COS1501 Theoretical Computer Science 1
Practice fill-in paper exam solutions

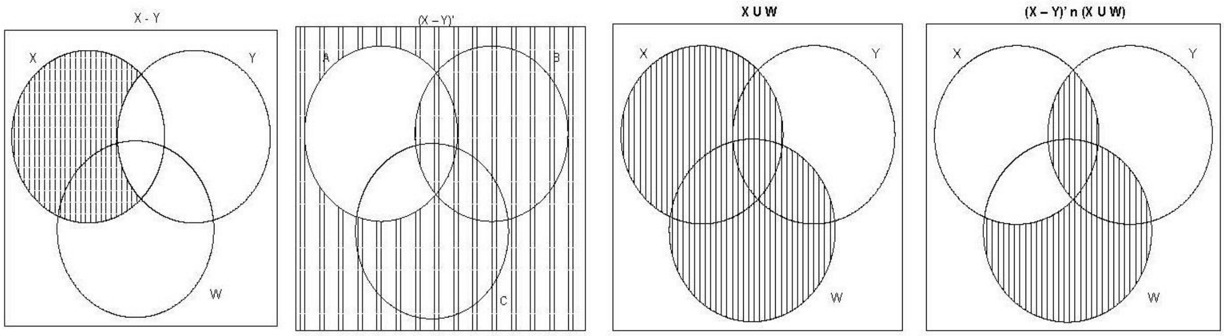
We have indicated where marks could be allocated in a real exam. Use these as guidelines to see where you would lose marks if you leave out steps, or have incorrect mathematical format.

SECTION 1 (Multiple-Choice Questions)
SETS AND RELATIONS
Each question comprises 2 marks. [16 marks]

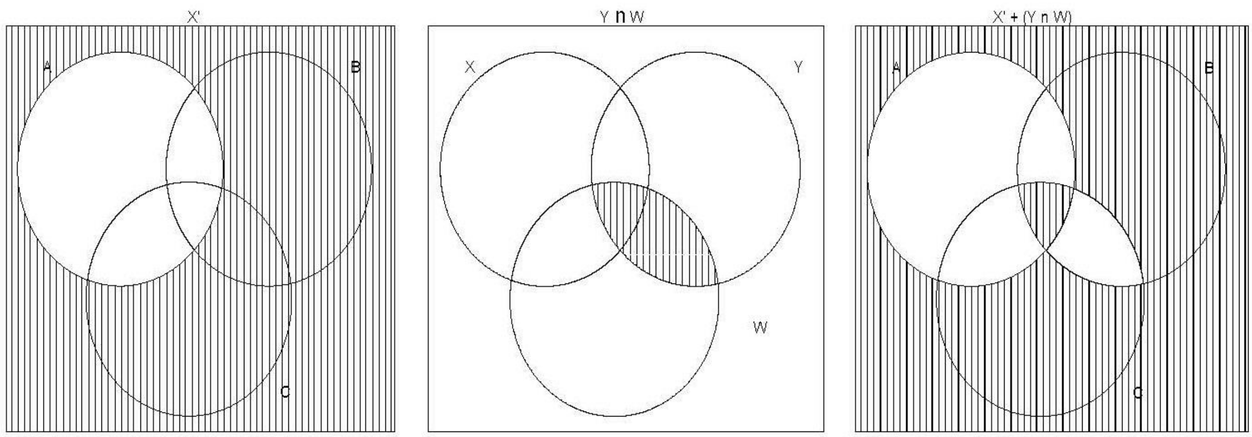
Question 1.1 Alternative 3
Question 1.2 Alternative 2
Question 1.3 Alternative 3
Question 1.4 Alternative 3
Question 1.5 Alternative 1
Question 1.6 Alternative 3
Question 1.7 Alternative 4
Question 1.8 Alternative 4

SECTION 2
SET THEORY [19 marks]

Question 2.1 (7 marks)
LHS: $(X - Y)' \cap (X \cup W)$ ✓✓✓✓



RHS: $X \cap (Y' \cup W)$ ✓✓✓



Question 2.2 (4 marks)

$x \in X \cap (Y' - W)$
iff $x \in X$ and ✓$^{1/2}$ $x \in (Y' - W)$ ✓$^{1/2}$
iff $x \in X$ and ($x \in Y'$ and ✓$^{1/2}$ $x \notin W$ ✓$^{1/2}$)
iff $x \in X$ and ($x \in Y'$ and $x \in W'$ ✓)
iff $x \in X \cap (Y' \cap W')$ ✓

Thus $X \cap (Y' - W) = X \cap (Y' \cap W')$ for all subsets X, Y and W.

