Tutorial Letter 205/2/2017

Special Relativity and Riemannian Geometry

APM3713

Semester 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains the solutions to Assignment 05.

BAR CODE

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Memo for Assignment 5 S2 2017

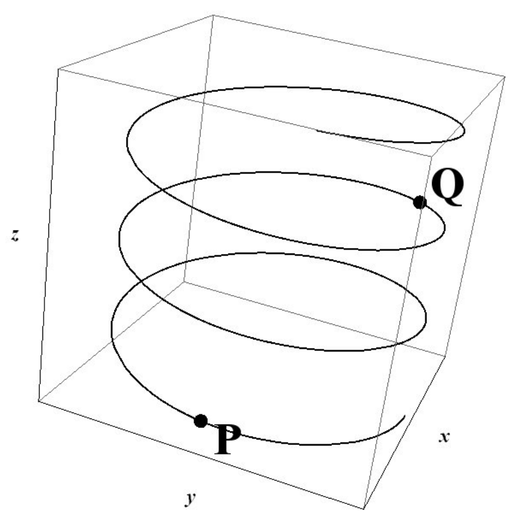
Basics of differential geometry (§ 3)

Question 1: Arc length

The equation for a length of a curve in an Euclidean plane can easily be generalized to give the length of a curve that exists in Euclidean (normal) three dimensional space. The length of such a space curve is given by

$$L\left(\mathrm{P},\,\mathrm{Q}\right) = \int_{P}^{Q} dl = \int_{u_{\mathrm{P}}}^{u_{\mathrm{Q}}} \left(\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2 \right)^{1/2} du$$

Consider the circular helix described by $x = \sin u$, $y = \cos u$ and $z = u/10$ with the points P and Q defined by the points where $u_{\mathrm{P}} = \pi/2$ and $u_{\mathrm{Q}} = 15\pi/4$ as shown in the figure below.



What is the length of the curve between points P and Q in arbitrary units?

- 18.79
- 13.42

- **10.26***
- 10.31
- 1.02

First we calculate the derivatives of the Cartesian coordinates with respect the the parameter u.

$$\begin{aligned}
\frac{dx}{du} &= \frac{d}{du}(\sin u) = \cos u \\
\frac{dy}{du} &= \frac{d}{du}(\cos u) = -\sin u \\
\frac{dz}{du} &= \frac{d}{du}\left(\frac{u}{10}\right) = \frac{1}{10}
\end{aligned}$$

The length of the helix is then given by

$$\begin{aligned}
L(\mathrm{P},\,\mathrm{Q}) &= \int_{u_\mathrm{P}}^{u_\mathrm{Q}} \left(\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2\right)^{1/2} du \\
&= \int_{\pi/2}^{15\pi/4} \left(\cos^2 u + \sin^2 u + \frac{1}{100}\right)^{1/2} du \\
&= \int_{\pi/2}^{15\pi/4} \left(1 + \frac{1}{100}\right)^{1/2} du \\
&= \int_{\pi/2}^{15\pi/4} \left(\frac{101}{100}\right)^{1/2} du \\
&= \int_{\pi/2}^{15\pi/4} \frac{\sqrt{101}}{10}\, du \\
&= \left[\frac{\sqrt{101}}{10} u\right]_{u=\pi/2}^{u=15\pi/4} \\
&= \frac{\sqrt{101}}{10}\left(\frac{15\pi}{4}\right) - \frac{\sqrt{101}}{10}\left(\frac{\pi}{2}\right) \\
&= \frac{13\pi\sqrt{101}}{40} \\
&= 10.26
\end{aligned}$$

Question 2: Metric tensor

The line element for a certain two dimensional Riemann space is given by

$$dl^2 = dr^2 + 2r\sin\phi\, dr d\phi + r^2 d\phi^2\,.$$

What is the metric tensor of this space?

