# Tutorial Letter 201/2/2014

## Numerical Methods II

## APM3711

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains solutions to assignment 01.

BAR CODE

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**ONLY FOR SEMESTER 2 STUDENTS**
**ASSIGNMENT 01**

**FIXED CLOSING DATE: 15 AUGUST 2014**
**Unique Number: 897324**

### Question 1

Solve the differential equation

$$\frac{dy}{dx} = 3x + 2y + xy, \quad y(0) = -1$$

by means of the Taylor–series expansion to get the value of $y$ at $x = 0.1$. Use terms up to $x^6$.

### Solution:

We approximate $y\,(0.1)$ by the truncated Taylor series

$$y\,(0.1) \simeq y\,(0) + (0.1)\,y'\,(0) + \frac{(0.1)^2}{2} y''\,(0) + \cdots + \frac{(0.1)^6}{6!} y^{(6)}(0) \tag{1}$$

The derivatives at $x = 0$ are calculated in the way illustrated on p. 398 [354] of *Gerald*:

$$\begin{array}{ll}
y'\,(x) = 3x + (2 + x)\,y, & y'\,(0) = 2\,(-1) = -2 \\
y''\,(x) = 3 + y + (2 + x)\,y', & y''\,(0) = 3 - 1 + 2\,(-2) = -2 \\
y'''\,(x) = 2y' + (2 + x)\,y'', & y'''\,(0) = 2\,(-2) + 2\,(-2) = -8 \\
y^{(4)}\,(x) = 3y'' + (2 + x)\,y''', & y^{(4)}\,(0) = 3\,(-2) + 2\,(-8) = -22 \\
y^{(5)}\,(x) = 4y''' + (2 + x)\,y^{(4)}, & y^{(5)}\,(0) = 4\,(-8) + 2\,(-22) = -76 \\
y^{(6)}\,(x) = 5y^{(4)} + (2 + x)\,y^{(5)}, & y^{(6)}\,(0) = -5 \cdot 22 - 2 \cdot 76 = -262
\end{array}$$

Substituting these values in (1) we get

$$y\,(0.1) \simeq -1.211431697.$$

The exact value, rounded to 10 significant numbers, can be calculated by adding more terms of the Taylor series to (1) until it is clear that further addition of terms will not alter the tenth digit. You may verify that this gives

$$y\,(0.1) = -1.211431726.$$

**Question 2**

Consider the system of coupled second-order differential equations

$$\begin{aligned} u'' - (t+1)(u')^2 + 2uv - u^3 &= \cos t \\ 2v'' + (\sin t)\, u'v' - 6u &= 2t + 3 \end{aligned}$$

with initial conditions

$$u(0) = 1, \quad u'(0) = 2, \;\; v(0) = 3, \;\; v'(0) = 4.$$

Use the second-order Runge-Kutta method with $h = 0.2$ and $a = 2/3$, $b = 1/3$, $\alpha = \beta = 3/2$, to find $u$, $u'$, $v$ and $v'$ at $t = 0.2$.

**Solution:**

**Given**

$$u'' - (t+1)(u')^2 + 2uv - u^3 = \cos t$$
$$2v'' + (\sin t)\, u'v' - 6u = 2t + 3$$

$$u(0) = 1, \quad u'(0) = 2, \quad v(0) = 3, \quad v'(0) = 4$$

First, we convert the system of second–order differential equations to a system of first–order differential equations, by introducing two new variables which equal the derivatives of the original variables. Let

$$\begin{aligned} u' &= w, \\ v' &= x, \end{aligned}$$

then the original differential equations can be written as

$$\begin{aligned} w' &= (t+1)\, w^2 - 2uv + u^3 + \cos t \\ x' &= 3u - \frac{1}{2}(\sin t)\, wx + t + \frac{3}{2} \end{aligned}$$

