50$) the ratio will be $e^{(0.0052)(50)} \approx 1.3$

4. We use the following equations to calculate

$$120\ 000 = P(0)e^{-k(1980)}$$

$$500\ 000 = P(0)e^{-k(2000)}$$

taking ratios: we have

$$(1)/(2): \quad \frac{120\ 000}{500\ 000} = e^{k(-1980+2000)}$$

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$$\frac{12}{50} = e^{K20}$$

$$\therefore K20 = \ln\left(\frac{12}{50}\right)$$

$$\therefore K = \frac{1}{20}\ln\left(\frac{12}{50}\right) = -0.071$$

Now at 1990, we have

$$X = 120\,000\, e^{0.071(1990-1980)}$$

$$= 120000\, e^{0.071(10)}$$

$$\therefore X = 244080$$

5. (a) From: $$P(t) = \frac{a}{b + \left(\frac{a}{P_0} - b\right)e^{-at}}$$

have: $$P(1) = \frac{a}{b + \left(\frac{a}{P_0} - b\right)e^{-a(1)}}$$

$$= \frac{0.2}{0.05 + \left(\frac{0.2}{100} - 0.05\right)e^{-(0.2)}}$$

$$= 18.6900$$

and $$P(5) = \frac{0.2}{0.05 + \left(\frac{0.2}{100} - 0.05\right)e^{(-0.2)(5)}}$$

$$= 6.1840$$

b) $P(0) = 100.000$, $P(1) = 18.6900$, $P(5) = 6.1840$

$$\therefore \frac{dP(t)}{dt} = aP(t) - bP^2(t)$$

at $t=0$; $$\frac{dP(0)}{dt} = 0.2(100) - (0.05)(100)^2$$
$$= -480$$

$t=1$; $$\frac{dP(1)}{dt} = 0.2(18.6900) - (0.05)(18.6900)^2$$
$$= -13.7278$$

and at

$t=5$: $$\frac{dP(5)}{dt} = 0.2(6.1840) - (0.05)(6.1840)^2$$
$$= -0.6753$$

c) Phase line: The logistic differential equation

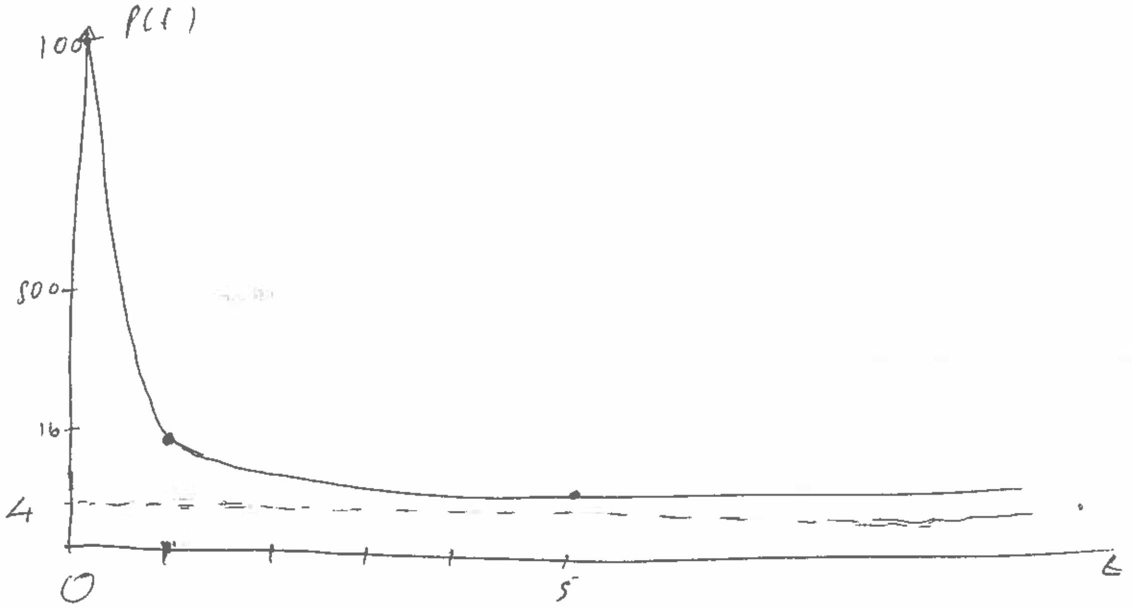
$$\frac{dP}{dt} = aP - bP^2 = P(a - bP)$$

