Tutorial letter 203/1/2017

Theoretical Computer Science 1

COS1501

Semester 1

School of Computing

Discussion of assignment 03

Dear Student,

By this time you should have received the tutorial matter listed below. These can be downloaded from *my*Unisa.

- MO001/4/2017 Learning units for COS1501 as on myUnisa;
- COSALLP/301/4/2017 General information regarding the School of Computing including lecturers' information;
- COS1501/101/3/2017 General information about the module and the assignments;
- COS1501/201/1/2017 Solutions to the first assignment, and **examination information**;
- COS1501/202/1/2017 Solutions to the second assignment;
- COS1501/203/1/2017 This tutorial letter;
- COS1501/102/3/2017 Solutions to self-assessment questions in assignments 02 and 03, example assignments & solutions, example examination paper & solutions, and extra questions & solutions.

Everything of the best with the exam!
Regards
COS1501 team

Discussion of assignment 03 semester 1.

Suppose A = {2, 4, 6, a, c, e} and B = {2, 6, a, c, d, e}.
Answer questions 1 and 2 by using the given sets A and B.

Question 1

Which one of the following possible relations on **A** is functional?

1. {(2, 4), (4, 6), (6, a), (a, c), (c, e), (2, e)}
2. {(2, 4), (a, 6), (6, 6), (e, e), (4, c), (c, 4)}
3. {(6, 2), (6, 4), (6, 6), (6, a), (6, c), (6, e)}
4. {(2, 4), (4, 6), (6, c), (a, d), (c, d), (e, 2)}

Answer: Alternative 2

Discussion

First we look at the definition for functionality:

Suppose R ⊆ B × C is a binary relation from a set B to a set C. We may call R functional if the elements of B that appear as first coordinates of ordered pairs in R do not appear in more than one ordered pair of R.

We consider the relations provided in the different alternatives:

1. Let L = {(2, 4), (4, 6), (6, a), (a, c), (c, e), (2, e)} (say). L is not functional since it contains the ordered pairs (2, 4) and (2, e), ie 2 appears more than once as first coordinate.

2. Let M = {(2, 4), (a, 6), (6, 6), (e, e), (4, c), (c, 4)} (say).
   M is a relation on A = {2, 4, 6, a, c, e } = dom(M). Ran(M) = {4, 6, c, e)} ⊆ A and furthermore each first coordinate appears only once as first coordinate in ordered pairs of M, thus M is functional. This alternative should therefore be selected.

3. Let N = {(6, 2), (6, 4), (6, 6), (6, a), (6, c), (6, e)} (say).
   N is not functional, because 6 appears as the first coordinate in each ordered pair in the relation N, ie 6 appears in more than one ordered pair as first coordinate.

4. Let S = {(2, 4), (4, 6), (6, c), (a, d), (c, d), (e, 2)} (say).
   Dom(S) = {2, 4, 6, a, c, e } = A. But ran(S) = {2, 4, 6, c, d} ⊈ A = {2, 4, 6, a, c, e} since d ∈ ran(S*)*. Therefore S is not a relation on A since ran(S) ⊈ A.

From the arguments provided we can deduce that alternative 2 should be selected.

Refer to study guide, p 98.

Question 2

Which one of the following alternatives does **not** provide an **injective function** from **B** to **A**?

1. {(e, e), (c, 2), (6, 4), (2, c), (a, a), (d, 6)}
2. {(e, e), (2, 2), (6, 6), (a, a), (d, c), (c, 4)}
3. {(d, 2), (2, 4), (a, e), (e, a), (6, 6), (c, c)}
4. {(2, e), (4, d), (6, c), (e, 2), (c, 6), (a, c)}

Answer: Alternative 4

Discussion

We look at the definitions for a function and for injectivity:

Suppose R ⊆ B × C is a binary relation from a set B to a set C. We may call R a **function** from B to C if every element of B appears exactly once as a first coordinate of an ordered pair in R (i.e. f is functional), and the domain of R is exactly the set B, ie dom(R) = B.

A function R from B to C is **injective** iff R has the property that whenever $f(a_1) = f(a_2)$ then $a_1 = a_2$. Alternatively, R from B to C is **injective** iff R has the property that whenever $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

We have A = {2, 4, 6, a, c, e} and B = {2, 6, a, c, d, e} and the relations from **B** to **A** in the different alternatives.

