# Tutorial Letter 201/2/2015

## Special Relativity and Riemannian Geometry

# APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 1.

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# Memo for Assignment 1 S2 2015

## Special relativity basics (§ 1.1 - 1.2)

## Consequences of Lorentz transformations (§ 1.3)

### Question 1: Lorentz transformation

Alice sees an explosion happening and measures the spacetime coordinates of the explosion to be $(t, x, y, z) = (1.5\,\text{ns}, 2\,\text{m}, 1\,\text{m}, 0\,\text{m})$. Bob is riding past in a train at a constant speed of $V = 0.4c$ in the positive $x$-direction. Use the Lorentz transformation equations to determine what time Bob measures the explosion taking place. (Hint: $1\,\text{ns}$ (nanosecond) $= 10^{-9}\text{s}$).

- $1.5\,\text{ns}$
- $1.407 \times 10^{-9}\,\text{s}$
- $-1.17 \times 10^{-9}\,\text{s}$
- $1.5\,\text{s}$
- $-1.41\,\text{ns}$*

The train is moving at $V = 0.4c$, so the Lorentz factor between Alice's frame ($S$) and Bob's frame ($S'$) is

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1 - V^2/c^2}} \\
&= \frac{1}{\sqrt{1 - (0.4c)^2/c^2}} \\
&= \frac{1}{\sqrt{1 - 0.16}} \\
&= \frac{1}{\sqrt{0.84}} \\
&= 1.09
\end{aligned}$$

The Lorentz transformation equation for the time coordinate is

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Using this, we transform Alice's measurements into Bob's frame as follows

$$\begin{aligned}
t' &= \gamma\left(t - \frac{vx}{c^2}\right) \\
&= 1.09\left(1.5\,\mathrm{ns} - \frac{(0.4c)\,(2\,\mathrm{m})}{c^2}\right) \\
&= 1.09\left(1.5\,\mathrm{ns} - \frac{(0.4)\,(2\,\mathrm{m})}{c}\right) \\
&= 1.09\left(1.5\,\mathrm{ns} - \frac{(0.4)\,(2\,\mathrm{m})}{3\times 10^8\mathrm{ms}^{-1}}\right) \\
&= 1.09\left(1.5\,\mathrm{ns} - 2.27\times 10^{-9}\mathrm{s}\right) \\
&= 1.09\,(1.5\,\mathrm{ns} - 2.27\,\mathrm{ns}) \\
&= 1.09\,(-1.17\,\mathrm{ns}) \\
&= -1.41\,\mathrm{ns}
\end{aligned}$$

The negative time coordinate just means that Bob sees the explosion happening at a time before the arbitrarily chosen zero time.

## Question 2: Time dilation

A group of astronauts take on a mission to travel to a nearby planet. According to the people on Earth, the spaceship takes 100 years to reach its destination, but the astronauts on the spaceship only aged 30 years during the journey. How fast was the spaceship travelling, assuming that it was moving at a constant velocity?

- $0.83c$
- **0.95c***
- $3.48c$
- $0.91c$

- $1.04c$

The astronauts measure their own proper time. Then the time that the people on Earth measure for the journey ($\Delta t_E$) will be related to the time that the astronauts measure ($\Delta t_A$) by the time dilation formula so that

$$\begin{aligned}
\Delta t_E &= \gamma \Delta t_A \\
100\,\text{years} &= \gamma \times 30\,\text{years} \\
\gamma &= 3.33 \\
\frac{1}{\sqrt{1 - V^2/c^2}} &= 3.33 \\
1 - V^2/c^2 &= 0.09 \\
V^2/c^2 &= 0.91 \\
V &= 0.95c
\end{aligned}$$

Remember, nothing with mass can travel faster than the speed of light $c$. If you ever get an answer where a speed is greater than $c$, you made a mistake somewhere.

## Question 3: Summation

The sum

$$B_\alpha = \sum_{\beta=0}^{3} \eta_{\beta\alpha} B^\beta$$

written out in full is

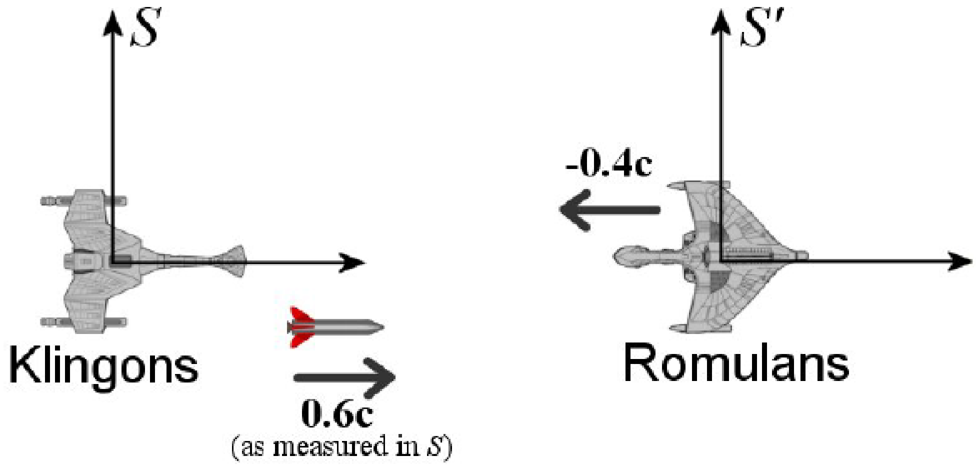
- $B_\alpha = 3\eta_{\beta\alpha} B^\beta$
- $B_\alpha = \eta_{\beta\alpha} B^\beta$
- $B_\alpha = \eta_{\beta\alpha} B^\beta + \eta_{\beta\alpha} B^\beta + \eta_{\beta\alpha} B^\beta + \eta_{\beta\alpha} B^\beta$
- $\mathbf{B}_\alpha = \eta_{\mathbf{0}\alpha} \mathbf{B}^\mathbf{0} + \eta_{\mathbf{1}\alpha} \mathbf{B}^\mathbf{1} + \eta_{\mathbf{2}\alpha} \mathbf{B}^\mathbf{2} + \eta_{\mathbf{3}\alpha} \mathbf{B}^{\mathbf{3}*}$
- $B_\alpha = \eta_{00} B^0 + \eta_{11} B^1 + \eta_{22} B^2 + \eta_{33} B^3$

## Question 4: Velocity addition

Two alien races, the Klingons and the Romulans are having a battle in space. During the battle, two of the spaceships approach each other head on at a speed of $0.4c$. The Klingon ship shoots a torpedo in the direction of the Romulan ship. The Klingons measure the torpedo leaving their ship at a speed of $0.6c$. How fast does the Romulans measure the torpedo to be approaching them?

- $c$
- $0.26c$
- **0.81c***
- $1.32c$
- $0.73c$

When in doubt, draw a picture! It will help you to disentangle the problem and organise your thoughts. It can also show the marker that you have insight into the problem. Below is a rough sketch of the situation.




The two spaceships are approaching each other, moving with a speed $0.4c$ relative to each other. We can interpret this in a few ways mathematically, where the physical situation remains unchanged. For example, we can say that the Klingon ship is moving at $0.2c$ in the positive $x$-direction, while the Romulan ship is moving at $0.2c$ in the negative $x$-direction.

Or we can say that the Romulan ship is stationary, with the Klingon ship moving at $0.4c$ in the positive $x$-direction.

As indicated in the figure above, we will approach this problem by taking the Klingon ship as being stationary, and the Romulan ship travelling towards in at $0.4c$ in the *negative $x$-direction*. If we call the frame in which the Klingon ship is stationary $S$ (if the figure above were a photograph, the "photographer" would also be stationary in this frame), and the frame in which the Romulan ship is stationary $S'$. Considering the standard configuration, we can now identify the speed between the frames as $V = -0.4c$.

Now the Klingon ship shoots a torpedo towards the Romulan ship (in the positive $x$-direction) and they (in the $S$ frame) measures its speed as $v_x = 0.6c$. The question requires you to calculate the speed of the torpedo as measured by the Romulans (in $S'$). So we use the velocity transformation equation for the $x$-direction to transform $v_x$ to $v'_x$.

$$
\begin{aligned}
v'_x &= \frac{v_x - V}{1 - v_x V/c^2} \\
&= \frac{(0.6c) - (-0.4c)}{1 - (0.6c)(-0.4c)/c^2} \\
&= \frac{c}{1 + 0.24} \\
&= 0.81c
\end{aligned}
$$

So the Romulans measure the speed of the torpedo to be $0.81c$ in the positive $x$-direction.

# Tutorial Letter 202/2/2015

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 2.

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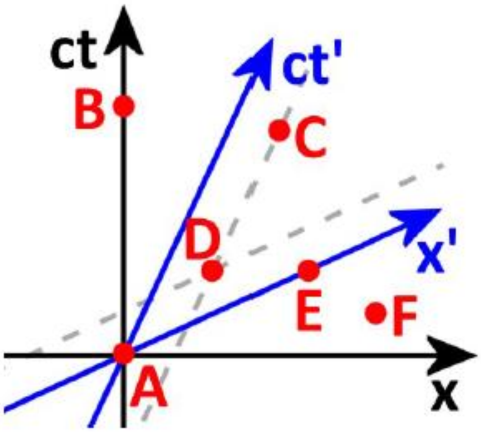
# Memo for Assignment 2 S2 2015

## Minkowski spacetime (§ 1.4)

## Physical laws in relativity (§ 2.1 - 2.2.3)

### Questions 1 - 3: Spacetime diagrams

Consider the spacetime diagram below to answer the following questions




1. In the $S'$ frame, the following two events occur at the same position

- A and B
- A and E
- A and D
- **D and C***
- D and E

The $S'$ frame is indicated by the blue coordinate axes. For two events to occur at the same position, they must occur at the same space coordinate. With the help of the dashed line that is parallel to the $x'$ axis, it is clear that events C and D have the same $x'$ coordinate.

2. To which point would observers in both the $S$ and $S'$ frame assign the same spacetime coordinates?

- **A***
- B
- C
- E
- None of the points

The only point to which observers in both $S$ and $S'$ will assign the same spacetime coordinates is the origin, event A where $(x, ct) = (x', ct') = (0, 0)$.

3. Which of the following statements are true *for all frames*?

- **Event A happens before event C***
- Events E and F are causally related
- Events A and B occur at the same position
- Events A and B occur at the same time
- Event C caused event F

Event A will happen at time $ct = ct' = 0$. It is not possible to draw a coordinate axes where Event C will be below the space axis. The spacetime diagram will have to look something like the figure below, which is not a valid Lorentz transformation. The axes of a Lorentz transformation will always be symmetric about the line $ct = x$ (shown as a dashed line in the figure).

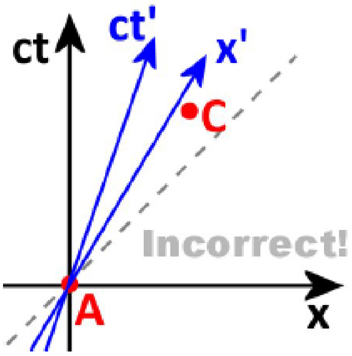
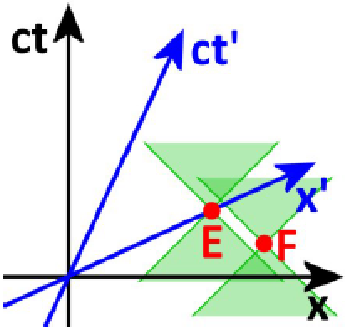



Figure 1: Not a correct Lorentz transformation!

Events E and F will be causally related if they can be related by a signal that travels slower than the speed of light, or equivalently, if the two events are in each others lightcones. From the figure below it is clear that they are not in each others lightcones (or, a signal connecting the two events will have to move faster than the speed of light), so one event cannot cause the other and they are not causally related.




Events A and B occur at the same position in $S$ ($x = 0$), but this is not true for all frames. In $S'$ (and all other inertial frames with a positive $V$ relative to $S$, A will occur at the origin and B will occur at some negative value of $x$.

Events A and B do not occur at the same time in $S$, so they do not occur at the same time in all frames, since you can give a counter example.

Events C and F are causally related, since they can be connected with a signal that moves slower than $c$, or are in each others light cones. But, event F will occur before C in all frames, so there is no way in which C could cause F, although it would be possible for F to cause C.

## Question 4: Mass energy

What is the mass energy of a proton with rest mass $m_p = 1.67 \times 10^{-27}$ kg?

- $5.01 \times 10^{-19}$ joules
- **$1.50 \times 10^{-10}$ joules***
- $1.50 \times 10^{44}$ joules
- $1.67 \times 10^{-27}$ joules
- $9 \times 10^{16}$ joules

The mass energy for the proton is

$$E = mc^2 = \left(1.67 \times 10^{-27}\,\text{kg}\right)\left(3 \times 10^{8}\right)^2 = 1.50 \times 10^{-10}\,\text{J}$$

## Question 5: Momentum

How fast must a body be travelling so that its correct relativistic momentum is 1 % greater than the classical momentum?

- $0.04c$
- $0.42c$
- **0.14c***
- $1.41c$
- $1.35c$

For the relativistic momentum to be 1% greater than the classical momentum, we must have

$$\begin{aligned} p_{rel} &= 1.01 p_{clas} \\ \gamma m u &= 1.01 m u \\ \gamma &= 1.01 \\ \sqrt{1 - V^2/c^2} &= 0.990 \\ 1 - V^2/c^2 &= 0.980 \\ V^2/c^2 &= 0.02 \\ V &= 0.141c \end{aligned}$$

## Question 6: Collisions

A proton and neutron collide in an elastic collision. Before the collision, the neutron is stationary and the proton has momentum $\mathbf{p}_p = (0.4, -0.2, 0.8)$ MeV/$c$ and the proton's momentum after the collision is $(-0.2, -0.5, 0.6)$ MeV/$c$. What is the neutron's momentum after the collision?

- $(0.2, 0.5, -0.6)$ MeV/$c$
- $(0.2, -0.7, 1.4)$ MeV/$c$
- **(0.6, 0.3, 0.2) MeV/c***
- $(0, 0, 0)$ MeV/$c$
- $(-0.6, 0.7, 0.2)$ MeV/$c$

The collision is elastic so that the kinetic energy is conserved. This is not really relevant to solve this problem, but be sure to know what it means. Total energy and momentum is always conserved. This means that the total momentum before the collision should be equal to the total momentum after the collision. A stationary object has no speed, and therefore no momentum. We can write this as

$$\begin{aligned} \mathbf{p}_n^{before} + \mathbf{p}_p^{before} &= \mathbf{p}_n^{after} + \mathbf{p}_p^{after} \\ \mathbf{p}_n^{after} &= \mathbf{p}_n^{before} + \mathbf{p}_p^{before} - \mathbf{p}_p^{after} \\ &= (0, 0, 0) + (0.4, -0.2, 0.8) - (-0.2, -0.5, 0.6) \\ &= (0.6, 0.3, 0.2) \text{ MeV}/c \end{aligned}$$

## Question 7: Kinetic energy

A proton (mass $m_p = 938.3\,\text{MeV}/c^2$) is moving with speed $0.4c$ along the $x$-axis relative to the laboratory frame. What is its kinetic energy?

- $7.6 \times 10^{18}\,\text{MeV}$
- $1023\,\text{MeV}$
- $178.7\,\text{MeV}$
- $273\,\text{MeV}$
- **85.39 MeV***

Take the laboratory frame to be $S$ and let the proton be stationary in the $S'$ frame. The Lorentz factor for the two frames is

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-0.4^2}} \\ &= 1.091\end{aligned}$$

The kinetic energy is then

$$\begin{aligned}E_K &= (\gamma - 1)\,mc^2 \\ &= (1.091 - 1)\,(938.3\,\text{MeV}) \\ &= 85.39\,\text{MeV}\end{aligned}$$

# Tutorial Letter 203/2/2015

## Special Relativity and Riemannian Geometry

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**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 3.

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# Memo for Assignment 3 S2 2015

## Four vectors and tensors (§ 2.2.4 - 2.3.5 (excluding 2.3.1 - 2.3.4))

### Question 1: Tensor notation

Consider the following in Minkowski spacetime

$$\sum_{\mu} \frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0$$

How many equations does this represent?

- 1
- 2
- **4***
- 6
- 16

Four equations are represented, one for each possible value of $\mu$. The other index, $\nu$ is a dummy index and is being summed over in each of the equations. Written out it full, the four equations are

$$\begin{aligned}
\sum_{\mu} \frac{\partial G^{0\nu}}{\partial x^{\nu}} = \frac{\partial G^{00}}{\partial x^{0}} + \frac{\partial G^{01}}{\partial x^{1}} + \frac{\partial G^{02}}{\partial x^{2}} + \frac{\partial G^{03}}{\partial x^{3}} &= 0 \\
\sum_{\mu} \frac{\partial G^{1\nu}}{\partial x^{\nu}} = \frac{\partial G^{10}}{\partial x^{0}} + \frac{\partial G^{11}}{\partial x^{1}} + \frac{\partial G^{12}}{\partial x^{2}} + \frac{\partial G^{13}}{\partial x^{3}} &= 0 \\
\sum_{\mu} \frac{\partial G^{2\nu}}{\partial x^{\nu}} = \frac{\partial G^{20}}{\partial x^{0}} + \frac{\partial G^{21}}{\partial x^{1}} + \frac{\partial G^{22}}{\partial x^{2}} + \frac{\partial G^{23}}{\partial x^{3}} &= 0 \\
\sum_{\mu} \frac{\partial G^{3\nu}}{\partial x^{\nu}} = \frac{\partial G^{30}}{\partial x^{0}} + \frac{\partial G^{31}}{\partial x^{1}} + \frac{\partial G^{32}}{\partial x^{2}} + \frac{\partial G^{33}}{\partial x^{3}} &= 0
\end{aligned}$$

Now you can see how much more condensed tensor notation is.

## Question 2: Tensors, vectors and scalars

The four momentum is given by

$$[P^\mu] = \left(P^0,\, P^1,\, P^2,\, P^3\right) = (E/c,\, \mathbf{p})\ .$$

Consider the following quantities from the equation:

a) $[P^\mu]$

b) $P^3$

c) $(P^0,\, P^1,\, P^2,\, P^3)$

d) $E$

e) $(E/c,\, \mathbf{p})$

f) $\mathbf{p}$

Which of the following statements are true?

- a and c are tensors, b is a vector and d is a scalar
- **c and e are tensors, f is a vector and b is a scalar***
- a and b are tensors, f is a vector and d is a scalar
- a and e are tensors, d is a vector and f is a scalar
- c and f are tensors, e is a vector and b is a scalar

The differences between tensors, vectors and scalars are important. In the same way that you can't equate a matrix to a number, you can't equate a tensor to a scalar. From the given list, (a), (c) and (e) are all tensors. $[P^\mu]$ is definitely a tensor, so anything equal to it will also be a tensor. The components of $[P^\mu]$, namely $P^0$, $P^1$, $P^2$ and $P^3$, are all scalars, just like the components of a vector are scalars. The energy (d) is a scalar. The normal three momentum $\mathbf{p}$ is a vector with 3 components.

## Question 3: Energy-momentum relation

An electron (mass $m_e = 0.511\,\text{MeV/c}^2$) is has an energy of $0.850\,\text{MeV}$. What is its momentum?

- $0.850\,\text{MeV}/c$
- $0.461\,\text{MeV}/c$
- $0.339\,\text{MeV}/c$
- $0.582\,\text{MeV}/c$
- **0.679 MeV/c***

Using the energy-momentum relation

$$\begin{aligned} E^2 &= \left(mc^2\right)^2 + (pc)^2 \\ (0.850\,\text{MeV})^2 &= (0.511\,\text{MeV})^2 + (pc)^2 \\ (pc)^2 &= (0.850\,\text{MeV})^2 - (0.511\,\text{MeV})^2 \\ (pc)^2 &= 0.461\,\text{MeV}^2 \\ p &= 0.679\,\text{MeV}/c \end{aligned}$$

## Question 4: Transformation of energy

An electron (mass $m_e = 0.511\,\text{MeV/c}^2$) is moving along the $x$-axis of an inertial reference frame $S$ with speed $v = 0.8c$, momentum $0.682\,\text{MeV/c}$ and total energy $0.852\,\text{MeV}$. What is its total energy in an inertial frame $S'$ that is moving in the standard configuration with speed $0.6c$ relative to $S$?

- $0.852\,\text{MeV}$
- $0.738\,\text{MeV}$
- **0.554 MeV***
- $0.511\,\text{MeV}$

- 0.443 MeV

The Lorentz factor for the two frames is

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-0.6^2}} \\ &= \frac{5}{4}\end{aligned}$$

Using the transformation equation for the total energy in $S'$ gives

$$\begin{aligned}E' &= \gamma\,(E - Vp_x) \\ &= \frac{5}{4}\,(0.852\,\text{MeV} - (0.6c)\,(0.682\,\text{MeV/c})) \\ &= \frac{5}{4}\,(0.852\,\text{MeV} - 0.409\,\text{MeV}) \\ &= 0.554\,\text{MeV}\end{aligned}$$

## Question 5: Four momentum

A photon with measured momentum $0.210\,\text{MeV}/c$ is moving along the $y$-axis relative to the laboratory frame. What is the value of the its four-momentum $[P^\mu]$ in $\text{MeV}/c$?

- **(0.21, 0, 0.21, 0)***
- (0, 0.21, 0, 0)
- (0.21, 0.21, 0, 0)
- (0, 0, 0.21, 0)
- (0.21, 0.21, 0.21, 0.21)

The four momentum is given by

$$[P^\mu] = (E/c,\, \mathbf{p}) = (E/c,\, p_x,\, p_y,\, p_z)\ .$$

From the question we know that $p_x = p_z = 0$ and $p_y = 0.21\,\text{MeV}/c$. It remains to calculate the energy of the photon. Since photons are massless, we cannot use the equation $E = \gamma mc^2$. We therefore use the energy momentum relation with $m = 0$ to get

$$\begin{aligned} E^2 &= \left(mc^2\right)^2 + (pc)^2 \\ E &= pc \\ E &= 0.21\,\text{MeV} \end{aligned}$$

The four momentum is then given by

$$[P^\mu] = (0.21,\ 0,\ 0.21,\ 0)\ \text{MeV}/c$$

## Question 6: Transformation of tensors

Using equation (2.110) in the textbook, how would a contravariant tensor of rank 1 $A^\nu$ transform in general?

- $A'^\mu = \sum_{\nu=0}^{3} \frac{\partial x'^\nu}{\partial x^\mu} A^\mu$
- $A'^\mu = \sum_{\nu=0}^{3} \frac{\partial x^\mu}{\partial x'^\nu} A_\nu$
- $A'^\mu = \sum_{\nu=0}^{3} \frac{\partial x^\mu}{\partial x'^\nu} A^\nu$
- $\mathbf{A'^\mu = \sum_{\nu=0}^{3} \frac{\partial x'^\mu}{\partial x^\nu} A^\nu}$ *
- $A'^\mu = \sum_{\nu=0}^{3} \frac{\partial x^\nu}{\partial x'^\mu} A^\mu$

# Tutorial Letter 204/2/2015

## Special Relativity and Riemannian Geometry

## APM3713

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## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 4.

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# Memo for Assignment 4 S2 2015

## Chapters 1 & 2

## Question 1

Bob, standing at the rear end of a railroad car, shoots an arrow toward the front end of the car. The velocity of the arrow as measured by Bob is $1/5c$. The length of the car as measured by Bob is 150 meters. Alice, standing on the station platform observes all of this as the train passes by her with a velocity of $3/5c$. What values does Alice measure for the following quantities:

(a) The length of the railroad car.

(b) The velocity of the arrow.

(c) The amount of the time the arrow is in the air.

(d) The distance that the arrow travels.

### Solution

#### Part A

Let's call the frame where Bob is at rest $S'$ and the frame where Alice is at rest $S$. The Lorentz factor between the two frames are

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\\ &= \frac{1}{\sqrt{1-\frac{9}{25}}}\\ &= \frac{1}{\sqrt{\frac{16}{25}}}\\ &= \frac{5}{4}\end{aligned}$$

Since Bob is at rest with respect to the railroad car, he will measure the proper length $L_P$. The length that Alice measures $L$ will be contracted according to

$$\begin{aligned} L &= \frac{L_P}{\gamma} \\ &= \frac{4}{5}150\,\mathrm{m} \\ &= 120\,\mathrm{m} \end{aligned}$$

So Alice will measure the railroad car to be $120\,\mathrm{m}$ long.

**Part B**

To determine the speed of the arrow as measured by Alice $v$ from the speed as measured by Bob $v'$, we use the velocity transformation equation. The velocity transformation equations given in the textbook on p30 is given as

$$v' = \frac{v - V}{1 - vV/c^2}$$

where $V$ is the relative speed between the two frames ($V = 3/5c$ in this case) and $v$ is the speed of the moving object (arrow) as measured in the $S$ frame and $v'$ is the the speed of the moving object as measured in the $S'$ frame, where the two frames are in the standard configuration.

To solve this problem, we are actually looking for the inverse velocity transformation, since we know $v'$ and want to calculate $v$. You can argue that $S$ is moving in the negative $x$-direction wrt $S'$, so we can just replace $V$ with $-V$ and $v$ with $v'$ (similar to obtaining the inverse Lorentz transformation equations). We can check this approach with a little algebra:

$$\begin{aligned} v' &= \frac{v - V}{1 - vV/c^2} \\ v'\left(1 - vV/c^2\right) &= v - V \\ v' - v'vV/c^2 &= v - V \\ v + v'vV/c^2 &= V + v' \\ v\left(1 + v'V/c^2\right) &= V + v' \\ v &= \frac{V + v'}{1 + v'V/c^2} \end{aligned}$$

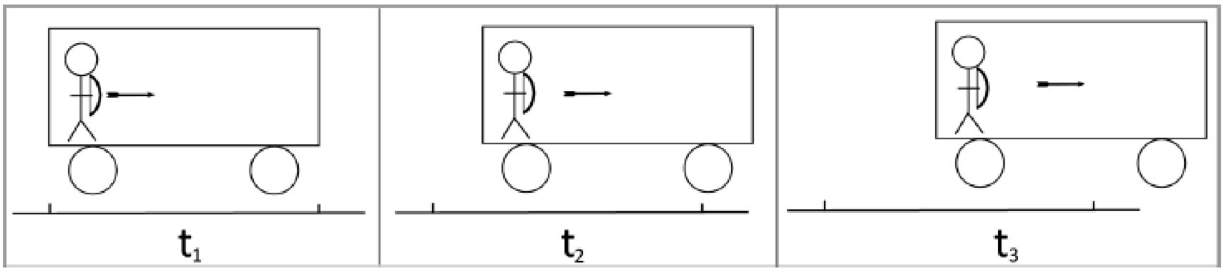
From here it is simple to determine the speed that Alice measures for the arrow by plugging in the known values

$$\begin{aligned} v &= \frac{V + v'}{1 + v'V/c^2} \\ &= \frac{3c/5 + c/5}{1 + (1c/5)\,(3c/5)\,/c^2} \\ &= \frac{4c/5}{28/25} \\ &= \frac{4c}{5} \times \frac{25}{28} \\ &= \frac{5}{7}c \end{aligned}$$

So Alice measures the arrow's speed to be $v = 5/7c$.

**Part C**

The important thing to recognise when doing this question is that the arrow is moving in the same direction as the arrow, so the arrow will travel further than just the length of the train car. The figure below illustrates this from Alice's point of view.




The markers at the bottom of each panel in the figure shows the length of the train car. The arrow will stop travelling once it hits the far wall of the car. As the arrow travels to the right, the far wall of the car is also moving to the right, so the arrow will eventually travel further than the length of the car.

The total distance that the arrow will travel, will be equal to the length of the car plus the distance the car has travelled in the in the time the arrow is in the air. We write this as (all quantities as measured in $S$)

$$\text{distance arrow travels} = (\text{length of car}) + (\text{distance car travels})$$

$$\Delta x_{arrow} = L + \Delta x_{train}$$

We use the formula

$$v = \frac{\Delta x}{\Delta t}$$

which is always valid *in a single frame* if the speed is constant. It is important to note that this formula is only valid for intervals of $x$ and $t$, not for single coordinates.

Both of the intervals of distance we are considering is travelled in the same amount of time, which is the time that the arrow is in the air, $\Delta t_{arrow}$. Now we can write

$$\begin{aligned}
v_{arrow}\Delta t_{arrow} &= L + v_{train}\Delta t_{arrow} \\
v_{arrow}\Delta t_{arrow} &= L + V\Delta t_{arrow} \\
\Delta t_{arrow}\left(v_{arrow} - V\right) &= L \\
\Delta t_{arrow} &= \frac{L}{v_{arrow} - V} \\
&= \frac{120\,\mathrm{m}}{5c/7 - 3c/5} \\
&= \frac{35 \times 120\,\mathrm{m}}{4 \times 3 \times 10^{8}\,\mathrm{ms}^{-1}} \\
&= 3.5 \times 10^{-6}\,\mathrm{s}
\end{aligned}$$

So, according to Alice, the arrow is in the air for $3.5 \times 10^{-6}$ seconds.