- $\begin{pmatrix} 1 & 2r\sin\phi \\ 2r\sin\phi & r^2 \end{pmatrix}$
- $\begin{pmatrix} 1 & r\sin\phi \\ r\sin\phi & r^2 \end{pmatrix}$ *
- $\begin{pmatrix} r\sin\phi & 1 \\ 1 & r\sin\phi \end{pmatrix}$
- $\begin{pmatrix} r^2 & 2r\sin\phi \\ 0 & 1 \end{pmatrix}$
- $\begin{pmatrix} 1 & 0 \\ 2r\sin\phi & r^2 \end{pmatrix}$

The line element for a general Riemann space is given by

$$dl^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$$

Since we are considering a two dimensional space ($n = 2$), we can can expand this to

$$dl^2 = g_{11}dx^1dx^1 + g_{12}dx^1dx^2 + g_{21}dx^2dx^1 + g_{22}dx^2dx^2$$

Choosing $x^1 = r$ and $x^2 = \phi$, we get

$$dl^2 = g_{11}dr^2 + g_{12}drd\phi + g_{21}d\phi dr + g_{22}d\phi^2$$

The metric tensor must be symmetric. (This ensures that the distance from the point P to the point Q will be the same as the distance from point Q to point P.) This means that we must have $g_{12} = g_{21}$. From the given line element, we can see that we have $g_{11} = 1$, $g_{12} = g_{21} = r\sin\phi$ and $g_{22} = r^2$. Putting this in array format gives

$$\begin{pmatrix} 1 & r\sin\phi \\ r\sin\phi & r^2 \end{pmatrix}.$$

Question 3: Kronecker delta

The sum

$$\sum_{i=1}^{3} \delta_{ii}$$

is equal to...

- 0
- 1
- 2
- **3***
- 4

The definition of the Kroneker delta is

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

It is a very common mistake to say that this sum is equal to 1. But it is a sum over number of components equal to 1 and the answer will depend on the number of dimensions you are working in. In this case

$$\begin{aligned} \sum_{i=1}^{3} \delta_{ii} &= \delta_{11} + \delta_{22} + \delta_{33} \\ &= 1 + 1 + 1 \\ &= 3 \end{aligned}$$

Question 4: Covariant and contravariant forms of vectors

Equation 2.70 in the textbook is written for four dimensional Minkowski space and gives a rule to determine the covariant form of a vector if the metric and contravariant form is known. This same equation written for a general two dimensional space is

$$A_j = \sum_{i=1}^{2} g_{ij} A^i \,.$$

Use this to determine the covariant form of $[A^i]$ in two dimensional space described by the surface of a unit sphere. The metric tensor (with $x^1 = \theta$ and $x^2 = \phi$) for this space is

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

and let

$$\left[A^i\right] = \begin{pmatrix} \pi \\ \pi/4 \end{pmatrix} .$$

- $[A_i] = \begin{pmatrix} \pi \\ 1/2 \end{pmatrix}$
- $[A_i] = \begin{pmatrix} \pi \\ \pi/\left(4\sqrt{2}\right) \end{pmatrix}$
- $[A_i] = \begin{pmatrix} \pi \\ \pi/2 \end{pmatrix}$
- $[\mathbf{A_i}] = \begin{pmatrix} \pi \\ \pi/8 \end{pmatrix}$ *
- $[A_i] = \begin{pmatrix} \pi \\ \pi/4 \end{pmatrix}$

Expanding the given equation for determining the covariant components of $[A^i]$ gives

$$\begin{aligned} A_j &= \sum_{i=1}^{2} g_{ij} A^i \\ &= g_{1j} A^1 + g_{2j} A^2 \end{aligned}$$

From the information given in the question, we know that $g_{11} = 1$, $g_{12} = g_{21} = 0$, $g_{22} = \sin^2\theta$, $A^1 = \pi$ and $A^2 = \pi/4$. The covariant components are then given by

$$\begin{aligned} A_1 &= g_{11}A^1 + g_{21}A^2 \\ &= (1)(\pi) + (0)(\pi/4) \\ &= \pi \end{aligned}$$

$$\begin{aligned} A_2 &= g_{12}A^1 + g_{22}A^2 \\ &= (0)(\pi) + \left(\sin^2\theta\right)(\pi/4) \\ &= \frac{\pi}{4}\sin^2\theta \end{aligned}$$

