Tutorial Letter 207/2/2018

Special Relativity and Riemannian Geometry

APM3713

Semester 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains the solutions to Assignment 07.

BAR CODE

Memo for Assignment 7 S2 2018

Chapters 3 & 4

Question 1

The right helicoid can be parametrized as

$$\begin{aligned} x(u, v) &= u\cos v \\ y(u, v) &= u\sin v \\ z(u, v) &= av \end{aligned}$$

where a is a constant.

(a) Find the line element for the surface.

(b) What is the metric tensor and the dual metric tensor?

(c) Determine the values of all the Christoffel coefficients of the surface.

(d) What is the value of the component $R^1{}_{212}$ of the Riemann curvature tensor?

(e) What is the Ricci tensor for the surface?

(f) What is the curvature scalar R for the surface?

(g) What is the Gaussian curvature of the surface?

(h) Is the surface Euclidean? Explain your answer.

(i) Suppose that the surface is filled with non-interacting particles, or dust. Use the two dimensional version of the energy-momentum tensor for dust and Einstein's field equation to find an expression for the Einstein constant κ for this surface.

Solution

Part A

In Cartesian coordinates, the line element is given by

$$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2 \,.$$

We have

$$\begin{aligned} x\,(u,\, v) &= u\cos v \\ y\,(u,\, v) &= u\sin v \\ z\,(u,\, v) &= av \end{aligned}$$

so that

$$\begin{aligned} dx &= \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \\ &= \frac{\partial}{\partial u}\left(u\cos v\right)\,du + \frac{\partial}{\partial v}\left(u\cos v\right)\,dv \\ &= \cos v\,du - u\sin v\,dv \end{aligned}$$

Similarly, we get

$$\begin{aligned} dy &= \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv \\ &= \frac{\partial}{\partial u}\left(u\sin v\right)\,du + \frac{\partial}{\partial v}\left(u\sin v\right)\,dv \\ &= \sin v\,du + u\cos v\,dv \end{aligned}$$

$$\begin{aligned} dz &= \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv \\ &= \frac{\partial}{\partial u}\left(av\right)\,du + \frac{\partial}{\partial v}\left(av\right)\,dv \\ &= a\,dv \end{aligned}$$

Substituting this into the Cartesian line element and simplifying gives

$$\begin{aligned} (dl)^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= \left(\cos v\,du - u\sin v\,dv\right)^2 + \left(\sin v\,du + u\cos v\,dv\right)^2 + \left(a\,dv\right)^2 \end{aligned}$$

$$
\begin{aligned}
&= \cos^2 v\, du^2 - u\cos v \sin v\, dudv + u^2 \sin^2 v\, dv^2 + \sin^2 v\, du^2 + u \cos v \sin v\, dudv \\
&\qquad +u^2\cos^2 v\, dv^2 + a^2\, dv^2 \\
&= \left(\cos^2 v + \sin^2 v\right)\, du^2 + \left[u^2\left(\sin^2 v + \cos^2 v\right) + a^2\right]\, dv^2 \\
&= du^2 + \left(u^2 + a^2\right)\, dv^2
\end{aligned}
$$

Part B

We know that the line element has the form

$$dl^2 = \sum_{i,j=1}^{n} g_{ij} dx^i dx^j$$

If we choose $x^1 = u$ and $x^2 = v$, this reduces to

$$
\begin{aligned}
dl^2 &= \sum_{i,j=1}^{2} g_{ij} dx^i dx^j \\
&= g_{11} dx^1 dx^1 + 2g_{12} dx^1 dx^2 + g_{22} dx^2 dx^2 \\
&= g_{11}\,(du)^2 + 2g_{12} dudv + g_{22}\,(dv)^2
\end{aligned}
$$

Above we used the fact that the metric tensor is symmetric $g_{ij} = g_{ji}$. Comparing this to the line element calculated in Part A allows us to identify

$$g_{11} = 1, \quad g_{12} = 0, \quad g_{22} = u^2 + a^2$$

so that the metric tensor for the surface is

$$[g_{ij}] = \begin{pmatrix} 1 & 0 \\ 0 & u^2 + a^2 \end{pmatrix}$$

We know that we must have

$$\sum_k g^{ik} g_{kj} = \delta^i_j$$

so that the dual metric $[g^{ij}]$ is just the matrix inverse of $[g_{ij}]$. We find

$$\left[g^{ij}\right] = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{u^2+a^2} \end{pmatrix}$$