We consider the relations provided in the different alternatives:

1. Let L = {(e, e), (c, 2), (6, 4), (2, c), (a, a), (d, 6)} (say).

It is the case that dom(L) = B and that ran(L) = A. It is also the case that for any two first coordinates in L that are not equal, it is also the case that the second coordinates for these first coordinates are not equal. (Look at the alternative definition of injectivity). We can conclude that L is injective.

2. Let M = {(e, e), (2, 2), (6, 6), (a, a), (d, c), (c, 4)} (say).

It is the case that dom(M) = B and that ran(M) = A. It is also the case that for any two first coordinates in L that are not equal, it is also the case that the second coordinates for these first coordinates are not equal. (Look at the alternative definition for injectivity). We can conclude that M is injective.

3. Let N = {(d, 2), (2, 4), (a, e), (e, a), (6, 6), (c, c)} (say).

It is the case that dom(N) = B and that ran(N) = A. It is also the case that for any two first coordinates in N that are not equal, it is also the case that the second coordinates for these first coordinates are not equal. (Look at the alternative definition for injectivity). We may conclude that N is injective.

4. Let S = {(2, e), (4, d), (6, c), (e, 2), (c, a), (a, c)} (say).
(4, d) $\in$ S but 4 $\notin$ B and d $\notin$ A, thus S is not a function from B to A. Therefore S is also not an injective function from B to A.

From the arguments provided we can deduce that alternative 4 should be selected.

Refer to study guide, p 106.

Question 3
Let R and S be relations on A = {a, e, f, g} with

R = {(a, e), (f, g), (g, e), (a, a)} and S = {(f, a), (e, f), (e, a), (g, f)}.

Which one of the following alternatives represents the relation R ○ S = S; R?

1. {(a, f), (g, f), (g, a)}
2. {(a, f), (a, a), (f, f), (g, f), (g, a)}
3. {(f, e), (f, a), (e, g), (e, e), (e, a), (g, g)}
4. {(f, e), (f, a), (e, g), (e, a)}

Answer: Alternative 3

Discussion
We first look at a definition of a composition relation:
Given relation P from A to B and R from B to C, the composition of P followed by R (R ○ P or P; R) is the relation from A to C defined by

R ○ P = P; R = {(a, c) | *there is some b $\in$ B such that (a, b) $\in$ P and (b, c) $\in$ R}.*

R = {(a, e), (f, g), (g, e), (a, a)} and S = {(f, a), (e, f), (e, a), (g, f)} are defined on A = {a, e, f, g}.

To determine S; R we start with the pair (f, **a**) of S, and then we look for a pair in R that has as first coordinate an **a**, and then see where it takes us.

Link (f, **a**) of S with (**a**, e) of R, then (f, e) $\in$ S; R,
Link (f, **a**) of S with (**a**, a) of R, then (f, a) $\in$ S; R,
Link (e, **f**) of S with (**f**, g) of R, then (e, g) $\in$ S; R,
Link (e, **a**) of S with (**a**, e) of R, then (e, e) $\in$ S; R,
Link (e, **a**) of S with (**a**, a) of R, then (e, a) $\in$ S; R and
Link (g, **f**) of S with (**f**, g) of R, then (g, g) $\in$ S; R.

No other pairs can be linked, so S; R = {(f, e), (f, a), (e, g), (e, e), (e, a), (g, g)}. Thus alternative 3 should be selected.

Refer to study guide, pp 79, 108, 109.

Let g be a function from $\mathbb{Z}$ (the set of integers) to $\mathbb{Q}$ (the set of rational numbers) defined by

$(x, y) \in g$ iff $y = 4x + 3$ ($g \subseteq \mathbb{Z} \times \mathbb{Q}$) and

let f be a function on $\mathbb{Z}$ defined by

$(x, y) \in f$ iff $y = x^2 + 2x - 35$ ($f \subseteq \mathbb{Z} \times \mathbb{Z}$).

Answer questions 4 to 7 by using the given functions g and f.

Question 4

Consider the function f on $\mathbb{Z}$. For which values of x is it the case that $x^2 + 2x - 35 < 0$?

Hint: Solve $x^2 + 2x - 35 < 0$ and keep in mind that $x \in \mathbb{Z}$.

