UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

APM3701

October/November 2016

PARTIAL DIFFERENTIAL EQUATIONS

Duration 2 Hours

100 Marks

EXAMINERS
FIRST PROF H JAFARI
EXTERNAL DR JN MWAMBAKANA

Closed book examination

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This examination question paper remains the property of the University of South Africa and may not be removed from the examination venue

This paper consists of 9 pages

Answer **ALL** questions

An appendix is attached, containing some properties of the Fourier (cosine and sine) series, Fourier transform, general solution of the parametric Bessel differential equation

$$r\frac{d^2R}{dr^2} + \frac{dR}{dr} + \lambda^2 rR = 0,$$

identities for the Bessel function, and the Fourier–Bessel series, and a table of suitable Fourier transforms for given boundary conditions and intervals for partial differential equations is also given

[TURN OVER]

QUESTION 1

Consider the following partial differential equations

$$
\begin{aligned}
&(i) \quad u_{xx} + u_{xy} - 2u_{yy} = 0\\
&(ii) \quad 2u_{xx} + u_{xy} + 4u_{yy} = 0,\\
&(iii) \quad 4u_{xx} + 4u_{xy} + u_{yy} = 0,\\
&(iv) \quad y\,u_{xx} + 2x\,u_{xy} + y\,u_{yy} = 0,
\end{aligned}
$$

a Determine their nature (i e whether elliptic, parabolic or hyperbolic) (8)

b Obtain general solution of (i), (ii) and (iii) (12)

[20 MARKS]

QUESTION 2

Solve the following Heat equation by the method of separation of variables

$$
\begin{aligned}
&u_t = k\,u_{xx}, \qquad 0 < x < 1,\ t > 0,\\
&u(0,t) = u(1,t) = 0, \qquad u(x,0) = 10
\end{aligned}
$$

[20 MARKS]

QUESTION 3

Consider the Dirichlet problem for the wave equation

$$
\frac{\partial^2 u}{\partial t^2} = C^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < L, \quad t > 0
$$

$$
\begin{aligned}
u(0,t) = 0 &= u(L,t)\\
u(x,0) &= u_0(x)\\
\frac{\partial u}{\partial t}(x,0) &= u_1(x)
\end{aligned}
$$

Suppose that there exists a solution to the above problem and $u_0(x) \in C[0,L]$ with respect to x and $u_1(x) \in C^1[0,L]$ with respect to t

Use the energy integral

$$
E(t) = \frac{1}{2}\int_0^L \left\{ C^2 \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2 \right\} dx
$$

to show that the solution u is unique

[20 MARKS]

[TURN OVER]

QUESTION 4

Solve the following partial differential equations by choosing suitable Fourier transforms

a

$$u_{xx} + u_{yy} = 0, \qquad x,\ y > 0,$$

$$\lim_{x\to\infty} u(x,y) = 0, \quad \lim_{x\to\infty} u(x,y) = 0,$$

$$u_x(0,y) = 0, \qquad u(x,0) = e^{-x},$$

(10)

b

$$u_t - c^2 u_{xx} = 0, \qquad 0 < x < \infty,\ t > 0,$$

$$u(x,0) = \tfrac{1}{x}, \qquad u(0,t) = 0,$$

$$\lim_{x\to\infty} u(x,t) = 0,$$

(10)

[20 Marks]

QUESTION 5

When there is heat transfer from the lateral side of an infinite long cylinder of radius a into a surrounding medium, the temperature inside the rod depends upon the time t and the distance r from its longitudinal axis (i e the axis through the centre and parallel to the lateral side)

a Write down the partial differential equation that models this problem (2)

b Suppose that the surrounding medium is at temperature zero and the initial temperature is constant at every point, derive the initial and boundary conditions

[**Hint** For the boundary condition use Newton's law of cooling] (4)

c Solve the initial boundary value problem obtained in (a) and (b) (14)

[20 MARKS]

TOTAL: [100 MARKS]

[TURN OVER]

INFORMATION SHEET

1 Fourier cosine series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \qquad (0 < x < L)$$

with

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx \qquad (n = 0, 1, 2, \ldots)$$

2 Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \qquad (0 < x < L)$$

with

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx \qquad (n = 1, 2, 3, \ldots)$$