has two equilibrium points, $P=0$ and $P=\frac{a}{b}$. Also, $dP/dt > 0$ for $0 < P < \frac{a}{b}$ and $dP/dt < 0$ for $P<0$, $P > \frac{a}{b}$. In this question,

$$\frac{a}{b} = \frac{0.2}{0.05} = 4$$

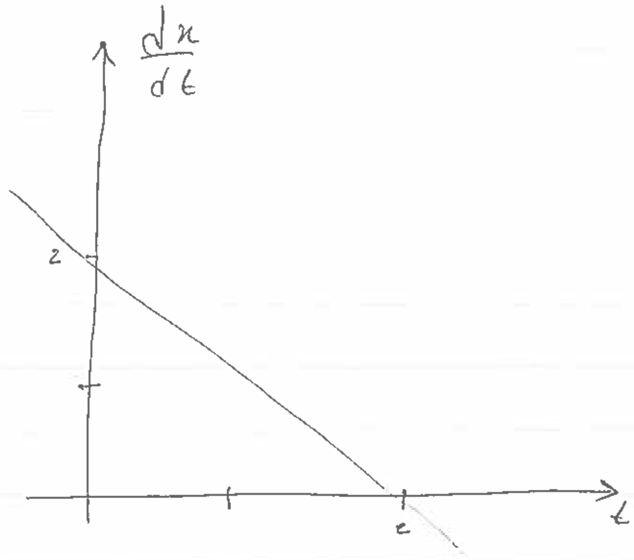
and therefore the phase line looks like this:

d) The initial value and (a) tell us that the solution curve $P(t)$ should go through the points $(t=0, 100)$, $(t=1, 18.6900)$, and $(t=2, P=6.1840)$. The answer to (b) further tells us what the slopes of the solution curve should be at these points: -480, -13.7278, -0.6753. Finally, since the initial point is larger than the equilibrium point 4, (c) tells that the solution will decrease, approaching asymptotically the value $P=4$ as $t \to \infty$. In conclusion, the solution curve looks as follows:



6. Given: $\frac{dx}{dt} = 2 - x$

(i) Sketch of $\frac{dx}{dt}$:



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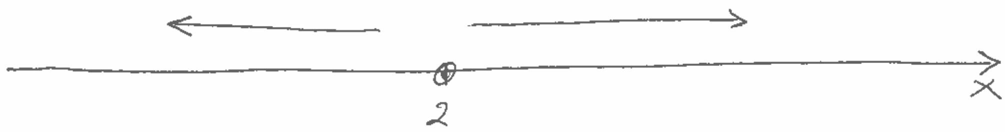
ii) <u>Phase line:</u>

equilibrium points: $\frac{dx}{dt} = 0$, $\therefore$ $0 = 2 - x$;
$\therefore x = 2$

slopes: $\frac{dx}{dt} > 0$ iff $x < 2$

$\frac{dx}{dt} < 0$ iff $x > 2$.