1. $-7 > x > 5, x \in \mathbb{Z}$
2. $-7 > x < 5, x \in \mathbb{Z}$
3. $-7 < x > 5, x \in \mathbb{Z}$
4. $-7 < x < 5, x \in \mathbb{Z}$

Answer: Alternative 4

We solve for x:

$x^2 + 2x - 35 < 0$

ie $(x - 5)(x + 7) < 0$

ie $[(x - 5) > 0$ and $(x + 7) < 0]$ OR $[(x - 5) < 0$ and $(x + 7) > 0]$

ie $[x > 5$ and $x < -7$ (which is impossible] OR $[x < 5$ and $x > -7]$

ie $x < 5$ and $x > -7$

ie $-7 < x < 5$.

From the above it is clear that alternative 4 should be selected.

Question 5

Which one of the following is NOT an ordered pair belonging to f?

1. (–5, 20)
2. (10, 85)
3. (-4, -27)
4. (1, -32)

Answer: Alternative 1

Discussion

The first and second coordinates of elements (x, y) of f are elements of $\mathbb{Z}$.

We consider the ordered pairs provided in the different alternatives:

1. $(x, y) \in f$ iff $y = x^2 + 2x - 35$. Is $(-5, 20) \in f$?
   Let x = -5 then
   $y = (-5)^2 + 2(-5) - 35$
   $= 25 - 10 - 35$
   $= -20$
   Thus $(-5, -20) \in f$ but $(-5, 20) \notin f$.

2. Is $(10, 85) \in f$?
   Let x = 10 then
   $y = (10)^2 + 2(10) - 35$
   $= 100 + 20 - 35$
   $= 85$
   Thus $(10, 85) \in f$.

3. Is $(-4, -27) \in f$?
   Let x = -4 then
   $y = (-4)^2 + 2(-4) - 35$
   $= 16 - 8 - 35$
   $= -27$
   Thus $(-4, -27) \in f$.

4. Is $(1, -32) \in f$?
   Let x = 1 then
   $y = (1)^2 + 2(1) - 35$
   $= -32$
   Thus $(1, -32) \in f$.

From the arguments provided we can deduce that alternative 1 should be selected.

Refer to study guide, pp 71, 72.