**Part D**

Again we use the definition of speed

$$v = \frac{\Delta x}{\Delta t}$$

We have already calculated the speed and the time interval that the arrow is in the air according to Alice, so we get for the distance the arrow travels

$$\begin{aligned}
\Delta x_{arrow} &= v_{arrow}\Delta t_{arrow} \\
&= \left(\frac{5c}{7}\right)\left(3.5 \times 10^{-6}\,\mathrm{s}\right) \\
&= \left(\frac{5}{7} \times 3 \times 10^{8}\,\mathrm{ms}^{-1}\right)\left(3.5 \times 10^{-6}\,\mathrm{s}\right) \\
&= 750\,\mathrm{m}
\end{aligned}$$

Or you could use the formula constructed in Part C:

$$\begin{aligned}\Delta x_{arrow} &= L + V\Delta t_{arrow} \\ &= 120\,\mathrm{m} + \frac{3c}{5}\left(3.5\times 10^{-6}\,\mathrm{s}\right) \\ &= 120\,\mathrm{m} + \frac{3}{5}\left(3\times 10^{8}\,\mathrm{ms}^{-1}\right)\left(3.5\times 10^{-6}\,\mathrm{s}\right) \\ &= 750\,\mathrm{m}\end{aligned}$$

# Question 2

Maxwell's wave equation for an electric field propagating in the $x$-direction is

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2},$$

where $E(x, t)$ is the amplitude of the electric field. Show that this equation is invariant under a Lorentz transformation to a reference frame moving with relative speed $v$ along the $x$-axis.

## Solution

The relevant Lorentz transformations are given by

$$\begin{aligned}x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right)\end{aligned}$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

Note that $x' = x'(x, t)$ and $t' = t'(x, t)$, so that we use the chain rule to obtain for a wave function

$$\begin{aligned}\frac{\partial E}{\partial x} &= \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial x} = \gamma\frac{\partial E}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial e}{\partial t'} \\ \frac{\partial E}{\partial t} &= \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial t} = -\gamma v\frac{\partial E}{\partial x'} + \gamma\frac{\partial E}{\partial t'}\end{aligned}$$

Therefore we have

$$\begin{aligned}\frac{\partial^2 E}{\partial x^2} &= \left(\gamma\frac{\partial E}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial E}{\partial t'}\right)\left(\gamma\frac{\partial E}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial E}{\partial t'}\right) \\ &= \gamma^2\frac{\partial^2 E}{\partial x'^2} - \frac{2\gamma^2 v}{c^2}\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'} + \frac{\gamma^2 v^2}{c^4}\frac{\partial^2 E}{\partial t'^2} \\ \frac{\partial^2 E}{\partial t^2} &= \left(-\gamma v\frac{\partial E}{\partial x'} + \gamma\frac{\partial E}{\partial t'}\right)\left(-\gamma v\frac{\partial E}{\partial x'} + \gamma\frac{\partial E}{\partial t'}\right) \\ &= \gamma^2 v^2\frac{\partial^2 E}{\partial x'^2} - 2\gamma^2 v\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'} + \gamma^2\frac{\partial^2 E}{\partial t'^2}\end{aligned}$$

Substituting this into the wave equation

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

and rearranging gives

$$\begin{aligned}\gamma^2\frac{\partial^2 E}{\partial x'^2} - \frac{\gamma^2 v^2}{c^2}\frac{\partial^2 E}{\partial x'^2} + \frac{\gamma^2 v^2}{c^4}\frac{\partial^2 E}{\partial t'^2} - \frac{\gamma^2}{c^2}\frac{\partial^2 E}{\partial t'^2} &= \left(\frac{2\gamma^2 v}{c^2}\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'} - \frac{2\gamma^2 v}{c^2}\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'}\right) \\ \gamma^2\frac{\partial^2 E}{\partial x'^2}\left(1 - \frac{v^2}{c^2}\right) - \frac{\gamma^2}{c^2}\frac{\partial^2 E}{\partial t'^2}\left(1 - \frac{v^2}{c^2}\right) &= 0 \\ \frac{\partial^2 E}{\partial x'^2} &= \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2}\end{aligned}$$

Therefore, the wave equation is invariant under a Lorentz transformation.

# Question 3

A physics professor claims in court that the reason he went through the red light ( $\lambda = 650\,\mathrm{nm}$) was that, due to his motion, the red color was Doppler shifted to green ( $\lambda = 550\,\mathrm{nm}$). How must he have been going for his story to be true? *Hint*: The relation between frequency $f$ and wavelength $\lambda$ of light is given by $c = \lambda f$ .

### Solution

We use the relativistic Doppler formula. It is clear from the problem that the professor is approaching the traffic light. The receiver in this case is the professor's eyes and the emitter is the light of the traffic light. So for the professor to receive green light when red light was emitted, we have

$$\begin{aligned}
f_{rec} &= f_{em}\sqrt{\frac{c+V}{c-V}} \\
\frac{c}{\lambda_{rec}} &= \frac{c}{\lambda_{em}}\sqrt{\frac{c+V}{c-V}} \\
\frac{c}{550\,\text{nm}} &= \frac{c}{650\,\text{nm}}\sqrt{\frac{c+V}{c-V}} \\
\frac{650\,\text{nm}}{550\,\text{nm}} &= \sqrt{\frac{c+V}{c-V}} \\
\left(\frac{13}{11}\right)^2 &= \frac{c+V}{c-V} \\
\frac{169c}{121} - \frac{169}{121}V &= c+V \\
-V\left(\frac{169}{121}+1\right) &= c - \frac{169c}{121} \\
V &= -\left(\frac{121}{290}\right)\left(-\frac{48c}{121}\right) \\
&= 0.17c
\end{aligned}$$

The professor must have been travelling at $0.17c$ for his story to be true.

## Question 4

Prove that the interval between two events in 2-dimentional spacetime is Lorentz invariant, that is, prove that

$$\sum_\mu \Delta x'_\mu \Delta x'^\mu = \sum_\mu \Delta x_\mu \Delta x^\mu$$

where $\Delta x'^\mu = \sum_\nu \Lambda^\mu_\nu \Delta x^\nu$.

### Solution

The spacetime interval in the $S'$ frame is given by

$$\sum_\mu \Delta x'_\mu \Delta x'^\mu$$

Substituting the Lorentz transformations gives

$$\begin{aligned}
\sum_\mu \Delta x'_\mu \Delta x'^\mu &= \left(\sum_\nu \Lambda^\nu_\mu \Delta x_\nu\right)\left(\sum_\nu \Lambda^\mu_\nu \Delta x^\nu\right) \\
&= \sum_\nu \left(\Lambda^\nu_\mu \Lambda^\mu_\nu\right)\left(\Delta x_\nu \Delta x^\nu\right) \\
&= \sum_\nu \delta^\nu_\mu \left(\Delta x_\nu \Delta x^\nu\right) \\
&= \sum_\mu \Delta x_\mu \Delta x^\mu
\end{aligned}$$

## Question 5

In the context of special relativity, a contravariant four-vector can be constructed from the charge density $\rho$ and the current density $\mathbf{j}$ as follows $[J^\mu] = (c\rho,\, j_x,\, j_y,\, j_z)$ where $j_x$, $j_y$ and $j_z$ are the components of $\mathbf{j}$ in the $x$, $y$ and $z$ directions, respectively. *Hint*: To answer the questions below, use the properties of four-vectors. Do not try to solve this using electromagnetism.

(a) Determine the transformation equations of $J^\mu$ to a frame $S'$ that is moving with a constant speed $V$ in the positive $x$-direction.

(b) Construct a quantity using the components of $J^\mu$ that is a Lorentz invariant in Minkowski spacetime.

(c) Imagine you are in a reference frame in which $\rho = 2/c$ and $j_x = j_y = j_z = 2$. Determine $[J'^\mu]$ as measured by someone moving at a velocity $V = \sqrt{3/4}c$ along the $x$-direction with respect to your reference frame.

## Solution

### Part A

$[J^\mu]$ is a contravariant four-vector and the relative movement of $S'$ describes the standard configuration in special relativity. So it's components transform as

$$\begin{aligned} J'^0 &= \gamma\left(J^0 - VJ^1/c\right) \\ &= \gamma\left(c\rho - Vj_x/c\right) \end{aligned}$$

$$\begin{aligned} J'^1 &= \gamma\left(J^1 - VJ^0/c\right) \\ &= \gamma\left(j_x - V\rho\right) \end{aligned}$$

$$\begin{aligned} J'^2 &= J^2 = j_y \\ J'^3 &= J^3 = j_z \end{aligned}$$

### Part B

The quantity

$$\sum_{\mu=0}^{3} J^\mu J_\mu$$

will be invariant under a Lorentz transformation. (Can you show this explicitly for each of its components?)

### Part C

First we calculate the Lorentz factor $\gamma$ for $V = \sqrt{3/4}c$

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-\frac{3}{4}}} \\ &= \frac{1}{\sqrt{\frac{1}{4}}} \end{aligned}$$

$$= \quad 2$$

We use the transformation equations from Part A and substitute $\gamma = 2$, $\rho = 2/c$, $j_x = j_y = j_z = 2$ and $V = \sqrt{3/4}c$ to get

$$\begin{aligned} J'^0 &= \gamma\left(c\rho - V j_x/c\right) \\ &= 2\left(2 - 2\sqrt{\frac{3}{4}}\right) \\ &= 0.54 \end{aligned}$$

$$\begin{aligned} J'^1 &= \gamma\left(j_x - V c\rho/c\right) \\ &= 2\left(2 - 2\sqrt{\frac{3}{4}}\right) \\ &= 0.54 \end{aligned}$$

$$\begin{aligned} J'^2 &= j_y = 2 \\ J'^3 &= j_z = 2 \end{aligned}$$

So that we have $[J'^\mu] = (0.54,\, 0.54,\, 2,\, 2)$.

# Question 6

An electron $e^-$ with kinetic energy 1 MeV makes a head-on collision with a positron $e^+$ that is at rest. (A positron is an antimatter particle that has the same mass as an electron, but opposite charge.) In the collision the two particles annihilate each other and are replaced by two photons $\gamma$ of equal energy. The reaction can be written as

$$e^- + e^+ \to 2\gamma\,.$$

Determine the energy, momentum and speed of each photon.

## Solution

By conservation of energy, the energy of the system before and after the collision will be the same.

$$E_e + E_p = 2E_\gamma$$

The total energy of the electron and positron before the collision is

$$\begin{aligned} E_e + E_p &= E_{Ke} + m_e c^2 + m_p c^2 \\ &= 1\,\text{MeV} + 2\left(0.511\,\text{MeV}/c^2\right)c^2 \\ &= 2.02\,\text{MeV} \end{aligned}$$

By conservation of energy, we find

$$\begin{aligned} E_\gamma &= \frac{E_e + E_p}{2} \\ &= \frac{2.02\,\text{MeV}}{2} \\ &= 1.01\,\text{MeV} \end{aligned}$$

The momentum of the photons will then be

$$\begin{aligned} p_\gamma &= \frac{E}{c} \\ &= 1.01\,\text{MeV}/c \end{aligned}$$

The speed of both photons will be $v_\gamma = c$.

> **Note:** It might be tempting to use conservation of momentum to solve this problem, rather than conservation of energy, i.e.
>
> $$\mathbf{p}_e + \mathbf{p}_p = 2\mathbf{p}_\gamma$$
>
> In this case, this approach wouldn't work, because we only know the directions of the photons velocities, and therefore their momenta. Since the energies of the photons are the same, the magnitudes of their momenta will be equal, but not their momenta in vector form, since they will move in different directions, thus $\mathbf{p}_{\gamma 1} \neq \mathbf{p}_{\gamma 2}$.

# Question 7

In special relativity, the energy, momentum and mass of a particle are all closely related to one another.

(a) Derive the relation $E^2 = c^2p^2 + m^2c^4$ by starting from the relativistic definitions of $E$ and $p$, i.e. $E = \gamma mc^2$ and $p = \gamma mv$.

(b) Use the equation derived in part (a) to show that the mass of a particle can be expressed as

$$m = \frac{c^2p^2 - E_k^2}{2E_kc^2}$$

where $E_k$ is the kinetic energy of the particle.

## Solution

### Part A

Squaring the definitions of $E$ and $p$

$$\begin{aligned} E^2 &= \gamma^2 m^2 c^4 \\ p^2 &= \gamma^2 m^2 u^2 \end{aligned}$$

Multiplying the last equation by $c^2$ and subtracting gives

$$\begin{aligned} E^2 - p^2c^2 &= \gamma^2 m_0^2 c^4 - \gamma^2 m_0^2 u^2 c^2 \\ &= m_0^2 c^4 \left(\gamma^2 - \gamma^2 \frac{u^2}{c^2}\right) \end{aligned}$$

Using

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

we show that

$$\gamma^2 - \gamma^2 \frac{u^2}{c^2} = \frac{1}{1 - u^2/c^2} - \frac{u^2/c^2}{1 - u^2/c^2}$$

$$
\begin{aligned}
&= \frac{1-u^2/c^2}{1-u^2/c^2} \\
&= 1
\end{aligned}
$$

It follows that

$$E^2 = p^2c^2 + m_0^2c^4$$

**Part B**

The total energy is equal to the kinetic energy plus the mass energy

$$E = E_k + mc^2$$

Squaring both sides gives

$$E^2 = E_k^2 + 2E_kmc^2 + m^2c^4$$

Using the result from the previous question we get

$$c^2p^2 = E_k^2 + 2E_kmc^2$$

Solving for $m$ gives

$$m = \frac{c^2p^2 - E_k^2}{2E_kc^2}$$

as required.

# Tutorial Letter 205/2/2015

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 5.

BAR CODE

# Memo for Assignment 5 S2 2015

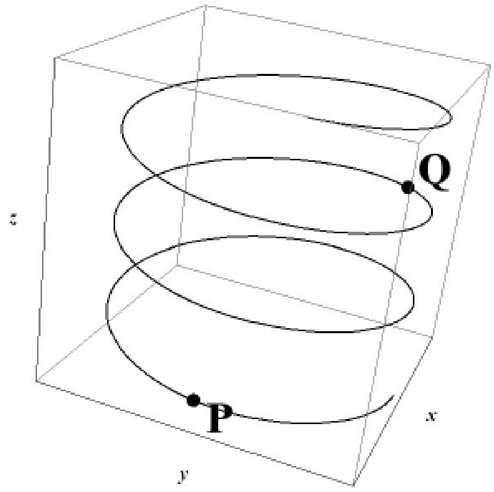
## Basics of differential geometry (§ 3)

### Question 1: Arc length

The equation for a length of a curve in an Euclidean plane can easily be generalized to give the length of a curve that exists in Euclidean (normal) three dimensional space. The length of such a space curve is given by

$$L\left(\mathrm{P},\,\mathrm{Q}\right)=\int_{P}^{Q}dl=\int_{u_{\mathrm{P}}}^{u_{\mathrm{Q}}}\left(\left(\frac{dx}{du}\right)^{2}+\left(\frac{dy}{du}\right)^{2}+\left(\frac{dz}{du}\right)^{2}\right)^{1/2}du$$

Consider the circular helix described by $x=\sin u$, $y=\cos u$ and $z=u/10$ with the points P and Q defined by the points where $u_{\mathrm{P}}=\pi/2$ and $u_{\mathrm{Q}}=15\pi/4$ as shown in the figure below.




What is the length of the curve between points P and Q in arbitrary units?

- 18.79
- 13.42
- **10.26***

- 10.31
- 1.02

First we calculate the derivatives of the Cartesian coordinates with respect the the parameter $u$.

$$\begin{aligned}\frac{dx}{du} &= \frac{d}{du}(\sin u) = \cos u\\ \frac{dy}{du} &= \frac{d}{du}(\cos u) = -\sin u\\ \frac{dz}{du} &= \frac{d}{du}\left(\frac{u}{10}\right) = \frac{1}{10}\end{aligned}$$

The length of the helix is then given by

$$\begin{aligned}L(\mathrm{P},\,\mathrm{Q}) &= \int_{u_\mathrm{P}}^{u_\mathrm{Q}}\left(\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2\right)^{1/2}du\\ &= \int_{\pi/2}^{15\pi/4}\left(\cos^2 u+\sin^2 u+\frac{1}{100}\right)^{1/2}du\\ &= \int_{\pi/2}^{15\pi/4}\left(1+\frac{1}{100}\right)^{1/2}du\\ &= \int_{\pi/2}^{15\pi/4}\left(\frac{101}{100}\right)^{1/2}du\\ &= \int_{\pi/2}^{15\pi/4}\frac{\sqrt{101}}{10}\,du\\ &= \left[\frac{\sqrt{101}}{10}u\right]_{u=\pi/2}^{u=15\pi/4}\\ &= \frac{\sqrt{101}}{10}\left(\frac{15\pi}{4}\right)-\frac{\sqrt{101}}{10}\left(\frac{\pi}{2}\right)\\ &= \frac{13\pi\sqrt{101}}{40}\\ &= 10.26\end{aligned}$$

## Question 2: Metric tensor

The line element for a certain two dimensional Riemann space is given by

$$dl^2 = dr^2 + 2r\sin\phi\, dr\, d\phi + r^2 d\phi^2\,.$$

What is the metric tensor of this space?

- $\begin{pmatrix} 1 & 2r\sin\phi \\ 2r\sin\phi & r^2 \end{pmatrix}$
- $\begin{pmatrix} 1 & r\sin\phi \\ r\sin\phi & r^2 \end{pmatrix}$*
- $\begin{pmatrix} r\sin\phi & 1 \\ 1 & r\sin\phi \end{pmatrix}$
- $\begin{pmatrix} r^2 & 2r\sin\phi \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ 2r\sin\phi & r^2 \end{pmatrix}$

The line element for a general Riemann space is given by

$$dl^2 = \sum_{i,j=1}^{n} g_{ij}dx^i dx^j$$

Since we are considering a two dimensional space ($n = 2$), we can can expand this to

$$dl^2 = g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + g_{21}dx^2dx^1 + g_{22}dx^2dx^2$$

Choosing $x^1 = r$ and $x^2 = \phi$, we get

$$dl^2 = g_{11}dr^2 + g_{12}drd\phi + g_{21}d\phi dr + g_{22}d\phi^2$$

The metric tensor must be symmetric. (This ensures that the distance from the point P to the point Q will be the same as the distance from point Q to point P.) This means that we must have $g_{12} = g_{21}$. From the given line element, we can see that we have $g_{11} = 1$, $g_{12} = g_{21} = r\sin\phi$ and $g_{22} = r^2$. Putting this in array format gives

$$\begin{pmatrix} 1 & r\sin\phi \\ r\sin\phi & r^2 \end{pmatrix}.$$

## Question 3: Kronecker delta

The sum

$$\sum_{i=1}^{3} \delta_{ii}$$

is equal to...

- 0
- 1
- 2
- **3***
- 4

The definition of the Kroneker delta is

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

It is a very common mistake to say that this sum is equal to 1. But it is a sum over number of components equal to 1 and the answer will depend on the number of dimensions you are working in. In this case

$$\begin{aligned} \sum_{i=1}^{3} \delta_{ii} &= \delta_{11} + \delta_{22} + \delta_{33} \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

## Question 4: Covariant and contravariant forms of vectors

Equation 2.70 in the textbook is written for four dimensional Minkowski space and gives a rule to determine the covariant form of a vector if the metric and contravariant form is known. This same equation written for a general two dimensional space is

$$A_j = \sum_{i=1}^{2} g_{ij} A^i \, .$$

Use this to determine the covariant form of $[A^i]$ in two dimensional space described by the surface of a unit sphere. The metric tensor (with $x^1 = \theta$ and $x^2 = \phi$) for this space is

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

and let

$$[A^i] = \begin{pmatrix} \pi \\ \pi/4 \end{pmatrix}.$$

- $[A_i] = \begin{pmatrix} \pi \\ 1/2 \end{pmatrix}$
- $[A_i] = \begin{pmatrix} \pi \\ \pi/\left(4\sqrt{2}\right) \end{pmatrix}$
- $[A_i] = \begin{pmatrix} \pi \\ \pi/2 \end{pmatrix}$
- $[\mathbf{A_i}] = \begin{pmatrix} \pi \\ \pi/8 \end{pmatrix}*$
- $[A_i] = \begin{pmatrix} \pi \\ \pi/4 \end{pmatrix}$

Expanding the given equation for determining the covariant components of $[A^i]$ gives

$$\begin{aligned} A_j &= \sum_{i=1}^{2} g_{ij}A^i \\ &= g_{1j}A^1 + g_{2j}A^2 \end{aligned}$$

From the information given in the question, we know that $g_{11} = 1$, $g_{12} = g_{21} = 0$, $g_{22} = \sin^2\theta$, $A^1 = \pi$ and $A^2 = \pi/4$. The covariant components are then given by

$$\begin{aligned} A_1 &= g_{11}A^1 + g_{21}A^2 \\ &= (1)(\pi) + (0)(\pi/4) \\ &= \pi \end{aligned}$$

$$\begin{aligned} A_2 &= g_{12}A^1 + g_{22}A^2 \\ &= (0)(\pi) + (\sin^2\theta)(\pi/4) \\ &= \frac{\pi}{4}\sin^2\theta \end{aligned}$$

You could also have computed it with

$$\begin{aligned} [A_i] &= [g_{ij}]\,[A^i] \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} \begin{pmatrix} \pi \\ \frac{\pi}{4} \end{pmatrix} \\ &= \begin{pmatrix} \pi \\ \frac{\pi}{4}\sin^2\theta \end{pmatrix} \end{aligned}$$

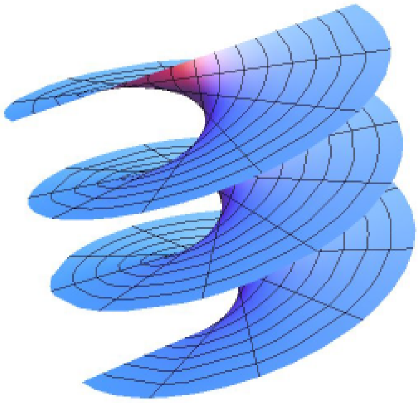
## Question 5: Riemann tensor

Calculate $R^1_{\ 221}$ of the right helicoid shown below that is parametrized as

$$\begin{aligned} x &= u\cos v \\ y &= u\sin v \\ z &= cv \end{aligned}$$

where $c$ is a constant and $x^1 = u$ and $x^2 = v$ if it is given that the only non-zero Christoffel coefficients for the surface are

$$\begin{aligned} \Gamma^2_{\ 12} &= \Gamma^2_{\ 21} = \frac{u}{u^2 + c^2} \\ \Gamma^1_{\ 22} &= \frac{-u}{u^2 + c^2} \end{aligned}$$



- $\mathbf{(c^2 - 2u^2)\,(u^2 + c^2)^{-2}}$*
- $-c^2\,(u^2 + c^2)^{-2}$
- $c^2\,(u^2 + c^2)^{-2}$
- $-u\,(c^2 + u + u^2)\,(u^2 + c^2)^{-2}$
- $-u^2\,(u^2 + c^2)^{-2}$

Using equation 3.35 in the textbook, the element $R^1_{\ 221}$ of the Riemann tensor is given by

$$\begin{aligned} R^1_{\ 221} &= \frac{\partial \Gamma^1_{\ 21}}{\partial x^2} - \frac{\partial \Gamma^1_{\ 22}}{\partial x^1} + \sum_m \Gamma^m_{\ 21}\Gamma^1_{\ m2} - \sum_m \Gamma^m_{\ 22}\Gamma^1_{\ m1} \\ &= \frac{\partial \Gamma^1_{\ 21}}{\partial x^2} - \frac{\partial \Gamma^1_{\ 22}}{\partial x^1} + \left(\Gamma^1_{\ 21}\Gamma^1_{\ 12} + \Gamma^2_{\ 21}\Gamma^1_{\ 22}\right) - \left(\Gamma^1_{\ 22}\Gamma^1_{\ 11} + \Gamma^2_{\ 22}\Gamma^1_{\ 21}\right) \end{aligned}$$

Substituting all the zero Christoffel symbols, this reduces to

$$R^1_{\ 221} = -\frac{\partial \Gamma^1_{\ 22}}{\partial x^1} + \Gamma^2_{\ 21}\Gamma^1_{\ 22}$$

Substituting the given values for the non-zero Christoffel symbols and using $x^1 = u$ gives

$$\begin{aligned} R^1_{\ 221} &= -\frac{\partial}{\partial u}\left(\frac{-u}{u^2 + c^2}\right) + \left(\frac{u}{u^2 + c^2}\right)\left(\frac{-u}{u^2 + c^2}\right) \\ &= \frac{c^2 - u^2}{(u^2 + c^2)^2} - \frac{u^2}{(u^2 + c^2)^2} \\ &= \frac{c^2 - 2u^2}{(u^2 + c^2)^2} \end{aligned}$$

# Tutorial Letter 206/2/2015

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 6.

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# Memo for Assignment 6 S2 2015

## Equivalence and Tensor Algebra (§ 4)

### Question 1: Tensor transformations

The transformation equations for transforming a contravariant tensor of rank one from polar to Cartesian coordinates are

$$\begin{aligned} A'^1 &= A^1 \cos\theta - A^2 r \sin\theta \\ A'^2 &= A^1 \sin\theta + A^2 r \cos\theta \end{aligned}$$

where $x^i = (r,\, \theta)$ (derived in Exercise 4.2 in the textbook). Use these to transform the tensor described by $[A^i] = (1/\cos\theta,\, r)$ to Cartesian coordinates $x'^i = (x,\, y)$. What is the value of $A'^2$?

- $\cos^2\theta\,(1 + r^2)$
- $\tan\theta + \mathbf{r^2}\cos\theta$*
- $1 - r^2 \sin\theta$
- $\cos\theta - r\sin\theta$
- $r\cos\theta - r\tan\theta$

The transformation equations for transforming a contravariant tensor of rank one from polar to Cartesian coordinates are

$$\begin{aligned} A'^1 &= A^1 \cos\theta - A^2 r \sin\theta \\ A'^2 &= A^1 \sin\theta + A^2 r \cos\theta \end{aligned}$$

The components of the tensor we want to transform is given as $A^1 = 1/\cos\theta$ and $A^2 = r$. Substituting this into the above equations give

$$A'^1 = \frac{\cos\theta}{\cos\theta} - r^2 \sin\theta$$

$$
\begin{aligned}
&= 1 - r^2 \sin\theta \\
A'^2 &= \frac{\sin\theta}{\cos\theta} + r^2\cos\theta \\
&= \tan\theta + r^2\cos\theta
\end{aligned}
$$

## Question 2: Tensor expressions

Which of the following tensor expressions is incorrect?

- $A^i = \sum_j g^{ij} A_j = \sum_{j,k} g^{ij} g_{jk} A^k$
- $\bar{A}^i{}_{kl} = \sum_{p,r,s} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} A^p{}_{rs}$
- $\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)$
- $\sum_{i=1}^3 \delta^i{}_i = 3$
- $\mathbf{\Gamma^i{}_{jk} = \Gamma^j{}_{ik}}$ *

**Option (1):** Correct. The rules of raising and lowering an index is followed. We can also write

$$
\begin{aligned}
\sum_{j,k} g^{ij} g_{jk} A^k &= \sum_k \delta^i{}_k A^k \\
&= \delta^i{}_1 A^1 + \delta^i{}_2 A^2 + \ldots + \delta^i{}_i A^i + \ldots \delta^i{}_n A^n
\end{aligned}
$$

In the last step the sum has been expanded and sums $k$ from 1 to $n$. Remember that $\delta^i{}_k$ is defined so that it is equal to 1 if $i = k$, and zero if $i \neq k$. So all the terms will be equal to zero, except the term where $i = k$, and in that case we have $\delta^i{}_i = 1$ so that we can write

$$
\sum_{j,k} g^{ij} g_{jk} A^k = A^i
$$

**Option (2):** This is the correct transformation for a tensor of this form. Here bars were used in stead of primes to indicate the other coordinate frame. The textbook uses primes, but this can sometimes become unclear, especially when writing by hand. Using either is fine, as long as you are consistent.

**Option (3):** This is correct. Lowering the first index of the Christoffel coefficient gives

$$\begin{aligned}
\Gamma_{\alpha\beta\gamma} &= g_{\alpha\eta}\Gamma^{\eta}{}_{\beta\gamma} \\
&= \frac{1}{2}g_{\alpha\eta}\sum_{\epsilon} g^{\eta\epsilon}\left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}}\right) \\
&= \frac{1}{2}\delta^{1}{}_{\alpha}\left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}}\right) + .... + \frac{1}{2}\delta^{\alpha}{}_{\alpha}\left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}}\right) + ...
\end{aligned}$$

In the last step we used

$$g_{\alpha\eta}g^{\eta\epsilon} = \delta^{\epsilon}{}_{\alpha}$$

The only non-vanishing term will be the one for which $\eta = \alpha$ so that

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\alpha}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

**Option (4):** This is correct. Expanding the sum gives

$$\begin{aligned}
\sum_{i=1}^{3}\delta^{i}{}_{i} &= \delta^{1}_{1} + \delta^{2}_{2} + \delta^{3}_{3} \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}$$

**Option (5):** This is incorrect. The Christoffel coefficients are symmetric in their lower indices, but not in the upper and lower index.

The symmetry in their lower indices is a consequence of the symmetry of the metric tensor. We can write

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\sum_{l} g^{il}\left(\frac{\partial g_{lk}}{\partial x^{j}} + \frac{\partial g_{jl}}{\partial x^{k}} - \frac{\partial g_{jk}}{\partial x^{l}}\right)$$

Since $g_{\alpha\beta} = g_{\beta\alpha}$, we can interchange the indices of all the metrics within the sum.

