Tutorial Letter 204/2/2018

Special Relativity and Riemannian Geometry

APM3713

Semester 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains the solutions to Assignment 04.

BAR CODE

Memo for Assignment 4 S2 2018

Chapters 1 & 2

Question 1

Bob, standing at the rear end of a railroad car, shoots an arrow toward the front end of the car. The velocity of the arrow as measured by Bob is $1/5c$. The length of the car as measured by Bob is 150 meters. Alice, standing on the station platform observes all of this as the train passes by her with a velocity of $3/5c$. What values does Alice measure for the following quantities:

(a) The length of the railroad car.

(b) The velocity of the arrow.

(c) The amount of the time the arrow is in the air.

(d) The distance that the arrow travels.

Solution

Part A

Let's call the frame where Bob is at rest S' and the frame where Alice is at rest S. The Lorentz factor between the two frames are

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\frac{V^2}{c^2}}} \\ &= \frac{1}{\sqrt{1-\frac{9}{25}}} \\ &= \frac{1}{\sqrt{\frac{16}{25}}} \\ &= \frac{5}{4}\end{aligned}$$

Since Bob is at rest with respect to the railroad car, he will measure the proper length L_P. The length that Alice measures L will be contracted according to

$$\begin{aligned} L &= \frac{L_P}{\gamma} \\ &= \frac{4}{5} 150\,\mathrm{m} \\ &= 120\,\mathrm{m} \end{aligned}$$

So Alice will measure the railroad car to be 120 m long.

Part B

To determine the speed of the arrow as measured by Alice v from the speed as measured by Bob v', we use the velocity transformation equation. The velocity transformation equations given in the textbook on p30 is given as

$$v' = \frac{v - V}{1 - vV/c^2}$$

where V is the relative speed between the two frames ($V = 3/5c$ in this case) and v is the speed of the moving object (arrow) as measured in the S frame and v' is the the speed of the moving object as measured in the S' frame, where the two frames are in the standard configuration.

To solve this problem, we are actually looking for the inverse velocity transformation, since we know v' and want to calculate v. You can argue that S is moving in the negative x-direction wrt S', so we can just replace V with $-V$ and v with v' (similar to obtaining the inverse Lorentz transformation equations). We can check this approach with a little algebra:

$$\begin{aligned} v' &= \frac{v - V}{1 - vV/c^2} \\ v'\left(1 - vV/c^2\right) &= v - V \\ v' - v'vV/c^2 &= v - V \\ v + v'vV/c^2 &= V + v' \\ v\left(1 + v'V/c^2\right) &= V + v' \\ v &= \frac{V + v'}{1 + v'V/c^2} \end{aligned}$$

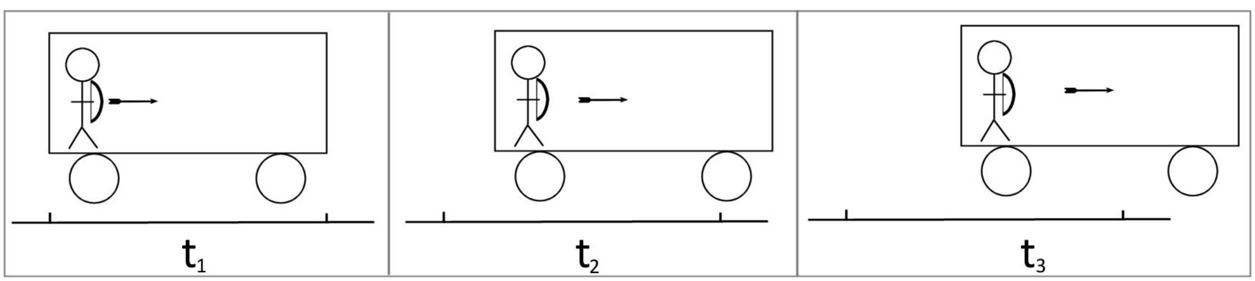
From here it is simple to determine the speed that Alice measures for the arrow by plugging in the known values

$$\begin{aligned} v &= \frac{V + v'}{1 + v'V/c^2} \\ &= \frac{3c/5 + c/5}{1 + (1c/5)(3c/5)/c^2} \\ &= \frac{4c/5}{28/25} \\ &= \frac{4c}{5} \times \frac{25}{28} \\ &= \frac{5}{7}c \end{aligned}$$

So Alice measures the arrow's speed to be $v = 5/7c$.