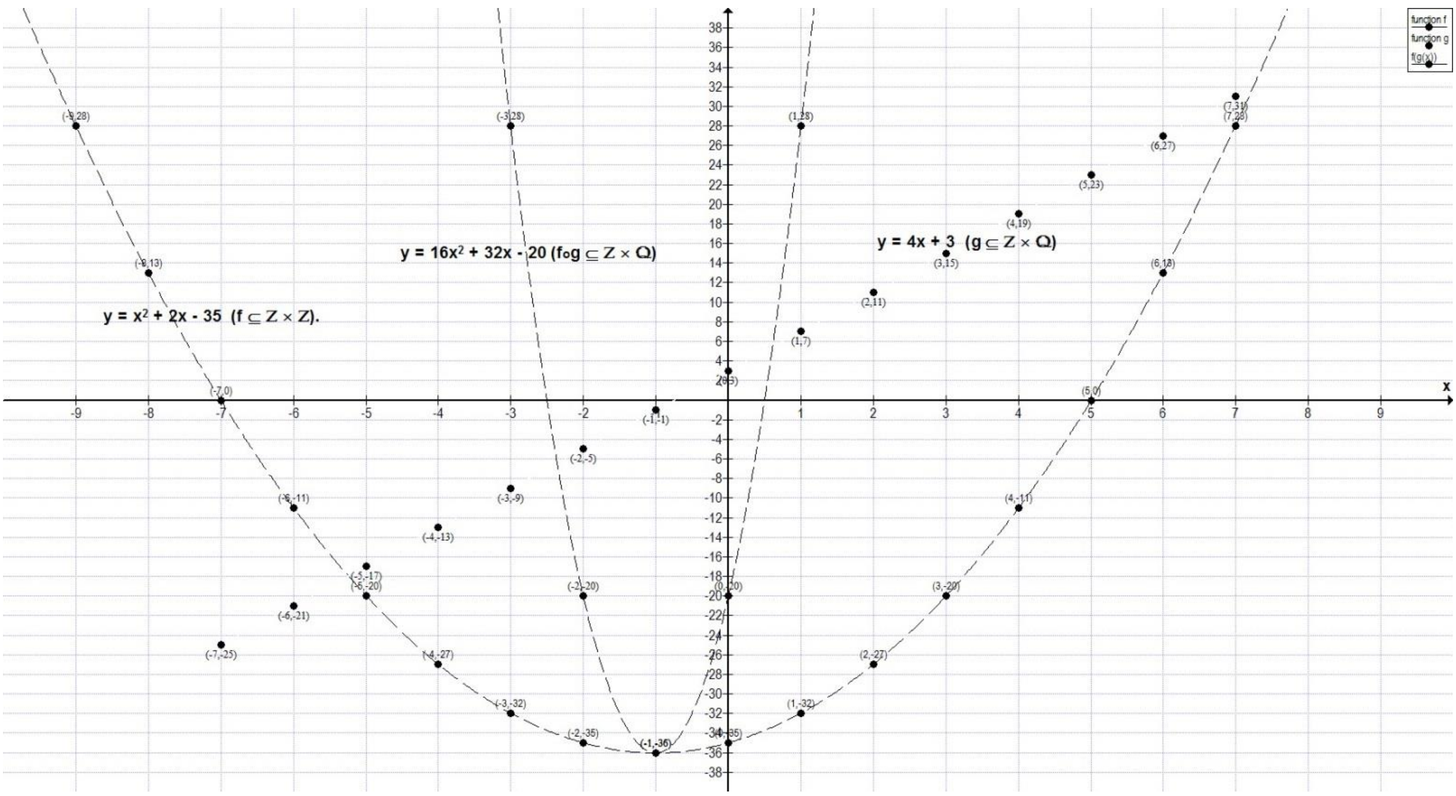
Question 6

Which one of the following alternatives represents the image of x under $f \circ g$ (ie $f \circ g(x)$))?

1. $4x^2 + 8x - 137$
2. $16x^2 + 32x - 20$
3. $4\mathbf{x}^2 + 32x + 15$
4. $4x^2 + 8\mathbf{x} - 140$

Answer: Alternative 2

We draw the graphs for functions f, g and f ○ g below for reference. (You are not expected to draw graphs in the assignments or in the exam). Note that we use dotted lines for f and f ○ g so that you can see the graphs clearly, but remember that only the ordered pairs indicated by dots are ordered pairs of these functions, because according to the definitions of these functions, dom(f) = ℤ**,** dom(g) = ℤ and dom(f ○ g) = ℤ:



Discussion

Given the functions g: A → B and f: B → C the composite function

f ○ g: A → C is defined by f ○ g(x) = f(g(x)).

f ○ g(x)

= f(g(x))

= f(4x + 3)

= $(4x + 3)^2 + 2(4x + 3) – 35$

= $16x^2 + 24x + 9 + 8x + 6 – 35$

= $16x^2 + 32x – 20$

From the above we can deduce that alternative 2 should be selected.

Refer to study guide, p 110.

Question 7

Which one of the following alternatives represents the range of g (ie ran(g))?

1. $\mathbb{Q}$
2. {y | 4x + 3 $\in \mathbb{Q}$ }
3. {y | for some y $\in \mathbb{Q}$, y = 4x + 3 $\in \mathbb{Q}$}
4. {y | (y – 3) / 4 $\in \mathbb{Z}$ }

Answer: Alternative 4

Alternative 1:*Discussion*

By the definition of the range of a function

ran(g) = {y | for some x $\in \mathbb{Z}$, (x, y) $\in$ g}
= {y | for some x $\in \mathbb{Z}$, y = 4x + 3 }
= {y | for some **x** $\in \mathbb{Z}$, **x** = (y – 3) / 4 }
= {y | (y – 3) / 4 $\in \mathbb{Z}$ }

Ordered pairs (y – 3) / 4 , y) are elements of g. By the definition of g it is the case that ran(g) $\subseteq$ $\mathbb{Q}$. It is important to make sure that the first coordinates, which are elements of $\mathbb{Z}$, are paired with second coordinates that are elements of $\mathbb{Q}$ since g is a function from $\mathbb{Z}$ to $\mathbb{Q}$.

From the arguments provided, alternative 4 should be selected.