Part C

The Christoffel coefficients are defined by

$$\Gamma^h_{\ ij} = \sum_k \frac{1}{2} g^{hk} \left(g_{ki,j} + g_{jk,i} - g_{ij,k}\right)$$

$$\Gamma^1_{\ 11} = \frac{1}{2} g^{11} \left(g_{11,1} + g_{11,1} - g_{11,1}\right) + \frac{1}{2} g^{12} \left(g_{21,1} + g_{12,1} - g_{11,2}\right)$$

All the g_{ik} and g^{ik} where $i \neq k$ will be zero, so their derivatives will also be zero. Remembering this will reduce the calculations a lot. So we have

$$\begin{aligned} v\Gamma^1_{\ 11} &= \frac{1}{2} g^{11} \left(g_{11,1} + g_{11,1} - g_{11,1}\right) + \frac{1}{2} g^{12} \left(g_{21,1} + g_{12,1} - g_{11,2}\right) \\ &= \frac{1}{2} g^{11} g_{11,1} \\ &= \frac{1}{2} (1) \frac{d}{du} (1) \\ &= 0 \end{aligned}$$

Using the symmetric property of the Christoffel coefficients $\Gamma^h_{\ ij} = \Gamma^h_{\ ji}$ will also cut down on calculations

$$\begin{aligned} \Gamma^1_{\ 12} = \Gamma^1_{\ 21} &= \frac{1}{2} g^{11} \left(g_{11,2} + g_{21,1} - g_{12,1}\right) + \frac{1}{2} g^{12} \left(g_{21,2} + g_{22,1} - g_{12,2}\right) \\ &= \frac{1}{2} g^{11} g_{11,2} \\ &= \frac{1}{2} (1) \frac{\partial}{\partial v} \left(u^2 + a^2\right) \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Gamma^1_{\ 22} &= \frac{1}{2} g^{11} \left(g_{12,2} + g_{21,2} - g_{22,1}\right) + \frac{1}{2} g^{12} \left(g_{22,2} + g_{22,2} - g_{22,2}\right) \\ &= -\frac{1}{2} g^{11} g_{22,1} \\ &= -\frac{1}{2} (1) \frac{\partial}{\partial u} \left(u^2 + a^2\right) \\ &= -u \end{aligned}$$

$$\begin{aligned}
\Gamma^2_{\ 11} &= \frac{1}{2}g^{21}\left(g_{11,1}+g_{11,1}-g_{11,1}\right)+\frac{1}{2}g^{22}\left(g_{21,1}+g_{12,1}-g_{11,2}\right)\\
&= -\frac{1}{2}g^{22}g_{11,2}\\
&= -\frac{1}{2}\left(\frac{1}{u^2+a^2}\right)\frac{\partial}{\partial v}\left(u^2+a^2\right)\\
&= 0
\end{aligned}$$

$$\begin{aligned}
\Gamma^2_{\ 12}=\Gamma^2_{\ 21} &= \frac{1}{2}g^{21}\left(g_{12,1}+g_{11,2}-g_{21,1}\right)+\frac{1}{2}g^{22}\left(g_{22,1}+g_{12,2}-g_{21,2}\right)\\
&= \frac{1}{2}g^{22}g_{22,1}\\
&= \frac{1}{2}\left(\frac{1}{u^2+a^2}\right)\frac{\partial}{\partial u}\left(u^2+a^2\right)\\
&= \frac{u}{u^2+a^2}
\end{aligned}$$

$$\begin{aligned}
\Gamma^2_{\ 22} &= \frac{1}{2}g^{21}\left(g_{12,2}+g_{21,2}-g_{22,1}\right)+\frac{1}{2}g^{22}\left(g_{22,2}+g_{22,2}-g_{22,2}\right)\\
&= \frac{1}{2}g^{22}g_{22,2}\\
&= \frac{1}{2}\left(\frac{1}{u^2+a^2}\right)\frac{\partial}{\partial v}\left(u^2+a^2\right)\\
&= 0
\end{aligned}$$

In summary, the only non-zero Christoffel coefficients that we have are $\Gamma^1_{\ 22}=-u$ and $\Gamma^2_{\ 12}=\Gamma^2_{\ 21}=u/\left(u^2+a^2\right)$.