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\sum_{l} g^{il}\left(\frac{\partial g_{kl}}{\partial x^{j}} + \frac{\partial g_{lj}}{\partial x^{k}} - \frac{\partial g_{kj}}{\partial x^{l}}\right)$$

The first two terms can also be switched to give

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\sum_{l} g^{il}\left(\frac{\partial g_{lj}}{\partial x^{k}} + \frac{\partial g_{kl}}{\partial x^{j}} - \frac{\partial g_{kj}}{\partial x^{l}}\right)$$

which is exactly the definition for ${\Gamma^i}_{kj}$. So therefore ${\Gamma^i}_{jk} = {\Gamma^i}_{kj}$, but it does not hold that ${\Gamma^i}_{jk} = {\Gamma^j}_{ik}$.

## Question 3: Contraction

Which of the following expressions are correct?

- $\sum_m \delta_l^m g_{km} = 3g_{km}$
- $\sum_m \delta_l^m g_{km} = g^{kl}$
- $\sum_m \delta_l^m g_{km} = g^{km}$
- $\mathbf{\sum_m \delta_l^m g_{km} = g_{kl}}$*
- $\sum_m \delta_l^m g_{km} = g_{km}$

In the sum

$$\sum_m \delta^m_{\ l}\, g_{km}$$

all the terms in the sum where $m \neq l$, will be zero (since $\delta_l^m = 0$ if $m \neq l$). The term where $m = l$ will be equal to $g_{kl}$, i.e.

$$\begin{aligned}
\sum_m \delta^m_{\ l}\, g_{km} &= \delta^0_{\ l} g_{k0} + \delta^1_{\ l} g_{k1} + \ldots + \delta^l_{\ l} g_{kl} + \ldots + \delta^N_{\ l}\, g_{kN} \\
&= (0)\, g_{k0} + (0)\, g_{k1} + \ldots + (1)\, g_{kl} + \ldots + (0)\, g_{kN} \\
&= g_{kl}
\end{aligned}$$

So the Kronecker delta can effectively be used to replace one index with another.

## Question 4: Einstein field equations

How many equations does the following expression represent?

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\kappa T_{\mu\nu}$$

- 1
- 2
- 4
- 8
- **16***

The expression represents 16 different equations. One for each possible combination of $\mu$ and $\nu$, where both indices can have values from **0** to **3**, since this equation is written for four dimensional spacetime. The 16 possible combinations are

| # | $\mu$ | $\nu$ | | # | $\mu$ | $\nu$ | | # | $\mu$ | $\nu$ | | # | $\mu$ | $\nu$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 0 | 0 | | **5** | 1 | 0 | | **9** | 2 | 0 | | **13** | 3 | 0 |
| **2** | 0 | 1 | | **6** | 1 | 1 | | **10** | 2 | 1 | | **14** | 3 | 1 |
| **3** | 0 | 2 | | **7** | 1 | 2 | | **11** | 2 | 2 | | **15** | 3 | 2 |
| **4** | 0 | 3 | | **8** | 1 | 3 | | **12** | 2 | 3 | | **16** | 3 | 3 |

# Tutorial Letter 207/2/2015

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 7.

BAR CODE

# Memo for Assignment 7 S2 2015

# Chapters 3 & 4

## Question 1

The right helicoid can be parametrized as

$$\begin{aligned} x(u, v) &= u\cos v \\ y(u, v) &= u\sin v \\ z(u, v) &= av \end{aligned}$$

where $a$ is a constant.

(a) Find the line element for the surface.

(b) What is the metric tensor and the dual metric tensor?

(c) Determine the values of all the Christoffel coefficients of the surface.

(d) What is the value of the component $R^1_{212}$ of the Riemann curvature tensor?

(e) What is the Ricci tensor for the surface?

(f) What is the curvature scalar $R$ for the surface?

(g) What is the Gaussian curvature of the surface?

(h) Is the surface Euclidean? Explain your answer.

(i) Suppose that the surface is filled with non-interacting particles, or dust. Use the two dimensional version of the energy-momentum tensor for dust and Einstein's field equation to find an expression for the Einstein constant $\kappa$ for this surface.

# Solution

## Part A

In Cartesian coordinates, the line element is given by

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2 .$$

We have

$$\begin{aligned} x(u,\, v) &= u\cos v \\ y(u,\, v) &= u\sin v \\ z(u,\, v) &= av \end{aligned}$$

so that

$$\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ &= \frac{\partial}{\partial u}(u\cos v)\ du + \frac{\partial}{\partial v}(u\cos v)\ dv \\ &= \cos v\, du - u\sin v\, dv \end{aligned}$$

Similarly, we get

$$\begin{aligned} dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \\ &= \frac{\partial}{\partial u}(u\sin v)\ du + \frac{\partial}{\partial v}(u\sin v)\ dv \\ &= \sin v\, du + u\cos v\, dv \end{aligned}$$

$$\begin{aligned} dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \\ &= \frac{\partial}{\partial u}(av)\ du + \frac{\partial}{\partial v}(av)\ dv \\ &= a\, dv \end{aligned}$$

Substituting this into the Cartesian line element and simplifying gives

$$\begin{aligned} (dl)^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= (\cos v\, du - u\sin v\, dv)^2 + (\sin v\, du + u\cos v\, dv)^2 + (a\, dv)^2 \end{aligned}$$

$$
\begin{aligned}
&= \cos^2 v\, du^2 - u\cos v \sin v\, dudv + u^2 \sin^2 v\, dv^2 + \sin^2 v\, du^2 + u \cos v \sin v\, dudv \\
&\qquad\qquad + u^2 \cos^2 v\, dv^2 + a^2\, dv^2 \\
&= \left(\cos^2 v + \sin^2 v\right) du^2 + \left[u^2\left(\sin^2 v + \cos^2 v\right) + a^2\right] dv^2 \\
&= du^2 + \left(u^2 + a^2\right) dv^2
\end{aligned}
$$

**Part B**

We know that the line element has the form

$$dl^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$$

If we choose $x^1 = u$ and $x^2 = v$, this reduces to

$$
\begin{aligned}
dl^2 &= \sum_{i,j=1}^{2} g_{ij} dx^i dx^j \\
&= g_{11} dx^1 dx^1 + 2 g_{12} dx^1 dx^2 + g_{22} dx^2 dx^2 \\
&= g_{11} (du)^2 + 2 g_{12} dudv + g_{22} (dv)^2
\end{aligned}
$$

Above we used the fact that the metric tensor is symmetric $g_{ij} = g_{ji}$. Comparing this to the line element calculated in Part A allows us to identify

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = u^2 + a^2$$

so that the metric tensor for the surface is

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & u^2 + a^2 \end{pmatrix}$$

We know that we must have

$$\sum_k g^{ik} g_{kj} = \delta^i_j$$

so that the dual metric $[g^{ij}]$ is just the matrix inverse of $[g_{ij}]$. We find

$$\left[g^{ij}\right] = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{u^2+a^2} \end{pmatrix}$$

**Part C**

The Christoffel coefficients are defined by

$$\Gamma^h_{ij} = \sum_k \frac{1}{2} g^{hk} \left( g_{ki,j} + g_{jk,i} - g_{ij,k} \right)$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} \left( g_{11,1} + g_{11,1} - g_{11,1} \right) + \frac{1}{2} g^{12} \left( g_{21,1} + g_{12,1} - g_{11,2} \right)$$

All the $g_{ik}$ and $g^{ik}$ where $i \neq k$ will be zero, so their derivatives will also be zero. Remembering this will reduce the calculations a lot. So we have

$$\begin{aligned} v\Gamma^1_{11} &= \frac{1}{2} g^{11} \left( g_{11,1} + g_{11,1} - g_{11,1} \right) + \frac{1}{2} g^{12} \left( g_{21,1} + g_{12,1} - g_{11,2} \right) \\ &= \frac{1}{2} g^{11} g_{11,1} \\ &= \frac{1}{2} (1) \frac{d}{du} (1) \\ &= 0 \end{aligned}$$

Using the symmetric property of the Christoffel coefficients $\Gamma^h_{ij} = \Gamma^h_{ji}$ will also cut down on calculations

$$\begin{aligned} \Gamma^1_{12} = \Gamma^1_{21} &= \frac{1}{2} g^{11} \left( g_{11,2} + g_{21,1} - g_{12,1} \right) + \frac{1}{2} g^{12} \left( g_{21,2} + g_{22,1} - g_{12,2} \right) \\ &= \frac{1}{2} g^{11} g_{11,2} \\ &= \frac{1}{2} (1) \frac{\partial}{\partial v} \left( u^2 + a^2 \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Gamma^1_{22} &= \frac{1}{2} g^{11} \left( g_{12,2} + g_{21,2} - g_{22,1} \right) + \frac{1}{2} g^{12} \left( g_{22,2} + g_{22,2} - g_{22,2} \right) \\ &= -\frac{1}{2} g^{11} g_{22,1} \\ &= -\frac{1}{2} (1) \frac{\partial}{\partial u} \left( u^2 + a^2 \right) \\ &= -u \end{aligned}$$

$$
\begin{aligned}
\Gamma^2_{\ 11} &= \frac{1}{2}g^{21}\left(g_{11,1}+g_{11,1}-g_{11,1}\right)+\frac{1}{2}g^{22}\left(g_{21,1}+g_{12,1}-g_{11,2}\right)\\
&= -\frac{1}{2}g^{22}g_{11,2}\\
&= -\frac{1}{2}\left(\frac{1}{u^2+a^2}\right)\frac{\partial}{\partial v}\left(u^2+a^2\right)\\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2_{\ 12}=\Gamma^2_{\ 21} &= \frac{1}{2}g^{21}\left(g_{12,1}+g_{11,2}-g_{21,1}\right)+\frac{1}{2}g^{22}\left(g_{22,1}+g_{12,2}-g_{21,2}\right)\\
&= \frac{1}{2}g^{22}g_{22,1}\\
&= \frac{1}{2}\left(\frac{1}{u^2+a^2}\right)\frac{\partial}{\partial u}\left(u^2+a^2\right)\\
&= \frac{u}{u^2+a^2}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma^2_{\ 22} &= \frac{1}{2}g^{21}\left(g_{12,2}+g_{21,2}-g_{22,1}\right)+\frac{1}{2}g^{22}\left(g_{22,2}+g_{22,2}-g_{22,2}\right)\\
&= \frac{1}{2}g^{22}g_{22,2}\\
&= \frac{1}{2}\left(\frac{1}{u^2+a^2}\right)\frac{\partial}{\partial v}\left(u^2+a^2\right)\\
&= 0
\end{aligned}
$$

In summary, the only non-zero Christoffel coefficients that we have are $\Gamma^1_{\ 22}=-u$ and $\Gamma^2_{\ 12}=\Gamma^2_{\ 21}=u/\left(u^2+a^2\right)$.

**Part D**

The Riemann Curvature tensor is defined by

$$
R^l_{\ ijk}\equiv\frac{\partial\Gamma^l_{\ ik}}{\partial x^j}-\frac{\partial\Gamma^l_{\ ij}}{\partial x^k}+\sum_m\Gamma^m_{\ ik}\Gamma^l_{\ mj}-\sum_m\Gamma^m_{\ ij}\Gamma^l_{\ mk}
$$

since we are dealing with a two dimensional surface, the only independent entry will be $R^1{}_{212}$, so it will be sufficient to only calculate this. We have

$$
\begin{aligned}
R^1{}_{212} &= \frac{\partial \Gamma^1{}_{22}}{\partial x^1} - \frac{\partial \Gamma^1{}_{21}}{\partial x^2} + \sum_m \Gamma^m{}_{22}\Gamma^1{}_{m1} - \sum_m \Gamma^m{}_{21}\Gamma^1{}_{m2} \\
&= \frac{\partial \Gamma^1{}_{22}}{\partial u} - \frac{\partial \Gamma^1{}_{21}}{\partial v} + \Gamma^1{}_{22}\Gamma^1{}_{11} + \Gamma^2{}_{22}\Gamma^1{}_{21} - \Gamma^1{}_{21}\Gamma^1{}_{12} - \Gamma^2{}_{21}\Gamma^1{}_{22} \\
&= \frac{\partial \Gamma^1{}_{22}}{\partial u} - \Gamma^2{}_{21}\Gamma^1{}_{22} \\
&= \frac{\partial}{\partial u}(-u) - \left(\frac{u}{u^2+a^2}\right)(-u) \\
&= -1 + \frac{u^2}{u^2+a^2} \\
&= \frac{-u^2 - a^2 + u^2}{u^2+a^2} \\
&= \frac{-a^2}{u^2+a^2}
\end{aligned}
$$

For the Riemann curvature tensor we have

$$R^1{}_{212} = R^2{}_{121} = \frac{-a^2}{u^2+a^2}$$

$$R^1{}_{221} = R^2{}_{112} = \frac{a^2}{u^2+a^2}$$

With all other entries equal to zero.

**Part E**

The Ricci tensor is defined by

$$R_{ij} \equiv \sum_k R^k{}_{ijk}$$

Using the fact that the Ricci tensor is symmetric we find the 4 entries of the Ricci tensor

$$
\begin{aligned}
R_{11} &= R^1{}_{111} + R^2{}_{112} \\
&= \frac{a^2}{u^2+a^2}
\end{aligned}
$$

$$
\begin{aligned}
R_{12} = R_{21} &= R^1{}_{121} + R^2{}_{122} \\
&= 0
\end{aligned}
$$

$$\begin{aligned} R_{22} &= R^1{}_{221} + R^2{}_{222} \\ &= \frac{a^2}{u^2 + a^2} \end{aligned}$$

**Part F**

The Ricci scalar is defined by

$$R \equiv \sum_{i,j} g^{ij} R_{ij}$$

So we have for the helicoid

$$\begin{aligned} R &= g^{11} R_{11} + g^{12} R_{12} + g^{21} R_{21} + g^{22} R_{22} \\ &= g^{11} R_{11} + g^{22} R_{22} \\ &= (1)\left(\frac{-a^2}{u^2 + a^2}\right) + \left(\frac{1}{u^2 + a^2}\right)\left(\frac{-a^2}{u^2 + a^2}\right) \\ &= \frac{-a^2 u^2 - a^4 - a^2}{(u^2 + a^2)^2} \\ &= \frac{-a^2 (u^2 + a^2 + 1)}{(u^2 + a^2)^2} \end{aligned}$$

**Part G**

The Gaussian curvature of a two dimensional surface is given by

$$K = \frac{R_{1212}}{g}$$

where $g = \det [g_{ij}]$ (see Exercise 3.16 p105).

The determinant of a diagonal matrix is just the product of its diagonal entries so that

$$\begin{aligned} g &= \prod_i g_{ii} \\ &= (1)\left(u^2 + a^2\right) \\ &= u^2 + a^2 \end{aligned}$$

$R_{1212}$ is the element of the Riemann curvature tensor with an index lowered, i.e.

$$R_{1212} = \sum_i g_{i1} R^i{}_{212}$$

$$
\begin{aligned}
&= g_{11}R^1{}_{212} + g_{21}R^2{}_{212} \\
&= (1)\left(\frac{-a^2}{u^2+a^2}\right) \\
&= \frac{-a^2}{u^2+a^2}
\end{aligned}
$$

So we have for the Gaussian curvature

$$
\begin{aligned}
K &= \frac{R_{1212}}{g} \\
&= \left(\frac{-a^2}{u^2+a^2}\right)\left(\frac{1}{u^2+a^2}\right) \\
&= \frac{-a^2}{(u^2+a^2)^2}
\end{aligned}
$$

**Part H**

No, the helicoid is not Euclidean (flat). The necessary and sufficient condition for a surface to be flat is that the Riemann curvature tensor (all its components) should vanish (be equal to zero) at all points on the surface. This is not true for all values of $u$ and $v$.

**Part I**

Einstein's field equation for two dimensions is

$$
R_{ij} - \frac{1}{2}Rg_{ij} = -\kappa T_{ij}
$$

where $i$ and $j$ can take the values of 1 or 2, as with the rest of the calculations regarding the surface above. The only non-zero component of the energy-momentum tensor $[T^{ij}]$ for dust is $T^{11} = \rho c^2$.

$[T^{ij}]$ is related to $[T_{ij}]$ by

$$
T_{ij} = \sum_{m,n} g_{im}g_{jn}T^{mn}
$$

Clearly, the only non-zero component of $[T_{ij}]$ will be $T_{11}$. We find

$$\begin{aligned} T_{11} &= \sum_{m,n} g_{1m} g_{1n} T^{mn} \\ &= g_{11} g_{11} T^{11} + g_{11} g_{12} T^{12} + g_{12} g_{11} T^{21} + g_{12} g_{12} T^{22} \\ &= g_{11} g_{11} T^{11} \\ &= \rho c^2 \end{aligned}$$

Now all the quantities in the Einstein field equation are known. We substitute and solve for $\kappa$

$$\begin{aligned} R_{11} - \frac{1}{2} R g_{11} &= -\kappa T_{11} \\ \frac{-a^2}{u^2+a^2} - \frac{1}{2}\left(\frac{-a^2\left(u^2+a^2+1\right)}{\left(u^2+a^2\right)^2}\right)(1) &= -\kappa \rho c^2 \\ \frac{-2a^2u^2 - 2a^4 + a^2u^2 + a^4 + a^2}{2\left(u^2+a^2\right)^2} &= -\kappa \rho c^2 \\ \frac{-a^2u^2 - a^4 + a^2}{2\left(u^2+a^2\right)^2} &= -\kappa \rho c^2 \\ \kappa &= \frac{a^2\left(u^2+a^2-1\right)}{2\rho c^2\left(u^2+a^2\right)^2} \end{aligned}$$

# Question 2

Show that the contracted Christoffel symbol $\sum_i \Gamma^i_{ik}$ is given by

$$\sum_i \Gamma^i_{ik} = \sum_i \sum_m \frac{g^{im}}{2} \frac{\partial g_{mi}}{\partial x^k}\,.$$

## Solution

The Christoffel coefficients are defined by

$$\Gamma^i{}_{jk} = \sum_m \frac{1}{2} g^{im}\left(g_{mj,k} + g_{km,j} - g_{jk,m}\right)$$

If we contract the Christoffel coefficients we have

$$\sum_i \Gamma^i{}_{ik} = \sum_i \sum_m \frac{1}{2} g^{im} \left( g_{mi,k} + g_{km,i} - g_{ik,m} \right)$$

Since $i$ and $m$ are just dummy indices being summed over the same range, they can be interchanged without changing the meaning of the expression. We interchange them in the last term in brackets to get

$$\sum_i \Gamma^i{}_{ik} = \sum_i \sum_m \frac{1}{2} g^{im} \left( g_{mi,k} + g_{km,i} - g_{mk,i} \right)$$

The metric is symmetric, so that $g_{km} = g_{mk}$,

$$\begin{aligned} \sum_i \Gamma^i{}_{ik} &= \sum_i \sum_m \frac{1}{2} g^{im} \left( g_{mi,k} + g_{km,i} - g_{km,i} \right) \\ &= \sum_i \sum_m \frac{1}{2} g^{im} g_{mi,k} \\ &= \sum_i \sum_m \frac{g^{im}}{2} \frac{\partial g_{mi}}{\partial x^k} \end{aligned}$$

# Question 3

Verify that if a tensor is symmetric in one frame, it will be symmetric in all coordinate frames. That is, show that if it is given that $X^{ij} = X^{ji}$ in frame $S$, then it will be true that $\bar{X}^{ij} = \bar{X}^{ji}$ in a coordinate frame $\bar{S}$.

## Solution

If $X^{ij} = X^{ji}$, then

Since $X^{ij}$ is a tensor, we know that it transforms as follows

$$\bar{X}^{ab} = \sum_i \sum_j \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} X^{ij}$$

On the RHS both $i$ and $j$ are just dummy indices, i.e. they are being summed over. This means that the two indices can be replaced by any other indices without changing the meaning of the expression, since they are just counters to be summed over, i.e.

$$\sum_i \sum_j \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} X^{ij} = \sum_\alpha \sum_\beta \frac{\partial \bar{x}^a}{\partial x^\alpha} \frac{\partial \bar{x}^b}{\partial x^\beta} X^{\alpha\beta} = \sum_r \sum_s \frac{\partial \bar{x}^a}{\partial x^r} \frac{\partial \bar{x}^b}{\partial x^s} X^{rs}$$

In particular, we can replace $j$ with $i$ and $i$ with $j$, so that

$$\begin{aligned} \bar{X}^{ab} &= \sum_i \sum_j \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} X^{ij} \\ &= \sum_j \sum_i \frac{\partial \bar{x}^a}{\partial x^j} \frac{\partial \bar{x}^b}{\partial x^i} X^{ji} \\ &= \sum_j \sum_i \frac{\partial \bar{x}^a}{\partial x^j} \frac{\partial \bar{x}^b}{\partial x^i} X^{ij} \end{aligned}$$

In the last step we used the symmetry of property $X^{ij} = X^{ji}$. This is the transformation expression for a second order contravariant tensor where $x^i \to \bar{x}^b$ and $x^j \to \bar{x}^a$ so we have

$$\begin{aligned} \bar{X}^{ab} &= \sum_j \sum_i \frac{\partial \bar{x}^a}{\partial x^j} \frac{\partial \bar{x}^b}{\partial x^i} X^{ij} \\ &= \bar{X}^{ba} \end{aligned}$$

Thus we have shown that if a tensor is symmetric in one coordinate frame, i.e. $X^{ij} = X^{ji}$ in $S$, then it is also symmetric in any other arbitrary coordinate frame $\bar{S}$.

## Question 4

Suppose that $R_{iklm} = K\left(g_{il}g_{km} - g_{im}g_{kl}\right)$ on some four dimensional Riemannian space. Show that for the curvature scalar we have $R = -12K$.

## Solution

From tensor contraction we can write

$$\begin{aligned} R_{kl} &= \sum_i \sum_m g^{im} R_{iklm} \\ &= \sum_i \sum_m K g^{im} \left( g_{il} g_{km} - g_{im} g_{kl} \right) \\ &= \sum_i \sum_m K \left( g^{im} g_{il} g_{km} - g^{im} g_{im} g_{kl} \right) \end{aligned}$$

Remember that tensors are not commutative, so be mindful of the order of the multiplication. We use the property of the metric tensor $\sum_i g^{im} g_{in} = \delta^m_n$ to get

$$\begin{aligned} R_{kl} &= \sum_m K \left( \delta^m_l g_{km} - \delta^m_m g_{kl} \right) \\ &= K \left( g_{kl} - 4 g_{kl} \right) \\ &= -3K g_{kl} \end{aligned}$$

In the second step above we used the definition the Kronecker delta. If we have

$$\sum_m \delta^m_l g_{km}$$

all the terms in the sum where $m \neq l$, will be zero, where the term where $m = l$ will be equal to $g_{kl}$, i.e.

$$\begin{aligned} \sum_m \delta^m_l g_{km} &= \delta^0_l g_{k0} + \delta^1_l g_{k1} + \ldots + \delta^l_l g_{kl} + \ldots + \delta^N_l g_{kN} \\ &= (0)\, g_{k0} + (0)\, g_{k1} + \ldots + (1)\, g_{kl} + \ldots + (0)\, g_{kN} \\ &= g_{kl} \end{aligned}$$

So the Kronecker delta can effectively be used to replace one index with another. On the other hand, if the two indices of the Kronecker delta are the same, i.e. $\sum_m \delta^m_m$, the result is *not equal to one* because of the summation. Then we have

$$\sum_m \delta^m_m g_{kl} = \delta^0_0 g_{kl} + \delta^1_1 g_{kl} + \ldots + \delta^m_m g_{kl} + \ldots + \delta^N_N g_{kl}$$

$$
\begin{aligned}
&= (1)\, g_{k0} + (1)\, g_{k1} + \ldots + (1)\, g_{kl} + \ldots + (1)\, g_{kN} \\
&= N g_{kl}
\end{aligned}
$$

In this case we are working in a 4 dimensional space, so that $N = 4$ and $\sum_m \delta^m_m g_{kl} = 4g_{kl}$

For the curvature scalar, we contract our result for $R_{kl}$

$$
\begin{aligned}
R &= \sum_k \sum_l g^{kl} R_{kl} \\
&= -3K \sum_k \sum_l g^{kl} g_{kl} \\
&= -3K \sum_k \delta^k_k \\
&= -12K
\end{aligned}
$$

# Question 5

Two $N$-dimensional Riemann spaces $M$ and $\bar{M}$ have the metric tensors $g_{ij}$ and $\bar{g}_{ij}$ respectively, and

$$\bar{g}_{ij} = k g_{ij}$$

where $k$ is a constant. What are the relationships between the curvature tensors, Ricci tensors, curvature scalar and Einstein tensors of the two spaces?

## Solution

We have

$$\bar{g}_{ij} = k g_{ij}$$

and therefore

$$\bar{g}^{ij} = \frac{1}{k} g^{ij}$$

The transformation of an arbitrary Christoffel symbol from $M$ to $\bar{M}$ gives

$$\Gamma^h_{ij} = \sum_k \frac{1}{2} g^{hk} \left( g_{ki,j} + g_{jk,i} - g_{ij,k} \right)$$

$$
\begin{aligned}
&= \sum_k \frac{k}{2}\bar{g}^{hk}\left(\frac{1}{k}\bar{g}_{ki,j} + \frac{1}{k}\bar{g}_{jk,i} - \frac{1}{k}\bar{g}_{ij,k}\right) \\
&= \sum_k \frac{1}{2}\bar{g}^{hk}\left(\bar{g}_{ki,j} + \bar{g}_{jk,i} - \bar{g}_{ij,k}\right) \\
&= \bar{\Gamma}^h_{ij}
\end{aligned}
$$

Using this we get for the curvature tensor

$$
\begin{aligned}
R^i{}_{jhk} &= \Gamma^i{}_{jh,k} - \Gamma^i{}_{jk,h} + \sum_m \Gamma^i{}_{mk}\Gamma^m_{jh} - \sum_m \Gamma^i{}_{mh}\Gamma^m_{jk} \\
&= \bar{\Gamma}^i{}_{jh,k} - \bar{\Gamma}^i{}_{jk,h} + \sum_m \bar{\Gamma}^i{}_{mk}\bar{\Gamma}^m_{jh} - \sum_m \bar{\Gamma}^i{}_{mh}\bar{\Gamma}^m_{jk} \\
&= \bar{R}^i{}_{jhk}
\end{aligned}
$$

Then Ricci tensor becomes

$$
\begin{aligned}
R_{jk} &= \sum_h \sum_i \sum_m g^{mh} g_{im} R^i{}_{jhk} \\
&= \sum_h \sum_i \sum_m \left(k\bar{g}^{mh}\right)\left(\frac{1}{k}\bar{g}_{im}\right)\bar{R}^i{}_{jhk} \\
&= \bar{R}_{jk}
\end{aligned}
$$

For the Curvature scalar

$$
\begin{aligned}
R &= \sum_i \sum_j g^{ij} R_{ij} \\
&= \sum_i \sum_j k\bar{g}^{ij}\bar{R}_{ij} \\
&= k\bar{R}
\end{aligned}
$$

And the relationship between the Einstein tensors is

$$
\begin{aligned}
G_{ij} &= R_{ij} - \frac{1}{2}g_{ij}R \\
&= \bar{R}_{ij} - \frac{1}{2}\frac{1}{k}\bar{g}_{ij}k\bar{R} \\
&= \bar{R}_{ij} - \frac{1}{2}\bar{g}_{ij}\bar{R} \\
&= \bar{G}_{ij}
\end{aligned}
$$

# Tutorial Letter 201/1/2016

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 1

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 01.

BAR CODE

# Memo for Assignment 1 S1 2016

## Special relativity basics (§ 1.1 - 1.2)

## Consequences of Lorentz transformations (§ 1.3)

### Question 1: Lorentz factor

What is the Lorentz factor ($\gamma$) when the relative speed between two coordinate frames is 60 % the speed of light?

- 0.8
- 1.58
- **1.25***
- 0.64
- 1.56

A frame moving at 60 % the speed of light is moving at $V = 0.6c$. Therefore, the Lorentz factor is

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1 - V^2/c^2}} \\
&= \frac{1}{\sqrt{1 - (0.6c)^2/c^2}} \\
&= \frac{1}{\sqrt{1 - 0.36}} \\
&= \frac{1}{\sqrt{0.64}} \\
&= 1.25
\end{aligned}$$

## Question 2: Lorentz transformation

Alice travels past Bob in a spaceship at a speed of $0.6c$. Take $S$ to be the coordinate frame where Bob is stationary and $S'$ to be the coordinate frame that moves with the Alice. Using the standard configuration, Alice is traveling in the positive $x$ (and $x'$) direction. Bob sets off an explosion and in his frame it explodes at time $t = 5\,\mathrm{s}$ at the point $x = 10^6\,\mathrm{km}$ in his frame. Use the Lorentz transformation equations to determine the coordinate $x'$ that Alice measures in her frame for the explosion.