Question 2.3 (4 marks)
Let {1, 2, 3, 4} be a universal set with subsets X = {1, 2} and Y = {3, 4} and W = {2, 4}.
Use the sets X, Y and W to show that:

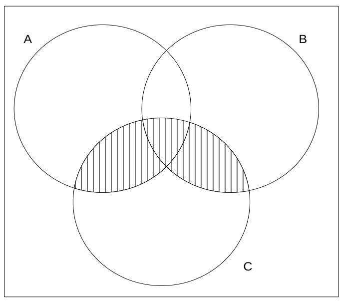
$$[X \cup (Y \cap (W' + X'))] = [X \cup (Y \cap W)].$$

We substitute the given sets for X, Y and W in the LHS and RHS separately :

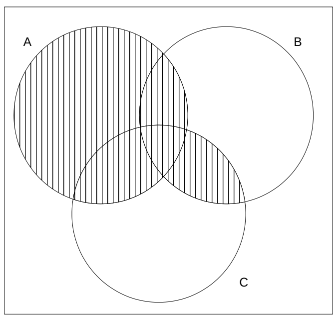
LHS: $[X \cup (Y \cap (W' + X'))] = [\{1, 2\} \cup (\{3, 4\} \cap (\{1, 3\} + \{3, 4\})]$ ✓$^{1/2}$
$= [\{1, 2\} \cup (\{3, 4\} \cap \{1, 4\}$ ✓$)]$
$= [\{1, 2\} \cup \{4\}$ ✓$^{1/2}]$
$= \{1, 2, 4\}$ ✓$^{1/2}$
RHS: $[X \cup (Y \cap W)] = [\{1, 2\} \cup (\{3, 4\} \cap \{2, 4\})]$ ✓$^{1/2}$
$= [\{1, 2\} \cup \{4\}]$ ✓$^{1/2}$
$= \{1, 2, 4\}$ ✓$^{1/2}$
We have shown that for the given sets X, Y and W that LHS = RHS.

Question 2.4 (4 marks)
(There could be more than one correct answer for (a)-(b).

a) $(A \cup B) \cap C$ ✓✓
or $(A \cap B) \cup (A \cap C)$



a) $A \cup (B \cap C)$ ✓✓



SECTION 3

RELATIONS AND FUNCTIONS [26 marks]

Question 3.1 (12 marks)

a) A bijective function is injective✓ and surjective.✓ **(**or a correct description of what surjectivity and injectivity is).

b) (i) A function $f: A \rightarrow B$ ✓$^{1/2}$ is injective iff f has the property ✓$^{1/2}$ that whenever $a_1 \neq a_2$ ✓$^{1/2}$ then $f(a_2) \neq f(a_2)$. ✓$^{1/2}$ (When we ask for a definition, you have to write it down as in the study guide, ie in the correct mathematical format).

(ii) {(1, a), (2, b), (3, c)} (or any other correct one-to-one function) ✓✓ (You will lose -1/2 mark here for incorrect brackets, or for each wrong ordered pair. Pay attention to the mathematical format).

c) T = {(3, 4)} ✓✓ (or any other strict partial order example : (A **strict partial order** is **irreflexive, antisymmetric & transitive). (**A few more examples of such a relation: T = {(3, 4), (3, 5)}; T = {(3, 4), (3, 5), (4, 5)})

d) B = {0, 1} partition: {{0}, {1}} ✓
justification:
- {0} and {1} are two non-empty subsets of B, ✓
- $\{0\} \cap \{1\} = \emptyset$, ✓and
- $\{0\} \cup \{1\} = B$. ✓ (Pay attention to the correct brackets and set representation)

Question 3.2 (5 marks)

a) Suppose $(x, y) \in R$ ✓$^{1/2}$ and $(y, z) \in R$ ✓$^{1/2}$ then
$y – x = 7m$ and $z – y = 7k$ ✓$^{1/2}$ (Note that you have to use two different values m and k)
i.e. $z – x = 7m + 7k$ ✓$^{1/2}$
i.e. $z – x = 7(m + k)$ ✓$^{1/2}$
Hence $(x, z) \in R$. ✓$^{1/2}$

→ How do we get this? We solve the two equations simultaneously:
$y – x = 7m$
$z – y = 7k$

$y – x + z – y = 7m + 7k$, which is $z – x = 7(m + k)$.

b) [0] ✓ = {….,-21, -14, -7, 0, 7, 14, 21, …}✓
Other examples: [1] = {…, 20, -13, -6, 1, 8, 15, 22, …}
[3] = {…, -18, -11, -4, 3, 10, 17, 24, } (any of the 7 equivalence classes correct). See the description of equivalence classes in the Study guide pp.91-92.

Question 3.3 (9 marks)

a)
(i) $ran(g) = \{y \mid \text{for some } x \in Z, (x, y) \in g\}$ ✓$^{1/2}$ (or $\{g(x) \mid x \in Z\}$
$= \{y \mid \text{for some } x \in Z, y = –x + 3\}$ ✓$^{1/2}$
$= \{y \mid x = –y + 3 \text{ is an integer}\}$ ✓
$= Z$✓

(ii) g is surjective ✓since ran(g) = Z✓

b) (x, y) ∈ f iff $y = x^2 + 2x$
(x, y) ∈ g iff y = –x + 3.