Part D

The Riemann Curvature tensor is defined by

$$R^l_{\ ijk}\equiv\frac{\partial\Gamma^l_{\ ik}}{\partial x^j}-\frac{\partial\Gamma^l_{\ ij}}{\partial x^k}+\sum_m\Gamma^m_{\ ik}\Gamma^l_{\ mj}-\sum_m\Gamma^m_{\ ij}\Gamma^l_{\ mk}$$

since we are dealing with a two dimensional surface, the only independent entry will be $R^1{}_{212}$, so it will be sufficient to only calculate this. We have

$$\begin{aligned}
R^1{}_{212} &= \frac{\partial \Gamma^1{}_{22}}{\partial x^1} - \frac{\partial \Gamma^1{}_{21}}{\partial x^2} + \sum_m \Gamma^m_{22}\Gamma^1{}_{m1} - \sum_m \Gamma^m_{21}\Gamma^1{}_{m2} \\
&= \frac{\partial \Gamma^1{}_{22}}{\partial u} - \frac{\partial \Gamma^1{}_{21}}{\partial v} + \Gamma^1{}_{22}\Gamma^1{}_{11} + \Gamma^2_{22}\Gamma^1{}_{21} - \Gamma^1{}_{21}\Gamma^1{}_{12} - \Gamma^2_{21}\Gamma^1{}_{22} \\
&= \frac{\partial \Gamma^1{}_{22}}{\partial u} - \Gamma^2_{21}\Gamma^1{}_{22} \\
&= \frac{\partial}{\partial u}(-u) - \left(\frac{u}{u^2+a^2}\right)(-u) \\
&= -1 + \frac{u^2}{u^2+a^2} \\
&= \frac{-u^2 - a^2 + u^2}{u^2+a^2} \\
&= \frac{-a^2}{u^2+a^2}
\end{aligned}$$

For the Riemann curvature tensor we have

$$R^1{}_{212} = R^2{}_{121} = \frac{-a^2}{u^2+a^2}$$

$$R^1{}_{221} = R^2{}_{112} = \frac{a^2}{u^2+a^2}$$

With all other entries equal to zero.

Part E

The Ricci tensor is defined by

$$R_{ij} \equiv \sum_k R^k{}_{ijk}$$

Using the fact that the Ricci tensor is symmetric we find the 4 entries of the Ricci tensor

$$\begin{aligned}
R_{11} &= R^1{}_{111} + R^2{}_{112} \\
&= \frac{a^2}{u^2+a^2}
\end{aligned}$$

$$\begin{aligned}
R_{12} = R_{21} &= R^1{}_{121} + R^2{}_{122} \\
&= 0
\end{aligned}$$

$$\begin{aligned} R_{22} &= R^1{}_{221} + R^2{}_{222} \\ &= \frac{a^2}{u^2 + a^2} \end{aligned}$$

Part F

The Ricci scalar is defined by

$$R \equiv \sum_{i,j} g^{ij} R_{ij}$$

So we have for the helicoid

$$\begin{aligned} R &= g^{11}R_{11} + g^{12}R_{12} + g^{21}R_{21} + g^{22}R_{22} \\ &= g^{11}R_{11} + g^{22}R_{22} \\ &= (1)\left(\frac{-a^2}{u^2 + a^2}\right) + \left(\frac{1}{u^2 + a^2}\right)\left(\frac{-a^2}{u^2 + a^2}\right) \\ &= \frac{-a^2u^2 - a^4 - a^2}{(u^2 + a^2)^2} \\ &= \frac{-a^2(u^2 + a^2 + 1)}{(u^2 + a^2)^2} \end{aligned}$$

Part G

The Gaussian curvature of a two dimensional surface is given by

$$K = \frac{R_{1212}}{g}$$

where $g = \det[g_{ij}]$ (see Exercise 3.16 p105).