3 General Fourier series

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \qquad (a < x < a + 2L)$$

with

$$a_n = \frac{1}{L}\int_a^{a+2L} f(x)\cos\frac{n\pi x}{L}dx, \qquad (n = 0, 1, 2, \ldots)$$

$$b_n = \frac{1}{L}\int_a^{a+2L} f(x)\sin\frac{n\pi x}{L}dx \qquad (n = 1, 2, 3, \ldots)$$

4 Fourier transform

$$\{\mathcal{F}(f(x))\}(\alpha) = \hat{f}(\alpha) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}dx$$

where

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(\alpha)\, e^{i\alpha x}dx$$

$$\mathcal{F}(u_{xx})(\alpha) = -\alpha^2 \hat{u}(\alpha)$$

[TURN OVER]

5 Fourier cosine transform

$$\{\mathcal{F}_c(f(x))\}(\alpha) = \hat{f}_c(\alpha) = \frac{2}{\pi}\int_0^\infty f(x)\cos\alpha x dx$$

with

$$f(x) = \int_0^\infty \hat{f}_c(\alpha)\cos\alpha x dx$$

$$\{\mathcal{F}_c(u_{xx})\}(\alpha) = -\frac{2}{\pi}u_x(0) - a^2\hat{u}_c(\alpha)$$

6 Fourier sine transform

$$\{\mathcal{F}_s(f(x))\}(\alpha) = \hat{f}_s(\alpha) = \frac{2}{\pi}\int_0^\infty f(x)\sin\alpha x dx$$

with

$$f(x) = \int_0^\infty \hat{f}_s(\alpha)\sin\alpha x dx$$

$$\{\mathcal{F}_x(u_{xx})\}(\alpha) = \frac{2}{\pi}\alpha u(0) - \alpha^2\hat{u}_s(\alpha)$$

7 Table of suitable Fourier transform for a given boundary condition and interval

Interval	**Boundary conditions**	**Suitable Fourier transform**	**Transformation**
$-\infty < x < \infty$	$\lim_{\lvert x\rvert\to\infty} u(x,y) = 0$ $u(x,0) = f(x)$	Fourier transform	$u_{xx}(x,y) \to -\alpha U(\alpha,y)$ $u(x,0) = f(x) \to U(\alpha,0) = F(\alpha)$
$0 < x < \infty$	$\lim_{x\to\infty} u(x,y) = 0$ $u_x(0,y) = f(y)$	Cosine Fourier transform	$u_{xx}(x,y) \to -\alpha^2 U_c(\alpha,y) - \frac{2}{\pi}f(y)$
$0 < x < \infty$	$\lim_{x\to\infty} u(x,y) = 0$ $u(0,y) = f(y)$	Sine Fourier transform	$u_{xx}(x,y) \to -\alpha^2 U_s(\alpha,y) + \alpha\frac{2}{\pi}f(y)$ $u(0,y) = f(y) \to U_s(0,y)$

8 Table of Fourier Transforms

Function	**Fourier Transform**
$f(x)$	$F(\alpha)$
$f(ax-b)$	$\frac{e^{-\frac{i\alpha b}{a}}}{\lvert a\rvert}F\left(\frac{\alpha}{a}\right)$ or $\frac{e^{i\frac{\alpha b}{a}}}{\lvert a\rvert}F\left(\frac{\alpha}{a}\right)$
$f*g$ (convolution)	$F(\alpha)\ \ G(\alpha)$
$e^{-a\lvert x\rvert}$	$\sqrt{\frac{2}{\pi}}\frac{a}{\alpha^2}$
$\frac{b^{-1}\sin\frac{a\pi}{b}}{\cosh\frac{\pi x}{b}+\cos\frac{\pi a}{b}}$	$\sqrt{\frac{2}{\pi}}\frac{\sinh a\alpha}{\sinh b\alpha}$
$e^{-b^2x^2}$	$\frac{1}{\sqrt{2}b}e^{-\frac{\alpha^2}{4b^2}}$
$f^{(n)}(x)$	$i^n\alpha^nF(\alpha)$ or $(-i)^n\alpha^nF(\alpha)$
$x^nf(x)$	$i^n\frac{d^nF}{d\alpha^n}$ or $(-i)^n\frac{d^nF}{d\alpha^n}$
$f(a)e^{ibx}$	$\frac{1}{\lvert a\rvert}F\left(\frac{\alpha-b}{a}\right)$