Part C

The important thing to recognise when doing this question is that the arrow is moving in the same direction as the arrow, so the arrow will travel further than just the length of the train car. The figure below illustrates this from Alice's point of view.



The markers at the bottom of each panel in the figure shows the length of the train car. The arrow will stop travelling once it hits the far wall of the car. As the arrow travels to the right, the far wall of the car is also moving to the right, so the arrow will eventually travel further than the length of the car.

The total distance that the arrow will travel, will be equal to the length of the car plus the distance the car has travelled in the in the time the arrow is in the air. We write this as (all quantities as measured in S)

$$\text{distance arrow travels} = (\text{length of car}) + (\text{distance car travels})$$

$$\Delta x_{arrow} = L + \Delta x_{train}$$

We use the formula

$$v = \frac{\Delta x}{\Delta t}$$

which is always valid *in a single frame* if the speed is constant. It is important to note that this formula is only valid for intervals of x and t, not for single coordinates.

Both of the intervals of distance we are considering is travelled in the same amount of time, which is the time that the arrow is in the air, Δt_{arrow}. Now we can write

$$\begin{aligned}
v_{arrow}\Delta t_{arrow} &= L + v_{train}\Delta t_{arrow} \\
v_{arrow}\Delta t_{arrow} &= L + V\Delta t_{arrow} \\
\Delta t_{arrow}\left(v_{arrow} - V\right) &= L \\
\Delta t_{arrow} &= \frac{L}{v_{arrow} - V} \\
&= \frac{120\,\mathrm{m}}{5c/7 - 3c/5} \\
&= \frac{35 \times 120\,\mathrm{m}}{4 \times 3 \times 10^{8}\,\mathrm{ms}^{-1}} \\
&= 3.5 \times 10^{-6}\,\mathrm{s}
\end{aligned}$$

So, according to Alice, the arrow is in the air for 3.5×10^{-6} seconds.

Part D

Again we use the definition of speed

$$v = \frac{\Delta x}{\Delta t}$$

We have already calculated the speed and the time interval that the arrow is in the air according to Alice, so we get for the distance the arrow travels

$$\begin{aligned}
\Delta x_{arrow} &= v_{arrow}\Delta t_{arrow} \\
&= \left(\frac{5c}{7}\right)\left(3.5 \times 10^{-6}\,\mathrm{s}\right) \\
&= \left(\frac{5}{7} \times 3 \times 10^{8}\,\mathrm{ms}^{-1}\right)\left(3.5 \times 10^{-6}\,\mathrm{s}\right) \\
&= 750\,\mathrm{m}
\end{aligned}$$

Or you could use the formula constructed in Part C:

$$\begin{aligned}
\Delta x_{arrow} &= L + V\Delta t_{arrow} \\
&= 120\,\mathrm{m} + \frac{3c}{5}\left(3.5\times10^{-6}\,\mathrm{s}\right) \\
&= 120\,\mathrm{m} + \frac{3}{5}\left(3\times10^{8}\,\mathrm{ms^{-1}}\right)\left(3.5\times10^{-6}\,\mathrm{s}\right) \\
&= 750\,\mathrm{m}
\end{aligned}$$

Question 2

Maxwell's wave equation for an electric field propagating in the x-direction is

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}\,,$$

where $E\,(x,\,t)$ is the amplitude of the electric field. Show that this equation is invariant under a Lorentz transformation to a reference frame moving with relative speed v along the x-axis.

Solution

The relevant Lorentz transformations are given by

$$\begin{aligned}
x' &= \gamma\,(x - vt) \\
t' &= \gamma\left(t - \frac{vx}{c^2}\right)
\end{aligned}$$

where $\gamma = \left(1 - v^2/c^2\right)^{-1/2}$.