The determinant of a diagonal matrix is just the product of its diagonal entries so that

$$\begin{aligned} g &= \prod_i g_{ii} \\ &= (1)\left(u^2 + a^2\right) \\ &= u^2 + a^2 \end{aligned}$$

R_{1212} is the element of the Riemann curvature tensor with an index lowered, i.e.

$$R_{1212} = \sum_i g_{i1} R^i{}_{212}$$

$$\begin{aligned} &= g_{11}R^1{}_{212} + g_{21}R^2{}_{212} \\ &= (1)\left(\frac{-a^2}{u^2+a^2}\right) \\ &= \frac{-a^2}{u^2+a^2} \end{aligned}$$

So we have for the Gaussian curvature

$$\begin{aligned} K &= \frac{R_{1212}}{g} \\ &= \left(\frac{-a^2}{u^2+a^2}\right)\left(\frac{1}{u^2+a^2}\right) \\ &= \frac{-a^2}{(u^2+a^2)^2} \end{aligned}$$

Part H

No, the helicoid is not Euclidean (flat). The necessary and sufficient condition for a surface to be flat is that the Riemann curvature tensor (all its components) should vanish (be equal to zero) at all points on the surface. This is not true for all values of u and v.

Part I

Einstein's field equation for two dimensions is

$$R_{ij} - \frac{1}{2}Rg_{ij} = -\kappa T_{ij}$$

where i and j can take the values of 1 or 2, as with the rest of the calculations regarding the surface above. The only non-zero component of the energy-momentum tensor $[T^{ij}]$ for dust is $T^{11} = \rho c^2$.

$[T^{ij}]$ is related to $[T_{ij}]$ by

$$T_{ij} = \sum_{m,n} g_{im}g_{jn}T^{mn}$$

Clearly, the only non-zero component of $[T_{ij}]$ will be T_{11}. We find

$$\begin{aligned} T_{11} &= \sum_{m,n} g_{1m} g_{1n} T^{mn} \\ &= g_{11}g_{11}T^{11} + g_{11}g_{12}T^{12} + g_{12}g_{11}T^{21} + g_{12}g_{12}T^{22} \\ &= g_{11}g_{11}T^{11} \\ &= \rho c^2 \end{aligned}$$

Now all the quantities in the Einstein field equation are known. We substitute and solve for κ

$$\begin{aligned} R_{11} - \frac{1}{2} R g_{11} &= -\kappa T_{11} \\ \frac{-a^2}{u^2+a^2} - \frac{1}{2}\left(\frac{-a^2\left(u^2+a^2+1\right)}{\left(u^2+a^2\right)^2}\right)(1) &= -\kappa \rho c^2 \\ \frac{-2a^2u^2 - 2a^4 + a^2u^2 + a^4 + a^2}{2\left(u^2+a^2\right)^2} &= -\kappa \rho c^2 \\ \frac{-a^2u^2 - a^4 + a^2}{2\left(u^2+a^2\right)^2} &= -\kappa \rho c^2 \\ \kappa &= \frac{a^2\left(u^2+a^2-1\right)}{2\rho c^2\left(u^2+a^2\right)^2} \end{aligned}$$

Question 2

Show that the contracted Christoffel symbol $\sum_i \Gamma^i_{ik}$ is given by

$$\sum_i \Gamma^i_{ik} = \sum_i \sum_m \frac{g^{im}}{2} \frac{\partial g_{mi}}{\partial x^k} .$$

Solution

The Christoffel coefficients are defined by

$$\Gamma^i_{\ jk} = \sum_m \frac{1}{2} g^{im} \left(g_{mj,k} + g_{km,j} - g_{jk,m}\right)$$

If we contract the Christoffel coefficients we have

$$\sum_i \Gamma^i{}_{ik} = \sum_i \sum_m \frac{1}{2} g^{im} \left(g_{mi,k} + g_{km,i} - g_{ik,m} \right)$$

Since i and m are just dummy indices being summed over the same range, they can be interchanged without changing the meaning of the expression. We interchange them in the last term in brackets to get

$$\sum_i \Gamma^i{}_{ik} = \sum_i \sum_m \frac{1}{2} g^{im} \left(g_{mi,k} + g_{km,i} - g_{mk,i} \right)$$

The metric is symmetric, so that $g_{km} = g_{mk}$,

$$\begin{aligned} \sum_i \Gamma^i{}_{ik} &= \sum_i \sum_m \frac{1}{2} g^{im} \left(g_{mi,k} + g_{km,i} - g_{km,i} \right) \\ &= \sum_i \sum_m \frac{1}{2} g^{im} g_{mi,k} \\ &= \sum_i \sum_m \frac{g^{im}}{2} \frac{\partial g_{mi}}{\partial x^k} \end{aligned}$$

Question 3

Verify that if a tensor is symmetric in one frame, it will be symmetric in all coordinate frames. That is, show that if it is given that $X^{ij} = X^{ji}$ in frame S, then it will be true that $\bar{X}^{ij} = \bar{X}^{ji}$ in a coordinate frame $\bar{S}$.