For the position of the explosion, Alice measures $x'$ equal to

- $10^6\,\mathrm{km}$
- $-1.12 \times 10^9\,\mathrm{m}$
- $10^8\,\mathrm{m}$
- $\mathbf{1.25 \times 10^5\,km}$*
- $0\,\mathrm{m}$

If we want to use $c = 3 \times 10^8\,\mathrm{ms}^{-1}$, we need to convert all the quantities to SI units. Thus we use $x = 10^6\,\mathrm{km} = 10^9\,\mathrm{m}$. The Lorentz factor ($\gamma$) has already been calculated in the previous question. Using the appropriate Lorentz transformation gives

$$\begin{aligned} x' &= \gamma\,(x - Vt) \\ &= 1.25\left(10^9 - 0.6\left(3 \times 10^8\right)(5)\right) \\ &= 1.25\left(10^9 - 9 \times 10^8\right) \\ &= 1.25 \times 10^8\,\mathrm{m} \\ &= 1.25 \times 10^5\,\mathrm{km} \end{aligned}$$

## Question 3: Time dilation

A group of astronauts take on a mission to travel to a nearby planet at a constant speed of $0.8c$. According to the people on Earth, the spaceship takes 88 years to reach its destination. How long did the journey take according to the astronauts?

- **52.7 years***
- 147 years
- 39.3 years
- 197.1 years
- 31.7 years

The Lorentz factor between the Earth's reference frame and the reference frame of the astronauts is given by

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1 - V^2/c^2}} \\ &= \frac{1}{\sqrt{1 - (0.8c)^2/c^2}} \\ &= \frac{1}{\sqrt{0.36}} \\ &= 1.67\end{aligned}$$

The astronauts measure their own proper time. The time that the people on Earth measure for the journey ($\Delta t_E$) will be related to the time that the astronauts measure ($\Delta t_A$) by the time dilation formula so that

$$\begin{aligned}\Delta t_E &= \gamma \Delta t_A \\ 88\,\text{years} &= 1.67 \times \Delta t_A \\ \Delta t_A &= 52.7\,\text{years}\end{aligned}$$

## Question 4: Summation

The sum

$$A_\alpha = \sum_{\beta=0}^{3} \eta_{\beta\alpha} A^\beta$$

written out in full is

- $A_\alpha = 3\eta_{\beta\alpha}A^\beta$
- $A_\alpha = \eta_{\beta\alpha}A^\beta$
- $A_\alpha = \eta_{\beta\alpha}A^\beta + \eta_{\beta\alpha}A^\beta + \eta_{\beta\alpha}A^\beta + \eta_{\beta\alpha}A^\beta$
- $\mathbf{A}_\alpha = \eta_{\mathbf{0}\alpha}\mathbf{A}^\mathbf{0} + \eta_{\mathbf{1}\alpha}\mathbf{A}^\mathbf{1} + \eta_{\mathbf{2}\alpha}\mathbf{A}^\mathbf{2} + \eta_{\mathbf{3}\alpha}\mathbf{A}^{\mathbf{3}*}$
- $A_\alpha = \eta_{00}A^0 + \eta_{11}A^1 + \eta_{22}A^2 + \eta_{33}A^3$

## Question 5: Relativistic Doppler shift

A spaceship moving toward the Earth at a speed of $0.35c$ communicates with Earth by transmitting on a frequency (measured in the spaceship rest frame) of 100 MHz. To what frequency must Earth receivers be tuned to receive these signals?

- 69.4 MHz
- **144** MHz*
- 48.1 MHz
- 207 MHz
- 167 MHz

Since the source of the radiation is approaching the receiver we use

$$\begin{aligned} f_{rec} &= f_{em}\sqrt{\frac{c+V}{c-V}} \\ &= 100\,\text{MHz} \times \sqrt{\frac{c+0.35c}{c-0.35c}} \\ &= 100\,\text{MHz} \times \sqrt{\frac{1.35c}{0.65c}} \\ &= 100\,\text{MHz} \times \sqrt{2.077} \\ &= 144\,\text{MHz} \end{aligned}$$

# Tutorial Letter 201/2/2016

## Special Relativity and Riemannian Geometry

# APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 01.

BAR CODE

# Memo for Assignment 1 S2 2016

## Special relativity basics (§ 1.1 - 1.2)

## Consequences of Lorentz transformations (§ 1.3)

### Question 1: Lorentz factor

What is the Lorentz factor ($\gamma$) when the relative speed between two coordinate frames is 90 % the speed of light?

- **2.29***
- 3.16
- 0.44
- 5.26
- 0.74

A frame moving at 90 % the speed of light is moving at $V = 0.9c$. Therefore, the Lorentz factor is

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1 - V^2/c^2}} \\ &= \frac{1}{\sqrt{1 - (0.9c)^2/c^2}} \\ &= \frac{1}{\sqrt{1 - 0.81}} \\ &= \frac{1}{\sqrt{0.19}} \\ &= 2.29\end{aligned}$$

## Question 2: Lorentz transformation

Alice sees an explosion happening and measures the spacetime coordinates of the explosion to be $(t, x, y, z) = (1.5\,\text{ns}, 2\,\text{m}, 1\,\text{m}, 0\,\text{m})$. Bob is riding past in a train at a constant speed of $V = 0.4c$ in the positive $x$-direction. Use the Lorentz transformation equations to determine what time Bob measures the explosion taking place. (Hint: $1\,\text{ns}$ (nanosecond) $= 10^{-9}$s.)

- $1.5\,\text{ns}$
- $1.41 \times 10^{-9}\,\text{s}$
- $-1.17 \times 10^{-9}\,\text{s}$
- $1.5\,\text{s}$
- $-1.41\,\text{ns}$*

The train is moving at $V = 0.4c$, so the Lorentz factor between Alice's frame $(S)$ and Bob's frame $(S')$ is

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1 - V^2/c^2}} \\
&= \frac{1}{\sqrt{1 - (0.4c)^2/c^2}} \\
&= \frac{1}{\sqrt{1 - 0.16}} \\
&= \frac{1}{\sqrt{0.84}} \\
&= 1.09
\end{aligned}$$

The Lorentz transformation equation for the time coordinate is

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

Using this, we transform Alice's measurements into Bob's frame as follows

$$t' = \gamma\left(t - \frac{vx}{c^2}\right)$$

$$
\begin{aligned}
&= 1.09\left(1.5\,\text{ns} - \frac{(0.4c)\,(2\,\text{m})}{c^2}\right) \\
&= 1.09\left(1.5\,\text{ns} - \frac{(0.4)\,(2\,\text{m})}{c}\right) \\
&= 1.09\left(1.5\,\text{ns} - \frac{(0.4)\,(2\,\text{m})}{3\times 10^8\text{ms}^{-1}}\right) \\
&= 1.09\left(1.5\,\text{ns} - 2.27\times 10^{-9}\text{s}\right) \\
&= 1.09\,(1.5\,\text{ns} - 2.27\,\text{ns}) \\
&= 1.09\,(-1.17\,\text{ns}) \\
&= -1.41\,\text{ns}
\end{aligned}
$$

The negative time coordinate just means that Bob sees the explosion happening at a time before the arbitrarily chosen zero time.

## Question 3: Length contraction

A spaceship is moving at such a speed past the Earth that the people on Earth measure its length to be one third of its proper length. How fast is the spaceship moving relative to the Earth?

- $2.67 \times 10^8\,\text{ms}^{-1}$
- $-2.67 \times 10^8\,\text{ms}^{-1}$
- $8 \times 10^{16}\,\text{ms}^{-1}$
- $8.49 \times 10^8\,\text{ms}^{-1}$
- $\mathbf{2.83 \times 10^8}\,\text{ms}^{-1}*$

The measured length of the spaceship is one third of its proper length, so we have

$$L = \frac{L_P}{3}$$

We also know from the length contraction formula that

$$L = \frac{L_P}{\gamma}$$

so that we can conclude that $\gamma = 3$ between Alice's and Bob's frames. From this we can easily get the relative speed between the frames (and thus the speed of the spaceship).

$$\begin{aligned}
\gamma &= 3 \\
\frac{1}{\sqrt{1 - V^2/c^2}} &= 3 \\
1 - V^2/c^2 &= \frac{1}{9} \\
V^2/c^2 &= \frac{8}{9} \\
V &= \frac{2\sqrt{2}}{3}c \\
&= \frac{2\sqrt{2}}{3}\left(3 \times 10^8\right) \\
&= 2.83 \times 10^8 \,\mathrm{ms}^{-1}
\end{aligned}$$

Remember, nothing with mass can travel faster than the speed of light $c$. If you ever get an answer where a speed is greater than $c$, you made a mistake somewhere.

## Question 4: Summation

The sum

$$x' = \sum_{\nu=0}^{3} \Lambda^{\mu}{}_{\nu} x^{\nu}$$

written out in full is

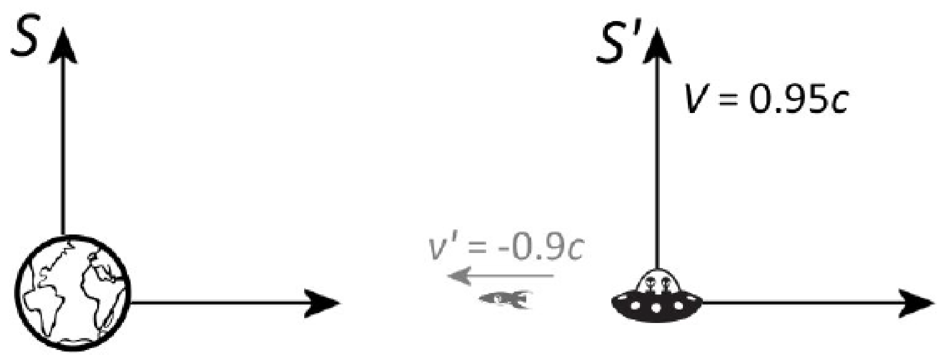
- $x' = \Lambda^{\mu}{}_{\nu} x^{\nu} + \Lambda^{\mu}{}_{\nu} x^{\nu} + \Lambda^{\mu}{}_{\nu} x^{\nu} + \Lambda^{\mu}{}_{\nu} x^{\nu}$
- $\mathbf{x}' = \mathbf{\Lambda}^{\mu}{}_{\mathbf{0}} \mathbf{x}^{\mathbf{0}} + \mathbf{\Lambda}^{\mu}{}_{\mathbf{1}} \mathbf{x}^{\mathbf{1}} + \mathbf{\Lambda}^{\mu}{}_{\mathbf{2}} \mathbf{x}^{\mathbf{2}} + \mathbf{\Lambda}^{\mu}{}_{\mathbf{3}} \mathbf{x}^{\mathbf{3}}*$
- $x' = \Lambda^{0}{}_{0} x^{0} + \Lambda^{1}{}_{1} x^{1} + \Lambda^{2}{}_{2} x^{2} + \Lambda^{3}{}_{3} x^{3}$
- $x' = 3\Lambda^{\mu}{}_{\nu} x^{\nu}$
- $x' = \Lambda^{\mu}{}_{\nu} x^{\nu}$

## Question 5: Velocity addition

A spaceship moves at a speed of $0.95c$ away from the Earth. It shoots a torpedo toward the Earth at a speed of $0.9c$ relative to the ship. What is the velocity of the torpedo relative to the Earth? (Take the direction in which the spaceship moves is the positive direction.)

- $-0.345c$
- **0.345c***
- $0.06c$
- $-0.06c$
- $0.9c$

When in doubt, draw a picture! It will help you to disentangle the problem and organise your thoughts. It can also show the marker that you have insight into the problem. Below is a rough sketch of the situation.




In the figure, we have chosen the reference frame of Earth to be $S$ and the spaceship's reference frame to be $S'$, which is moving at $V = 0.95c$ in the positive $x$-direction with respect to $S'$. The torpedo shot from the spaceship moves at $v' = -0.9c$ in the negative $x$-direction as measured in $S'$. We what to calculate the speed of the torpedo measured by the people on Earth, that is the speed of the torpedo in $S$. We call this speed $v$. We use the velocity transformation equation given in the textbook to obtain

$$
\begin{aligned}
v' &= \frac{v - V}{1 - vV/c^2} \\
-0.9c &= \frac{v - (0.95c)}{1 - v\,(0.95c)\,/c^2} \\
(-0.9c)\left(1 - v\,(0.95c)\,/c^2\right) &= v - 0.95c \\
-0.9c + 0.855v &= v - 0.95c \\
0.145v &= 0.05c \\
v &= 0.345c
\end{aligned}
$$

Even though the spaceship shot the torpedo towards Earth, since the speed of the torpedo in their frame was slower than their speed with respect to Earth, it will never reach the Earth and the planet is safe.

This might seem strange at first, but this is not some strange relativistic effect, but one we know from everyday experience. Suppose you are in a car driving on the highway in the positive $x$-direction. If you drive past a car moving slower than yourself, the car will appear to move in the negative $x$-direction as you go past it from your perspective. But to a person standing next to the highway, both of you will be moving in the same direction.

# Tutorial Letter 202/1/2016

## Special Relativity and Riemannian Geometry

## APM3713

### Semester 1

### Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 02.

BAR CODE

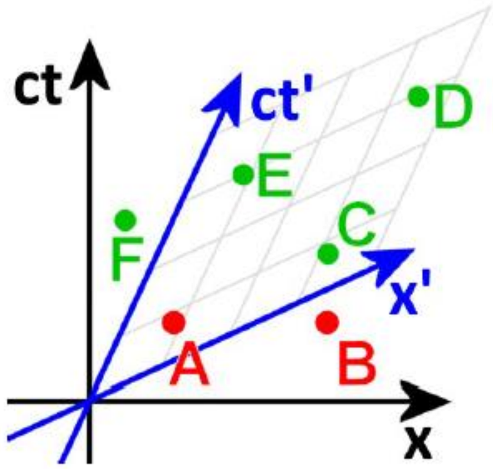
# Memo for Assignment 2 S1 2016

## Minkowski spacetime (§ 1.4)

## Physical laws in relativity (§ 2.1 - 2.2.3)

### Questions 1 - 3: Minkowski diagrams

An observer in $S$ measures two simultaneous events, Event A and Event B. Another observer is at rest in frame $S'$, which moves in the standard configuration with respect to $S$ at a speed comparable to the speed of light. The situation is indicated in the Minkowski diagram below. Refer to the diagram to answer questions 1 to 3.




### Question 1

Would an observer in $S'$also observe Events A and B to be simultaneous?

- Yes.
- No, he would observe Event A before Event B.
- **No, he would observe Event B before Event A.***

- It depends on the exact relative speed of $S$ and $S'$.
- It depends on the exact time measurement that the observer in $S$ measures for the two events.

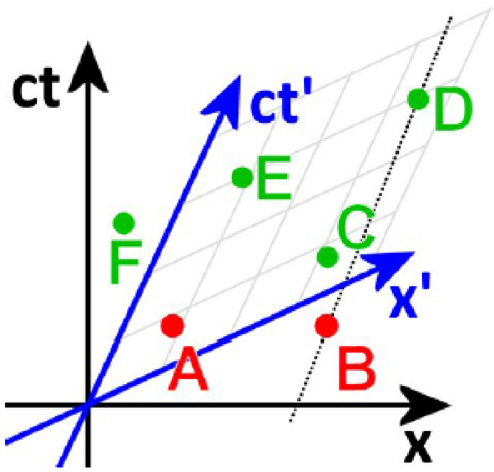
From the diagram, we can see that Event B has a negative time value in $S'$ (it is below the $x'$axis), whereas Event A has got a positive value of $ct'$. Therefore, the two events are not simultaneous, and Event B occurs first in frame $S'$. Event C is an example of an event that would be observed to occur simultaneously with Event A in $S'$, since they will have the same value on the $ct'$ axis.

## Question 2

According to the observer in $S'$, which event indicated on the diagram will occur at the same position as Event B?

- A
- C
- **D**
- F
- There is no such event indicated on the diagram

For the two events to have the same position in $S'$, they must have the same $x'$ value. Event D meets this criteria, as indicated by the dotted line in the diagram below. Event C occurs at the same position in $S$, but not in $S'$

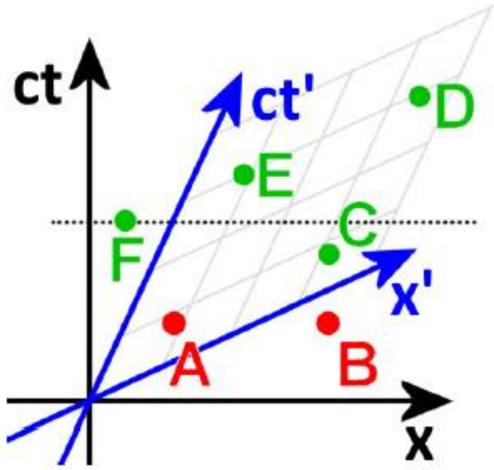


## Question 3

According to the observer in $S$, which event indicated on the diagram will occur at the same time as Event F?

- C
- D
- E
- More than one of the above
- **There is no such event indicated on the diagram**

For an event to occur at the same time as Event F in $S$, the event has to have the same value of $ct$ as Event F. As indicated in the diagram below, none of the indicated events meets this criteria.




## Question 4: Mass energy

What is the mass energy of 1 g of matter?

- $3 \times 10^{8}$ joules
- $\mathbf{9 \times 10^{13}}$ **joules***

- $3 \times 10^5$ joules
- $9 \times 10^{19}$ joules
- $9 \times 10^{16}$ joules

The mass energy of 1 g of matter is

$$E = mc^2 = \left(1 \times 10^{-3}\,\text{kg}\right)\left(3 \times 10^8\right)^2 = 9 \times 10^{13}\,\text{J}$$

This is about the energy generated by a large power plant in one year. In nuclear power plants, it is the mass energy that is being converted to heat so that it can be utilized. If the lump of metal in this question were uranium 235, about 1 part in 1000 of the rest energy can be converted by nuclear fission. Therefore, 1 kg of $U^{235}$ can yield $9 \times 10^{13}$ joules.

## Question 5: Momentum

At what speed is the relativistic momentum of a particle twice as great as the result obtained from the non-relativistic expression?

- $2c$
- **0.87c***
- $0.75c$
- $0.55c$
- $0.50c$

For the relativistic momentum to be twice as great than the classical momentum, we must have

$$\begin{aligned} p_{rel} &= 2p_{clas} \\ \gamma m u &= 2mu \\ \gamma &= 2 \\ \sqrt{1 - V^2/c^2} &= 0.5 \end{aligned}$$

$$\begin{aligned} 1 - V^2/c^2 &= 0.25 \\ V^2/c^2 &= 0.75 \\ V &= 0.87c \end{aligned}$$

## Question 6: Collisions

A proton and neutron collide in an elastic collision. Before the collision, the neutron is stationary and the proton has momentum $\mathbf{p}_p = (-0.3,\ 0.1,\ -0.5)$ MeV/$c$ and the proton's momentum after the collision is $(0.2,\ -0.2,\ 0.3)$ MeV/$c$. What is the neutron's momentum after the collision?

- **$(-0.5,\ 0.3,\ -0.8)$ MeV/c***
- $(-0.1,\ -0.1,\ -0.2)$ MeV/$c$
- $(0,\ 0,\ 0)$ MeV/$c$
- $(0.5,\ -0.3,\ 0.8)$ MeV/$c$
- $(0.1,\ 0.1,\ 0.2)$ MeV/$c$

The collision is elastic so that the kinetic energy is conserved. This is not really relevant to solve this problem, but be sure to know what it means. Total energy and momentum is always conserved. This means that the total momentum before the collision should be equal to the total momentum after the collision. A stationary object has no speed, and therefore no momentum. We can write this as

$$\begin{aligned} \mathbf{p}_n^{before} + \mathbf{p}_p^{before} &= \mathbf{p}_n^{after} + \mathbf{p}_p^{after} \\ \mathbf{p}_n^{after} &= \mathbf{p}_n^{before} + \mathbf{p}_p^{before} - \mathbf{p}_p^{after} \\ &= (0,\ 0,\ 0) + (-0.3,\ 0.1,\ -0.5) - (0.2,\ -0.2,\ 0.3) \\ &= (-0.5,\ 0.3,\ -0.8)\ \text{MeV}/c \end{aligned}$$

## Question 7: Kinetic energy

What is the kinetic energy of an electron (mass $m_e = 0.511\,\text{MeV}/c^2$) that has a total relativistic energy of $2\,\text{MeV}$?

- 2.511 MeV
- 2 MeV
- 1.739 MeV
- **1.489 MeV***
- 0.511 MeV

The total energy is the sum of the mass energy and the kinetic energy:

$$\begin{aligned} E &= E_0 + E_K \\ E_K &= E - mc^2 \\ &= 2\,\text{MeV} - \left(0.511\,\text{MeV}/c^2\right)c^2 \\ &= 1.489\,\text{MeV} \end{aligned}$$

# Tutorial Letter 202/2/2016

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 02.

BAR CODE

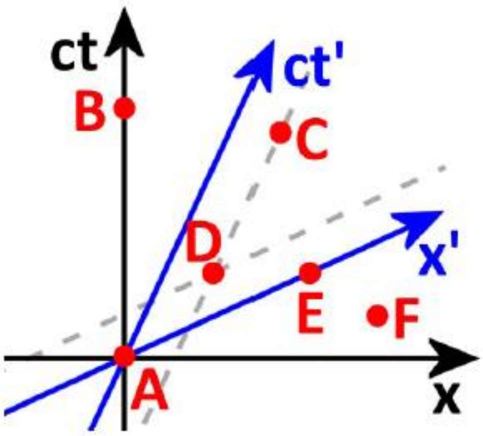
# Memo for Assignment 2 S2 2016

## Minkowski spacetime (§ 1.4)

## Physical laws in relativity (§ 2.1 - 2.2.3)

### Questions 1 - 3: Minkowski diagrams

Consider the spacetime diagram below to answer the following questions




1. In the $S'$ frame, the following two events occur at the same position

- A and B
- A and E
- A and D
- **D and C***
- D and E

The $S'$ frame is indicated by the blue coordinate axes. For two events to occur at the same position, they must occur at the same space coordinate. With the help of the dashed line that is parallel to the $x'$ axis, it is clear that events C and D have the same $x'$ coordinate.

2. To which point would observers in both the $S$ and $S'$ frame assign the same spacetime coordinates?

- **A***
- B
- C
- E
- None of the points

The only point to which observers in both $S$ and $S'$ will assign the same spacetime coordinates is the origin, event A where $(x, ct) = (x', ct') = (0, 0)$.

3. Which of the following statements are true *for all frames*?

- **Event A happens before event C***
- Events E and F are causally related
- Events A and B occur at the same position
- Events A and B occur at the same time
- Event C caused event F

Event A will happen at time $ct = ct' = 0$. It is not possible to draw a coordinate axes where Event C will be below the space axis. The spacetime diagram will have to look something like the figure below, which is not a valid Lorentz transformation. The axes of a Lorentz transformation will always be symmetric about the line $ct = x$ (shown as a dashed line in the figure).

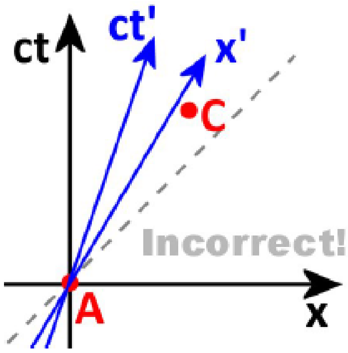
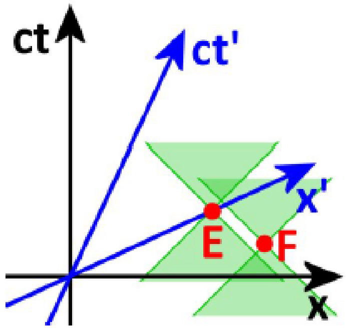



Figure 1: Not a correct Lorentz transformation!

Events E and F will be causally related if they can be related by a signal that travels slower than the speed of light, or equivalently, if the two events are in each others lightcones. From the figure below it is clear that they are not in each others lightcones (or, a signal connecting the two events will have to move faster than the speed of light), so one event cannot cause the other and they are not causally related.




Events A and B occur at the same position in $S$ ($x = 0$), but this is not true for all frames. In $S'$ (and all other inertial frames with a positive $V$ relative to $S$, A will occur at the origin and B will occur at some negative value of $x$.

Events A and B do not occur at the same time in $S$, so they do not occur at the same time in all frames, since you can give a counter example.

Events C and F are causally related, since they can be connected with a signal that moves slower than $c$, or are in each others light cones. But, event F will occur before C in all frames, so there is no way in which C could cause F, although it would be possible for F to cause C.

## Question 4: Mass energy

What is the mass energy of a litre of water? (One litre of water weighs 1 kg.)

- $3 \times 10^8$ joules
- $9 \times 10^8$ joules
- $3 \times 10^{16}$ joules
- **$9 \times 10^{16}$ joules***
- $9 \times 10^{19}$ joules

The mass energy of a 1 kg of water (or 1 kg of any material) is

$$E = mc^2 = (1\,\text{kg})\left(3 \times 10^8\right)^2 = 9 \times 10^{16}\text{ J}$$

This is about the energy used by South Africans in a month. In nuclear power plants, it is the mass energy that is being converted to heat so that it can be utilized. All of the mass energy cannot be used as in the question above. In the case of uranium 235, about 1 part in 1000 of the mass energy can be converted to energy by nuclear fission. Therefore, 1 kg of $U^{235}$ can yield $9 \times 10^{13}$ joules.

## Question 5: Momentum

At what speed is the magnitude of the relativistic momentum of a particle three times the magnitude of the nonrelativistic momentum?

- $3c$
- $1.05c$
- **0.94c***
- $0.87c$
- $0.33c$

For the relativistic momentum to be three times as great than the classical momentum, we must have

$$\begin{aligned} p_{rel} &= 3p_{clas} \\ \gamma mu &= 2mu \\ \gamma &= 3 \\ \sqrt{1 - V^2/c^2} &= \frac{1}{3} \\ 1 - V^2/c^2 &= \frac{1}{9} \\ V^2/c^2 &= \frac{8}{9} \\ V &= 0.94c \end{aligned}$$

## Question 6: Collisions

A proton and neutron collide in an elastic collision. Before the collision, the neutron is stationary and the proton has momentum $\mathbf{p}_p = (0.4, -0.2, 0.8)$ MeV/$c$ and the proton's momentum after the collision is $(-0.2, -0.5, 0.6)$ MeV/$c$. What is the neutron's momentum after the collision?

- $(0.2, 0.5, -0.6)$ MeV/$c$
- **(0.6, 0.3, 0.2) MeV/c***
- $(0.2, -0.7, 1.4)$ MeV/$c$
- $(0, 0, 0)$ MeV/$c$
- $(-0.6, 0.7, 0.2)$ MeV/$c$

The collision is elastic so that the kinetic energy is conserved. This is not really relevant to solve this problem, but be sure to know what it means. Total energy and momentum is always conserved. This means that the total momentum before the collision should be equal to the total momentum after the collision. A stationary object has no speed, and therefore no momentum. We can write this as

$$\mathbf{p}_n^{before} + \mathbf{p}_p^{before} = \mathbf{p}_n^{after} + \mathbf{p}_p^{after}$$

$$\begin{aligned}
\mathbf{p}_n^{after} &= \mathbf{p}_n^{before} + \mathbf{p}_p^{before} - \mathbf{p}_p^{after} \\
&= (0,\ 0,\ 0) + (0.4,\ -0.2,\ 0.8) - (-0.2,\ -0.5,\ 0.6) \\
&= (0.6,\ 0.3,\ 0.2)\ \mathrm{MeV}/c
\end{aligned}$$

## Question 7: Kinetic energy

A proton (mass $m_p = 938.3\,\mathrm{MeV}/c^2$) is moving with speed $0.4c$ along the $x$-axis relative to the laboratory frame. What is its kinetic energy?

- $7.6 \times 10^{18}\,\mathrm{MeV}$
- $1023\,\mathrm{MeV}$
- $273\,\mathrm{MeV}$
- $178.7\,\mathrm{MeV}$
- **85.39 MeV***

Take the laboratory frame to be $S$ and let the proton be stationary in the $S'$ frame. The Lorentz factor for the two frames is

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \\
&= \frac{1}{\sqrt{1 - 0.4^2}} \\
&= 1.091
\end{aligned}$$

The kinetic energy is then

$$\begin{aligned}
E_K &= (\gamma - 1)\, mc^2 \\
&= (1.091 - 1)\,(938.3\,\mathrm{MeV}) \\
&= 85.39\,\mathrm{MeV}
\end{aligned}$$

# Tutorial Letter 203/1/2016

## Special Relativity and Riemannian Geometry

# APM3713

## Semester 1

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 03.

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# Memo for Assignment 3 S1 2016

## Four vectors and tensors (§ 2.2.4 - 2.3.5 (excluding 2.3.1 - 2.3.4))

### Question 1: Tensor notation

Consider the following equation in Minkowski spacetime

$$A_\mu = \sum_\nu \eta_{\mu\nu} A^\nu$$

How many equations does this represent?

- 1
- 2
- **4***
- 6
- 16

Minkowski space time is four dimensional, so that we have $\mu = 0, 1, 2, 3$. Therefore, four equations are represented, one for each possible value of $\mu$. The other index, $\nu$ is a dummy index and is being summed over in each of the equations. Written out it full, the four equations are

$$\begin{aligned}
A_0 &= \sum_\nu \eta_{0\nu} A^\nu = \eta_{00} A^0 + \eta_{01} A^1 + \eta_{02} A^2 + \eta_{03} A^3 \\
A_1 &= \sum_\nu \eta_{1\nu} A^\nu = \eta_{10} A^0 + \eta_{11} A^1 + \eta_{12} A^2 + \eta_{13} A^3 \\
A_2 &= \sum_\nu \eta_{2\nu} A^\nu = \eta_{20} A^0 + \eta_{21} A^1 + \eta_{22} A^2 + \eta_{23} A^3 \\
A_3 &= \sum_\nu \eta_{3\nu} A^\nu = \eta_{30} A^0 + \eta_{31} A^1 + \eta_{32} A^2 + \eta_{33} A^3
\end{aligned}$$

Now you can see how much more condensed tensor notation is.