Note that $x' = x'\,(x,\,t)$ and $t' = t'\,(x,\,t)$, so that we use the chain rule to obtain for a wave function

$$\begin{aligned}
\frac{\partial E}{\partial x} &= \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial x} = \gamma\frac{\partial E}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial e}{\partial t'} \\
\frac{\partial E}{\partial t} &= \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial t} = -\gamma v\frac{\partial E}{\partial x'} + \gamma\frac{\partial E}{\partial t'}
\end{aligned}$$

Therefore we have

$$\begin{aligned}
\frac{\partial^2 E}{\partial x^2} &= \left(\gamma\frac{\partial E}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial E}{\partial t'}\right)\left(\gamma\frac{\partial E}{\partial x'} - \frac{\gamma v}{c^2}\frac{\partial E}{\partial t'}\right) \\
&= \gamma^2\frac{\partial^2 E}{\partial x'^2} - \frac{2\gamma^2 v}{c^2}\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'} + \frac{\gamma^2 v^2}{c^4}\frac{\partial^2 E}{\partial t'^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 E}{\partial t^2} &= \left(-\gamma v\frac{\partial E}{\partial x'} + \gamma\frac{\partial E}{\partial t'}\right)\left(-\gamma v\frac{\partial E}{\partial x'} + \gamma\frac{\partial E}{\partial t'}\right) \\
&= \gamma^2 v^2\frac{\partial^2 E}{\partial x'^2} - 2\gamma^2 v\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'} + \gamma^2\frac{\partial^2 E}{\partial t'^2}
\end{aligned}$$

Substituting this into the wave equation

$$\frac{\partial^2 E}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}$$

and rearranging gives

$$\begin{aligned}
\gamma^2\frac{\partial^2 E}{\partial x'^2} - \frac{\gamma^2 v^2}{c^2}\frac{\partial^2 E}{\partial x'^2} + \frac{\gamma^2 v^2}{c^4}\frac{\partial^2 E}{\partial t'^2} - \frac{\gamma^2}{c^2}\frac{\partial^2 E}{\partial t'^2} &= \left(\frac{2\gamma^2 v}{c^2}\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'} - \frac{2\gamma^2 v}{c^2}\frac{\partial E}{\partial x'}\frac{\partial E}{\partial t'}\right) \\
\gamma^2\frac{\partial^2 E}{\partial x'^2}\left(1 - \frac{v^2}{c^2}\right) - \frac{\gamma^2}{c^2}\frac{\partial^2 E}{\partial t'^2}\left(1 - \frac{v^2}{c^2}\right) &= 0 \\
\frac{\partial^2 E}{\partial x'^2} &= \frac{1}{c^2}\frac{\partial^2 E}{\partial t'^2}
\end{aligned}$$

Therefore, the wave equation is invariant under a Lorentz transformation.

Question 3

A physics professor claims in court that the reason he went through the red light ($\lambda = 650\,\text{nm}$) was that, due to his motion, the red color was Doppler shifted to green ($\lambda = 550\,\text{nm}$). How must he have been going for his story to be true? *Hint*: The relation between frequency f and wavelength λ of light is given by $c = \lambda f$.

Solution

We use the relativistic Doppler formula. It is clear from the problem that the professor is approaching the traffic light. The receiver in this case is the professor's eyes and the emitter is the light of the traffic light. So for the professor to receive green light when red light was emitted, we have

$$\begin{aligned}
f_{rec} &= f_{em}\sqrt{\frac{c+V}{c-V}} \\
\frac{c}{\lambda_{rec}} &= \frac{c}{\lambda_{em}}\sqrt{\frac{c+V}{c-V}} \\
\frac{c}{550\,\mathrm{nm}} &= \frac{c}{650\,\mathrm{nm}}\sqrt{\frac{c+V}{c-V}} \\
\frac{650\,\mathrm{nm}}{550\,\mathrm{nm}} &= \sqrt{\frac{c+V}{c-V}} \\
\left(\frac{13}{11}\right)^2 &= \frac{c+V}{c-V} \\
\frac{169c}{121} - \frac{169}{121}V &= c+V \\
-V\left(\frac{169}{121}+1\right) &= c - \frac{169c}{121} \\
V &= -\left(\frac{121}{290}\right)\left(-\frac{48c}{121}\right) \\
&= 0.17c
\end{aligned}$$

The professor must have been travelling at $0.17c$ for his story to be true.