Therefore, the corresponding system of first–order differential equations is

$$\left\{ \begin{array}{lll} u' = & w & u(0) = \; 1 \\ v' = & x & v(0) = \; 3 \\ w' = & (t+1)\, w^2 - 2uv + u^3 + \cos t & w(0) = \; 2 \\ x' = & 3u - \frac{1}{2}(\sin t)\, wx + t + \frac{3}{2} & x(0) = \; 4 \end{array} \right. \tag{1}$$

To see how the algorithms can be adapted to systems such as (1), note first that the system (1) can be written as

$$\left\{ \begin{array}{lll} u' = & U(t,\, u,\, v,\, w,\, x) & u(0) = \; 1 \\ v' = & V(t,\, u,\, v,\, w,\, x) & v(0) = \; 3 \\ w' = & W(t,\, u,\, v,\, w,\, x) & w(0) = \; 2 \\ x' = & X(t,\, u,\, v,\, w,\, x) & x(0) = \; 4 \end{array} \right. \tag{2}$$

where

$$\begin{cases} U(t,\, u,\, v,\, w,\, x) = & w, \\ V(t,\, u,\, v,\, w,\, x) = & x, \\ W(t,\, u,\, v,\, w,\, x) = & (t+1)\, w^2 - 2uv + u^3 + \cos t, \\ X(t,\, u,\, v,\, w,\, x) = & 3u - \frac{1}{2}(\sin t)\, wx + t + \frac{3}{2}. \end{cases}$$

The basic idea is to adapt the algorithm for solving problems of the form

$$y' = f(t, y), \quad y(t_0) = y_0 \tag{3}$$

to solving problems of the form (2), by replacing $y$ by $u$, $v$, $w$, and $x$, and replacing $f$ by $U$, $V$, $W$ and $X$. Note that the algorithm is **not** carried out separately for each unknown, but simultaneously for all four.

In this case of the second–order Runge–Kutta method with $a = \dfrac{2}{3}$, $b = \dfrac{1}{3}$, $\alpha = \beta = \dfrac{3}{2}$, the algorithm for (3) is

$$\begin{aligned} y_{n+1} &= y_n + \frac{2}{3}k_1 + \frac{1}{3}k_2, \\ k_1 &= hf(t_n, y_n), \\ k_2 &= hf\left(t_n + \frac{3}{2}h,\ y_n + \frac{3}{2}k_1\right). \end{aligned}$$

Adapted to the system (2), this gives us the algorithm

$$\begin{aligned} u_{n+1} &= u_n + \frac{2}{3}k_{u1} + \frac{1}{3}k_{u2} \\ v_{n+1} &= v_n + \frac{2}{3}k_{v1} + \frac{1}{3}k_{v2} \\ w_{n+1} &= w_n + \frac{2}{3}k_{w1} + \frac{1}{3}k_{w2} \\ x_{n+1} &= x_n + \frac{2}{3}k_{x1} + \frac{1}{3}k_{x2} \end{aligned}$$

$$\begin{aligned} k_{u1} &= hU(t_n,\, u_n,\, v_n,\, w_n,\, x_n) \\ k_{v1} &= hV(t_n,\, u_n,\, v_n,\, w_n,\, x_n) \\ k_{w1} &= hW(t_n,\, u_n,\, v_n,\, w_n,\, x_n) \\ k_{x1} &= hX(t_n,\, u_n,\, v_n,\, w_n,\, x_n) \end{aligned}$$

$$
\begin{aligned}
k_{u2} &= hU\left(t_n+\frac{3}{2}h,\ u_n+\frac{3}{2}k_{v1},\ v_n+\frac{3}{2}k_{v1},\ w_n+\frac{3}{2}k_{w1},\ x_n+\frac{3}{2}k_{x1}\right)\\
k_{v2} &= hV\left(t_n+\frac{3}{2}h,\ u_n+\frac{3}{2}k_{v1},\ v_n+\frac{3}{2}k_{v1},\ w_n+\frac{3}{2}k_{w1},\ x_n+\frac{3}{2}k_{x1}\right)\\
k_{w2} &= hW\left(t_n+\frac{3}{2}h,\ u_n+\frac{3}{2}k_{v1},\ v_n+\frac{3}{2}k_{v1},\ w_n+\frac{3}{2}k_{w1},\ x_n+\frac{3}{2}k_{x1}\right)\\
k_{x2} &= hX\left(t_n+\frac{3}{2}h,\ u_n+\frac{3}{2}k_{v1},\ v_n+\frac{3}{2}k_{v1},\ w_n+\frac{3}{2}k_{w1},\ x_n+\frac{3}{2}k_{x1}\right).
\end{aligned}
$$