## Question 2: Tensors, vectors and scalars

The four velocity is given by

$$[U^\mu] = \left(U^0,\, U^1,\, U^2,\, U^3\right) = (c\gamma,\, \gamma\mathbf{v})\ .$$

Consider the following quantities from the equation:

a) $[U^\mu]$

b) $U^2$

c) $(U^0,\, U^1,\, U^2,\, U^3)$

d) $\gamma$

e) $(c\gamma,\, \gamma\mathbf{v})$

f) $\gamma\mathbf{v}$

Which of the following statements are true?

- a and b are tensors, f is a vector and d is a scalar
- a and c are tensors, b is a vector and d is a scalar
- a and e are tensors, d is a vector and f is a scalar
- **c and e are tensors, f is a vector and b is a scalar***
- c and f are tensors, e is a vector and b is a scalar

The differences between tensors, vectors and scalars are important. In the same way that you can't equate a matrix to a number, you can't equate a tensor to a scalar. From the given list, (a), (c) and (e) are all tensors. $[U^\mu]$ is a four-vector and therefore a tensor, and anything equal to it will also be a tensor. The components of $[U^\mu]$, namely $U^0$, $U^1$, $U^2$ and $U^3$, are all scalars, just like the components of a vector are scalars. The Lorentz factor (d) is of course a scalar, since it is just a number. The normal three-velocity $\mathbf{v}$ is a vector with 3 components, and a vector multiplied by a scalar as in (f) is a vector.

## Question 3: Energy-momentum relation

A muon is a subatomic particle with a rest mass of $m = 105.7\,\mathrm{MeV/c^2}$. Consider a muon with a momentum of $50\,\mathrm{MeV}/c$. What is its energy?

- 50 MeV
- **116.9 MeV***
- 13.7 GeV
- 105.7 MeV
- 12.5 MeV

Using the energy-momentum relation

$$\begin{aligned} E^2 &= \left(mc^2\right)^2 + (pc)^2 \\ E^2 &= (105.7\,\mathrm{MeV})^2 + (50\,\mathrm{MeV})^2 \\ E^2 &= 13\,672.5\,\mathrm{MeV}^2 \\ E &= 116.9\,\mathrm{MeV} \end{aligned}$$

## Question 4: Massless particles

A photon is measured to have an energy of 14 MeV. What is its momentum?

- **14 MeV/c***
- $0\,\mathrm{MeV}/c$
- $196\,\mathrm{MeV}/c$
- $42\,\mathrm{MeV}/c$
- $3.7\,\mathrm{MeV}/c$

The momentum and energy of any massless particle is related by $E = pc$. This is a direct consequence of putting $m = 0$ into the energy-momentum relation used in the previous question. A common mistake of students is to say that a massless particle has no momentum, citing the equation $p = \gamma mv$ as the reason. This equation does not work for massless particles. To see why, consider calculating the Lorentz factor for a massless particle. All massless particles move at the speed of light $c$, in all reference frames. So the Lorentz factor would be

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-\frac{c^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-1}} \\ &= \frac{1}{\sqrt{0}}\end{aligned}$$

As you can see, this leads to division by zero, which renders the equation $p = \gamma mv$ meaningless for massless particles.

The idea that massless particles have momentum without having mass might seem odd, but this is very real and measurable. One application is solar sails, which is a cost effective form of propulsion where photons form the Sun is used to move spacecraft. Basically, photons from the Sun collide with the sails and are reflected in the opposite direction. The photons transfer some of their momentum to the sails (conservation of momentum) and cause the spacecraft to move.

Getting back to the question at hand, the actual momentum of the photon is simply

$$\begin{aligned}p &= E/c \\ &= 14\,\mathrm{MeV}/c\end{aligned}$$

## Question 5: Transformation of momentum

An electron (mass $m_e = 0.511\,\mathrm{MeV/c^2}$) is moving along the $x$-axis of an inertial reference frame $S$ with speed $v = 0.8c$, momentum $0.682\,\mathrm{MeV/c}$ and total energy $0.852\,\mathrm{MeV}$. What is its momentum as measured in an inertial frame $S'$ that is moving in the standard configuration with speed $0.6c$ relative to $S$?

- $0.682\,\mathrm{MeV}/c$
- $0.593\,\mathrm{MeV}/c$
- $0.285\,\mathrm{MeV}/c$
- **0.214 MeV/c***
- $0.046\,\mathrm{MeV}/c$

In this scenario, you have two observers in $S$ and $S'$ respectively observing a particle that is moving in both frames. Since $S$ and $S'$ is moving with respect to each other, they will measure different speeds (and therefore momenta and energies) for the particle. The observer in $S$ measures the speed of the particle as $v = 0.8c$. The speed of the particle $v'$ as measured in $S'$ is related to the speed $v$ measured in $S$ by the velocity transformation equation (1.43 in the textbook).

When using equations like $p = \gamma(v)\,mv$ and $E = \gamma(v)\,mc^2$, the Lorentz factors depend on the relative speed between the observer's frame and the rest frame of the particle. That is why you use the speed of the particle $v$ as measured by the observer to determine $\gamma$. In this case, the energy and momentum as measured by the observer in $S'$would be $p' = \gamma(v')\,mv'$ and $E' = \gamma(v')\,mc^2$. You can solve this problem by getting $v'$ from the velocity transformation equation and then using it in $p' = \gamma(v')\,mv'$, but there is a much simpler way. The momentum in the $x$-direction is a component of the four-momentum. The transformation for the components of the four-momentum is given in equations 2.34 - 2.37 in the textbook.

We know what one observer measures for the momentum in his frame ($S$), and want to transform this quantity to the other observer's frame of reference ($S'$). To to this, we need the Lorentz factor that relates $S$ and $S'$, we therefore need to calculate the Lorentz factor using the relative speed between $S$ and $S'$, $V = 0.6c$.

The Lorentz factor relating the two frames is

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\\ &= \frac{1}{\sqrt{1-0.6^2}}\\ &= \frac{5}{4}\end{aligned}$$

Since the electron is moving in the $x$-direction, $p_x$ is the only non-zero component of the momentum vector. The appropriate transformation equation is

$$\begin{aligned} p'_x &= \gamma\left(p_x - VE/c^2\right) \\ &= \frac{5}{4}\left[0.682\,\mathrm{MeV}/c - (0.6c)\,(0.852\,\mathrm{MeV})\,/c^2\right] \\ &= \frac{5}{4}\left[0.682\,\mathrm{MeV}/c - 0.511\,\mathrm{MeV}/c^2\right] \\ &= 0.214\,\mathrm{MeV}/c \end{aligned}$$

## Question 6: Four-momentum

A proton (mass $m_p = 938.3\,\mathrm{MeV}/c^2$) is moving with speed $0.4c$ along the $x$-axis relative to the laboratory frame. What is the value of the first component of the four-momentum $P^0$ for the proton?

- $1210\,\mathrm{MeV}/c$
- $84.45\,\mathrm{MeV}/c$
- $409.1\,\mathrm{MeV}/c$
- $1117\,\mathrm{MeV}/c$
- **1023 MeV/c***

The Lorentz factor relating the laboratory frame and the rest frame of the proton is

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-0.4^2}} \\ &= 1.09 \end{aligned}$$

The value of $P^0$ is given by $E/c$, where $E$ is the total energy of the proton. For the total energy, we get

$$\begin{aligned} E &= \gamma mc^2 \\ &= (1.09)\left(938.3\,\mathrm{MeV}/c^2\right)c^2 \\ &= 1023\,\mathrm{MeV} \end{aligned}$$

Therefore, we have $P^0 = 1023\,\mathrm{MeV/c}$.

## Question 7: Transformation of tensors

Using equation (2.110) in the textbook, how would a covariant tensor of rank 1 $A_\nu$ transform in general?

- $A'_\mu = \sum_{\nu=0}^{3} \frac{\partial x'^\nu}{\partial x^\mu} A_\mu$
- $\mathbf{A}'_\mu = \sum_{\nu=0}^{3} \frac{\partial \mathbf{x}^\nu}{\partial \mathbf{x}'^\mu} \mathbf{A}_\nu$ *
- $A'_\mu = \sum_{\nu=0}^{3} \frac{\partial x'^\mu}{\partial x^\nu} A^\nu$
- $A'_\mu = \sum_{\nu=0}^{3} \frac{\partial x^\mu}{\partial x'^\nu} A_\nu$
- $A'_\mu = \sum_{\nu=0}^{3} \frac{\partial x^\nu}{\partial x'^\mu} A_\mu$

# Tutorial Letter 203/2/2016

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 03.

BAR CODE

# Memo for Assignment 3 S2 2016

## Four vectors and tensors (§ 2.2.4 - 2.3.5 (excluding 2.3.1 - 2.3.4))

### Question 1: Tensor notation

Consider the following equation in Minkowski spacetime

$$\sum_{\mu} \frac{\partial G^{\mu\nu}}{\partial x^{\nu}} = 0$$

How many equations does this represent?

- 1
- 2
- **4***
- 6
- 16

Minkowski space time is four dimensional, so that we have $\mu = 0, 1, 2, 3$. Therefore, four equations are represented, one for each possible value of $\mu$. The other index, $\nu$, is a dummy index and is being summed over in each of the equations. Written out it full, the four equations are

$$\begin{aligned}
\sum_{\mu} \frac{\partial G^{0\nu}}{\partial x^{\nu}} = \frac{\partial G^{00}}{\partial x^{0}} + \frac{\partial G^{01}}{\partial x^{1}} + \frac{\partial G^{02}}{\partial x^{2}} + \frac{\partial G^{03}}{\partial x^{3}} &= 0 \\
\sum_{\mu} \frac{\partial G^{1\nu}}{\partial x^{\nu}} = \frac{\partial G^{10}}{\partial x^{0}} + \frac{\partial G^{11}}{\partial x^{1}} + \frac{\partial G^{12}}{\partial x^{2}} + \frac{\partial G^{13}}{\partial x^{3}} &= 0 \\
\sum_{\mu} \frac{\partial G^{2\nu}}{\partial x^{\nu}} = \frac{\partial G^{20}}{\partial x^{0}} + \frac{\partial G^{21}}{\partial x^{1}} + \frac{\partial G^{22}}{\partial x^{2}} + \frac{\partial G^{23}}{\partial x^{3}} &= 0 \\
\sum_{\mu} \frac{\partial G^{3\nu}}{\partial x^{\nu}} = \frac{\partial G^{30}}{\partial x^{0}} + \frac{\partial G^{31}}{\partial x^{1}} + \frac{\partial G^{32}}{\partial x^{2}} + \frac{\partial G^{33}}{\partial x^{3}} &= 0
\end{aligned}$$

Now you can see how much more condensed tensor notation is.

## Question 2: Tensors, vectors and scalars

The four momentum is given by

$$[P^\mu] = \left(P^0,\, P^1,\, P^2,\, P^3\right) = (E/c,\, \mathbf{p})\ .$$

Consider the following quantities from the equation:

a) $[P^\mu]$

b) $P^3$

c) $(P^0,\, P^1,\, P^2,\, P^3)$

d) $E$

e) $(E/c,\, \mathbf{p})$

f) $\mathbf{p}$

Which of the following statements are true?

- a and c are tensors, b is a vector and d is a scalar
- **c and e are tensors, f is a vector and b is a scalar***
- a and b are tensors, f is a vector and d is a scalar
- a and e are tensors, d is a vector and f is a scalar
- c and f are tensors, e is a vector and b is a scalar

The differences between tensors, vectors and scalars are important. In the same way that you can't equate a matrix to a number, you can't equate a tensor to a scalar. From the given list, (a), (c) and (e) are all tensors. $[P^\mu]$ is a four-vector and therefore a tensor, and anything equal to it will also be a tensor. The components of $[P^\mu]$, namely $P^0$, $P^1$, $P^2$ and $P^3$, are all scalars, just like the components of a vector are scalars. The energy (d) is a scalar. The normal three-momentum $\mathbf{p}$ is a vector with 3 components.

## Question 3: Energy-momentum relation

An electron (mass $m_e = 0.511\,\mathrm{MeV/c^2}$) is has an energy of $0.850\,\mathrm{MeV}$. What is its momentum?

- $0.850\,\mathrm{MeV}/c$
- $0.461\,\mathrm{MeV}/c$
- $0.339\,\mathrm{MeV}/c$
- $0.582\,\mathrm{MeV}/c$
- **0.679 MeV/c***

Using the energy-momentum relation

$$\begin{aligned}
E^2 &= \left(mc^2\right)^2 + (pc)^2 \\
(0.850\,\mathrm{MeV})^2 &= (0.511\,\mathrm{MeV})^2 + (pc)^2 \\
(pc)^2 &= (0.850\,\mathrm{MeV})^2 - (0.511\,\mathrm{MeV})^2 \\
(pc)^2 &= 0.461\,\mathrm{MeV}^2 \\
p &= 0.679\,\mathrm{MeV}/c
\end{aligned}$$

## Question 4: Massless particles

A photon is measured to have an energy of $9\,\mathrm{MeV}$. What is its momentum?

- **9 MeV/c***
- $0\,\mathrm{MeV}/c$
- $81\,\mathrm{MeV}/c$
- $42\,\mathrm{MeV}/c$
- $3\,\mathrm{MeV}/c$

The momentum and energy of any massless particle is related by $E = pc$. This is a direct consequence of putting $m = 0$ into the energy-momentum relation used in the previous question. A commom mistake of students is to say that a massless particle has no momentum, citing the equation $p = \gamma mv$ as the reason. This equation does not work for massless particles. To see why, consider calculating the Lorentz factor for a massless particle. All massless paticles move at the speed of light $c$, in all reference frames. So the Lorentz factor would be

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-\frac{c^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-1}} \\ &= \frac{1}{\sqrt{0}}\end{aligned}$$

As you can see, this leads to division by zero, which renders the equation $p = \gamma mv$ meaningless for massless particles.

The idea that massless particles have momentum without having mass might seem odd, but this is very real and measurable. One application is solar sails, which is a cost effective form of propulsion where photons form the Sun is used to move spacecraft. Basically, photons from the Sun collide with the sails and are reflected back. The photons transfer some of their momentum to the sails (conservation of momentum) and cause the spacecraft to move.

Getting back to the question at hand, the actual momentum of the photon is simply

$$\begin{aligned}p &= E/c \\ &= 9\,\mathrm{MeV}/c\end{aligned}$$

## Question 5: Transformation of energy

An electron (mass $m_e = 0.511\,\mathrm{MeV/c^2}$) is moving along the $x$-axis of an inertial reference frame $S$ with speed $v = 0.8c$, momentum $0.682\,\mathrm{MeV/c}$ and total energy $0.852\,\mathrm{MeV}$. What is its total energy in an inertial frame $S'$ that is moving in the standard configuration with speed $0.6c$ relative to $S$?

- 0.852 MeV
- 0.738 MeV
- **0.554 MeV***
- 0.511 MeV
- 0.443 MeV

In this scenario, you have two observers in $S$ and $S'$ respectively observing a particle that is moving in both frames. Since $S$ and $S'$ is moving with respect to each other, they will measure different speeds (and therefore momenta and energies) for the particle. The observer in $S$ measures the speed of the particle as $v = 0.8c$. The speed of the particle $v'$ as measured in $S'$ is related to the speed $v$ measured in $S$ by the velocity transformation equation (1.43 in the textbook).

When using equations like $p = \gamma(v)\,mv$ and $E = \gamma(v)\,mc^2$, the Lorentz factors depend on the relative speed between the observer's frame and the rest frame of the particle. That is why you use the speed of the particle $v$ as measured by the observer to determine $\gamma$. In this case, the energy and momentum as measured by the observer in $S'$would be $p' = \gamma(v')\,mv'$ and $E' = \gamma(v')\,mc^2$. You can solve this problem by getting $v'$ from the velocity transformation equation and then using it in $E' = \gamma(v')\,mc^2$, but there is a much simpler way. The energy (multiplied by a constant) is a component of the four-momentum. The transformation for the components of the four-momentum is given in equations 2.34 - 2.37 in the textbook.

We know what one observer measures for the energy in his frame ($S$), and want to transform this quantity to the other observer's frame of reference ($S'$). To to this, we need the Lorentz factor that relates $S$ and $S'$, we therefore need to calculate the Lorentz factor using the relative speed between $S$ and $S'$, $V = 0.6c$.

The Lorentz factor for the two frames is

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\\ &= \frac{1}{\sqrt{1-0.6^2}}\\ &= \frac{5}{4}\end{aligned}$$

Using the transformation equation for the total energy in $S'$ gives

$$\begin{aligned} E' &= \gamma\,(E - Vp_x) \\ &= \frac{5}{4}\,(0.852\,\mathrm{MeV} - (0.6c)\,(0.682\,\mathrm{MeV/c})) \\ &= \frac{5}{4}\,(0.852\,\mathrm{MeV} - 0.409\,\mathrm{MeV}) \\ &= 0.554\,\mathrm{MeV} \end{aligned}$$

## Question 6: Four-momentum

A photon with measured momentum $0.210\,\mathrm{MeV}/c$ is moving along the $y$-axis relative to the laboratory frame. What is the value of the its four-momentum $[P^\mu]$ in $\mathrm{MeV}/c$?

- $(0.21,\,0.21,\,0,\,0)$ MeV/$c$
- $(0,\,0,\,0.21,\,0)$ MeV/$c$
- $(0,\,0.21,\,0,\,0)$ MeV/$c$
- **(0.21, 0, 0.21, 0) MeV/c***
- $(0.21,\,0.21,\,0.21,\,0.21)$ MeV/$c$

The four-momentum is given by

$$[P^\mu] = (E/c,\,\mathbf{p}) = (E/c,\,p_x,\,p_y,\,p_z)\;.$$

From the question we know that $p_x = p_z = 0\,\mathrm{MeV}/c$ and $p_y = 0.21\,\mathrm{MeV}/c$. It remains to calculate the energy of the photon. Since photons are massless, we cannot use the equation $E = \gamma mc^2$. We therefore use the energy-momentum relation with $m = 0$ to get

$$\begin{aligned} E^2 &= \left(mc^2\right)^2 + (pc)^2 \\ E &= pc \\ E &= 0.21\,\mathrm{MeV} \end{aligned}$$

The four-momentum is then given by

$$[P^\mu] = (0.21,\,0,\,0.21,\,0)\ \mathrm{MeV}/c$$

## Question 7: Transformation of tensors

Using equation (2.110) in the textbook, how would a contravariant tensor of rank 1 $A^{\nu}$ transform in general?

- $A'^{\mu} = \sum_{\nu=0}^{3} \frac{\partial x'^{\nu}}{\partial x^{\mu}} A^{\mu}$
- $A'^{\mu} = \sum_{\nu=0}^{3} \frac{\partial x^{\mu}}{\partial x'^{\nu}} A_{\nu}$
- $A'^{\mu} = \sum_{\nu=0}^{3} \frac{\partial x^{\mu}}{\partial x'^{\nu}} A^{\nu}$
- $\mathbf{A}'^{\mu} = \sum_{\nu=0}^{\mathbf{3}} \frac{\partial \mathbf{x}'^{\mu}}{\partial \mathbf{x}^{\nu}} \mathbf{A}^{\nu} *$
- $A'^{\mu} = \sum_{\nu=0}^{3} \frac{\partial x^{\nu}}{\partial x'^{\mu}} A^{\mu}$

# Tutorial Letter 204/1/2016

## Special Relativity and Riemannian Geometry

# APM3713

## Semester 1

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 04.

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# Memo for Assignment 4 S1 2016

# Chapters 1 & 2

## Question 1

Two astronauts, Alice and Bob, leave the Earth and travel to a distant planet 12 lightyears away, as measured from Earth. Assume that the planet and Earth are at rest with respect to each other. The astronauts depart at the same time on different spaceships. Alice travels at a speed of $0.9c$, and Bob travels at $0.5c$. (Hint: A lightyear is the distance traveled by light in one year, which is just $c$ multiplied by one year, or $9.46 \times 10^{12}$ km. In many problems it is simpler to write it as $1c \cdot \text{year}$, since $c$ often cancels out.)

(a) What is the distance of the journey according to Alice?

(b) What is the duration of Alice's journey according to the people on Earth?

(c) What is the duration of the journey according to Alice?

(d) What is the speed of Alice's spaceship, as measured by Bob?

### Solution

#### Part A

The Lorentz factor between the frame where the planets are stationary ($S$) and Alice's frame ($S'$) is

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \\ &= \frac{1}{\sqrt{1-(0.9)^2}} \\ &= \frac{1}{\sqrt{1-0.81}} \\ &= 2.29\end{aligned}$$

The proper distance of the journey is the one measured by the people on Earth, since they are stationary with respect to the beginning and end points of the distance. We call this distance $\Delta x = 12\,c \cdot \text{year}$. To get the distance as measured in $S'$, $\Delta x'$, we use the length contraction formula

$$\begin{aligned}\Delta x' &= \frac{\Delta x}{\gamma} \\ &= \frac{12\,c \cdot \text{years}}{2.29} \\ &= 5.24\,c \cdot \text{years}\end{aligned}$$

You can call these quantities $L$ and $L_P$ if that makes more sense to you. But using primed and unprimed symbols ($\Delta x$, $\Delta x'$,$\Delta t$ and$\Delta t'$) helps you keep track of who measures what in a natural way.

**Part B**

We want to know the duration of the journey $\Delta t$ as measured in $S$. We already know the distance measured in that frame ($\Delta x = 12\,c \cdot \text{years}$). Since $\Delta x$ and $\Delta t$ is measured *in the same frame* $S$, we can use

$$\begin{aligned}\Delta t &= \frac{\Delta x}{V} \\ &= \frac{12\,c \cdot \text{years}}{0.9c} \\ &= 13.3\,c \cdot \text{years}\end{aligned}$$

This equation is only valid if all the quantities is measured in the same reference frame. This is not the proper time, but the diluted time. Any observer measures his own proper time, so the proper time will be $\Delta\tau = \Delta t'$ which we will calculate in the next question. So for questions like these, the proper length is measured in one frame, and the proper time in the other.

### Part C

At this point, we have enough information to calculate the duration according to Alice in two ways. Since we already know the distance of the journey in $S'$ (Part A), we can use

$$\begin{aligned} \Delta t' &= \frac{\Delta x'}{V} \\ &= \frac{5.24\, c \cdot \text{years}}{0.9c} \\ &= 5.8\,\text{years} \end{aligned}$$

We can also use the duration as measured in $S$ (Part B) and use the time dilation formula

$$\begin{aligned} \Delta \tau &= \frac{\Delta T}{\gamma} \\ &= \frac{13.3\, c \cdot \text{years}}{2.29} \\ &= 5.8\,\text{years} \end{aligned}$$

### Part D

The two spaceships are moving in the same direction, so we use the velocity transformation formula for the $x$-direction. Bob's velocity as measured in $S$ is $0.5c$ so that $V = 0.5c$. Alice's speed as measured in $S$ is $v_A = 0.9c$, so in $S''$ (Bob's frame) it is

$$\begin{aligned} v''_A &= \frac{v_A - V}{1 - v_A V/c^2} \\ &= \frac{0.9c - 0.5c}{1 - (0.9c)\,(0.5c)\,/c^2} \\ &= \frac{0.4c}{1 - 0.45} \\ &= 0.73c \end{aligned}$$

## Question 2

The space and time coordinates of two events as measured in an inertial frame $S$ are as follows:

| | | | |
|---|---|---|---|
| Event A | $x_A = a$ | $t_A = a/(2c)$ | $y_A = z_A = 0$ |
| Event B | $x_B = 2a$ | $t_B = 5a/(6c)$ | $y_B = z_B = 0$ |

where $a$ is some constant. There is an inertial frame $S'$, in the standard configuration with $S$, in which these two events are simultaneous.

(a) Draw a rough Minkowski diagram to indicate these two events in both the $S$ frame. Also indicate the $x'$ and $ct'$ axes on your diagram.

(b) Use the Lorentz transformation equations to find the speed of the $S'$ frame relative to $S$.

(c) At what time do these events occur in the $S'$ frame?

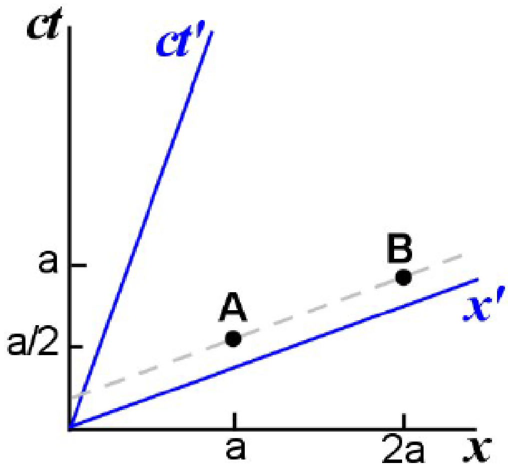
(d) Calculate the spacetime separation between these two events.

(e) Is the spacetime separation between the two events time-like, space-like or light-like?

(f) Is there a frame where Event A caused Event B?

## Solution

### Part A




If the two events are simultaneous in $S'$, it means that they will have the same $t'$ coordinate. Similar to a frame where the coordinate axes are represented by perpendicular axes, all equal values of $t'$ will lie on a line parallel to the $x'$axis.

**Part B**

Since the two events occur simultaneously in $S'$, we have

$$\begin{aligned}
t'_A &= t'_B \\
\gamma\left(t_A - \frac{vx_A}{c^2}\right) &= \gamma\left(t_B - \frac{vx_B}{c^2}\right) \\
\frac{a}{2c} - \frac{va}{c^2} &= \frac{5a}{6c} - \frac{2va}{c^2} \\
\frac{c}{2} - v &= \frac{5c}{6} - 2v \\
v &= \frac{5c}{6} - \frac{c}{2} \\
&= \frac{c}{3}
\end{aligned}$$

**Part C**

For the Lorentz factor we have

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\left(\frac{v}{c}\right)^2}} \\
&= \frac{1}{\sqrt{1-\frac{1}{9}}} \\
&= \frac{1}{\sqrt{\frac{8}{9}}} \\
&= \frac{3}{2\sqrt{2}} \\
&= 1.06
\end{aligned}$$

We can calculate either $t'_A$ or $t'_B$:

$$\begin{aligned}
t'_A &= \gamma\left(t_A - \frac{vx_A}{c^2}\right) \\
&= \frac{3}{2\sqrt{2}}\left(\frac{a}{2c} - \frac{ca}{3c^2}\right) \\
&= \frac{3}{2\sqrt{2}}\left(\frac{a}{2c} - \frac{a}{3c}\right) \\
&= \frac{3}{2\sqrt{2}}\left(\frac{3a}{6c} - \frac{2a}{6c}\right) \\
&= \frac{a}{4c\sqrt{2}}
\end{aligned}$$

OR

$$\begin{aligned}
t'_B &= \gamma\left(t_B - \frac{vx_B}{c^2}\right)\\
&= \frac{3}{2\sqrt{2}}\left(\frac{5a}{6c} - \frac{2ac}{3c^2}\right)\\
&= \frac{3}{2\sqrt{2}}\left(\frac{5a}{6c} - \frac{4a}{6c}\right)\\
&= \frac{a}{4c\sqrt{2}}
\end{aligned}$$

**Part D**

The spacetime separation is given by

$$\begin{aligned}
(\Delta s)^2 &= (c\Delta t)^2 - (\Delta x)^2\\
&= (ct_B - ct_A)^2 - (x_B - x_A)^2\\
&= \left(\frac{5a}{6} - \frac{a}{2}\right)^2 - (2a - a)^2\\
&= \left(\frac{2a}{6}\right)^2 - a^2\\
&= \frac{a^2}{9} - a^2\\
&= -\frac{8a^2}{9}
\end{aligned}$$

**Part E**

Space-like (since$(\Delta s)^2 < 0$)

**Part F**

No. Events that have space-like spacetime separations are not causally related. This means that any particle (light signal, information, anything) that is a result of the first event would have had to travel faster than the speed of light to be at the later event and to cause it in some way. This is not possible according to the postulates of special relativity, so it is impossible for the two events to have caused each other in any frame.

# Question 3

Using the Lorentz transformation equations for intervals, show that the spacetime separation is invariant. That is, show that $(\Delta s)^2 = (\Delta s')^2$.