Refer to study guide, pp 98, 104-105

Let A = {□, ◊, ⌂} and let # be a binary operation from A × A to A presented by the following table:

#	□	◊	⌂
□	◊	□	◊
◊	□	◊	⌂
⌂	⌂	⌂	◊

Answer questions 8 and 9 by referring to the table for #.

Question 8

Which one of the following statements pertaining to the binary operation # is TRUE?

1. ◊ is the identity element for #.
2. # is commutative, but not associative.
3. # is associative, but not commutative.

4. [(⌂ # □) # ⌂] = [⌂ # (□ # ⌂)]

Answer: Alternative 1

We consider the statements provided in the different alternatives:

1. *Definition of an identity element of a binary operation:*
   An element e of X is an identity element in respect of the binary operation
   $* : X \times X \rightarrow X$ *iff e * **x** = **x** * e = **x** for all x* $\in$ *X. (Note that the output is **x**.)*

   Is it possible to identify an element *e* in A such that *e* # x = x # *e* = x for all x $\in$ A?
   Yes, e = ◊ is such an element of A:

	e # x	= x # e	= x
x = □:	□ # ◊	= ◊ # □	= □,
x = ◊:	◊ # ◊	= ◊ # ◊	= ◊, and
x = ⌂:	⌂ # ◊	= ◊ # ⌂	= ⌂.

   So ◊ acts as an identity element for #.

   The highlighted row and column in the table confirm that ◊ is the identity element for #. For an explanatory example, refer to study guide, p 121.

2. The binary operation # is not commutative. We give a counterexample:
   (⌂ # □) = ⌂ and (□ # ⌂) = ◊**,** but ⌂ ≠ ◊.

3. The binary operation # is not associative. We give a counterexample:
   (⌂ # ⌂) # □ = ◊ # □ = □**,** and ⌂ # (⌂ # □) = ⌂ # ⌂ = ◊**,** but □ ≠ ◊.

4. We determine [(⌂ # □) # ⌂] and [⌂ # (□ # ⌂)] then compare the results:
   [(⌂ # □) # ⌂]
   = ⌂ # ⌂
   = ◊

   [⌂ # (□ # ⌂)]
   = ⌂ # ◊
   = ⌂

   We see that [(⌂ # □) # ⌂] ≠ [⌂ # (□ # ⌂)] since ◊ ≠ ⌂.
   This alternative is therefore not the one to select. This alternative also proves that # is not associative.

From the above we can deduce that alternative 1 should be selected.

Refer to study guide, pp 116-122.

Question 9

can be written in list notation. Which one of the following ordered pairs is NOT an element of the list notation set representing #?

1. ((⌂, ⌂), ◊)
2. ((◊, ◊), ◊)
3. ((□, ◊), ◊)
4. ((□, ⌂), ◊)

Answer: Alternative 3

We consider the ordered pairs provided in the different alternatives:

1. From the table (⌂ # ⌂) = ◊
   thus ((⌂, ⌂), ◊) ∈ # .

2. ◊ # ◊ = ◊
   thus ((◊, ◊), ◊) ∈ # .

3. From the table □ # ◊ = □
   thus ((□, ◊), □) ∈ # , but ((□, ◊), ◊) ∉ #.

4. From the table □ # ⌂ = ◊
thus ((□, ⌂), ◊) ∈ # .

From the arguments provided we can deduce that alternative 3 should be selected.

Refer to study guide, pp 118, 119.

Question 10

Perform the following matrix multiplication operation:

$$\begin{bmatrix} 1 & -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ 4 & -1 \end{bmatrix}$$

Which one of the following alternatives represents the correct answer to the above operation?

1. The operation is not possible.
2. $\begin{bmatrix} 18 & -15 \end{bmatrix}$
3. $\begin{bmatrix} 2 & 13 & 9 \end{bmatrix}$

4. $\begin{bmatrix} 2 & 0 \\ 4 & -12 \\ 12 & -3 \end{bmatrix}$

Answer: Alternative 2

Discussion
*A 1 × **3** matrix multiplied by a **3** × 2 matrix gives a 1 × 2 matrix.*

We determine a_{ij}:
a_{11} = 1(2) + (-4)(−1) + 3(4) = 18
a_{12} = 1(0) + (-4)(3) + 3(-1) = -15
Thus the answer to the multiplication of the given matrices is

$$\begin{bmatrix} 18 & -15 \end{bmatrix}$$

From the above we can deduce that alternative 2 should be selected.