Question 4

Prove that the interval between two events in 2-dimentional spacetime is Lorentz invariant, that is, prove that

$$\sum_{\mu} \Delta x'_{\mu}\Delta x'^{\mu} = \sum_{\mu} \Delta x_{\mu}\Delta x^{\mu}$$

where $\Delta x'^{\mu} = \sum_{\nu} \Lambda^{\mu}_{\nu}\Delta x^{\nu}$.

Solution

The spacetime interval in the S' frame is given by

$$\sum_{\mu} \Delta x'_{\mu} \Delta x'^{\mu}$$

Substituting the Lorentz transformations gives

$$\begin{aligned}
\sum_{\mu} \Delta x'_{\mu} \Delta x'^{\mu} &= \left(\sum_{\nu} \Lambda^{\nu}_{\mu} \Delta x_{\nu} \right) \left(\sum_{\nu} \Lambda^{\mu}_{\nu} \Delta x^{\nu} \right) \\
&= \sum_{\nu} \left(\Lambda^{\nu}_{\mu} \Lambda^{\mu}_{\nu} \right) \left(\Delta x_{\nu} \Delta x^{\nu} \right) \\
&= \sum_{\nu} \delta^{\nu}_{\mu} \left(\Delta x_{\nu} \Delta x^{\nu} \right) \\
&= \sum_{\mu} \Delta x_{\mu} \Delta x^{\mu}
\end{aligned}$$

Question 5

In the context of special relativity, a contravariant four-vector can be constructed from the charge density ρ and the current density $\mathbf{j}$ as follows $[J^{\mu}] = (c\rho,\, j_x,\, j_y,\, j_z)$ where j_x, j_y and j_z are the components of $\mathbf{j}$ in the x, y and z directions, respectively. *Hint*: To answer the questions below, use the properties of four-vectors. Do not try to solve this using electromagnetism.

(a) Determine the transformation equations of J^{μ} to a frame S' that is moving with a constant speed V in the positive x-direction.

(b) Construct a quantity using the components of J^{μ} that is a Lorentz invariant in Minkowski spacetime.

(c) Imagine you are in a reference frame in which $\rho = 2/c$ and $j_x = j_y = j_z = 2$. Determine $[J'^{\mu}]$ as measured by someone moving at a velocity $V = \sqrt{3/4}c$ along the x-direction with respect to your reference frame.

Solution

Part A

$[J^\mu]$ is a contravariant four-vector and the relative movement of S' describes the standard configuration in special relativity. So it's components transform as

$$\begin{aligned} J'^0 &= \gamma\left(J^0 - VJ^1/c\right) \\ &= \gamma\left(c\rho - Vj_x/c\right) \end{aligned}$$

$$\begin{aligned} J'^1 &= \gamma\left(J^1 - VJ^0/c\right) \\ &= \gamma\left(j_x - V\rho\right) \end{aligned}$$

$$\begin{aligned} J'^2 &= J^2 = j_y \\ J'^3 &= J^3 = j_z \end{aligned}$$

Part B

The quantity

$$\sum_{\mu=0}^{3} J^\mu J_\mu$$

will be invariant under a Lorentz transformation. (Can you show this explicitly for each of its components?)

Part C

First we calculate the Lorentz factor γ for $V = \sqrt{3/4}c$

$$\begin{aligned} \gamma &= \frac{1}{\sqrt{1 - \frac{V^2}{c^2}}} \\ &= \frac{1}{\sqrt{1 - \frac{3}{4}}} \\ &= \frac{1}{\sqrt{\frac{1}{4}}} \end{aligned}$$

$$= \quad 2$$

We use the transformation equations from Part A and substitute $\gamma = 2$, $\rho = 2/c$, $j_x = j_y = j_z = 2$ and $V = \sqrt{3/4}c$ to get

$$\begin{aligned} J'^0 &= \gamma\,(c\rho - V j_x/c) \\ &= 2\left(2 - 2\sqrt{\frac{3}{4}}\right) \\ &= 0.54 \end{aligned}$$

$$\begin{aligned} J'^1 &= \gamma\,(j_x - V c\rho/c) \\ &= 2\left(2 - 2\sqrt{\frac{3}{4}}\right) \\ &= 0.54 \end{aligned}$$

$$\begin{aligned} J'^2 &= j_y = 2 \\ J'^3 &= j_z = 2 \end{aligned}$$

So that we have $[J'^\mu] = (0.54,\, 0.54,\, 2,\, 2)$.