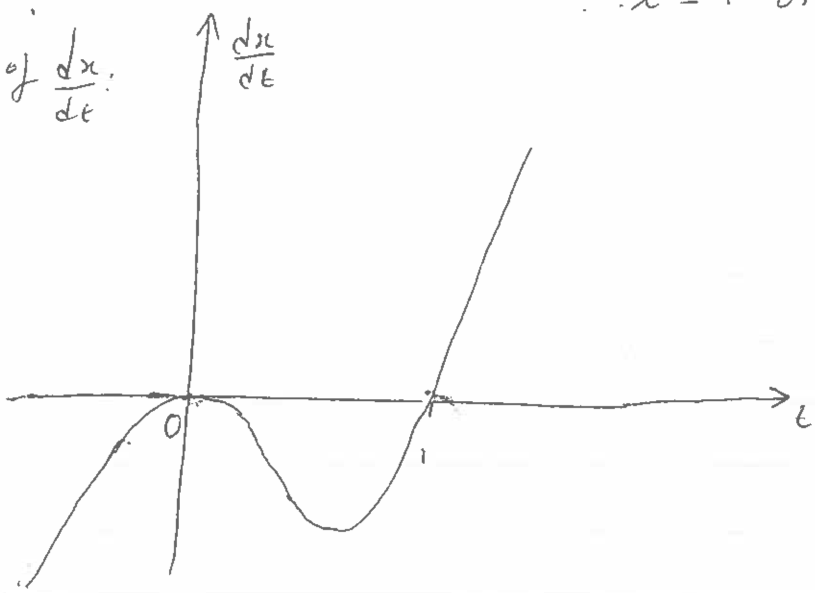


The equilibrium point here is unstable

if the system starts at $x(0) = 0.5$, it will not return to the equilibrium point $x=2$ but it move away from the equilibrium and will never return to it

(b) Given: $\frac{dx}{dt} = (x-1)x^2$
we can also find x-intercepts by letting $y = \frac{dx}{dt} = 0$
Thus we have: $\frac{dx}{dt} = 0$; $(x-1) = 0$ or $x^2 = 0$
$\therefore x = 1$ or $x = 0$

(i) sketch of $\frac{dx}{dt}$:



Assignment 2 solutions

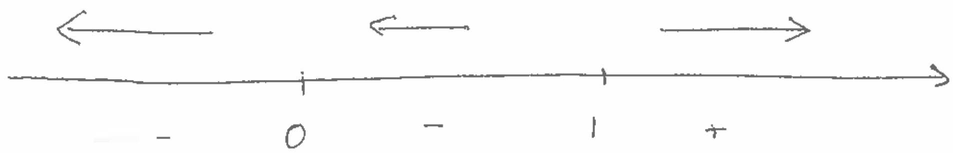
(ii) <u>Phase line:</u>

similarly, equilipoints: $\frac{dx}{dt} = (x-1)x^2 = 0$

$$\Rightarrow \quad (x-1)=0 \quad \text{OR} \quad x^2 = 0$$
$$x = 1 \quad \text{or} \quad x = 0$$

by since: $\frac{dx}{dt} > 0$ iff $x > 1$

and $\frac{dx}{dt} < 0$ iff $x < 1$



The equilibrium points are unstable.

if the system starts at $x(0) = 0.5$, it will not return to the equilibrium point $x=1$, but the system will approach the equilibrium point $x=0$ from the right. but the system will continue and pass this equibrium point and never return to it.

(c) Given: $\frac{dx}{dt} = (x-2)x(x+1)$

we can also find x-intercepts by letting $y = \frac{dx}{dt} = 0$
Thus we have: $(x-2) = 0$, or $x = 0$, $x+1 = 0$
$x = 2$, or $x = 0$; $x = -1$