Solution

If $X^{ij} = X^{ji}$, then

Since X^{ij} is a tensor, we know that it transforms as follows

$$\bar{X}^{ab} = \sum_i \sum_j \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} X^{ij}$$

On the RHS both i and j are just dummy indices, i.e. they are being summed over. This means that the two indices can be replaced by any other indices without changing the meaning of the expression, since they are just counters to be summed over, i.e.

$$\sum_i \sum_j \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} X^{ij} = \sum_\alpha \sum_\beta \frac{\partial \bar{x}^a}{\partial x^\alpha} \frac{\partial \bar{x}^b}{\partial x^\beta} X^{\alpha\beta} = \sum_r \sum_s \frac{\partial \bar{x}^a}{\partial x^r} \frac{\partial \bar{x}^b}{\partial x^s} X^{rs}$$

In particular, we can replace j with i and i with j, so that

$$\begin{aligned} \bar{X}^{ab} &= \sum_i \sum_j \frac{\partial \bar{x}^a}{\partial x^i} \frac{\partial \bar{x}^b}{\partial x^j} X^{ij} \\ &= \sum_j \sum_i \frac{\partial \bar{x}^a}{\partial x^j} \frac{\partial \bar{x}^b}{\partial x^i} X^{ji} \\ &= \sum_j \sum_i \frac{\partial \bar{x}^a}{\partial x^j} \frac{\partial \bar{x}^b}{\partial x^i} X^{ij} \end{aligned}$$

In the last step we used the symmetry of property $X^{ij} = X^{ji}$. This is the transformation expression for a second order contravariant tensor where $x^i \to \bar{x}^b$ and $x^j \to \bar{x}^a$ so we have

$$\begin{aligned} \bar{X}^{ab} &= \sum_j \sum_i \frac{\partial \bar{x}^a}{\partial x^j} \frac{\partial \bar{x}^b}{\partial x^i} X^{ij} \\ &= \bar{X}^{ba} \end{aligned}$$

Thus we have shown that if a tensor is symmetric in one coordinate frame, i.e. $X^{ij} = X^{ji}$ in S, then it is also symmetric in any other arbitrary coordinate frame $\bar{S}$.

Question 4

Suppose that $R_{iklm} = K\left(g_{il}g_{km} - g_{im}g_{kl}\right)$ on some four dimensional Riemannian space. Show that for the curvature scalar we have $R = -12K$.

Solution

From tensor contraction we can write

$$\begin{aligned} R_{kl} &= \sum_i \sum_m g^{im} R_{iklm} \\ &= \sum_i \sum_m K g^{im} \left(g_{il} g_{km} - g_{im} g_{kl} \right) \\ &= \sum_i \sum_m K \left(g^{im} g_{il} g_{km} - g^{im} g_{im} g_{kl} \right) \end{aligned}$$

Remember that tensors are not commutative, so be mindful of the order of the multiplication. We use the property of the metric tensor $\sum_i g^{im} g_{in} = \delta^m_n$ to get

$$\begin{aligned} R_{kl} &= \sum_m K \left(\delta^m_l g_{km} - \delta^m_m g_{kl} \right) \\ &= K \left(g_{kl} - 4 g_{kl} \right) \\ &= -3K g_{kl} \end{aligned}$$

In the second step above we used the definition the Kronecker delta. If we have

$$\sum_m \delta^m_l g_{km}$$

all the terms in the sum where $m \neq l$, will be zero, where the term where $m = l$ will be equal to g_{kl}, i.e.

$$\begin{aligned} \sum_m \delta^m_l g_{km} &= \delta^0_l g_{k0} + \delta^1_l g_{k1} + \ldots + \delta^l_l g_{kl} + \ldots + \delta^N_l g_{kN} \\ &= (0)\, g_{k0} + (0)\, g_{k1} + \ldots + (1)\, g_{kl} + \ldots + (0)\, g_{kN} \\ &= g_{kl} \end{aligned}$$