You could also have computed it with

$$\begin{aligned} [A_i] &= [g_{ij}]\left[A^i\right] \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}\begin{pmatrix} \pi \\ \frac{\pi}{4} \end{pmatrix} \\ &= \begin{pmatrix} \pi \\ \frac{\pi}{4}\sin^2\theta \end{pmatrix} \end{aligned}$$

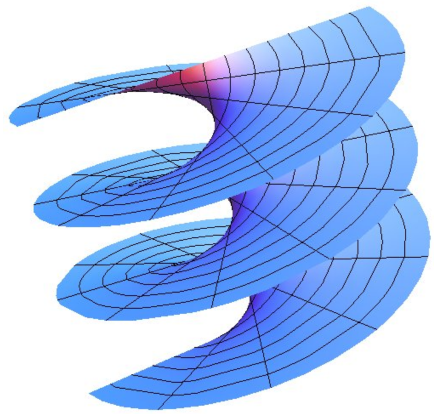
Question 5: Riemann tensor

Calculate $R^1{}_{221}$ of the right helicoid shown below that is parametrized as

$$\begin{aligned} x &= u\cos v \\ y &= u\sin v \\ z &= cv \end{aligned}$$

where c is a constant and $x^1 = u$ and $x^2 = v$ if it is given that the only non-zero Christoffel coefficients for the surface are

$$\begin{aligned} \Gamma^2{}_{12} &= \Gamma^2{}_{21} = \frac{u}{u^2 + c^2} \\ \Gamma^1{}_{22} &= \frac{-u}{u^2 + c^2} \end{aligned}$$



- $\mathbf{(c^2 - 2u^2)(u^2 + c^2)^{-2}}$*
- $-c^2 (u^2 + c^2)^{-2}$
- $c^2 (u^2 + c^2)^{-2}$
- $-u (c^2 + u + u^2)(u^2 + c^2)^{-2}$
- $-u^2 (u^2 + c^2)^{-2}$

Using equation 3.35 in the textbook, the element $R^1_{\ 221}$ of the Riemann tensor is given by

$$\begin{aligned} R^1_{\ 221} &= \frac{\partial \Gamma^1_{\ 21}}{\partial x^2} - \frac{\partial \Gamma^1_{\ 22}}{\partial x^1} + \sum_m \Gamma^m_{\ 21} \Gamma^1_{\ m2} - \sum_m \Gamma^m_{\ 22} \Gamma^1_{\ m1} \\ &= \frac{\partial \Gamma^1_{\ 21}}{\partial x^2} - \frac{\partial \Gamma^1_{\ 22}}{\partial x^1} + \left(\Gamma^1_{\ 21} \Gamma^1_{\ 12} + \Gamma^2_{\ 21} \Gamma^1_{\ 22} \right) - \left(\Gamma^1_{\ 22} \Gamma^1_{\ 11} + \Gamma^2_{\ 22} \Gamma^1_{\ 21} \right) \end{aligned}$$

Substituting all the zero Christoffel symbols, this reduces to

$$R^1_{\ 221} = -\frac{\partial \Gamma^1_{\ 22}}{\partial x^1} + \Gamma^2_{\ 21} \Gamma^1_{\ 22}$$

Substituting the given values for the non-zero Christoffel symbols and using $x^1 = u$ gives

$$\begin{aligned} R^1_{\ 221} &= -\frac{\partial}{\partial u} \left(\frac{-u}{u^2 + c^2} \right) + \left(\frac{u}{u^2 + c^2} \right) \left(\frac{-u}{u^2 + c^2} \right) \\ &= \frac{c^2 - u^2}{(u^2 + c^2)^2} - \frac{u^2}{(u^2 + c^2)^2} \\ &= \frac{c^2 - 2u^2}{(u^2 + c^2)^2} \end{aligned}$$