9 Table of Fourier Cosine Transforms

Function	**Fourier Cosine Transform**
$f(x)$	$F_c(\alpha)$
$\dfrac{1}{x^2+a^2}$	$\dfrac{2}{\pi}\dfrac{e^{-a\alpha}}{\lvert a\rvert}$
$e^{-ax} \quad (x>0)$	$\dfrac{4}{\pi^2}\sqrt{\dfrac{2}{\pi}}\dfrac{a}{(a^2+\alpha^2)}$
$e^{-ax^2} \quad (a>0)$	$\dfrac{1}{\pi\sqrt{a}}e^{-\frac{\alpha^2}{4a}}$
$x^{-\frac{1}{2}}$	$\sqrt{\dfrac{1}{\alpha}}$
$\sin ax^2$	$\dfrac{1}{\pi}\left(\cos\dfrac{\alpha^2}{4a}-\sin\dfrac{\alpha^2}{4a}\right)$
$\cos ax^2$	$\dfrac{1}{\pi}\left(\cos\dfrac{\alpha^2}{4a}+\sin\dfrac{\alpha^2}{4a}\right)$
$\dfrac{d^2f(x)}{dx}$	$-\alpha^2F_c(\alpha)-\sqrt{\dfrac{2}{\pi}}f'(0)$

10 Table of Fourier Sine Transforms

Function	Fourier Sine Transform
$f(x)$	$F_s(\alpha)$
x^{-1}	$\sqrt{\frac{2}{\pi}}$
$\frac{x}{x^2+a^2}$	$\sqrt{\frac{2}{\pi}}e^{-a\alpha}$
$e^{-ax}\,(x>0)$	$\frac{2}{\pi}\sqrt{\frac{2}{\pi}}\frac{\alpha}{(\alpha^2+a^2)}$
xe^{-ax^2}	$\frac{\sqrt{2}}{2\pi}\frac{\alpha e^{-\frac{\alpha^2}{4a}}}{a^{\frac{3}{2}}}$
$x^{-\frac{1}{2}}$	$\frac{1}{\sqrt{\alpha}}$
$\tan^{-1}\left(\frac{x}{a}\right)$	$\sqrt{\frac{2}{\pi}}\frac{e^{-a\alpha}}{\alpha}$
$\frac{d^2f(x)}{dx^2}$	$\sqrt{\frac{2}{\pi}}\alpha f(0)-\alpha^2F_s(\alpha)$

11 The general solution of the parametric Bessel differential equation

$$r\frac{d^2R}{dr^2}+\frac{dR}{dr}+\lambda^2rR=0$$

is

$$R=AJ_0(\lambda r)+BY_0(\lambda r)$$

where J_0 is the Bessel function of the first kind of order 0 and Y_0 is the Bessel function of the second kind of order 0

12 Identities for Bessel functions

(i) $J_n'(x)-\frac{n}{x}J_n(x)=-J_{n+1}(x)$

(ii) $\frac{d}{dx}\{xJ_1(x)\}=xJ_0(x)$

(iii) $J_0(0)=1$

13 The Fourier–Bessel series of a function f defined on the interval $(0, b)$ is given by

(i)

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{\infty} c_i J_n(\lambda_i x) \\
c_i &= \frac{2}{b^2 J_{n+1}^2(\lambda_i b)} \int_0^b x J_n(\lambda_i x) f(x)\, dx,
\end{aligned}
$$

where the λ_i are defined by $J_n(\lambda b) = 0$

(ii)

$$
\begin{aligned}
f(x) &= \sum_{i=1}^{\infty} c_i J_n(\lambda_i x) \\
c_i &= \frac{2\lambda_i^2}{\left(\lambda_i^2 b^2 - n^2 + h^2\right) J_n^2(\lambda_i b)} \int_0^b x J_n(\lambda_i x) f(x)\, dx,
\end{aligned}
$$

where the λ_i are defined by

$$h J_n(\lambda b) + \lambda b J_n'(\lambda b) = 0$$

(iii)

$$
\begin{aligned}
f(x) &= c_1 + \sum_{i=2}^{\infty} c_i J_0(\lambda, x) \\
c_1 &= \frac{2}{b^2} \int_0^b x f(x)\, dx \\
c_i &= \frac{2}{b^2 J_0^2(\lambda_i b)} \int_0^b x J_0(\lambda_i x) f(x)\, dx,
\end{aligned}
$$

where the λ_i are defined by

$$J_0'(\lambda b) = 0$$