Hence, with $h=0.2$, we calculate $u(0.2)$, $v(0.2)$, $u'(0.2)$ and $v'(0.2)$ from the given initial values as follows:

$$
\begin{aligned}
k_{u1} &= (0.2)\ U(0,\ 1,\ 3,\ 2,4) = (0.2)(2) = 0.4\\
\\
k_{v1} &= (0.2)\ V(0,\ 1,\ 3,\ 2,4) = (0.2)(4) = 0.8\\
\\
k_{w1} &= (0.2)\ W(0,\ 1,\ 3,\ 2,4)\\
&= (0.2)\left[(0+1)\,2^2-2\cdot 1\cdot 3+1^3+1\right] = 0\\
\\
k_{x1} &= (0.2)\ X(0,\ 1,\ 3,\ 2,4)\\
&= (0.2)(3\cdot 1-(0.5)\cdot 0\cdot 2\cdot 4+0+1.5) = 0.9\\
k_{u2} &= (0.2)\ U\left(0+\frac{3}{2}(0.2),\ 1+\frac{3}{2}(0.4),\ 3+\frac{3}{2}(0.8),\ 2+\frac{3}{2}(0),\ 4+\frac{3}{2}(0.9)\right)\\
&= (0.2)\ U(0.3,\ 1.6,\ 4.2,\ 2.0,\ 5.35)\\
&= (0.2)\cdot 2 = 0.4\\
\\
k_{v2} &= (0.2)\ V(0.3,\ 1.6,\ 4.2,\ 2.0,\ 5.35)\\
&= (0.2)\cdot 5.35 = 1.07\\
k_{w2} &= (0.2)\ W(0.3,\ 1.6,\ 4.2,\ 2.0,\ 5.35)\\
&= (0.2)\left[(0.3+1)\,2^2-2(1.6)(4.2)+(1.6)^3+\cos(0.3)\right]\\
&= -0.6377\\
k_{x2} &= (0.2)\ X(0.3,\ 1.6,\ 4.2,\ 2.0,\ 5.35)\\
&= (0.2)\left[3(1.6)-(0.5)\cdot\sin(0.3)\cdot 2\cdot(5.35)+0.3+1.5\right]\\
&= 1.0038
\end{aligned}
$$

$$
\begin{aligned}
u(0.2) &= 1+\frac{2}{3}(0.4)+\frac{1}{3}(0.4)\approx 1.4\\
v(0.2) &= 3+\frac{2}{3}(0.8)+\frac{1}{3}(1.07)\approx 3.89
\end{aligned}
$$

$$
\begin{aligned}
u'(0.2) &= w(0.2) = 2 + \frac{2}{3}(0) + \frac{1}{3}(-0.6377) \approx 1.79 \\
v'(0.2) &= x(0.2) = 4 + \frac{2}{3}(0.9) + \frac{1}{3}(1.0038)\,9 \approx 4.93
\end{aligned}
$$

## Question 3

The differential equation

$$y' = y - x^2 \qquad y(0) = 1$$

and starting values

$$y(0.2) = 1.2186, \quad y(0.4) = 1.4682, \quad y(0.6) = 1.7379.$$

Use the fourth–order **Adams–Moulton** predictor–corrector method with $h = 0.2$ to solve the equation up to $x = 1.2$.
Compare with the analytical solution,

$$y = x^2 + 2x + 2 - e^x.$$

## Solution:

Given

$$
\begin{aligned}
y' &= y - x^2 \\
y(0) &= 1;\ y(0.2) = 1.2186; \\
y(0.4) &= 1.4682;\ y(0.6) = 1.7379
\end{aligned}
$$