## Solution

The Lorentz transformations are

$$\begin{aligned}
\Delta t' &= \gamma\left(\Delta t - \left(V/c^2\right)\Delta x\right)\\
\Delta x' &= \gamma\left(\Delta x - V\Delta t\right)\\
\Delta y' &= \Delta y\\
\Delta z' &= \Delta z
\end{aligned}$$

The spacetime separation in some frame $S'$ is given by

$$(\Delta s')^2 = (c\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2$$

Substituting the Lorentz transformations gives

$$\begin{aligned}
(\Delta s')^2 &= \left(c\gamma\left(\Delta t - \left(V/c^2\right)\Delta x\right)\right)^2 - (\gamma(\Delta x - V\Delta t))^2 - (\Delta y)^2 - (\Delta z)^2\\
&= c^2\gamma^2\left(\Delta t^2 - 2\left(V/c^2\right)\Delta x\Delta t + \left(V/c^2\right)^2\Delta x^2\right) - \gamma^2\left(\Delta x^2 - 2V\Delta x\Delta t + V^2\Delta t^2\right) - (\Delta y)^2 - (\Delta z)^2\\
&= \left(\gamma^2 - \frac{\gamma^2V^2}{c^2}\right)(c\Delta t)^2 - \left(\gamma^2 - c^2\gamma^2\left(V/c^2\right)^2\right)\Delta x^2 - (\Delta y)^2 - (\Delta z)^2\\
&= \left(\gamma^2 - \frac{\gamma^2V^2}{c^2}\right)(c\Delta t)^2 - \left(\gamma^2 - \frac{\gamma^2V^2}{c^2}\right)\Delta x^2 - (\Delta y)^2 - (\Delta z)^2
\end{aligned}$$

Now

$$\begin{aligned}
\gamma^2 - \frac{\gamma^2V^2}{c^2} &= \gamma^2\left(1 - \frac{V^2}{c^2}\right)\\
&= \frac{1}{1-\frac{V^2}{c^2}}\left(1 - \frac{V^2}{c^2}\right)\\
&= 1
\end{aligned}$$

so that

$$\begin{aligned}(\Delta s')^2 &= (c\Delta t)^2 - \Delta x^2 - (\Delta y)^2 - (\Delta z)^2 \\ &= (\Delta s)^2\end{aligned}$$

# Question 4

A positron of mass $m_e = 0.511\,\text{MeV}/c^2$ moves in the $x$-direction at speed $v_1 = 0.6c$. It collides head-on with an electron (also of mass $m_e$ ) whose speed is $v_2 = -0.4c$. The two particles combine to form a larger particle called positronium.

(a) Use the law of conservation of momentum to determine the momentum of the positronium particle.

(b) Use the law of conservation of energy to determine the total relativistic energy of the positronium particle.

(c) What is the contravariant four-momentum $[P^{\mu}]$ of the electron?

(d) Assuming Minkowski spacetime, determine the covariant counterpart of the four-momentum $[P_{\mu}]$ of the electron.

(e) Transform the four-momentum of the electron that you wrote down in Part (c) to the rest frame of the positron.

## Solution

### Part A

First we calculate the momenta of the two initial particles. For the positron we find

$$\gamma_1 = \frac{1}{\sqrt{1 - v_1^2/c^2}} = \frac{1}{\sqrt{1 - (0.6c)^2/c^2}} = 1.25$$

$$\begin{aligned}p_1 &= \gamma_1 m_e v \\ &= (1.25)\left(0.511\,\text{MeV}/c^2\right)(0.6c) \\ &= 0.383\,\text{MeV}/c\end{aligned}$$

And for the electron

$$\gamma_2 = \frac{1}{\sqrt{1 - v_1^2/c^2}} = \frac{1}{\sqrt{1 - (-0.4c)^2/c^2}} = 1.09$$

$$\begin{aligned} p_2 &= \gamma_2 m_e v \\ &= (1.09)\left(0.511\,\mathrm{MeV}/c^2\right)(-0.4c) \\ &= -0.223\,\mathrm{MeV}/c \end{aligned}$$

Let the momentum of the positronium particle be $p_3$. By conservation of momentum, we have

$$\begin{aligned} p_3 &= p_1 + p_2 \\ &= 0.383\,\mathrm{MeV}/c - 0.223\,\mathrm{MeV}/c \\ &= 0.160\,\mathrm{MeV}/c \end{aligned}$$

**Part B**

The energies of the positron particle is

$$\begin{aligned} E_1 &= \gamma_1 m_e c^2 \\ &= (1.25)\left(0.511\,\mathrm{MeV}/c^2\right)c^2 \\ &= 0.639\,\mathrm{MeV} \end{aligned}$$

and for the electron we have

$$\begin{aligned} E_2 &= \gamma_2 m_e c^2 \\ &= (1.09)\left(0.511\,\mathrm{MeV}/c^2\right)c^2 \\ &= 0.557\,\mathrm{MeV} \end{aligned}$$

Call the energy of the positronium $E_3$. By conservation of energy we find

$$\begin{aligned} E_3 &= E_1 + E_2 \\ &= 0.639\,\mathrm{MeV} + 0.557\,\mathrm{MeV} \\ &= 1.196\,\mathrm{MeV} \end{aligned}$$

**Part C**

The four-momentum is given by

$$
\begin{aligned}
[P^\mu] &= (E/c,\, \mathbf{p}) \\
&= (E/c,\, p_x,\, p_y\, p_z) \\
&= (0.557,\, 0.383,\, 0,\, 0)\ \text{MeV}/c
\end{aligned}
$$

**Part D**

We can determine the covariant counterpart of the four-momentum $[P_\mu]$ by lowering the index of $[P^\mu]$.

We use

$$P_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} P^\mu$$

to get

$$P_\mu = \eta_{\mu 0} P^\mu + \eta_{\mu 1} P^\mu + \eta_{\mu 2} P^\mu + \eta_{\mu 3} P^\mu$$

We use the fact that $\eta_{\mu\nu} = 0$ if $\mu \neq \nu$ and $\eta_{00} = 1$, $\eta_{11} = \eta_{22} = \eta_{33} = -1$ to get

$$
\begin{aligned}
P_0 &= \eta_{00} P^0 = P^0 = 0.557\,\text{MeV}/c \\
P_1 &= \eta_{11} P^1 = -P^1 = -0.383\,\text{MeV}/c \\
P_2 &= \eta_{22} P^2 = -P^2 = 0 \\
P_3 &= \eta_{33} P^3 = -P^3 = 0
\end{aligned}
$$

So that we have

$$[P_\mu] = (0.557,\, -0.383,\, 0,\, 0)\ \text{MeV}/c$$

**Part E**

In this question we transform the four-momentum of the electron from the laboratory frame (the one we've been working in so implicitly up to now, let's call it $S$) to the rest frame of the positron, which we call $S'$. We already know the relative speed between the frames is $v_1 = V = -0.4c$. And the Lorentz factor relating the $S$ and $S'$, which we calculated in Part A as $\gamma = 1.25$.

The four-momentum is a contravariant four-vector, and therefore its transformation equations are given by 2.56 - 2.59 in the textbook (you can also refer to equations 2.34 - 2.37). We get

$$\begin{aligned} P'^0 &= \gamma\left(P^0 - VP^1/c\right) \\ &= \gamma\left(E/c - Vp_x/c\right) \\ &= 1.25\left[0.557\,\text{MeV}/c - (-0.4c)\left(-0.223\,\text{MeV}/c\right)/c\right] \\ &= 0.585\,\text{MeV}/c \end{aligned}$$

$$\begin{aligned} P'^1 &= \gamma\left(P^1 - VP^0/c\right) \\ &= \gamma\left(p_x - VE/c^2\right) \\ &= 1.25\left[-0.223\,\text{MeV}/c - (-0.4c)\left(0.557\,\text{MeV}\right)/c^2\right] \\ &= -2.5\times 10^{-4}\text{MeV}/c \end{aligned}$$

$$\begin{aligned} P'^2 &= P^2 = 0 \\ P'^3 &= P^3 = 0 \end{aligned}$$

So that we have $[P'^\mu] = (0.585,\, -2.5\times 10^{-4},\, 0,\, 0)\,\text{MeV}/c$.

# Question 5

Use the energy-momentum relation ($E^2 = c^2p^2 + m^2c^4$) to show that the mass of a particle can be expressed as

$$m = \frac{c^2p^2 - E_k^2}{2E_kc^2}$$

where $E_k$ is the kinetic energy of the particle.

## Solution

The total energy is equal to the kinetic energy plus the mass energy

$$E = E_k + mc^2$$

Squaring both sides gives

$$E^2 = E_k^2 + 2E_k mc^2 + m^2c^4$$

Substituting the energy-momentum relation we get

$$c^2p^2 = E_k^2 + 2E_k mc^2$$

Solving for $m$ gives

$$m = \frac{c^2p^2 - E_k^2}{2E_k c^2}$$

as required.

# Tutorial Letter 204/2/2016

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 04.

BAR CODE

# Memo for Assignment 4 S2 2016

# Chapters 1 & 2

## Question 1

In the future, NASA launches its first manned mission to Pluto. The distance between the Earth and Pluto is about 4 lighthours. This means that it takes a photon (travelling at speed $c$) 4 hours to travel from the Earth to Pluto. The spaceship departs from Earth and travels at a constant speed of $0.4c$ to Pluto. Assume that Earth and Pluto is stationary with respect to each other.

(Hint: A lightyear is the distance traveled by light in one year, which is just $c$ multiplied by one year, or $9.46 \times 10^{12}$ km. In many problems it is simpler to write it as $1\,c \cdot$ year, since $c$ often cancels out. In this case, you can write the distance between Earth and Pluto as $4\,c \cdot$ hour.)

(a) How long does the journey to Pluto take according to the people on Earth?

(b) How long does the journey take according to the astronauts on the spaceship?

(c) According to the astronauts in the spaceship, what will the distance of their journey be?

(d) During the journey, the astronauts communicate with the control centre on Earth by sending signals at a frequency (measured in the spaceship rest frame) of 300 MHz. To what frequency must Earth receivers be tuned to receive these signals?

## Solution

### Part A

Let's call the rest frame of the Earth $S$ and the rest frame of the spaceship $S'$. The distance between Earth and Pluto given in the question measured in $S$, so that $\Delta x = 4\,c \cdot$ hour.

This is also the proper length, since it was measured in the frame where the beginning and end points are stationary. We want to know the time that the people in $S$ measure for the duration of the journey $\Delta t$. Since $\Delta x$ and $\Delta t$ is measured *in the same frame* $S$, we can use

$$\begin{aligned}\Delta t &= \frac{\Delta x}{V} \\ &= \frac{4\,c\cdot\text{hour}}{0.4c} \\ &= 10\,\text{hours}\end{aligned}$$

So as measured in $S$, it takes the spaceship10 hours to reach Pluto.

**Part B**

Now we want to know the duration of the journey as measured by the astronauts in $S'$, namely$\Delta t'$. The astronauts will measure the proper time of the spaceship's travels, since any observer always measures his own proper time. First, we calculate the Lorentz factor relating the two frames.

$$\gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} = \frac{1}{\sqrt{1-0.4^2}} = \frac{1}{\sqrt{0.84}} = 1.09\,.$$

We use the time dilation formula and the answer from the previous question:

$$\begin{aligned}\Delta t' &= \frac{\Delta t}{\gamma} \\ &= \frac{10\,\text{hours}}{1.09} \\ &= 9.17\,\text{hours}\end{aligned}$$

So the astronauts will experience 9.17 hours of travel time before they reach Pluto.

**Part C**

There is two ways in which you can calculate this. You can use the length contraction formula to get

$$\Delta x' = \frac{\Delta x}{\gamma}$$

$$\begin{aligned} &= \frac{4\,c\cdot\text{hour}}{1.09} \\ &= 3.67\,c\cdot\text{hour} \end{aligned}$$

Or, since you already know the duration of the journey in $S'$, you can also use

$$\begin{aligned} \Delta x' &= V\Delta t' \\ &= (0.4c)\,(9.17\,\text{hours}) \\ &= 3.67\,c\cdot\text{hour} \end{aligned}$$

**Part D**

Since the source of the signal (the spaceship) is moving away from the receiver (Earth) we use

$$\begin{aligned} f_{rec} &= f_{em}\sqrt{\frac{c-V}{c+V}} \\ &= 300\,\text{MHz}\sqrt{\frac{c-0.4c}{c+0.4c}} \\ &= 300\,\text{MHz}\sqrt{\frac{0.6c}{1.4c}} \\ &= (300\,\text{MHz})\,(0.655) \\ &= 196.5\,\text{MHz} \end{aligned}$$

# Question 2

In frame $S$, event B occurs $2\,\mu\text{s}$ ($2\times10^{-6}\,\text{s}$) after event A and at $x_B = 1.5\,\text{km}$ from event A. Take event A to occur at time $t_A = 0$ and position $x_A = 0$ in frame $S$.

(a) How fast must an observer in frame $S'$ be moving along the positive $x$-axis so that events A and B occur simultaneously in his frame?

(b) Is it possible for event B to precede event A for some observer?

(c) Roughly draw a Minkowski diagram below for frames $S$ and $S'$. Indicate events A and B on your diagram. If you answered "yes" to part (b) indicate the axes $ct''$ and $x''$ of an inertial frame $S''$ for which event B occurs before event A. If you answered "no" to part (b), use the diagram to explain why.

(d) Compute the spacetime separation $(\Delta s)^2$ between the events.

(e) Are the two events causally related? Explain your answer.

## Solution

### Part A

These kinds of questions may seem a bit confusing on the first read, but they are usually very simple. Always start by writing down what you know and work from there. From the question, we know that $\Delta t = t_B - t_A = 2\,\mu\text{s}$ and $\Delta x = x_B - x_A = 1.5\,\text{km}$.

Part A of the question introduces an observer in frame $S'$ that moves past frame $S$ at an unknown speed $V$.

The origin of the frames are always arbitrary, so we can choose that to be anything. It is almost always simplest to choose the origins to be at the an event, or Event A in this case. Remember that an "event" is just something that happens that we can assign a specific set of coordinates to. With these kinds of problems that have only one spatial dimension, the sets of coordinates will consist of one spatial and one temporal coordinate. In this case we will choose the origin of both frames to coincide with Event A. This means that we assign $(x_A,\, t_A) = (x'_A,\, t'_A) = (0,\, 0)$.

Part A of this question mentions that in $S'$, Events A and B occur simultaneously so that $t'_A = t'_B$.

We can now construct a table with all the known spacetime coordinates.

| | Event A | Event B |
|---|---|---|
| Frame $S$ | $x_A = 0\,\text{km};\ t_A = 0\,\text{s}$ | $x_B = 1.5\,\text{km};\ t_B = 2\,\mu\text{s}$ |
| Frame $S'$ | $x'_A = 0\,\text{km};\ t'_A = 0\,\text{s}$ | $x'_B = ?;\ t'_B = 0\,\text{s}$ |

With some problems, it will also make sense to draw a picture of the situation to help visualize it.

This part of the question requires us to determine the speed $V$ of $S'$ relative to $S$. From the transformation rules for intervals, we have

$$
\begin{aligned}
\Delta\bar{t} &= \gamma\left(\Delta t - V\Delta x/c^2\right) \\
\bar{t}_B - \bar{t}_A &= \gamma\left((t_B - t_A) - V\left(x_B - x_A\right)/c^2\right) \\
0 &= \gamma\left(2\times 10^{-6}\,\mathrm{s} - V\left(1500\,\mathrm{m}\right)/c^2\right) \\
V\left(1500\,\mathrm{m}\right)/c^2 &= 2\times 10^{-6}\,\mathrm{s} \\
V &= \frac{\left(2\times 10^{-6}\,\mathrm{s}\right)c^2}{1500\,\mathrm{m}} \\
&= \frac{\left(2\times 10^{-6}\,\mathrm{s}\right)\left(3\times 10^{8}\,\mathrm{ms}^{-1}\right)^2}{1500\,\mathrm{m}} \\
&= 1.2\times 10^{8}\,\mathrm{ms}^{-1} \\
&= 0.4c
\end{aligned}
$$

Note that the coordinates were converted to SI units (seconds and metres) so that we get an answer in $\mathrm{ms}^{-1}$ (meters per seconds). If you kept them in the original units, you had to convert $3\times 10^{8}\,\mathrm{ms}^{-1}$ to kilometers per millisecond first.

So $\bar{S}$ is moving with a speed $V = 0.4c$ relative to $S$.

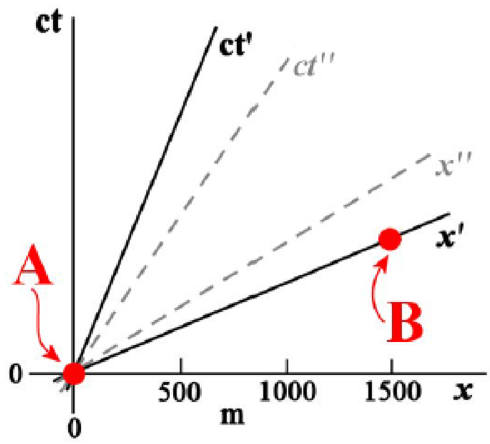
**Part B**

Yes.

**Part C**

Event A should be at the origin of both sets of coordinates and Event B should be above 1500 km on the $x$ axis and on the $x'$ axis, as it occurs at $t' = 0$.

The axes of the frame $S''$ has to "inside" the axes of $S'$, indicating that $S''$ travels at a speed greater that $0.4c$ relative to $S$. Event A is still on the origin of $S''$, so that occurs at time $t'' = 0\,\mathrm{s}$. Event B is below the $x''$axis, so that it will occur at a negative time, i.e. *before* $t'' = 0\,\mathrm{s}$. (Remember that the $x''$ axis indicates the line for which $t'' = 0\,\mathrm{s}$)




**Part D**

The spacetime separation is given by

$$\begin{aligned}(\Delta s)^2 &= (c\Delta t)^2 - (\Delta x)^2 \\ &= \left[\left(3\times 10^8\,\mathrm{ms}^{-1}\right)\left(2\times 10^{-6}\,\mathrm{s}\right)\right]^2 - (1500\,\mathrm{m})^2 \\ &= 3.6\times 10^5\,\mathrm{m}^2 - 2.25\times 10^6\,\mathrm{m}^2 \\ &= -1.89\times 10^6\,\mathrm{m}^2\end{aligned}$$

**Part E**

No. The spacetime separation between the two events is negative. You can also argue that in the calculations above we have shown that there exists a frame where Event A precedes Event B ($S$) and a frame where Event B precedes Event A ($S''$). So the one event could not have caused the other one.

## Question 3

Maxwell's wave equation for an electric field propagating in the $x$-direction is

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2},$$

where $E(x,t)$ is the amplitude of the electric field. Show that this equation is invariant under a Lorentz transformation to a reference frame moving with relative speed $v$ along the

$x$-axis.

## Solution

The relevant Lorentz transformations are given by

$$\begin{aligned} x' &= \gamma\,(x - vt) \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}$$

where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$.

Note that $x' = x'\,(x,\, t)$ and $t' = t'\,(x,\, t)$, so that we use the chain rule to obtain for a wave function

$$\begin{aligned} \frac{\partial E}{\partial x} &= \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial x} = \gamma\frac{\partial E}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial e}{\partial t'} \\ \frac{\partial E}{\partial t} &= \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial t} = -\gamma v\frac{\partial E}{\partial x'} + \gamma\frac{\partial E}{\partial t'} \end{aligned}$$

Therefore we have

$$\begin{aligned} \frac{\partial^2 E}{\partial x^2} &= \left(\gamma\frac{\partial E}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial E}{\partial t'}\right)\left(\gamma\frac{\partial E}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial E}{\partial t'}\right) \\ &= \gamma^2\frac{\partial^2 E}{\partial x'^2} - \frac{2\gamma^2 v}{c^2}\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'} + \frac{\gamma^2 v^2}{c^4}\frac{\partial^2 E}{\partial t'^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 E}{\partial t^2} &= \left(-\gamma v\frac{\partial E}{\partial x'} + \gamma\frac{\partial E}{\partial t'}\right)\left(-\gamma v\frac{\partial E}{\partial x'} + \gamma\frac{\partial E}{\partial t'}\right) \\ &= \gamma^2 v^2\frac{\partial^2 E}{\partial x'^2} - 2\gamma^2 v\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'} + \gamma^2\frac{\partial^2 E}{\partial t'^2} \end{aligned}$$

Substituting this into the wave equation

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

and rearranging gives

$$\gamma^2\frac{\partial^2 E}{\partial x'^2} - \frac{\gamma^2 v^2}{c^2}\frac{\partial^2 E}{\partial x'^2} + \frac{\gamma^2 v^2}{c^4}\frac{\partial^2 E}{\partial t'^2} - \frac{\gamma^2}{c^2}\frac{\partial^2 E}{\partial t'^2} = \left(\frac{2\gamma^2 v}{c^2}\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'} - \frac{2\gamma^2 v}{c^2}\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'}\right)$$

$$\gamma^2 \frac{\partial^2 E}{\partial x'^2}\left(1-\frac{v^2}{c^2}\right) - \frac{\gamma^2}{c^2}\frac{\partial^2 E}{\partial t'^2}\left(1-\frac{v^2}{c^2}\right) = 0$$

$$\frac{\partial^2 E}{\partial x'^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2}$$

Therefore, the wave equation is invariant under a Lorentz transformation.

You can follow the same approach using the Galilean transformation equations and you will find that the equation has a different form in $S'$than in $S$ . Therefore, the equation is *not* invariant under a Galilean transformation.

## Question 4

A particle is measured in an inertial frame $S$ to have a total energy of $E = 5\,\text{GeV}$ $(1\,\text{GeV} = 10^9\,\text{eV})$ and momentum of $p = 3\,\text{GeV}/c$.

(a) What is the mass of the particle, in $\text{GeV}/c^2$?

(b) What is the speed of the particle?

(c) What is the energy $E'$ of the particle in another inertial frame $S'$ in which the particle's momentum is $p' = 4\,\text{GeV}/c$?

(d) What is the kinetic energy of the particle in $S'$?

(e) What is the maximum momentum this particle can have, according to the limits set by special relativity?

### Solution

#### Part A

**Solution 1** Rearranging the equation

$$E^2 = p^2c^2 + m^2c^4$$

gives

$$m^2 = \frac{E^2}{c^4} - \frac{p^2}{c^2}$$

Substituting the given values for $E$ and $p$ gives

$$\begin{aligned} m^2 &= \frac{(5)^2}{c^4} - \frac{(3\,\mathrm{GeV}/c)^2}{c^2} \\ &= \frac{25\,\mathrm{GeV}^2}{c^4} - \frac{9\,\mathrm{GeV}^2}{c^4} \\ &= \frac{16\,\mathrm{GeV}^2}{c^4} \\ m &= 4\,\mathrm{GeV}/c^2 \end{aligned}$$

**Solution 2** If you prefer, you can convert all the quantities to SI units, but this tends to be unnecessarily tedious as the $c$'s don't cancel and the numbers are huge. We use the conversion factor $1\,\mathrm{eV} = 1.60 \times 10^{-19}\,\mathrm{J}$

$$\begin{aligned} m^2 &= \frac{E^2}{c^4} - \frac{p^2}{c^2} \\ &= \frac{(5 \times 10^9 \times 1.60 \times 10^{-19})^2}{c^4} - \frac{(3 \times 10^9 \times 1.60 \times 10^{-19}/c)^2}{c^2} \\ &= \frac{(8 \times 10^{-10})^2}{c^4} - \frac{(4.8 \times 10^{-10})^2}{c^4} \\ &= \frac{(8 \times 10^{-10})^2}{(3 \times 10^8)^4} - \frac{(4.8 \times 10^{-10})^2}{(3 \times 10^8)^4} \\ & \quad \frac{6.4 \times 10^{-19}}{8.1 \times 10^{33}} - \frac{2.3 \times 10^{-19}}{8.1 \times 10^{33}} \\ &= 7.90 \times 10^{-53} - 2.84 \times 10^{-53} \\ &= 5.06 \times 10^{-53} \\ m &= 7.11 \times 10^{-27}\,\mathrm{kg} \end{aligned}$$

## Part B

**Solution 1** The Lorentz factor $\gamma$ depends only on the speed, so if we can calculate $\gamma$, we can get $V$.

$$E = \gamma m c^2$$

$$
\begin{aligned}
\gamma &= \frac{E}{mc^2} \\
&= \frac{5\,\text{GeV}}{(4\,\text{GeV}/c^2)\,c^2} \\
&= \frac{5}{4} \\
\frac{1}{\sqrt{1-V^2/c^2}} &= \frac{5}{4} \\
1-V^2/c^2 &= \frac{16}{25} \\
V^2/c^2 &= 1-\frac{16}{25} \\
&= \frac{9}{25} \\
V &= \frac{3}{5}c
\end{aligned}
$$

**Solution 2** Or, if you insist on using SI units, you can get the Lorentz factor as follows

$$
\begin{aligned}
\gamma &= \frac{E}{mc^2} \\
&= \frac{8\times 10^{-10}}{(7.11\times 10^{-27})\,(3\times 10^8)^2} \\
&= \frac{8\times 10^{-10}}{6.4\times 10^{-10}} \\
&= \frac{5}{4}
\end{aligned}
$$

The rest of the solution is the same as for Solution 1.

## Part C

In any single inertial frame, the equations of special relativity hold, so we can calculate the energy $E'$ in the frame where the momentum is equal to $p'$as follows.

$$
\begin{aligned}
E'^2 &= p'^2c^2 + m^2c^4 \\
&= (4\,\text{GeV}/c)^2\,c^2 + \left(4\,\text{GeV}/c^2\right)^2 c^4 \\
&= 32\,\text{GeV}^2 \\
E' &= 4\sqrt{2}\,\text{GeV}
\end{aligned}
$$

Remember that the mass of the particle is invariant, so it is the same in all inertial frames.

**Part D**

The kinetic energy of the particle is given by

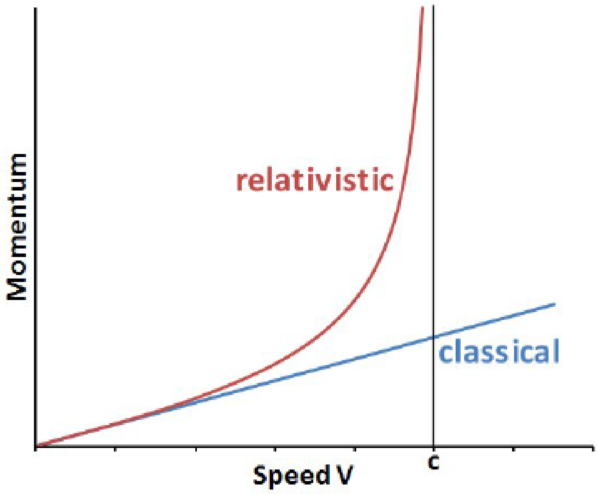
$$E'_K = (\gamma' - 1)\, mc^2$$

At this point *we do not know the speed of the particle in $S'$, so we do not know the value of $\gamma'$*. The Lorentz factor is not invariant. You can calculate the value of $\gamma'$ for $S'$using a similar method as we did in Part B, and substitute it into the above equation. Or you can so it like this:

$$\begin{aligned}
E'_K &= (\gamma' - 1)\, mc^2 \\
&= \gamma' mc^2 - mc^2 \\
&= E' - mc^2 \\
&= 4\sqrt{2}\,\text{GeV} - \left(4\,\text{GeV}/c^2\right) c^2 \\
&= 4\left(\sqrt{2} - 1\right)\,\text{GeV} \\
&= 1.66\,\text{GeV}
\end{aligned}$$

**Part E**

Special relativity places no upper limit on momentum. Below is a graph showing the classical (blue) and relativistic (red) momenta for a object at different speeds. In classical (Newtonian) mechanics, the momentum increases linearly with the speed. In special relativity, the Lorentz factor ensures that the momentum goes to infinity as the speed approaches $c$. Form the graph you can also see that the relativistic momentum approaches the Newtonian momentum at speeds much smaller than $c$.




# Question 5

Derive the energy-momentum relation $E^2 = c^2p^2 + m^2c^4$ by starting from the relativistic definitions of $E$ and $p$, i.e. $E = \gamma mc^2$ and $p = \gamma mv$.

## Solution

Squaring the definitions of $E$ and $p$

$$\begin{aligned} E^2 &= \gamma^2 m^2 c^4 \\ p^2 &= \gamma^2 m^2 u^2 \end{aligned}$$

Multiplying the last equation by $c^2$ and subtracting gives

$$\begin{aligned} E^2 - p^2c^2 &= \gamma^2 m^2 c^4 - \gamma^2 m^2 u^2 c^2 \\ &= m^2c^4 \left( \gamma^2 - \gamma^2 \frac{u^2}{c^2} \right) \end{aligned}$$

Using

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

we show that

$$\begin{aligned}\gamma^2 - \gamma^2\frac{u^2}{c^2} &= \frac{1}{1-u^2/c^2} - \frac{u^2/c^2}{1-u^2/c^2} \\ &= \frac{1-u^2/c^2}{1-u^2/c^2} \\ &= 1\end{aligned}$$

It follows that

$$E^2 = p^2c^2 + m^2c^4$$

# Tutorial Letter 205/1/2016

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 1

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 05.

BAR CODE

# Memo for Assignment 5 S1 2016

## Basics of differential geometry (§ 3)

### Question 1: Metric tensor

The line element for a certain two-dimensional Riemannian space is given by

$$dl^2 = d\theta^2 + 2\cos\theta d\theta d\phi + d\phi^2 \,.$$

Putting $x^1 = \theta$ and $x^2 = \phi$, what is the metric tensor of this space?

