Tutorial Letter 206/2/2018

Special Relativity and Riemannian Geometry

APM3713

Semester 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains the solutions to Assignment 06.

BAR CODE

Memo for Assignment 6 S2 2018

Equivalence and Tensor Algebra (Â§ 4)

Question 1: Tensor transformations

The transformation equations for transforming a contravariant tensor of rank one from polar to Cartesian coordinates are

$$\begin{aligned} A'^1 &= A^1 \cos\theta - A^2 r \sin\theta \\ A'^2 &= A^1 \sin\theta + A^2 r \cos\theta \end{aligned}$$

where $x^i = (r,\, \theta)$ (derived in Exercise 4.2 in the textbook). Use these to transform the tensor described by $[A^i] = (1/\cos\theta,\, r)$ to Cartesian coordinates $x'^i = (x,\, y)$. What is the value of A'^2?

- $\cos^2\theta\,(1 + r^2)$
- $\tan\theta + \mathbf{r^2}\cos\theta$*
- $1 - r^2 \sin\theta$
- $\cos\theta - r\sin\theta$
- $r\cos\theta - r\tan\theta$

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The components of the tensor we want to transform is given as $A^1 = 1/\cos\theta$ and $A^2 = r$. Substituting this into the above equations give

$$A'^1 = \frac{\cos\theta}{\cos\theta} - r^2 \sin\theta$$

$$
\begin{aligned}
&= 1 - r^2 \sin\theta \\
A'^2 &= \frac{\sin\theta}{\cos\theta} + r^2\cos\theta \\
&= \tan\theta + r^2\cos\theta
\end{aligned}
$$

Question 2: Tensor expressions

Which of the following tensor expressions is incorrect?

- $A^i = \sum_j g^{ij} A_j = \sum_{j,k} g^{ij} g_{jk} A^k$
- $\bar{A}^i{}_{kl} = \sum_{p,r,s} \frac{\partial \bar{x}^i}{\partial x^p} \frac{\partial x^r}{\partial \bar{x}^k} \frac{\partial x^s}{\partial \bar{x}^l} A^p{}_{rs}$
- $\Gamma_{\alpha\beta\gamma} = \frac{1}{2}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\alpha}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\alpha}\right)$
- $\sum_{i=1}^{3} \delta^i{}_i = 3$
- $\mathbf{\Gamma^i{}_{jk} = \Gamma^j{}_{ik}}$ *

Option (1): Correct. The rules of raising and lowering an index is followed. We can also write

$$
\begin{aligned}
\sum_{j,k} g^{ij} g_{jk} A^k &= \sum_k \delta^i{}_k A^k \\
&= \delta^i{}_1 A^1 + \delta^i{}_2 A^2 + ... + \delta^i{}_i A^i + ... \delta^i{}_n A^n
\end{aligned}
$$

In the last step the sum has been expanded and sums k from 1 to n. Remember that $\delta^i{}_k$ is defined so that it is equal to 1 if $i = k$, and zero if $i \neq k$. So all the terms will be equal to zero, except the term where $i = k$, and in that case we have $\delta^i{}_i = 1$ so that we can write

$$
\sum_{j,k} g^{ij} g_{jk} A^k = A^i
$$

Option (2): This is the correct transformation for a tensor of this form. Here bars were used in stead of primes to indicate the other coordinate frame. The textbook uses primes, but this can sometimes become unclear, especially when writing by hand. Using either is fine, as long as you are consistent.

Option (3): This is correct. Lowering the first index of the Christoffel coefficient gives

$$\begin{aligned}
\Gamma_{\alpha\beta\gamma} &= g_{\alpha\eta}\Gamma^{\eta}{}_{\beta\gamma} \\
&= \frac{1}{2} g_{\alpha\eta} \sum_{\epsilon} g^{\eta\epsilon} \left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}} \right) \\
&= \frac{1}{2} \delta^{1}{}_{\alpha} \left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}} \right) + ... + \frac{1}{2} \delta^{\alpha}{}_{\alpha} \left(\frac{\partial g_{\epsilon\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\epsilon}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\epsilon}} \right) + ...
\end{aligned}$$