**Analytical solution**

$$\frac{dy}{dx} - y = -x^2$$

This is a first order linear equation and can be put into the form

$$
\begin{aligned}
y' + P(x)y &= Q(x) \\
P(x) &= -1
\end{aligned}
$$

and

$$Q(x) = -x^2$$

The integrating factor is

$$e^{\int P(x)dx} = e^{\int -dx} = e^{-x}$$

$$\therefore \qquad y'e^{-x} - ye^{-x} = -x^2 e^{-x}$$

$$\begin{aligned} ye^{-x} &= -\int x^2 e^{-x} dx \\ &= x^2 e^{-x} + 2xe^{-x} + 2e^{-x} + c \end{aligned}$$

(using integration by parts)

$$\begin{aligned} \therefore \quad & y = x^2 + 2x + 2 + ce^x \\ & x = 0 : 1 = 2 + ce^0 \\ \therefore \quad & c = -1 \\ \therefore \quad & y = x^2 + 2x + 2 - e^x \end{aligned}$$

**Numerical solution**
The Adams–Moulton method is a **multistep method**, i.e. more than one past value is used to approximate the new value. In *Gerald* four multistep methods are discussed.

(1) The **3rd order Adams method** (see (5.13) [(5.10)]) has a local error of order 4 and a global error of order 3:

$$y_{i+1} = y_i + \frac{h}{12}(23f_i - 16f_{i-1} + 5f_{i-2})$$

where $f_i$ means $f(x_i, y_i)$.

(2) The **4th order Adams method** (also called the Adams–Bashforth method; see (5.15) [p. 363]) is

$$y_{i+1} = y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3})$$

(3) **Milne's predictor–corrector method** (see p. 413 [364]) is

$$\begin{aligned} y_p &= \frac{4h}{3}(2f_i - f_{i-1} + 2f_{i-2}) + y_{i-3} \\ y_{i+1} &= y_{i-1} + \frac{h}{3}(f(x_i + h, y_p) + 4f_i + f_{i-1}) \end{aligned}$$

(4) The **Adams–Moulton method** (also called the Adams predictor–corrector method; see (5.24) – (5.25) [(5.22)]) is

$$\begin{aligned} y_p &= y_i + \frac{h}{24}(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}) \\ y_{i+1} &= y_i + \frac{h}{24}(9f(x_i + h, y_p) + 19f_i - 5f_{i-1} + f_{i-2}) \end{aligned}$$

The last three methods all have a local error of order $h^5$ and a global error of order $h^4$, but Milne's method is sometimes unstable, and the error constant for the Adams–Moulton method is smaller than that of the 4th order Adams method (19/720 acs opposed to 251/720).
See pages 409, 414, 417–418 [361, 365, 369–370] for a discussion of the relative merits of these multistep methods and the Runge–Kutta methods.

The program and results follows. You should make sure that you can reproduce the table of results by using a calculator.