Refer to study guide, pp 131, 132.

Question 11
Let p, q and r be simple declarative statements. Which one of the following alternatives is equivalent to p ∨ (r → (q ∨ ¬r)) ?

1. p ∨ q ∨ r
2. p ∨ (¬r ∧ q)
3. p ∨ (p ∧ r) ∨(p ∧ ¬r)
4. p ∨ (q ∨¬r)

Correct alternative: 4

(a) Discussion

We start with statement p ∨ (r → (q ∨ ¬r)) and see where it takes us. We know that for any two declarative statements p and q it is true that p → q ≡ ¬p ∨ q. (If you are not convinced, draw truth tables for p → q and ¬p ∨ q). Let us apply this fact in the statement:

p ∨ (r → (q ∨ ¬r))

≡ p ∨ (**¬r ∨ (q ∨ ¬r)**)

≡ p ∨ ((**¬r ∨ q**) ∨ ¬r) (associative law)

$\equiv p \lor ((\mathbf{q \lor \neg r}) \lor \neg r)$ (commutative law)

$\equiv p \lor (q \lor \mathbf{(\neg r} \lor \neg r\mathbf{)})$ (associative law)

$\equiv p \lor (q \lor \neg r)$ (idempotent law)

From the above we can deduce that alternative 4 should be selected. If you are not convinced, draw truth tables for the different alternatives.

Refer to study guide, pp 140 – 147.

Question 12

Let p, q and r be simple declarative statements. Which alternative provides the truth values for the compound statement provided in the given table?

Hint: Determine the truth values of ¬q, (r ∧ p), and (¬q → (r ∧ p)) in separate columns before determining the truth values of [p → (¬q → (r ∧ p)]

p	**q**	**r**	**[p → (¬q → (r ∧ p)]**
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

1.

→
T
F
T
F
F
F
F
F

2.

→
T
T
T
T
F
F
T
T

3.

→
T
T
T
F
T
T
T
T

4.

→
T
F
T
T
T
T
T
T

We draw the complete truth table :

p	**q**	**r**	**¬q**	**(r ∧ p)**	**(¬q → (r ∧ p)**	**[p → (¬q → (r ∧ p)]**
T	T	T	F	T	T	T
T	T	F	F	F	T	T
T	F	T	T	T	T	T
T	F	F	T	F	F	F
F	T	T	F	F	T	T
F	T	F	F	F	T	T
F	F	T	T	F	F	T
F	F	F	T	F	F	T

It is clear that alternative 3 should be selected.

Refer to study guide, pp 140 – 147.

Question 13

Which one of the alternatives provides the negation of the statement

$$\forall x \in \mathbb{Z}^+, [((2x + 1) > 4) \vee (2x^2 \geq 2)]?$$

1. $\exists x \in \mathbb{Z}^+, [(2x < 3) \vee (2x^2 \leq 2)]$

2. $\exists x \in \mathbb{Z}^+, [((2x + 1) \leq 4) \vee (2x^2 < 2)]$

3. $\forall x \in \mathbb{Z}^+, [((2x + 1) \leq 4) \wedge (2x^2 < 2)]$

4. $\exists x \in \mathbb{Z}^+, [(2x \leq 3) \wedge (2x^2 < 2)]$

Correct alternative: 4

Discussion

Firstly we will derive the negation of the given statement step by step:

$\neg[\forall x \in \mathbb{Z}^+, [((2x + 1) > 4) \vee (2x^2 \geq 2)]]$ (use square brackets [ ] to clearly indicate what is negated)

$\equiv \exists x \in \mathbb{Z}^+, \neg [((2x + 1) > 4) \vee (2x^2 \geq 2)]$ (rules for negation of quantified statements)

$\equiv \exists x \in \mathbb{Z}^+, \neg((2x + 1) > 4) \wedge \neg(2x^2 \geq 2)$ (de Morgan's law)

≡ $\exists x \in \mathbb{Z}^+$, $((2x + 1) \not> 4) \wedge (2x^2 \not\geq 2)$ (note the use of brackets to clearly identify statements)