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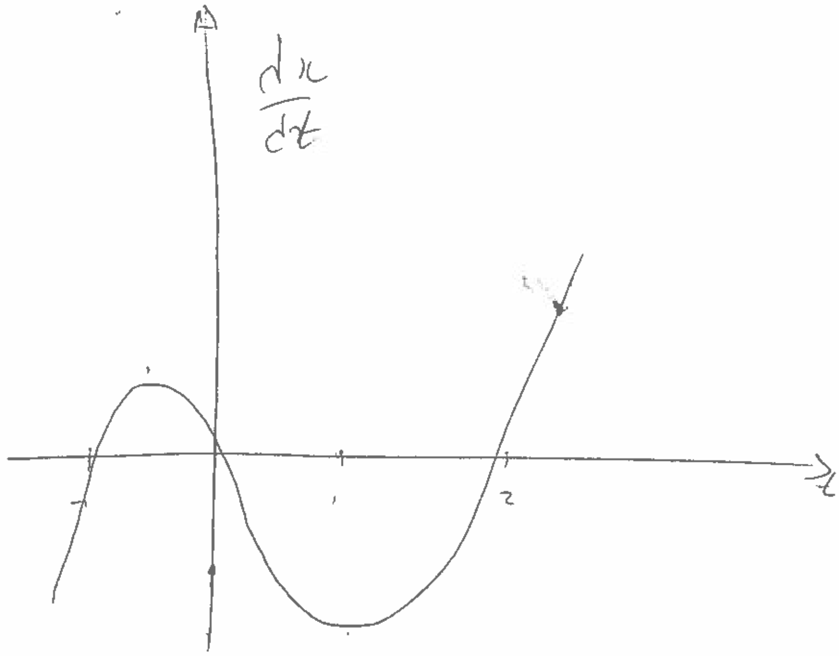
i) sketch:



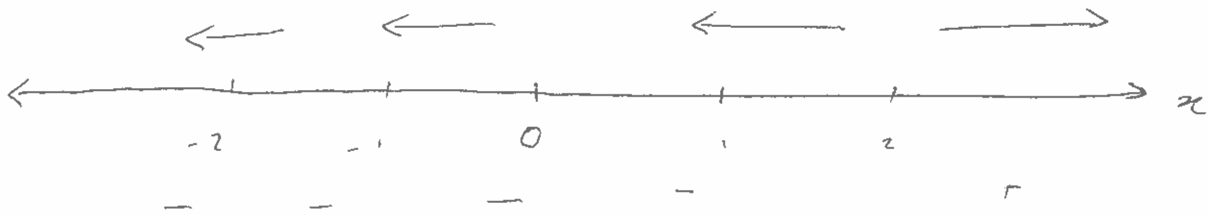
ii) Phase line:

similarly, for equilibrium point:

$$\frac{dx}{dt} = 0, \Rightarrow (x-2)=0, \text{ or } x=0, \text{ or } x+1=0$$
$$x=2, \; x=0, \; x=-1$$

now: $\frac{dx}{dt} > 0$ iff $x \geq 2$, $x \leq -1$

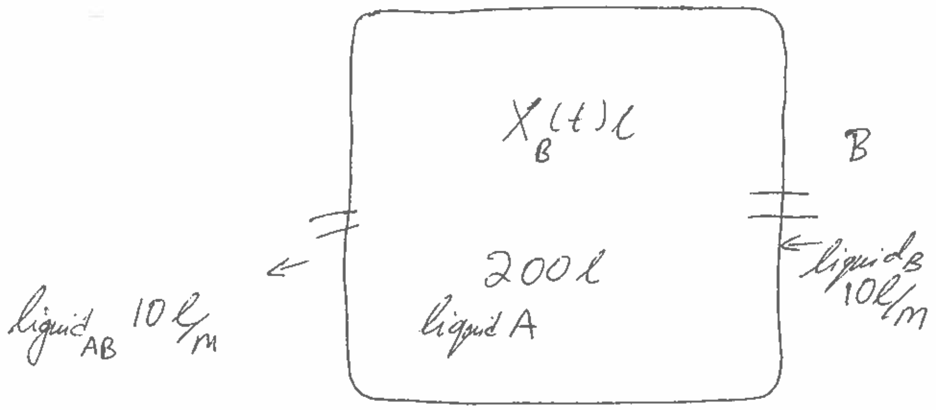
and $\frac{dx}{dt} < 0$ iff $-1 < x < 2$.