- $\begin{pmatrix} \cos\theta & 1 \\ 1 & \cos\theta \end{pmatrix}$
- $\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}*$
- $\begin{pmatrix} 1 & 2\cos\theta \\ 2\cos\theta & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 2\cos\theta \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ 2\cos\theta & 1 \end{pmatrix}$

The line element for a general Riemann space is given by

$$dl^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$$

Since we are considering a two dimensional space ($n = 2$), we can can expand this to

$$dl^2 = g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + g_{21}dx^2dx^1 + g_{22}dx^2dx^2$$

Putting $x^1 = \theta$ and $x^2 = \phi$, we get

$$dl^2 = g_{11}d\theta^2 + g_{12}d\theta d\phi + g_{21}d\phi d\theta + g_{22}d\phi^2$$

The metric tensor must be symmetric. (This ensures that the distance from the point P to the point Q will be the same as the distance from point Q to point P.) This means that we must have $g_{12} = g_{21}$. From the given line element, we can see that we have $g_{11} = 1$, $g_{12} = g_{21} = \cos\theta$ and $g_{22} = 1$. Putting this in array format gives

$$\begin{pmatrix} 1 & \cos\theta \\ \cos\theta & 1 \end{pmatrix}.$$

## Question 2: Arc length

Consider the curve described by

$$x = \cos^3 t, \qquad y = \sin^3 t$$

with the points P and Q defined by the points where $t_P = 0$ and $t_Q = \pi/2$.

What is the length of the curve between points P and Q in arbitrary units?

- 4.05
- 3
- 1.77
- $\mathbf{\frac{3}{2}}$*
- 1.21

First we calculate the derivatives of the Cartesian coordinates with respect the the parameter $u$.

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt}\left(\cos^3 t\right) = -3\cos^2 t\,\sin t \\ \frac{dy}{dt} &= \frac{d}{dt}\left(\sin^3 t\right) = 3\cos t\,\sin^2 t \end{aligned}$$

We find that

$$
\begin{aligned}
\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 &= \left(-3\cos^2 t\,\sin t\right)^2 + \left(3\cos t\,\sin^2 t\right)^2 \\
&= 9\cos^4 t\,\sin^2 t + 9\cos^2 t\,\sin^4 t \\
&= 9\cos^2 t\,\sin^2 t\left(\cos^2 t + \sin^2 t\right) \\
&= 9\cos^2 t\,\sin^2 t
\end{aligned}
$$

The length of the curve is then given by

$$
\begin{aligned}
L\left(\mathrm{P},\,\mathrm{Q}\right) &= \int_{t_{\mathrm{P}}}^{t_{\mathrm{Q}}} \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right)^{1/2} dt \\
&= \int_0^{\pi/2} \sqrt{9\cos^2 t\,\sin^2 t dt} \\
&= \int_0^{\pi/2} 3\cos t\,\sin t\,dt \\
&= \left.\frac{3}{2}\sin^2 t\right]_0^{\pi/2} \\
&= \frac{3}{2}\sin^2\frac{\pi}{2} - \frac{3}{2}\sin^2 0 \\
&= \frac{3}{2}
\end{aligned}
$$

## Question 3: Kronecker delta

The sum

$$\sum_{i=1}^{3} \delta_{ii}$$

is equal to...

- 0
- 1
- 2
- **3***
- 4

The definition of the Kroneker delta is

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

(The components of $\delta^i_{\ j}$ and $\delta_{ij}$ are the same for similar values of $i$ and $j$, even though the two tensors are not strictly equal since they are of different types.) It is a very common mistake to say that this sum is equal to 1. But it is a sum over number of components equal to 1 and the answer will depend on the number of dimensions you are working in. In this case

$$\begin{aligned} \sum_{i=1}^{3} \delta_{ii} &= \delta_{11} + \delta_{22} + \delta_{33} \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

## Question 4: Christoffel coefficients

Consider a surface with a metric tensor

$$g_{ij} = \begin{pmatrix} 1 + 4u^2 & 4uv \\ 4uv & 1 + 4v^2 \end{pmatrix}$$

where $x^1 = u$ and $x^2 = v$. What is the value of the connection coefficient $\Gamma^1_{22}$?

- $0$
- $(1 + 4u) / (u + 4u^2)$
- $\mathbf{4u / (1 + 4u^2 + 4v^2)} *$
- $v / (1 + u^2 + v^2)$
- $4u (1 + 4v^2) / (1 + 4u^2 + 4v^2)$

To compute the connection coefficients, we need the dual metric tensor. Since we must have

$$\sum_{i,j=1}^{3} g_{ij} g^{ij} = \sum_{i=1}^{3} \delta^i_{\ i}$$

this is just the inverse matrix of the metric tensor given by

$$g^{ij} = \frac{1}{1+4u^2+4v^2}\begin{pmatrix} 1+4v^2 & -4uv \\ -4uv & 1+4u^2 \end{pmatrix}.$$

The Christoffel (or connection) coefficients are defined by

$$\Gamma^h_{ij} = \sum_k \frac{1}{2} g^{hk}\left(g_{ki,j}+g_{jk,i}-g_{ij,k}\right)$$

To compute $\Gamma^1_{22}$, we let $h \to 1$, $i \to 2$ and $j \to 2$ in the above equation to get

$$\Gamma^1_{22} = \sum_k \frac{1}{2} g^{1k}\left(g_{k2,2}+g_{2k,2}-g_{22,k}\right)$$

Expanding the sum over $k = 1, 2$ (since we are working in a 2 dimensional space) gives

$$\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}g^{11}\left(g_{12,2}+g_{21,2}-g_{22,1}\right)+\frac{1}{2}g^{12}\left(g_{22,2}+g_{22,2}-g_{22,2}\right) \\
&= \frac{1}{2}g^{11}\left(2g_{12,2}-g_{22,1}\right)+\frac{1}{2}g^{12}g_{22,2}
\end{aligned}$$

It will always be the case that $g_{ik}$ is symmetric. We used this above to simplify the equation. Now we just substitute the values from the metric and dual metric tensor

$$\begin{aligned}
\Gamma^1_{22} &= \frac{1}{2}g^{11}\left(2g_{12,2}-g_{22,1}\right)+\frac{1}{2}g^{12}g_{22,2} \\
&= \frac{1}{2}g^{11}\left(2\frac{d}{dx^2}g_{12}-\frac{d}{dx^1}g_{22}\right)+\frac{1}{2}g^{12}\frac{d}{dx^2}g_{22} \\
&= \frac{1}{2}\left(\frac{1+4v^2}{1+4u^2+4v^2}\right)\left[2\frac{d}{dv}(4uv)-\frac{d}{du}\left(1+4v^2\right)\right]+\frac{1}{2}\left(\frac{-4uv}{1+4u^2+4v^2}\right)\frac{d}{dv}\left(1+4v^2\right) \\
&= \frac{4u\left(1+4v^2\right)}{1+4u^2+4v^2}-\frac{16uv^2}{1+4u^2+4v^2} \\
&= \frac{4u}{1+4u^2+4v^2}
\end{aligned}$$

# Tutorial Letter 205/2/2016

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 05.

BAR CODE

# Memo for Assignment 5 S2 2016

## Basics of differential geometry (§ 3)

### Question 1: Metric tensor

The line element for a certain two dimensional Riemannian space is given by

$$dl^2 = dr^2 + 2r\sin\phi\, dr\, d\phi + r^2 d\phi^2 \,.$$

Putting $x^1 = r$ and $x^2 = \phi$, what is the metric tensor of this space?

- $\begin{pmatrix} 1 & 2r\sin\phi \\ 2r\sin\phi & r^2 \end{pmatrix}$
- $\begin{pmatrix} r\sin\phi & 1 \\ 1 & r\sin\phi \end{pmatrix}$
- $\begin{pmatrix} 1 & r\sin\phi \\ r\sin\phi & r^2 \end{pmatrix}$*
- $\begin{pmatrix} 1 & 0 \\ 2r\sin\phi & r^2 \end{pmatrix}$
- $\begin{pmatrix} r^2 & 2r\sin\phi \\ 0 & 1 \end{pmatrix}$

The line element for a general Riemann space is given by

$$dl^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$$

Since we are considering a two dimensional space ($n = 2$), we can can expand this to

$$dl^2 = g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + g_{21}dx^2dx^1 + g_{22}dx^2dx^2$$

Choosing $x^1 = r$ and $x^2 = \phi$, we get

$$dl^2 = g_{11}dr^2 + g_{12}drd\phi + g_{21}d\phi dr + g_{22}d\phi^2$$

The metric tensor must be symmetric. (This ensures that the distance from the point P to the point Q will be the same as the distance from point Q to point P.) This means that we must have $g_{12} = g_{21}$. From the given line element, we can see that we have $g_{11} = 1$, $g_{12} = g_{21} = r\sin\phi$ and $g_{22} = r^2$. Putting this in array format gives

$$\begin{pmatrix} 1 & r\sin\phi \\ r\sin\phi & r^2 \end{pmatrix}.$$

## Question 2: Kronecker delta

The sum

$$\sum_{i,j=1}^{2} g_{ij}g^{ij}$$

is equal to...

- 0
- 1
- 2*
- 3
- 4

We have (see Equation 3.24 in your textbook, and use symmetry)

$$\sum_{i,j=1}^{2} g_{ij}g^{ij} = \sum_{i=1}^{2}\sum_{j=1}^{2} g_{ij}g^{ij} = \sum_{i=1}^{2} \delta^i_{\ i}$$

It is a very common mistake to say that this sum is equal to 1. The definition of the Kroneker delta is

$$\delta^i_{\ j} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

(The components of $\delta^i_{\ j}$ and $\delta_{ij}$ are the same for similar values of $i$ and $j$, even though the two tensors are not strictly equal since they are of different types.) The sum in the question is a sum over number of components all equal to 1 and the answer will depend on the number of dimensions you are working in. In this case $(N = 2)$

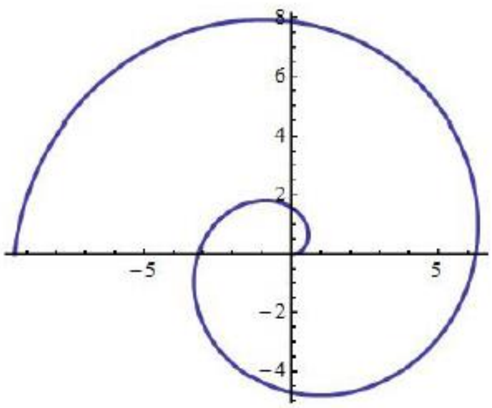
$$\begin{aligned}\sum_{i=1}^{2} \delta^i_{\ i} &= \delta^1_{\ 1} + \delta^2_{\ 2} \\ &= 1 + 1 \\ &= 2\end{aligned}$$

## Question 3: Curvature of a curve

Consider the curve with parametric equations

$$\begin{aligned} x &= t \cos t \\ y &= t \sin t \end{aligned}$$

The curve is shown below for $0 \leq t \leq 3\pi$.




What is the general equation for the curvature of this curve?

- 1
- $1/t$
- $t^2$

- $(1+2t)\,/\,(1+t)^3$
- $\mathbf{(2+t^2)\,/\,(1+t^2)^{3/2}*}$

We will use equation 3.32 in the textbook. Computing the derivatives of the parametric equations give

$$\begin{aligned}\dot{x} &= \cos t - t\sin t\\ \ddot{x} &= -t\cos t - 2\sin t\end{aligned}$$

$$\begin{aligned}\dot{y} &= t\cos t + \sin t\\ \ddot{y} &= 2\cos t - t\sin t\end{aligned}$$

Using this, we get

$$\begin{aligned}\dot{x}\ddot{y} - \dot{y}\ddot{x} &= (\cos t - t\sin t)(2\cos t - t\sin t) - (t\cos t + \sin t)(-t\cos t - 2\sin t)\\ &= \left(2\cos^2 t - 3t\cos t\,\sin t + t^2\sin^2 t\right) - \left(-t^2\cos^2 t - 3t\cos t\,\sin t - 2\sin^2 t\right)\\ &= 2\left(\cos^2 t + \sin^2 t\right) + t^2\left(\sin^2 t + \cos^2 t\right)\\ &= 2 + t^2\end{aligned}$$

and

$$\begin{aligned}\dot{x}^2 + \dot{y}^2 &= (\cos t - t\sin t)^2 + (t\cos t + \sin t)^2\\ &= \cos^2 t - 2t\cos t\,\sin t + t^2\sin^2 t + t^2\cos^2 t + 2t\cos t\,\sin t + \sin^2 t\\ &= \left(\cos^2 t + \sin^2 t\right) + t^2\left(\sin^2 t + \cos^2 t\right)\\ &= 1 + t^2\end{aligned}$$

Thus, for the curvature we get

$$\begin{aligned}k &= \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}\\ &= \frac{2 + t^2}{\left(1 + t^2\right)^{3/2}}\end{aligned}$$

## Question 4: Christoffel coefficients

Consider a surface with a metric tensor

$$g_{ij} = \begin{pmatrix} 1+v^2 & uv \\ uv & 1+u^2 \end{pmatrix}$$

where $x^1 = u$ and $x^2 = v$. What is the value of the connection coefficient $\Gamma^2_{\ 12}$?

- $v/\left(1+u^2+v^2\right)$
- $\mathbf{u/\left(1+u^2+v^2\right)*}$
- $-uv^2/\left(1+u^2+v^2\right)$
- $0$
- $\left(1+2u^2\right)/\left(u+u^3\right)$

To compute the connection coefficients, we need the dual metric tensor. Since we must have

$$\sum_{i,j=1}^{3} g_{ij}g^{ij} = \sum_{i=1}^{3} \delta^i_{\ i}$$

this is just the inverse matrix of the metric tensor given by

$$g^{ij} = \frac{1}{1+u^2+v^2}\begin{pmatrix} 1+u^2 & -uv \\ -uv & 1+v^2 \end{pmatrix}.$$

The Christoffel (or connection) coefficients are defined by

$$\Gamma^h_{\ ij} \ = \ \sum_k \frac{1}{2} g^{hk}\left(g_{ki,j} + g_{jk,i} - g_{ij,k}\right)$$

To compute $\Gamma^2_{\ 12}$, we let $h \to 2$, $i \to 1$ and $j \to 2$ in the above equation to get

$$\Gamma^2_{\ 12} \ = \ \sum_k \frac{1}{2} g^{2k}\left(g_{k1,2} + g_{2k,1} - g_{12,k}\right)$$

Expanding the sum over $k = 1, 2$ (since we are working in a 2 dimensional space) gives

$$\begin{aligned}\Gamma^2_{12} &= \frac{1}{2}g^{21}\left(g_{11,2} + g_{21,1} - g_{12,1}\right) + \frac{1}{2}g^{22}\left(g_{21,2} + g_{22,1} - g_{12,2}\right) \\ &= \frac{1}{2}g^{21}g_{11,2} + \frac{1}{2}g^{22}g_{22,1}\end{aligned}$$

It will always be the case that $g_{ik}$ is symmetric. We used this above to simplify the equation. Now we just substitute the values from the metric and dual metric tensor

$$\begin{aligned}\Gamma^2_{12} &= \frac{1}{2}g^{21}g_{11,2} + \frac{1}{2}g^{22}g_{22,1} \\ &= \frac{1}{2}\left(\frac{-uv}{1+u^2+v^2}\right)\frac{d}{dx^2}\left(1+v^2\right) + \frac{1}{2}\left(\frac{1+v^2}{1+u^2+v^2}\right)\frac{d}{dx^1}\left(1+u^2\right) \\ &= \frac{1}{2}\left(\frac{-uv}{1+u^2+v^2}\right)\frac{d}{dv}\left(1+v^2\right) + \frac{1}{2}\left(\frac{1+v^2}{1+u^2+v^2}\right)\frac{d}{du}\left(1+u^2\right) \\ &= \frac{-uv^2}{1+u^2+v^2} + \frac{u+uv^2}{1+u^2+v^2} \\ &= \frac{u}{1+u^2+v^2}\end{aligned}$$

# Tutorial Letter 206/1/2016

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 1

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 06.

BAR CODE

# Memo for Assignment 6 S1 2015

## General Relativity and Tensor Algebra (§ 4)

### Question 1: Tensor transformations

The transformation equations for transforming a contravariant tensor of rank one from polar to Cartesian coordinates are

$$\begin{aligned} A'^1 &= A^1 \cos\theta - A^2 r \sin\theta \\ A'^2 &= A^1 \sin\theta + A^2 r \cos\theta \end{aligned}$$

where $x^i = (r, \theta)$ (derived in Exercise 4.2 in the textbook). Use these to transform the tensor described by $[A^i] = (r^2, 1/\sin\theta)$ to Cartesian coordinates $x'^i = (x, y)$. What is the value of $A'^2$?

- $1 + r^3 \cos\theta$
- $1 + r \cot\theta$
- $r(\cos\theta - 1)$
- $\mathbf{r}(\sin\theta + \cot\theta)$*
- $\cot\theta - r^3 \sin\theta$

The transformation equations for transforming a contravariant tensor of rank one from polar to Cartesian coordinates are

$$\begin{aligned} A'^1 &= A^1 \cos\theta - A^2 r \sin\theta \\ A'^2 &= A^1 \sin\theta + A^2 r \cos\theta \end{aligned}$$

The components of the tensor we want to transform is given as $A^1 = r^2$ and $A^2 = 1/\sin\theta$. Substituting this into the above equations give

$$A'^1 = r^2 \cos\theta - \frac{r \sin\theta}{\sin\theta}$$

$$
\begin{aligned}
&= r(\cos\theta - 1) \\
A'^2 &= r^2 \sin\theta + \frac{r\cos\theta}{\sin\theta} \\
&= r(\sin\theta + \cot\theta)
\end{aligned}
$$

## Question 2: Tensor expressions

Which of the following tensor expressions is incorrect?

- $A^i = \sum_j g^{ij} A_j = \sum_{j,k} g^{ij} g_{jk} A^k$
- $\bar{A}^i_{\ kl} = \sum_{p,r,s} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} A^p_{\ rs}$
- $\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)$
- $\mathbf{\sum_{i=1}^{3} \delta^i_{\ i} = 1^*}$
- $\Gamma^i_{\ jk} = \Gamma^i_{\ kj}$

**Option (1):** Correct. The rules of raising and lowering an index is followed. We can also write

$$
\begin{aligned}
\sum_{j,k} g^{ij} g_{jk} A^k &= \sum_k \delta^i_{\ k} A^k \\
&= \delta^i_{\ 1} A^1 + \delta^i_{\ 2} A^2 + ... + \delta^i_{\ i} A^i + ... \delta^i_{\ n} A^n
\end{aligned}
$$

In the last step the sum has been expanded and sums $k$ from 1 to $n$. Remember that $\delta^i_{\ k}$ is defined so that it is equal to 1 if $i = k$, and zero if $i \neq k$. So all the terms will be equal to zero, except the term where $i = k$, and in that case we have $\delta^i_{\ i} = 1$ so that we can write

$$\sum_{j,k} g^{ij} g_{jk} A^k = A^i$$

**Option (2):** This is the correct transformation for a tensor of this form. Here bars were used in stead of primes to indicate the other coordinate frame. The textbook uses primes, but this can sometimes become unclear, especially when writing by hand. Using either is fine, as long as you are consistent.

**Option (3):** This is correct. Lowering the first index of the Christoffel coefficient gives

$$\begin{aligned}
\Gamma_{\alpha\beta\gamma} &= g_{\alpha\eta}\Gamma^{\eta}{}_{\beta\gamma} \\
&= \frac{1}{2}g_{\alpha\eta}\sum_{\epsilon} g^{\eta\epsilon}\left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}}\right) \\
&= \frac{1}{2}\delta^{1}{}_{\alpha}\left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}}\right) + .... + \frac{1}{2}\delta^{\alpha}{}_{\alpha}\left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}}\right) + ...
\end{aligned}$$

In the last step we used

$$g_{\alpha\eta}g^{\eta\epsilon} = \delta^{\epsilon}{}_{\alpha}$$

The only non-vanishing term will be the one for which $\eta = \alpha$ so that

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\alpha}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

**Option (4):** This is incorrect. Expanding the sum gives

$$\begin{aligned}
\sum_{i=1}^{3}\delta^{i}{}_{i} &= \delta^{1}_{1} + \delta^{2}_{2} + \delta^{3}_{3} \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}$$

**Option (5):** Correct. This is a consequence of the symmetry of the metric tensor. We can write

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\sum_{l} g^{il}\left(\frac{\partial g_{lk}}{\partial x^{j}} + \frac{\partial g_{jl}}{\partial x^{k}} - \frac{\partial g_{jk}}{\partial x^{l}}\right)$$

Since $g_{\alpha\beta} = g_{\beta\alpha}$, we can interchange the indices of all the metrics within the sum.

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\sum_{l} g^{il}\left(\frac{\partial g_{kl}}{\partial x^{j}} + \frac{\partial g_{lj}}{\partial x^{k}} - \frac{\partial g_{kj}}{\partial x^{l}}\right)$$

The first two terms can also be switched to give

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\sum_{l} g^{il}\left(\frac{\partial g_{lj}}{\partial x^{k}} + \frac{\partial g_{kl}}{\partial x^{j}} - \frac{\partial g_{kj}}{\partial x^{l}}\right)$$

which is exactly the definition for $\Gamma^{i}{}_{kj}$. So therefore $\Gamma^{i}{}_{jk} = \Gamma^{i}{}_{kj}$.

## Question 3: Raising and lowering indices

Which of the following expressions is correct?

- $R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = \sum_{\eta} g^{\eta\gamma} R^{\alpha}{}_{\beta\eta\delta}$ *
- $R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = \sum_{\gamma} g_{\eta\gamma} R^{\alpha}{}_{\beta\gamma\delta}$
- $R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = \sum_{\eta} g_{\eta\gamma} R^{\alpha}{}_{\beta\gamma\delta}$
- $R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = \sum_{\eta} g^{\eta\gamma} R^{\alpha}{}_{\beta\gamma\delta}$
- $R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} = \sum_{\alpha} g^{\alpha\gamma} R^{\alpha}{}_{\beta\gamma\delta}$

You can use the metric tensor to lower indices and the dual metric tensor to raise indices. In this case we want to raise the index $\gamma$, so we multiply $R^{\alpha}_{\beta\gamma\delta}$ with the dual metric tensor where one index is $\gamma$, and sum over it:

$$R^{\alpha}{}_{\beta}{}^{\gamma}{}_{\delta} \;=\; \sum_{\eta} g^{\eta\gamma} R^{\alpha}{}_{\beta\eta\delta}$$

## Question 4: Covariant and contravariant forms

Equation 2.70 in the textbook is written for four dimensional Minkowski space and gives a rule to determine the covariant form of a vector if the metric and contravariant form is known. This same equation written for a general two dimensional space is

$$A_j = \sum_{i=1}^{2} g_{ij} A^i \,.$$

Use this to determine the covariant form of $[A^i]$ in two dimensional space described by the surface of a paraboloid. The metric tensor for this space is

$$[g_{ij}] = \begin{pmatrix} 1 + a^2 r^2 & 0 \\ 0 & r^2 \end{pmatrix}$$

and let

$$\left[A^i\right] = \begin{pmatrix} 1 \\ a^2 \end{pmatrix} .$$

- $[A_i] = \begin{pmatrix} a^2 \\ 1 \end{pmatrix}$
- $[\mathbf{A_i}] = \begin{pmatrix} 1 + a^2r^2 \\ a^2r^2 \end{pmatrix}$*
- $[A_i] = \begin{pmatrix} a^2 + a^4r^2 \\ r^2 \end{pmatrix}$
- $[A_i] = \begin{pmatrix} a^2 \\ 1 + a^2r^2 \end{pmatrix}$
- $[A_i] = \begin{pmatrix} 1 + a^2r^2 + a^2 \\ 1 + a^2r^2 \end{pmatrix}$

Expanding the given equation for determining the covariant components of $[A^i]$ gives

$$\begin{aligned} A_j &= \sum_{i=1}^{2} g_{ij}A^i \\ &= g_{1j}A^1 + g_{2j}A^2 \end{aligned}$$

From the information given in the question, we know that $g_{11} = 1 + a^2r^2$, $g_{12} = g_{21} = 0$, $g_{22} = r^2$, $A^1 = 1$ and $A^2 = a^2$. The covariant components are then given by

$$\begin{aligned} A_1 &= g_{11}A^1 + g_{21}A^2 \\ &= \left(1 + a^2r^2\right)(1) + (0)\left(a^2\right) \\ &= 1 + a^2r^2 \end{aligned}$$

$$\begin{aligned} A_2 &= g_{12}A^1 + g_{22}A^2 \\ &= (0)(1) + \left(r^2\right)\left(a^2\right) \\ &= a^2r^2 \end{aligned}$$

You will notice that we could also have computed it with

$$\begin{aligned} [A_i] &= [g_{ij}]\left[A^i\right] \\ &= \begin{pmatrix} 1 + a^2r^2 & 0 \\ 0 & r^2 \end{pmatrix} \begin{pmatrix} 1 \\ a^2 \end{pmatrix} \end{aligned}$$

$$= \begin{pmatrix} 1 + a^2r^2 \\ a^2r^2 \end{pmatrix}$$

## Question 5: Einstein field equations

How many equations does the following expression represent?

$$T^{\mu\nu} = \left(\rho + p/c^2\right) U^\mu U^\nu - p g^{\mu\nu}$$

- 1
- 2
- 4
- 8
- **16***

The expression represents 16 different equations. One for each possible combination of $\mu$ and $\nu$, where both indices can have values from 0 to 3, since this equation is written for four dimensional spacetime. The 16 possible combinations are

| # | $\mu$ | $\nu$ | | # | $\mu$ | $\nu$ | | # | $\mu$ | $\nu$ | | # | $\mu$ | $\nu$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 0 | 0 | | **5** | 1 | 0 | | **9** | 2 | 0 | | **13** | 3 | 0 |
| **2** | 0 | 1 | | **6** | 1 | 1 | | **10** | 2 | 1 | | **14** | 3 | 1 |
| **3** | 0 | 2 | | **7** | 1 | 2 | | **11** | 2 | 2 | | **15** | 3 | 2 |
| **4** | 0 | 3 | | **8** | 1 | 3 | | **12** | 2 | 3 | | **16** | 3 | 3 |

# Tutorial Letter 206/2/2016

## Special Relativity and Riemannian Geometry

# APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 06.

BAR CODE

# Memo for Assignment 6 S2 2015

## General Relativity and Tensor Algebra (§ 4)

### Question 1: Tensor transformations

The transformation equations for transforming a contravariant tensor of rank one from polar to Cartesian coordinates are

$$\begin{aligned} A'^1 &= A^1 \cos\theta - A^2 r \sin\theta \\ A'^2 &= A^1 \sin\theta + A^2 r \cos\theta \end{aligned}$$

where $x^i = (r,\, \theta)$ (derived in Exercise 4.2 in the textbook). Use these to transform the tensor described by $[A^i] = (r^2,\, 1/\sin\theta)$ to Cartesian coordinates $x'^i = (x,\, y)$. What is the value of $A'^1$?

- $1 + r^3 \cos\theta$
- $1 + r \cot\theta$
- $\mathbf{r}\,(\cos\theta - \mathbf{1})$*
- $r\,(\sin\theta + \cot\theta)$
- $\cot\theta - r^3 \sin\theta$

The transformation equations for transforming a contravariant tensor of rank one from polar to Cartesian coordinates are

$$\begin{aligned} A'^1 &= A^1 \cos\theta - A^2 r \sin\theta \\ A'^2 &= A^1 \sin\theta + A^2 r \cos\theta \end{aligned}$$

The components of the tensor we want to transform is given as $A^1 = r^2$ and $A^2 = 1/\sin\theta$. Substituting this into the above equations give

$$A'^1 = r^2 \cos\theta - \frac{r\sin\theta}{\sin\theta}$$

$$
\begin{aligned}
&= r(\cos\theta - 1) \\
A'^2 &= r^2\sin\theta + \frac{r\cos\theta}{\sin\theta} \\
&= r(\sin\theta + \cot\theta)
\end{aligned}
$$

## Question 2: Tensor expressions

Which of the following tensor expressions is incorrect?

- $A^i = \sum_j g^{ij} A_j = \sum_{j,k} g^{ij} g_{jk} A^k$
- $\bar{A}^i{}_{kl} = \sum_{p,r,s} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} A^p{}_{rs}$
- $\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)$
- $\sum_{i=1}^{3} \delta^i{}_i = 3$
- $\mathbf{\Gamma^i{}_{jk} = \Gamma^j{}_{ik}}$*

**Option (1):** Correct. The rules of raising and lowering an index is followed. We can also write

$$
\begin{aligned}
\sum_{j,k} g^{ij} g_{jk} A^k &= \sum_k \delta^i{}_k A^k \\
&= \delta^i{}_1 A^1 + \delta^i{}_2 A^2 + \ldots + \delta^i{}_i A^i + \ldots \delta^i{}_n A^n
\end{aligned}
$$

In the last step the sum has been expanded and sums $k$ from 1 to $n$. Remember that $\delta^i{}_k$ is defined so that it is equal to 1 if $i = k$, and zero if $i \neq k$. So all the terms will be equal to zero, except the term where $i = k$, and in that case we have $\delta^i{}_i = 1$ so that we can write

$$
\sum_{j,k} g^{ij} g_{jk} A^k = A^i
$$

**Option (2):** This is the correct transformation for a tensor of this form. Here bars were used in stead of primes to indicate the other coordinate frame. The textbook uses primes, but this can sometimes become unclear, especially when writing by hand. Using either is fine, as long as you are consistent.