≡ $\exists x \in \mathbb{Z}^+$, $((2x + 1) \leq 4)) \wedge (2x^2 < 2)$

≡ $\exists x \in \mathbb{Z}^+$, $(2x \leq 3) \wedge (2x^2 < 2)$ (simplify)

Although this is not part of the question, let us investigate whether the statement or its negation is true. Take note of this, because you might be asked in the exam to determine whether the statement and/or its negation is true. We look at the statement first:

$$\forall x \in \mathbb{Z}^+, [((2x + 1) > 4) \vee (2x^2 \geq 2)]$$

This statement states that for all positive integers x, it is true that $((2x + 1) > 4)$ **or** $(2x^2 \geq 2)$. This means that $((2x + 1) > 4)$ **or** $(2x^2 \geq 2)$ must be true for the statement to be true. Both may also be true. (Refer to the truth table in the study guide for the disjunction '∨'.)

We substitute a few positive integers for x in both $((2x + 1) > 4)$ and $(2x^2 \geq 2)$ and see where that takes us:

x	$(2x + 1) > 4$	$2x^2 \geq 2$
1	$2(1) + 1 = 3 > 4$ which is not true	$2(1)^2 = 2 \geq 2$ which is true
2	$2(2) + 1= 5 > 4$ which is true	$2(2)^2 = 8 \geq 2$ which is true
3	$2(3) + 1= 7 > 4$ which is true	$2(3)^2 = 18 \geq 2$ which is true
	etc…	etc…

We prove that that $2x^2 \geq 2$ is true for all positive integers x:

$2x^2 \geq 2$ iff $x^2 \geq 1$. Thus $x^2 \geq 1$ for all $x \in \mathbb{Z}^+$. Thus $((2x + 1) > 4) \vee (2x^2 \geq 2)$ is true for all positive integers x.

What about the negation statement?

$$\exists x \in \mathbb{Z}^+, (2x \leq 3) \wedge (2x^2 < 2)$$

The negation statement states that there exists a positive integer x such that **both** $(2x \leq 3)$ **and** $(2x^2 < 2)$ are true. We give a counterexample: if $2x^2 < 2$, then $x^2 < 1$ which is not true for any $x \in \mathbb{Z}^+$. Therefore $(2x \leq 3) \wedge (2x^2 < 2)$ is not true for any $x \in \mathbb{Z}^+$. The negation statement is therefore not true.

From the discussion above, alternative 4 should be selected.

Refer to study guide, pp 152-158.

Question 14

Consider the following quantified statement:

$$\forall x \in \mathbb{Z}^+ [((2x^2 + 2x) \geq 4) \vee ((x^3 - 1) > 2)].$$

Which one of the alternatives provides a true statement regarding the given statement or its negation?

1. The negation of the statement is $\exists x \in \mathbb{Z}^+ [((2x^2 + 2x) \leq 4) \wedge ((x^3 - 1) < 2)]$.
2. The negation of the statement is $\exists x \in \mathbb{Z}^+ [((2x^2 + 2x) < 4) \wedge (x^3 \leq 3)]$.
3. The negation statement is true for x = 1.
4. x = 0 would be a counterexample to prove that the statement is not true.

Discussion

Let us first determine the negation of the statement:

$\neg [\forall x \in \mathbb{Z}^+, [(2x^2 + 2x \geq 4) \vee (x^3 - 1 > 2)]]$ (use square brackets [ ] to indicate what is negated)

$\equiv \exists x \in \mathbb{Z}^+, \neg[(2x^2 + 2x \geq 4) \vee (x^3 - 1 > 2)]$ (rules for negation of quantified statements)

$\equiv \exists x \in \mathbb{Z}^+, \neg(2x^2 + 2x \geq 4) \wedge \neg(x^3 - 1 > 2)]$ (de Morgans laws)

$\equiv \exists x \in \mathbb{Z}^+, (2x^2 + 2x \ngeq 4) \wedge (x^3 - 1 \ngtr 2)]$

$\equiv \exists x \in \mathbb{Z}^+, (2x^2 + 2x < 4) \wedge (x^3 - 1 \leq 2)]$

$\equiv \exists x \in \mathbb{Z}^+, (2x^2 + 2x < 4) \wedge (x^3 \leq 3)]$ (simplify)

Alternative 1:

From the discussion above it is clear that alternative 1 is not the negation of the statement.