The equilibrium points are unstable

If the system starts at $x(0) = 0.1$, it will not return to the equilibrium point $x=2$, but the system will approach the equilibrium points $x=0$, and $x=-1$ from the right and then it will continue and pass this equilibrium points and never return to it.

7.



a) $\dfrac{dX_B(t)}{dt}$ = rate of change of the mixture in the tank

= (rate of mixture entering tank) - (rate of mixture leaving tank)

ie rate of mixture entering tank = 10 l/m,

rate of mixture leaving tank = $10\,l/m \cdot \dfrac{X(t)\,l}{200\,l}$,

because, we know that liquid$_{AB}$ leaves the tank at 10 l per minute, and in the tank we have $\cancel{X}_B$ keeped at full capacity. Thus using ratios we have:

$10\,l/m$: liquid$_{AB}$

$200\,l$: $X_B\,l$

$$\text{liquid}_{AB} = \frac{10\,l/m \cdot X_B\,l}{200\,l}.$$

$$\therefore \frac{dX_{B\ell}}{dt} = 10\ell/m - \frac{10\ell/m \cdot x(t)\ell}{200\ell}$$, with initial

conditions: $X_B = 0$, because initially liquid B was not in the tank.

(b) $$\frac{dX_B}{dt} = 10\ell/m - \frac{X_B\,\ell/m}{20}$$

$$= \frac{1}{20}(200 - X_B)$$

$$\therefore \int \frac{dX_B}{200 - X_B} = \int \frac{1}{20}\,dt$$

$$-\ln|200 - X_B| = \frac{1}{20}t + c$$

$$\ln|200 - X_B| = -\left(\frac{1}{20}t + c\right)$$

$$\therefore |200 - X_B| = e^{-\left(\frac{1}{20}t + c\right)}$$

$$= Ae^{-\frac{1}{20}t} \quad (A = e^{-c})$$

$$\therefore (200 - X_B) = e^{-\left(\frac{1}{20}t + c\right)} \quad \text{OR} \quad (200 - X_B) = -Ae^{-\frac{1}{20}t}$$

$$-X_B = e^{-\left(\frac{1}{20}t + c\right)} - 200 \quad \text{OR} \quad -X_B = -Ae^{-\frac{1}{20}t} - 200$$

$$\therefore X_B = 200 - e^{-\left(\frac{1}{20}t + c\right)} \quad \text{— (1)} \qquad \therefore X_B = Ae^{-\frac{1}{20}t} + 200 \quad \text{— (2)}$$

But both (1) and (2) are valid because from (1) we have:

Assignment 2 solutions

$$-200 = -A\,e^{-\left(\frac{1}{20}t\right)}$$

$$\therefore A = 200$$

and thus $\quad X_B = 200 - 200e^{-\left(\frac{1}{20}t\right)} \qquad (3)$

and from (2) we have $X_B = 0$ (its initial value)

$$-200 = A e^{-\frac{1}{2}(t)}$$

$$\therefore A = -200$$

and thus $\quad X_B = -200e^{-\frac{1}{20}t} + 200 \quad — (4)$

which is the same as (3)

(c) From either (3) or (4), we have:

$$X_B = 200 - 200e^{-\frac{1}{20}t}$$

but $\quad \text{liquid}_A = \text{liquid}_B = \frac{200\ell}{2} = 100\ell.$

$$\therefore \quad X_B = 100 = 200 - 200e^{-\frac{1}{20}t}$$

$$1 = -2e^{-\frac{1}{20}t} + 2$$

$$-1 = -2e^{-\frac{1}{20}t}$$

$$\frac{1}{2} = e^{-\frac{1}{20}t}$$

$$\therefore -\frac{1}{20}t = \ln\left(\tfrac{1}{2}\right)$$

$$t = -20\ln\left(\tfrac{1}{2}\right)$$

$$= 13.86 \simeq 14 \text{ minutes}$$