So the Kronecker delta can effectively be used to replace one index with another. On the other hand, if the two indices of the Kronecker delta are the same, i.e. $\sum_m \delta^m_m$, the result is *not equal to one* because of the summation. Then we have

$$\sum_m \delta^m_m g_{kl} = \delta^0_0 g_{kl} + \delta^1_1 g_{kl} + \ldots + \delta^m_m g_{kl} + \ldots + \delta^N_N g_{kl}$$

$$\begin{aligned} &= (1)\, g_{k0} + (1)\, g_{k1} + \ldots + (1)\, g_{kl} + \ldots + (1)\, g_{kN} \\ &= N g_{kl} \end{aligned}$$

In this case we are working in a 4 dimensional space, so that $N = 4$ and $\sum_m \delta^m_m g_{kl} = 4g_{kl}$

For the curvature scalar, we contract our result for R_{kl}

$$\begin{aligned} R &= \sum_k \sum_l g^{kl} R_{kl} \\ &= -3K \sum_k \sum_l g^{kl} g_{kl} \\ &= -3K \sum_k \delta^k_k \\ &= -12K \end{aligned}$$

Question 5

Two N-dimensional Riemann spaces M and $\bar{M}$ have the metric tensors g_{ij} and $\bar{g}_{ij}$ respectively, and

$$\bar{g}_{ij} = k g_{ij}$$

where k is a constant. What are the relationships between the curvature tensors, Ricci tensors, curvature scalar and Einstein tensors of the two spaces?

Solution

We have

$$\bar{g}_{ij} = k g_{ij}$$

and therefore

$$\bar{g}^{ij} = \frac{1}{k} g^{ij}$$

The transformation of an arbitrary Christoffel symbol from M to $\bar{M}$ gives

$$\Gamma^h_{ij} = \sum_k \frac{1}{2} g^{hk} \left(g_{ki,j} + g_{jk,i} - g_{ij,k} \right)$$

$$
\begin{aligned}
&= \sum_k \frac{k}{2}\bar{g}^{hk}\left(\frac{1}{k}\bar{g}_{ki,j} + \frac{1}{k}\bar{g}_{jk,i} - \frac{1}{k}\bar{g}_{ij,k}\right) \\
&= \sum_k \frac{1}{2}\bar{g}^{hk}\left(\bar{g}_{ki,j} + \bar{g}_{jk,i} - \bar{g}_{ij,k}\right) \\
&= \bar{\Gamma}^h_{ij}
\end{aligned}
$$

Using this we get for the curvature tensor

$$
\begin{aligned}
R^i{}_{jhk} &= \Gamma^i{}_{jh,k} - \Gamma^i{}_{jk,h} + \sum_m \Gamma^i{}_{mk}\Gamma^m{}_{jh} - \sum_m \Gamma^i{}_{mh}\Gamma^m{}_{jk} \\
&= \bar{\Gamma}^i{}_{jh,k} - \bar{\Gamma}^i{}_{jk,h} + \sum_m \bar{\Gamma}^i{}_{mk}\bar{\Gamma}^m{}_{jh} - \sum_m \bar{\Gamma}^i{}_{mh}\bar{\Gamma}^m{}_{jk} \\
&= \bar{R}^i{}_{jhk}
\end{aligned}
$$

Then Ricci tensor becomes

$$
\begin{aligned}
R_{jk} &= \sum_h \sum_i \sum_m g^{mh} g_{im} R^i{}_{jhk} \\
&= \sum_h \sum_i \sum_m \left(k\bar{g}^{mh}\right)\left(\frac{1}{k}\bar{g}_{im}\right)\bar{R}^i{}_{jhk} \\
&= \bar{R}_{jk}
\end{aligned}
$$

For the Curvature scalar

$$
\begin{aligned}
R &= \sum_i \sum_j g^{ij} R_{ij} \\
&= \sum_i \sum_j k\bar{g}^{ij}\bar{R}_{ij} \\
&= k\bar{R}
\end{aligned}
$$

And the relationship between the Einstein tensors is

$$
\begin{aligned}
G_{ij} &= R_{ij} - \frac{1}{2} g_{ij} R \\
&= \bar{R}_{ij} - \frac{1}{2}\frac{1}{k}\bar{g}_{ij} k\bar{R} \\
&= \bar{R}_{ij} - \frac{1}{2}\bar{g}_{ij}\bar{R} \\
&= \bar{G}_{ij}
\end{aligned}
$$