PROGRAM A99_1_3(output);

```
CONST
size = 10;
VAR
i :  integer;
h, x, y, yp, fp, exact :  real;
fn :  array[0..size] of real;
z :  text;

FUNCTION f(x, y :  real) :  real;
BEGIN
f := y - x*x;
END; {f}

FUNCTION yexact(x :  real) :  real;
BEGIN
yexact := x*x + 2*x + 2 - exp(x);
END; {yexact}

BEGIN
assign(z,'AS99_1_3.DAT');
rewrite(z);
writeln(z);
writeln(z,'  *****  ASSIGNMENT 1,  ',
'QUESTION 3(b)  *****');
writeln(z);
writeln(z,'  ***********  ADAMS-MOULTON',
' METHOD  **********');
h := 0.2;
writeln(z,'  h = ',h:4:2);
writeln(z,'    x  exact y     y     error',
'  f(x,y)    yp  f(x+h,yp)');

fn[0] := f(0.0, 1.0);
fn[1] := f(0.2, 1.2186);
fn[2] := f(0.4, 1.4682);
x := 0.6;
y := 1.7379;
write(z,x:6:2,' ',y:7:4,' ',y:7:4,' ',
(0.0):7:4);

FOR i := 3 to 5 DO
BEGIN
fn[i] := f(x,y);
yp := y + h*(55*fn[i] - 59*fn[i-1]
```

```
+ 37*fn[i-2] - 9*fn[i-3])/24;
fp := f(x + h,yp);
y := y + h*(9*fp + 19*fn[i] - 5*fn[i-1]
+ fn[i-2])/24;
x := x + h;
writeln(z,' ',fn[i]:7:4,' ',yp:7:4,' ',
fp:7:4);
exact := yexact(x);
write(z,x:6:2,' ',exact:7:4,' ',y:7:4,' ',
(exact - y):7:4);
END; {for i}
writeln(z);
close(z);
END.
```

```
*****  ASSIGNMENT 1,  QUESTION 3(b)  *****

***********  ADAMS-MOULTON METHOD  **********
h = 0.20
x  exact y     y     error  f(x,y)    yp  f(x+h,yp)
0.60  1.7379  1.7379  0.0000  1.3779  2.0146  1.3746
0.80  2.0145  2.0145 -0.0000  1.3745  2.2819  1.2819
1.00  2.2817  2.2817 -0.0000  1.2817  2.5202  1.0802
1.20  2.5199  2.5199 -0.0000
```

## Question 4

(a) The first three Chebyshev polynomials are

$$\begin{aligned}
&T_0 = 1\\
&T_1 = x\\
&T_2 = 2x^2 - 1\\
&T_3(x) = 4x^3 - 3x\\
&T_4(x) = 8x^4 - 8x^2 + 1.
\end{aligned}$$

(i) Economize the truncated power series

$$p(x) = 1 + 2x - x^3 + 3x^4.$$

(ii) Give an upper limit for the absolute value of the difference between the original truncated power series and the economized one.

(b) Find the Padé approximation $R_4(x)$, with denominator of degree 2, to the function

$$f(x) = 4 + 4x^2 - 2x^6.$$

**Solution:**

(a) (i)

$$p(x) = 1 + 2x - x^3 + 3x^4$$

Subtract: $\frac{3}{4}T_4(x) = 3^4 - 3x^4 - 3x^2 + \frac{3}{8}$ from $p(x)$, in order to remove the highest order term as to get the economized series, $g(x) = -x^3 + 3x^2 + 2x + \frac{5}{8}$

(ii) For all Chebysshev polynomials $T_i$ we have $|T_i| \leq 1$ for all $x \in [-1, 1]$, and therefore $|p(x) - g(x)| = \left|\frac{3}{8}T_4(x)\right| \leq \frac{3}{8} = 0.375$

(b) The Padé approximation is

$$R_4(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2}$$

This gives

$$R_4(x) - f(x) = \frac{a_0 + a_1 x + a_2 x^2}{1 + b_1 x + b_2 x^2} - 4 - 4x^2 + 2x^6$$

which gives:

$$4 + 4b_1 x + 4(1 + b_2)x^2 + (4b_1 - a_3)x^3 + 4b_2 x^4 - a_0 - a_1 x - a_2 x^2.$$

Equate coefficients of like powers of $x$ :

$$\begin{aligned}
x^0 :&\quad 4 - a_0 = 0 \\
x^1 :&\quad 4b_1 - a_1 = 0 \\
x^2 :&\quad 4\left(1 + b_2\right) - a_2 = 0 \\
x^3 :&\quad 4b_1 = 0 \\
x^4 :&\quad 4b_2 = 0
\end{aligned}$$

Therefore,

$$b_1 = 0, \quad a_0 = 4, \quad a_1 = 0, \quad b_2 = 0, \quad a_2 = 4$$

Thus,

$$R_4\left(x\right) = 4 + 4x^2$$