**Option (3):** This is correct. Lowering the first index of the Christoffel coefficient gives

$$\begin{aligned}
\Gamma_{\alpha\beta\gamma} &= g_{\alpha\eta}\Gamma^{\eta}{}_{\beta\gamma} \\
&= \frac{1}{2}g_{\alpha\eta}\sum_{\epsilon} g^{\eta\epsilon}\left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}}\right) \\
&= \frac{1}{2}\delta^{1}{}_{\alpha}\left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}}\right) + .... + \frac{1}{2}\delta^{\alpha}{}_{\alpha}\left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}}\right) + ...
\end{aligned}$$

In the last step we used

$$g_{\alpha\eta}g^{\eta\epsilon} = \delta^{\epsilon}{}_{\alpha}$$

The only non-vanishing term will be the one for which $\eta = \alpha$ so that

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\alpha}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}}\right)$$

**Option (4):** This is correct. Expanding the sum gives

$$\begin{aligned}
\sum_{i=1}^{3}\delta^{i}{}_{i} &= \delta^{1}{}_{1} + \delta^{2}{}_{2} + \delta^{3}{}_{3} \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}$$

**Option (5):** This is incorrect. The Christoffel coefficients are symmetric in their lower indices, but not in the upper and lower index.

The symmetry in their lower indices is a consequence of the symmetry of the metric tensor. We can write

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\sum_{l} g^{il}\left(\frac{\partial g_{lk}}{\partial x^{j}} + \frac{\partial g_{jl}}{\partial x^{k}} - \frac{\partial g_{jk}}{\partial x^{l}}\right)$$

Since $g_{\alpha\beta} = g_{\beta\alpha}$, we can interchange the indices of all the metrics within the sum.

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\sum_{l} g^{il}\left(\frac{\partial g_{kl}}{\partial x^{j}} + \frac{\partial g_{lj}}{\partial x^{k}} - \frac{\partial g_{kj}}{\partial x^{l}}\right)$$

The first two terms can also be switched to give

$$\Gamma^{i}{}_{jk} = \frac{1}{2}\sum_{l} g^{il}\left(\frac{\partial g_{lj}}{\partial x^{k}} + \frac{\partial g_{kl}}{\partial x^{j}} - \frac{\partial g_{kj}}{\partial x^{l}}\right)$$

which is exactly the definition for ${\Gamma^i}_{kj}$. So therefore ${\Gamma^i}_{jk} = {\Gamma^i}_{kj}$, but it does not hold that ${\Gamma^i}_{jk} = {\Gamma^j}_{ik}$.

## Question 3: Kronecker delta tensor

Which of the following expressions are correct?

- $\sum_m \delta_l^m g_{km} = 3g_{km}$
- $\sum_m \delta_l^m g_{km} = g^{kl}$
- $\sum_m \delta_l^m g_{km} = g^{km}$
- $\mathbf{\sum_m \delta_l^m g_{km} = g_{kl}}$*
- $\sum_m \delta_l^m g_{km} = g_{km}$

In the sum

$$\sum_m \delta_l^m g_{km}$$

all the terms in the sum where $m \neq l$, will be zero (since $\delta_l^m = 0$ if $m \neq l$). The term where $m = l$ will be equal to $g_{kl}$, i.e.

$$\begin{aligned}
\sum_m \delta_l^m g_{km} &= \delta_l^0 g_{k0} + \delta_l^1 g_{k1} + \ldots + \delta_l^l g_{kl} + \ldots + \delta_l^N g_{kN} \\
&= (0)\, g_{k0} + (0)\, g_{k1} + \ldots + (1)\, g_{kl} + \ldots + (0)\, g_{kN} \\
&= g_{kl}
\end{aligned}$$

So the Kronecker delta can effectively be used to replace one index with another.

## Question 4: Covariant and contravariant forms

Equation 2.70 in the textbook is written for four dimensional Minkowski space and gives a rule to determine the covariant form of a vector if the metric and contravariant form is known. This same equation written for a general two dimensional space is

$$A_j = \sum_{i=1}^{2} g_{ij} A^i \,.$$

Use this to determine the covariant form of $[A^i]$ in two dimensional space described by the surface of a unit sphere. The metric tensor (with $x^1 = \theta$ and $x^2 = \phi$) for this space is

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

and let

$$[A^i] = \begin{pmatrix} \pi \\ \pi/4 \end{pmatrix}.$$

- $[A_i] = \begin{pmatrix} \pi \\ 1/2 \end{pmatrix}$
- $[A_i] = \begin{pmatrix} \pi \\ \pi/\left(4\sqrt{2}\right) \end{pmatrix}$
- $[A_i] = \begin{pmatrix} \pi \\ \pi/2 \end{pmatrix}$
- $[\mathbf{A_i}] = \begin{pmatrix} \pi \\ \pi/8 \end{pmatrix}$*
- $[A_i] = \begin{pmatrix} \pi \\ \pi/4 \end{pmatrix}$

Expanding the given equation for determining the covariant components of $[A^i]$ gives

$$\begin{aligned} A_j &= \sum_{i=1}^{2} g_{ij} A^i \\ &= g_{1j} A^1 + g_{2j} A^2 \end{aligned}$$

From the information given in the question, we know that $g_{11} = 1$, $g_{12} = g_{21} = 0$, $g_{22} = \sin^2\theta$, $A^1 = \pi$ and $A^2 = \pi/4$. The covariant components are then given by

$$\begin{aligned} A_1 &= g_{11} A^1 + g_{21} A^2 \\ &= (1)(\pi) + (0)(\pi/4) \\ &= \pi \end{aligned}$$

$$\begin{aligned} A_2 &= g_{12}A^1 + g_{22}A^2 \\ &= (0)(\pi) + \left(\sin^2\theta\right)(\pi/4) \\ &= \frac{\pi}{4}\sin^2\theta \end{aligned}$$

You could also have computed it with

$$\begin{aligned} [A_i] &= [g_{ij}]\left[A^i\right] \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}\begin{pmatrix} \pi \\ \frac{\pi}{4} \end{pmatrix} \\ &= \begin{pmatrix} \pi \\ \frac{\pi}{4}\sin^2\theta \end{pmatrix} \end{aligned}$$

## Question 5: Einstein field equations

How many equations does the following expression represent?

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -\kappa T_{\mu\nu}$$

- 1
- 2
- 4
- 8
- **16***

The expression represents 16 different equations. One for each possible combination of $\mu$ and $\nu$, where both indices can have values from 0 to 3, since this equation is written for four dimensional spacetime. The 16 possible combinations are

| # | $\mu$ | $\nu$ | | # | $\mu$ | $\nu$ | | # | $\mu$ | $\nu$ | | # | $\mu$ | $\nu$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **1** | 0 | 0 | | **5** | 1 | 0 | | **9** | 2 | 0 | | **13** | 3 | 0 |
| **2** | 0 | 1 | | **6** | 1 | 1 | | **10** | 2 | 1 | | **14** | 3 | 1 |
| **3** | 0 | 2 | | **7** | 1 | 2 | | **11** | 2 | 2 | | **15** | 3 | 2 |
| **4** | 0 | 3 | | **8** | 1 | 3 | | **12** | 2 | 3 | | **16** | 3 | 3 |

# Tutorial Letter 207/2/2016

## Special Relativity and Riemannian Geometry

## APM3713

## Semester 2

## Department of Mathematical Sciences

**IMPORTANT INFORMATION:**

This tutorial letter contains the solutions to Assignment 07.

BAR CODE

# Memo for Assignment 7 S2 2016

# Chapters 3 & 4

## Question 1

he right helicoid can be parametrized as

$$\begin{aligned} x(u, v) &= u\cos v \\ y(u, v) &= u\sin v \\ z(u, v) &= av \end{aligned}$$

where $a$ is a constant.

(a) Find the line element for the surface.

(b) What is the metric tensor and the dual metric tensor?

(c) Determine the values of all the Christoffel coefficients of the surface.

(d) What is the value of the component $R^1{}_{212}$ of the Riemann curvature tensor?

(e) What is the Ricci tensor for the surface? (Hint: $R^1{}_{212}$ is the only independent component of the Riemann tensor when working with a two dimensional space. It has the symmetries $R^1{}_{212} = R^2{}_{121} = -R^2{}_{112} = -R^1{}_{221}$ with all other components equal to zero.)

(f) What is the curvature scalar $R$ for the surface?

(g) What is the Gaussian curvature of the surface?

(h) Is the surface Euclidean? Explain your answer.

(i) Suppose that the surface is filled with non-interacting particles, or dust. Use the two dimensional version of the energy-momentum tensor for dust and Einstein's field equation to find an expression for the Einstein constant $\kappa$ for this surface.

## Solution

### Part A

In Cartesian coordinates, the line element is given by

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2 \,.$$

We have

$$\begin{aligned} x(u,\, v) &= u\cos v \\ y(u,\, v) &= u\sin v \\ z(u,\, v) &= av \end{aligned}$$

so that

$$\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ &= \frac{\partial}{\partial u}(u\cos v)\; du + \frac{\partial}{\partial v}(u\cos v)\; dv \\ &= \cos v\, du - u\sin v\, dv \end{aligned}$$

Similarly, we get

$$\begin{aligned} dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \\ &= \frac{\partial}{\partial u}(u\sin v)\; du + \frac{\partial}{\partial v}(u\sin v)\; dv \\ &= \sin v\, du + u\cos v\, dv \end{aligned}$$

$$\begin{aligned} dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \\ &= \frac{\partial}{\partial u}(av)\; du + \frac{\partial}{\partial v}(av)\; dv \\ &= a\, dv \end{aligned}$$

Substituting this into the Cartesian line element and simplifying gives

$$\begin{aligned} (dl)^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= (\cos v\, du - u\sin v\, dv)^2 + (\sin v\, du + u\cos v\, dv)^2 + (a\, dv)^2 \end{aligned}$$

$$\begin{aligned} &= \cos^2 v\, du^2 - u\cos v \sin v\, dudv + u^2 \sin^2 v\, dv^2 + \sin^2 v\, du^2 + u \cos v \sin v\, dudv \\ &\qquad +u^2 \cos^2 v\, dv^2 + a^2\, dv^2 \\ &= \left(\cos^2 v + \sin^2 v\right) du^2 + \left[u^2 \left(\sin^2 v + \cos^2 v\right) + a^2\right] dv^2 \\ &= du^2 + \left(u^2 + a^2\right) dv^2 \end{aligned}$$

**Part B**

We know that the line element has the form

$$dl^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$$

If we choose $x^1 = u$ and $x^2 = v$, this reduces to

$$\begin{aligned} dl^2 &= \sum_{i,j=1}^{2} g_{ij} dx^i dx^j \\ &= g_{11} dx^1 dx^1 + 2g_{12} dx^1 dx^2 + g_{22} dx^2 dx^2 \\ &= g_{11} (du)^2 + 2g_{12} dudv + g_{22} (dv)^2 \end{aligned}$$

Above we used the fact that the metric tensor is symmetric $g_{ij} = g_{ji}$. Comparing this to the line element calculated in Part A allows us to identify

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = u^2 + a^2$$

so that the metric tensor for the surface is

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & u^2 + a^2 \end{pmatrix}$$

We know that we must have

$$\sum_k g^{ik} g_{kj} = \delta^i_j$$

so that the dual metric $[g^{ij}]$ is just the matrix inverse of $[g_{ij}]$. We find

$$\left[g^{ij}\right] = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{u^2+a^2} \end{pmatrix}$$

**Part C**

The Christoffel coefficients are defined by

$$\Gamma^h_{ij} = \sum_k \frac{1}{2} g^{hk} \left( g_{ki,j} + g_{jk,i} - g_{ij,k} \right)$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} \left( g_{11,1} + g_{11,1} - g_{11,1} \right) + \frac{1}{2} g^{12} \left( g_{21,1} + g_{12,1} - g_{11,2} \right)$$

All the $g_{ik}$ and $g^{ik}$ where $i \neq k$ will be zero, so their derivatives will also be zero. Remembering this will reduce the calculations a lot. So we have

$$\begin{aligned} v\Gamma^1_{11} &= \frac{1}{2} g^{11} \left( g_{11,1} + g_{11,1} - g_{11,1} \right) + \frac{1}{2} g^{12} \left( g_{21,1} + g_{12,1} - g_{11,2} \right) \\ &= \frac{1}{2} g^{11} g_{11,1} \\ &= \frac{1}{2} (1) \frac{d}{du} (1) \\ &= 0 \end{aligned}$$

Using the symmetric property of the Christoffel coefficients $\Gamma^h_{ij} = \Gamma^h_{ji}$ will also cut down on calculations

$$\begin{aligned} \Gamma^1_{12} = \Gamma^1_{21} &= \frac{1}{2} g^{11} \left( g_{11,2} + g_{21,1} - g_{12,1} \right) + \frac{1}{2} g^{12} \left( g_{21,2} + g_{22,1} - g_{12,2} \right) \\ &= \frac{1}{2} g^{11} g_{11,2} \\ &= \frac{1}{2} (1) \frac{\partial}{\partial v} \left( u^2 + a^2 \right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Gamma^1_{22} &= \frac{1}{2} g^{11} \left( g_{12,2} + g_{21,2} - g_{22,1} \right) + \frac{1}{2} g^{12} \left( g_{22,2} + g_{22,2} - g_{22,2} \right) \\ &= -\frac{1}{2} g^{11} g_{22,1} \\ &= -\frac{1}{2} (1) \frac{\partial}{\partial u} \left( u^2 + a^2 \right) \\ &= -u \end{aligned}$$

$$\begin{aligned}
\Gamma^2{}_{11} &= \frac{1}{2}g^{21}\left(g_{11,1}+g_{11,1}-g_{11,1}\right)+\frac{1}{2}g^{22}\left(g_{21,1}+g_{12,1}-g_{11,2}\right)\\
&= -\frac{1}{2}g^{22}g_{11,2}\\
&= -\frac{1}{2}\left(\frac{1}{u^2+a^2}\right)\frac{\partial}{\partial v}\left(u^2+a^2\right)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^2{}_{12}=\Gamma^2{}_{21} &= \frac{1}{2}g^{21}\left(g_{12,1}+g_{11,2}-g_{21,1}\right)+\frac{1}{2}g^{22}\left(g_{22,1}+g_{12,2}-g_{21,2}\right)\\
&= \frac{1}{2}g^{22}g_{22,1}\\
&= \frac{1}{2}\left(\frac{1}{u^2+a^2}\right)\frac{\partial}{\partial u}\left(u^2+a^2\right)\\
&= \frac{u}{u^2+a^2}
\end{aligned}$$

$$\begin{aligned}
\Gamma^2{}_{22} &= \frac{1}{2}g^{21}\left(g_{12,2}+g_{21,2}-g_{22,1}\right)+\frac{1}{2}g^{22}\left(g_{22,2}+g_{22,2}-g_{22,2}\right)\\
&= \frac{1}{2}g^{22}g_{22,2}\\
&= \frac{1}{2}\left(\frac{1}{u^2+a^2}\right)\frac{\partial}{\partial v}\left(u^2+a^2\right)\\
&= 0
\end{aligned}$$

In summary, the only non-zero Christoffel coefficients that we have are $\Gamma^1{}_{22} = -u$ and $\Gamma^2{}_{12} = \Gamma^2{}_{21} = u/\left(u^2+a^2\right)$.

**Part D**

The Riemann Curvature tensor is defined by

$$R^l{}_{ijk} \equiv \frac{\partial \Gamma^l{}_{ik}}{\partial x^j} - \frac{\partial \Gamma^l{}_{ij}}{\partial x^k} + \sum_m \Gamma^m_{ik}\Gamma^l{}_{mj} - \sum_m \Gamma^m_{ij}\Gamma^l{}_{mk}$$

since we are dealing with a two dimensional surface, the only independent entry will be $R^1{}_{212}$, so it will be sufficient to only calculate this. We have

$$\begin{aligned} R^1{}_{212} &= \frac{\partial \Gamma^1{}_{22}}{\partial x^1} - \frac{\partial \Gamma^1{}_{21}}{\partial x^2} + \sum_m \Gamma^m{}_{22}\Gamma^1{}_{m1} - \sum_m \Gamma^m{}_{21}\Gamma^1{}_{m2} \\ &= \frac{\partial \Gamma^1{}_{22}}{\partial u} - \frac{\partial \Gamma^1{}_{21}}{\partial v} + \Gamma^1{}_{22}\Gamma^1{}_{11} + \Gamma^2{}_{22}\Gamma^1{}_{21} - \Gamma^1{}_{21}\Gamma^1{}_{12} - \Gamma^2{}_{21}\Gamma^1{}_{22} \\ &= \frac{\partial \Gamma^1{}_{22}}{\partial u} - \Gamma^2{}_{21}\Gamma^1{}_{22} \\ &= \frac{\partial}{\partial u}(-u) - \left(\frac{u}{u^2+a^2}\right)(-u) \\ &= -1 + \frac{u^2}{u^2+a^2} \\ &= \frac{-u^2 - a^2 + u^2}{u^2+a^2} \\ &= \frac{-a^2}{u^2+a^2} \end{aligned}$$

For the Riemann curvature tensor we have

$$R^1{}_{212} = R^2{}_{121} = \frac{-a^2}{u^2+a^2}$$

$$R^1{}_{221} = R^2{}_{112} = \frac{a^2}{u^2+a^2}$$

With all other entries equal to zero.

**Part E**

The Ricci tensor is defined by

$$R_{ij} \equiv \sum_k R^k{}_{ijk}$$

Using the fact that the Ricci tensor is symmetric we find the 4 entries of the Ricci tensor

$$\begin{aligned} R_{11} &= R^1{}_{111} + R^2{}_{112} \\ &= \frac{a^2}{u^2+a^2} \end{aligned}$$

$$\begin{aligned} R_{12} = R_{21} &= R^1{}_{121} + R^2{}_{122} \\ &= 0 \end{aligned}$$

$$\begin{aligned} R_{22} &= R^1{}_{221} + R^2{}_{222} \\ &= \frac{a^2}{u^2+a^2} \end{aligned}$$

**Part F**

The Ricci scalar is defined by

$$R \equiv \sum_{i,j} g^{ij} R_{ij}$$

So we have for the helicoid

$$\begin{aligned} R &= g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22} \\ &= g^{11}R_{11} + g^{22}R_{22} \\ &= (1)\left(\frac{-a^2}{u^2+a^2}\right) + \left(\frac{1}{u^2+a^2}\right)\left(\frac{-a^2}{u^2+a^2}\right) \\ &= \frac{-a^2u^2 - a^4 - a^2}{(u^2+a^2)^2} \\ &= \frac{-a^2\left(u^2+a^2+1\right)}{(u^2+a^2)^2} \end{aligned}$$

**Part G**

The Gaussian curvature of a two dimensional surface is given by

$$K = \frac{R_{1212}}{g}$$

where $g = \det[g_{ij}]$ (see Exercise 3.16 p105).

The determinant of a diagonal matrix is just the product of its diagonal entries so that

$$\begin{aligned} g &= \prod_i g_{ii} \\ &= (1)\left(u^2+a^2\right) \\ &= u^2+a^2 \end{aligned}$$

$R_{1212}$ is the element of the Riemann curvature tensor with an index lowered, i.e.

$$R_{1212} = \sum_i g_{i1} R^i{}_{212}$$

$$\begin{aligned} &= g_{11}R^{1}{}_{212} + g_{21}R^{2}{}_{212} \\ &= (1)\left(\frac{-a^2}{u^2+a^2}\right) \\ &= \frac{-a^2}{u^2+a^2} \end{aligned}$$

So we have for the Gaussian curvature

$$\begin{aligned} K &= \frac{R_{1212}}{g} \\ &= \left(\frac{-a^2}{u^2+a^2}\right)\left(\frac{1}{u^2+a^2}\right) \\ &= \frac{-a^2}{(u^2+a^2)^2} \end{aligned}$$

**Part H**

No, the helicoid is not Euclidean (flat). The necessary and sufficient condition for a surface to be flat is that the Riemann curvature tensor (all its components) should vanish (be equal to zero) at all points on the surface. This is not true for all values of $u$ and $v$.

**Part I**

Einstein's field equation for two dimensions is

$$R_{ij} - \frac{1}{2}Rg_{ij} = -\kappa T_{ij}$$

where $i$ and $j$ can take the values of 1 or 2, as with the rest of the calculations regarding the surface above. The only non-zero component of the energy-momentum tensor $[T^{ij}]$ for dust is $T^{11} = \rho c^2$.

$[T^{ij}]$ is related to $[T_{ij}]$ by

$$T_{ij} = \sum_{m,n} g_{im}g_{jn}T^{mn}$$

Clearly, the only non-zero component of $[T_{ij}]$ will be $T_{11}$. We find

$$\begin{aligned} T_{11} &= \sum_{m,n} g_{1m} g_{1n} T^{mn} \\ &= g_{11}g_{11}T^{11} + g_{11}g_{12}T^{12} + g_{12}g_{11}T^{21} + g_{12}g_{12}T^{22} \\ &= g_{11}g_{11}T^{11} \\ &= \rho c^2 \end{aligned}$$

Now all the quantities in the Einstein field equation are known. We substitute and solve for $\kappa$

$$\begin{aligned} R_{11} - \frac{1}{2}Rg_{11} &= -\kappa T_{11} \\ \frac{-a^2}{u^2+a^2} - \frac{1}{2}\left(\frac{-a^2\left(u^2+a^2+1\right)}{\left(u^2+a^2\right)^2}\right)(1) &= -\kappa\rho c^2 \\ \frac{-2a^2u^2 - 2a^4 + a^2u^2 + a^4 + a^2}{2\left(u^2+a^2\right)^2} &= -\kappa\rho c^2 \\ \frac{-a^2u^2 - a^4 + a^2}{2\left(u^2+a^2\right)^2} &= -\kappa\rho c^2 \\ \kappa &= \frac{a^2\left(u^2+a^2-1\right)}{2\rho c^2\left(u^2+a^2\right)^2} \end{aligned}$$

# Question 2

Show that

$$\Gamma_{ijk} + \Gamma_{kji} = \frac{\partial g_{ik}}{\partial x^j}\,.$$

## Solution

Lowering the contravariant index of $\Gamma^n{}_{jk}$ gives

$$\sum_n g_{in}\Gamma^n{}_{jk} = \frac{1}{2}\sum_n\sum_m g_{in}g^{nm}\left(g_{mj,k} + g_{km,j} - g_{jk,m}\right)$$

$$
\begin{aligned}
\Gamma_{ijk} &= \frac{1}{2}\sum_m \delta_i^m \left(g_{mj,k} + g_{km,j} - g_{jk,m}\right) \\
\Gamma_{ijk} &= \frac{1}{2}\left(g_{ij,k} + g_{ki,j} - g_{jk,i}\right)
\end{aligned}
$$

In the second step we used the property of the metric tensor $\sum_m g_{in} g^{nm} = \delta_i^m$ (equation 3.24 in the textbook).

In the third step above we used the definition of the Kronecker delta. Consider the multiplication of the Kronecker delta and a general tensor

$$
\sum_a \delta^b{}_a X^a
$$

all the terms in the sum where $a \neq b$, will be zero, where the term where $a = b$ will be equal to $X^b$, i.e.

$$
\begin{aligned}
\sum_a \delta^b{}_a X^a &= \delta^b{}_0 X^0 + \delta^b{}_1 X^1 + \ldots + \delta^b{}_b X^b + \ldots + \delta^b{}_N X^N \\
&= (0)\, X^0 + (0)\, X^1 + \ldots + (1)\, X^b + \ldots + (0)\, X^N \\
&= X^b
\end{aligned}
$$

So the Kronecker delta can effectively be used to substituted one index for another.

Similarly,

$$
\Gamma_{kji} = \frac{1}{2}\left(g_{kj,i} + g_{ik,j} - g_{ji,k}\right)
$$

Summing the above gives

$$
\begin{aligned}
\Gamma_{ijk} + \Gamma_{kji} &= \frac{1}{2}\left(g_{ij,k} + g_{ki,j} - g_{jk,i} + g_{kj,i} + g_{ik,j} - g_{ji,k}\right) \\
&= \frac{1}{2}\left(g_{ij,k} + g_{ki,j} - g_{jk,i} + g_{kj,i} + g_{ik,j} - g_{ji,k}\right) \\
&= \frac{1}{2}\left(0 + 0 + 2g_{ik,j}\right) \\
&= \frac{\partial g_{ik}}{\partial x^j}
\end{aligned}
$$

# Question 3

Verify that if a tensor is symmetric in one frame, it will be symmetric in all coordinate frames. That is, show that if it is given that $X^{ij} = X^{ji}$ in frame $S$, then it will be true that $\bar{X}^{ij} = \bar{X}^{ji}$ in a coordinate frame $\bar{S}$.

## Solution

If $X^{ij} = X^{ji}$, then

Since $X^{ij}$ is a tensor, we know that it transforms as follows

$$\bar{X}^{ab} = \sum_i \sum_j \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} X^{ij}$$

On the RHS both $i$ and $j$ are just dummy indices, i.e. they are being summed over. This means that the two indices can be replaced by any other indices without changing the meaning of the expression, since they are just counters to be summed over, i.e.

$$\sum_i \sum_j \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} X^{ij} = \sum_\alpha \sum_\beta \frac{\partial \bar{x}^a}{\partial x^\alpha} \frac{\partial \bar{x}^b}{\partial x^\beta} X^{\alpha\beta} = \sum_r \sum_s \frac{\partial \bar{x}^a}{\partial x^r} \frac{\partial \bar{x}^b}{\partial x^s} X^{rs}$$

In particular, we can replace $j$ with $i$ and $i$ with $j$, so that

$$\begin{aligned} \bar{X}^{ab} &= \sum_i \sum_j \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} X^{ij} \\ &= \sum_j \sum_i \frac{\partial \bar{x}^a}{\partial x^j} \frac{\partial \bar{x}^b}{\partial x^i} X^{ji} \\ &= \sum_j \sum_i \frac{\partial \bar{x}^a}{\partial x^j} \frac{\partial \bar{x}^b}{\partial x^i} X^{ij} \end{aligned}$$

In the last step we used the symmetry of property $X^{ij} = X^{ji}$. This is the transformation expression for a second order contravariant tensor where $x^i \to \bar{x}^b$ and $x^j \to \bar{x}^a$ so we have

$$\bar{X}^{ab} = \sum_j \sum_i \frac{\partial \bar{x}^a}{\partial x^j} \frac{\partial \bar{x}^b}{\partial x^i} X^{ij}$$

$$= \quad \bar{X}^{ba}$$

Thus we have shown that if a tensor is symmetric in one coordinate frame, i.e. $X^{ij} = X^{ji}$ in $S$, then it is also symmetric in any other arbitrary coordinate frame $\bar{S}$.

# Question 4

Using the definition of the Riemann curvature tensor (equation 3.35 in the textbook), prove the identity

$$R^l{}_{ijk} + R^l{}_{kij} + R^l{}_{jki} = 0\,.$$

## Solution

Equation 3.35 in the textbook gives

$$R^l{}_{ijk} = \frac{\partial \Gamma^l{}_{ik}}{\partial x^j} - \frac{\partial \Gamma^l{}_{ij}}{\partial x^k} + \sum_m \Gamma^m_{ik}\Gamma^l{}_{mj} - \sum_m \Gamma^m_{ij}\Gamma^l{}_{mk}$$

Since the labels of indices has no intrinsic meaning, we can write

$$R^l{}_{kij} = \frac{\partial \Gamma^l{}_{kj}}{\partial x^i} - \frac{\partial \Gamma^l{}_{ki}}{\partial x^j} + \sum_m \Gamma^m_{kj}\Gamma^l{}_{mi} - \sum_m \Gamma^m_{ki}\Gamma^l{}_{mj}$$

and

$$R^l{}_{jki} = \frac{\partial \Gamma^l{}_{ji}}{\partial x^k} - \frac{\partial \Gamma^l{}_{jk}}{\partial x^i} + \sum_m \Gamma^m_{ji}\Gamma^l{}_{mk} - \sum_m \Gamma^m_{jk}\Gamma^l{}_{mi}$$

Using the fact that the Christoffel coefficients are symmetric in the lower indices, i.e. $\Gamma^i{}_{jk} = \Gamma^i{}_{kj}$ we get

$$\begin{aligned} R^l{}_{ijk} + R^l{}_{kij} &= \left(\frac{\partial \Gamma^l{}_{ik}}{\partial x^j} - \frac{\partial \Gamma^l{}_{ki}}{\partial x^j}\right) + \left(\sum_m \Gamma^m_{ik}\Gamma^l{}_{mj} - \sum_m \Gamma^m_{ki}\Gamma^l{}_{mj}\right) - \sum_m \Gamma^m_{ij}\Gamma^l{}_{mk} + \sum_m \Gamma^m_{kj}\Gamma^l{}_{mi} \\ &= 0 - \frac{\partial \Gamma^l{}_{ij}}{\partial x^k} + \frac{\partial \Gamma^l{}_{kj}}{\partial x^i} + 0 - \sum_m \Gamma^m_{ij}\Gamma^l{}_{mk} + \sum_m \Gamma^m_{kj}\Gamma^l{}_{mi} \end{aligned}$$

and it follows that $R^l{}_{ijk} + R^l{}_{kij} = \left(-\frac{\partial \Gamma^l{}_{ij}}{\partial x^k} + \frac{\partial \Gamma^l{}_{ji}}{\partial x^k}\right) + \left(\frac{\partial \Gamma^l{}_{kj}}{\partial x^i} - \frac{\partial \Gamma^l{}_{jk}}{\partial x^i}\right) + \left(-\sum_m \Gamma^m_{ij}\Gamma^l{}_{mk} + \sum_m \Gamma^m_{ji}\Gamma^l{}_{mk}\right) + \left(\sum_m \Gamma^m_{kj}\Gamma^l{}_{mi} - \sum_m \Gamma^m_{jk}\Gamma^l{}_{mi}\right) = 0$