Alternative 2:

From the discussion above, alternative 2 is clearly the negation of the statement.

Alternative 3:

We look at $(2x^2 + 2x < 4)$ and $(x^3 \leq 3)$ separately. Remember that for the negation statement to be true, there must be at least 1 positive integer for which both $(2x^2 + 2x < 4)$ **and** $(x^3 \leq 3)$ must be true (see the definition of conjunction in the study guide).

$2x^2 + 2x < 4$ then $2(1)^2 + 2(1) = 4 \nless 4$. Because $(2x^2 + 2x < 4)$ is not true for x = 1, we can deduce that the negation statement is not true for x = 1. This alternative can therefore not be selected.

Alternative 4:

This statement states that we can use the value x = 0 as a counterexample to prove that the original statement is not true. This however is not true, because $0 \notin \mathbb{Z}^+$ and can therefore not be used in a counterexample.

Question 15

Which one of the alternatives provides a proof by contrapositive of the statement

"If $3x^2 + 6x + 4$ is even, then x even."

1. Proof:
   Suppose x is even. Let x = 2k, then we have to prove that $3x^2 + 6x + 4$ is even.
   $3x^2 + 6x + 4 = (2k)^2 + 6(2k) + 4$
   $= 3(2k)^2 + 12k + 4$
   $= 12k^2 + 12k + 4$
   $= 2(6k^2 + 6k + 2)$, which is even.

2. Proof:
   Assume that x is odd. Then we have to prove that $3x^2 + 6x + 4$ is odd.
   Let x = 2k + 1, then
   $3(2k+ 1)^2 + 6(2k + 1) + 4$
   $= 3(4k^2 + 4k + 1) + 12k + 6 + 4$
   $= 12k^2 + 24k + 13$
   $= 2(6k^2 + 12k + 6) + 1$ which is odd (even + odd = odd).

3. Proof:
   If $3x^2 + 6x + 4$ is even, then x can be odd or even. Let us assume x is odd.
   Let x = 2k + 1, then
   $3(2k+ 1)^2 + 6(2k + 1) + 4$
   $= 3(4k^2 + 4k + 1) + 12k + 6 + 4$
   $= 12k^2 + 24k + 13$
   $= 2(6k^2 + 12k + 6) + 1$ which is odd (even + odd = odd).
   But this is a contradiction, so our assumption that x is odd, is incorrect. X must therefore be even.

4. Proof:
   Suppose x is even. Let x = 2, then we have to prove that $3x^2 + 6x + 4$ is even.
   $3(2)^2 + 6(2) + 4$
   $= 12 + 12 + 4$
   $= 28$
   $= 2(14)$, which is even.

Correct alternative: 2

Discussion: It is very important that you know how to apply each of the proof methods discussed in the study guide.

We look at each of the alternatives:

1. The proof provided in this alternative is not a contrapositive proof. This proof assumes that the contrapositive of $p \to q$ is $q \to p$. Is this true? Look at the following truth table:

p	**q**	**$p \rightarrow q$**	**$q \rightarrow p$**
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Clearly $p \rightarrow q$ and $q \rightarrow p$ are not equivalent.

2. The proof provided in this alternative is a proof by contrapositive. Another way to look at this proof method is the following:
The contrapositive of $p \rightarrow q$ is $\neg q \rightarrow \neg p$. This means that $p \rightarrow q$ is logically equivalent to $\neg q \rightarrow \neg p$.

The contrapositive of the statement "If $3x^2 + 6x + 4$ is even, then x even."
is "If x odd (or not even) then $3x^2 + 6x + 4$ is odd (or not even)."
This is exactly what is proven in alternative 2. Thus this is the correct alternative to select.

3. This alternative provides a proof by contradiction. We assume the 'if' part of the given statement is true, ie "$3x^2 + 6x + 4$ is even" is true, then we assume the opposite of the 'then' part. The 'then' part states that "x is even", so we assume the opposite, ie "x is odd", and then try to get to a contradiction.
This alternative does not provide the required contrapositive proof.

4. This alternative is not a proof. One do not substitute specific values for x in a proof. One example (ie choosing a value for x and substituting it in the expression) does not provide a general proof to show that "If $3x^2 + 6x + 4$ is even then x is even".

Refer to study guide, pp 152 - 163.