In the last step we used

$$g_{\alpha\eta} g^{\eta\epsilon} = \delta^{\epsilon}{}_{\alpha}$$

The only non-vanishing term will be the one for which $\eta = \alpha$ so that

$$\Gamma_{\alpha\beta\gamma} = \frac{1}{2} \left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\alpha}}{\partial x^{\gamma}} - \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}} \right)$$

Option (4): This is correct. Expanding the sum gives

$$\begin{aligned}
\sum_{i=1}^{3} \delta^{i}{}_{i} &= \delta^{1}{}_{1} + \delta^{2}{}_{2} + \delta^{3}{}_{3} \\
&= 1 + 1 + 1 \\
&= 3
\end{aligned}$$

Option (5): This is incorrect. The Christoffel coefficients are symmetric in their lower indices, but not in the upper and lower index.

The symmetry in their lower indices is a consequence of the symmetry of the metric tensor. We can write

$$\Gamma^{i}{}_{jk} = \frac{1}{2} \sum_{l} g^{il} \left(\frac{\partial g_{lk}}{\partial x^{j}} + \frac{\partial g_{jl}}{\partial x^{k}} - \frac{\partial g_{jk}}{\partial x^{l}} \right)$$

Since $g_{\alpha\beta} = g_{\beta\alpha}$, we can interchange the indices of all the metrics within the sum.

$$\Gamma^{i}{}_{jk} = \frac{1}{2} \sum_{l} g^{il} \left(\frac{\partial g_{kl}}{\partial x^{j}} + \frac{\partial g_{lj}}{\partial x^{k}} - \frac{\partial g_{kj}}{\partial x^{l}} \right)$$

The first two terms can also be switched to give

$$\Gamma^{i}{}_{jk} = \frac{1}{2} \sum_{l} g^{il} \left(\frac{\partial g_{lj}}{\partial x^{k}} + \frac{\partial g_{kl}}{\partial x^{j}} - \frac{\partial g_{kj}}{\partial x^{l}} \right)$$

which is exactly the definition for ${\Gamma^i}_{kj}$. So therefore ${\Gamma^i}_{jk} = {\Gamma^i}_{kj}$, but it does not hold that ${\Gamma^i}_{jk} = {\Gamma^j}_{ik}$.

Question 3: Contraction

Which of the following expressions are correct?

- $\sum_m \delta_l^m g_{km} = 3g_{km}$
- $\sum_m \delta_l^m g_{km} = g^{kl}$
- $\sum_m \delta_l^m g_{km} = g^{km}$
- $\boldsymbol{\sum_m \delta_l^m g_{km} = g_{kl}}$*
- $\sum_m \delta_l^m g_{km} = g_{km}$

In the sum

$$\sum_m \delta^m_l g_{km}$$

all the terms in the sum where $m \neq l$, will be zero (since $\delta^m_l = 0$ if $m \neq l$). The term where $m = l$ will be equal to g_{kl}, i.e.

$$\begin{aligned}
\sum_m \delta^m_l g_{km} &= \delta^0_l g_{k0} + \delta^1_l g_{k1} + \ldots + \delta^l_l g_{kl} + \ldots + \delta^N_l g_{kN} \\
&= (0)\, g_{k0} + (0)\, g_{k1} + \ldots + (1)\, g_{kl} + \ldots + (0)\, g_{kN} \\
&= g_{kl}
\end{aligned}$$

So the Kronecker delta can effectively be used to replace one index with another.

Question 4: Einstein field equations

How many equations does the following expression represent?

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = -\kappa T_{\mu\nu}$$

- 1
- 2
- 4
- 8
- **16***

The expression represents 16 different equations. One for each possible combination of μ and ν, where both indices can have values from 0 to 3, since this equation is written for four dimensional spacetime. The 16 possible combinations are

#	μ	ν		#	μ	ν		#	μ	ν		#	μ	ν
1	0	0		**5**	1	0		**9**	2	0		**13**	3	0
2	0	1		**6**	1	1		**10**	2	1		**14**	3	1
3	0	2		**7**	1	2		**11**	2	2		**15**	3	2
4	0	3		**8**	1	3		**12**	2	3		**16**	3	3