Tutorial Letter 207/2/2017

Special Relativity and Riemannian Geometry

APM3713

Semester 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains the solutions to Assignment 07.

BAR CODE

Tutorial Letter 101/3/2017

Special Relativity And Riemannian Geometry

Semester 2

Assignment 1

1) $x^1 = x = a\cos v$, $x^2 = y = a\sin v$, $x^3 = z = u$

a) $ds^2 = g_{ab}\, dx^a dx^b$

$= dx^a dx_a$

$= dx^1 dx_1 + dx^2 dx_2 + dx^3 dx_3$

$= (-a\sin v\, dv)^2 + (a\cos v\, dv)^2 + (du)^2$

$= a^2 \sin^2 v\, dv^2 + a^2\cos^2 v\, dv^2 + du^2$

$ds^2 = a^2 dv^2 + du^2$

b) The metric tensor. $g_{ab} = \begin{pmatrix} a^2 & 0 \\ 0 & 1 \end{pmatrix}$

The dual metric tensor. $g^{ab} = \begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & 1 \end{pmatrix}$

c) The only non-zero components of the metric tensor are, $g_{11}, g_{22}, g^{11}, g^{22}$.

The Christoffel coefficients of the surface are given by,

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\delta}\left(\partial_\beta g_{\delta\gamma} + \partial_\gamma g_{\beta\delta} - \partial_\delta g_{\beta\gamma}\right)$$

$\therefore$ All partial derivatives of the metric vanish since it contains constant components.

$$\Gamma^{\alpha}{}_{\beta\gamma} = 0$$

$\longrightarrow$

d) The Riemann Curvature Tensor $R^{\alpha}{}_{\beta\lambda\delta} = 0$.

Since $R^{\alpha}{}_{\beta\lambda\delta} = \partial_{\lambda}\Gamma^{\alpha}{}_{\beta\delta} - \partial_{\delta}\Gamma^{\alpha}{}_{\beta\lambda} + \Gamma^{\gamma}{}_{\beta\delta}\Gamma^{\alpha}{}_{\gamma\lambda} - \Gamma^{\gamma}{}_{\beta\lambda}\Gamma^{\alpha}{}_{\gamma\delta}$

e) The Ricci tensor is $R_{\alpha\beta} = R^{\lambda}{}_{\alpha\lambda\beta} = 0$

f) The curvature scalar $R = R^{\alpha}{}_{\beta} = 0$

g) The Gaussian Curvature $K = \dfrac{R_{1212}}{g} = 0$

h) yes, since all the components of $R^{\alpha}{}_{\beta\gamma\delta} = 0$.

i) Since the Riemann Curvature Tensor is zero.

we have that $\kappa T_{\mu\nu} = 0$, Hence $\kappa = 0$

2.) The Christoffel symbol $\Gamma^{i}{}_{jk} = \frac{1}{2} g^{il}(\partial_j g_{lk} + \partial_k g_{lj} - \partial_l g_{jk})$

$$\therefore \Gamma^{i}{}_{ik} = \frac{1}{2} g^{il}(\partial_i g_{lk} + \partial_k g_{li} - \partial_l g_{ik})$$

$$= \frac{1}{2} g^{il} \partial_k g_{il}$$

$\longrightarrow$

3.) $X^{ij} = X^{ji}$ in S

in $\bar{S}$, $\bar{X}^{ij} = \partial_k \bar{x}^i \partial_\ell \bar{x}^j X^{k\ell}$

$$\bar{X}^{ij} = \partial_k \bar{x}^i \partial_\ell \bar{x}^j X^{\ell k} \quad \text{---- (1)}$$

then Also $\bar{X}^{ji} = \partial_k \bar{x}^j \partial_\ell \bar{x}^i X^{k\ell}$

$$\bar{X}^{ji} = \partial_k \bar{x}^j \partial_\ell \bar{x}^i X^{\ell k} \quad \text{--- (2)}$$

replace j with i, there $\bar{X}^{ij} = \bar{X}^{ji}$.

4.) $R_{ik\ell m} = K(g_{i\ell} g_{km} - g_{im} g_{k\ell})$

$$R_{km} = R^\ell{}_{k\ell m}$$
$$= K(4 g_{km} - g_{mk})$$
$$= 3K g_{km}$$

$R = R^m{}_m = 12K$ or $-12K$

5.) $\bar{g}_{ij} = K g_{ij}$, $K \bar{g}^{ij} = g^{ij}$

$$\Gamma^i_{jk} = \frac{1}{2} K \bar{g}^{i\ell}\left(\frac{1}{K}\partial_j \bar{g}_{\ell k} + \frac{1}{K}\partial_k \bar{g}_{\ell j} - \frac{1}{K}\partial_\ell \bar{g}_{jk}\right) = \bar{\Gamma}^i_{jk}$$

This means that the Curvature tensors are the same.

$$\therefore \bar{g}^{ij} \bar{g}_{ik} = \bar{\delta}^j_k = \frac{1}{K} g^{ij} K g_{ik} = \delta^i_j$$

$$R^{\alpha}{}_{\beta\gamma\delta} = \bar{R}^{\alpha}{}_{\beta\gamma\delta} \Rightarrow \bar{R}_{\alpha\beta} = R_{\alpha\beta}, \quad \bar{R} = \frac{R}{K}$$

$$\therefore \quad G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$$

$$\bar{G}_{\alpha\beta} = \bar{R}_{\alpha\beta} - \frac{1}{2} \bar{R} \bar{g}_{\alpha\beta}$$

$$= R_{\alpha\beta} - \frac{1}{2} \frac{R}{K} K g_{\alpha\beta}$$

$$= R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$$

$$= G_{\alpha\beta}$$

Total = 22

= 100%