Question 6

An electron e^- with kinetic energy 1 MeV makes a head-on collision with a positron e^+ that is at rest. (A positron is an antimatter particle that has the same mass as an electron, but opposite charge.) In the collision the two particles annihilate each other and are replaced by two photons γ of equal energy. The reaction can be written as

$$e^- + e^+ \rightarrow 2\gamma\,.$$

Determine the energy, momentum and speed of each photon.

Solution

By conservation of energy, the energy of the system before and after the collision will be the same.

$$E_e + E_p = 2E_\gamma$$

The total energy of the electron and positron before the collision is

$$\begin{aligned} E_e + E_p &= E_{Ke} + m_e c^2 + m_p c^2 \\ &= 1\,\mathrm{MeV} + 2\left(0.511\,\mathrm{MeV}/c^2\right)c^2 \\ &= 2.02\,\mathrm{MeV} \end{aligned}$$

By conservation of energy, we find

$$\begin{aligned} E_\gamma &= \frac{E_e + E_p}{2} \\ &= \frac{2.02\,\mathrm{MeV}}{2} \\ &= 1.01\,\mathrm{MeV} \end{aligned}$$

The momentum of the photons will then be

$$\begin{aligned} p_\gamma &= \frac{E}{c} \\ &= 1.01\,\mathrm{MeV}/c \end{aligned}$$

The speed of both photons will be $v_\gamma = c$.

> **Note:** It might be tempting to use conservation of momentum to solve this problem, rather than conservation of energy, i.e.
>
> $$\mathbf{p}_e + \mathbf{p}_p = 2\mathbf{p}_\gamma$$
>
> In this case, this approach wouldn't work, because we only know the directions of the photons velocities, and therefore their momenta. Since the energies of the photons are the same, the magnitudes of their momenta will be equal, but not their momenta in vector form, since they will move in different directions, thus $\mathbf{p}_{\gamma 1} \neq \mathbf{p}_{\gamma 2}$.

Question 7

In special relativity, the energy, momentum and mass of a particle are all closely related to one another.

(a) Derive the relation $E^2 = c^2p^2 + m^2c^4$ by starting from the relativistic definitions of E and p, i.e. $E = \gamma mc^2$ and $p = \gamma mv$.

(b) Use the equation derived in part (a) to show that the mass of a particle can be expressed as

$$m = \frac{c^2p^2 - E_k^2}{2E_kc^2}$$

where E_k is the kinetic energy of the particle.

Solution

Part A

Squaring the definitions of E and p

$$\begin{aligned} E^2 &= \gamma^2m^2c^4 \\ p^2 &= \gamma^2m^2u^2 \end{aligned}$$

Multiplying the last equation by c^2 and subtracting gives

$$\begin{aligned} E^2 - p^2c^2 &= \gamma^2m_0^2c^4 - \gamma^2m_0^2u^2c^2 \\ &= m_0^2c^4\left(\gamma^2 - \gamma^2\frac{u^2}{c^2}\right) \end{aligned}$$

Using

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}$$

we show that

$$\gamma^2 - \gamma^2\frac{u^2}{c^2} = \frac{1}{1 - u^2/c^2} - \frac{u^2/c^2}{1 - u^2/c^2}$$

$$\begin{aligned} &= \frac{1 - u^2/c^2}{1 - u^2/c^2} \\ &= 1 \end{aligned}$$

It follows that

$$E^2 = p^2c^2 + m_0^2c^4$$

Part B

The total energy is equal to the kinetic energy plus the mass energy

$$E = E_k + mc^2$$

Squaring both sides gives

$$E^2 = E_k^2 + 2E_kmc^2 + m^2c^4$$

Using the result from the previous question we get

$$c^2p^2 = E_k^2 + 2E_kmc^2$$

Solving for m gives

$$m = \frac{c^2p^2 - E_k^2}{2E_kc^2}$$

as required.