(i) $f \circ g(x) = f(g(x))$ ✓1/2
= f(–x + 3) ✓1/2
= $(–x + 3)^2 + 2(–x + 3)$ ✓
= $x^2 - 6x + 9 - 2x + 6$ ✓
= $x^2 - 8x + 15$ ✓

(ii) y = -x + 3
substitute -3 for x, then –(-3) + 3 = 6.
This means that the ordered pair (-3, 6) is in g, but not (-3, 0). ✓

SECTION 4
OPERATIONS AND MATRICES [10 marks]

Question 4.1 (3 marks)

✓✓✓ $$AB = \begin{bmatrix} -14 \\ -16 \\ -10 \end{bmatrix}.$$

(We have not shown the calculations here. Please see Study guide pp. 129-132)

Question 4.2 (5 marks)

(i)

*	A	b	c
a	**A**	**b**✓1/2	**c**✓1/2
b	**b**✓1/2	b	b
c	**c**✓1/2	b	b

For id element: 1 mark for each b and c in place.

(ii)

(a*b)*c = b*c = b ✓and a*(b*c) = a*b = b ✓ (these answers will differ depending on highlighted values in table. Highlighted values can be any letter a, b or c).

(iii)

No, ✓1/2 one cannot say that * is associative – all possibilities must be investigated. ✓1/2

Question 4.3 (2 marks)

Use the table below to determine whether a*(b*(b*c)) = (c*a)*(a*b). Show all the steps and state as part of your answer whether a*(b*(b*c)) = (c*a)*(a*b) is an identity.

*	**a**	**b**	**c**
a	a	c	a
b	b	b	c
c	c	c	b

We do the LHS and RHS separately:

LHS: a*(b*(b*c))
= a*(b*c) ✓1/2
= a*c
= a ✓1/2

RHS: (c*a)*(a*b)
= c*c
= b ✓1/2

Therefore LHS ≠ RHS , it is not an identity. ✓1/2

SECTION 5

LOGIC **[18 marks]**

Question 5.1 **(11 marks)**

a)

(i)	$(p \lor \neg(\neg(\neg q)) \equiv (\neg q \land p)$	T	F✓
(ii)	$(\neg(\neg(\neg p)) \rightarrow q) \equiv p \lor q$	T✓	F
(iii)	$\neg (q \rightarrow p) \equiv \neg p \rightarrow q$	T	F✓

b)

(i) Complete the truth table below. Do NOT add any rows. (*Hint:* Complete the highlighted column last).

✓✓ **(one mark per row)**

p	**q**	**r**	**¬q**	**¬r**	**(¬r ∨ p)**	**q → (¬r ∨ p)**	**q → (¬r ∨ p) ↔ (p ∨ ¬q) ∧ r**	**(p ∨ ¬q)**	**(p ∨ ¬q) ∧ r**
T	T	F	F	T	T	T	F	T	F
T	F	T	T	F	T	T	T	T	T

(iii) Complete the truth table for the following statement. (*Hint:* Complete the highlighted column last). (one mark each for indicated columns)

p	**r**	**¬p**	**¬r**	✓ **(p ∨ r)**	✓ **(¬r ∧ p)**	✓ **(p ∨ r) ∧ (¬r ∧ p)**	✓ **[(p ∨ r) ∧ (¬r ∧ p)] ↔(¬p → r)**	✓ **(¬p → r)**
T	T	F	F	T	F	F	F	T
T	F	F	T	T	T	T	T	T
F	T	T	F	T	F	F	F	T
F	F	T	T	F	F	F	T	F

(ii) Neither. ✓ For tautology highlighted column must be all **F**'s. For contradiction, highlighted column must be all **F**'s.

Question 5.2 **(7 marks)**

Consider the statement: $\exists\ x \in Z^+, [(x - 1 \geq 0) \land (x + 6 < 8)]$.

a) Is the given statement true? Justify your answer.

Yes ✓ the statement is only true for x = 1✓ . It only needs to be true for one $x \in Z^+$.

b)

$\neg [\forall x \in Z, [(x - 5 \leq -1) \vee (x^2 + 2x \neq 15)]]$

$\equiv \ \exists\ x \in Z,$ ✓ $\neg [(x - 5 \leq -1) \vee (x^2 + 2x \neq 15)]$ ✓

$\equiv \ \exists\ x \in Z, \neg (x - 5 \leq -1)$ ✓1/2 $\wedge$ ✓$\neg (x^2 + 2x \neq 15)$ ✓1/2

$\equiv \ \exists\ x \in Z, (x - 5 > -1)$ ✓1/2 $\wedge$ $(x^2 + 2x = 15)$ ✓1/2

Note: Make sure you use the correct mathematical format and the correct brackets. You need to include ≡ $\exists\ x \in Z$, in each step. Also write down all the steps. You get marks for each step that is correct. If you make a mistake in line 3 you can still get marks for the first two lines. But if you only write down the answer, and it is wrong, you lose all the marks.

SECTION 6

MATHEMATICAL PROOFS [11 marks]

Question 6.1 (4marks)

if $n^2 - 5n + 4 < 0$, then $n > 0$.

Direct proof:

$n^2 - 5n + 4 < 0$

ie $(n - 1)(n - 4) < 0$✓

ie $n < 1$ and $n > 4$ (impossible), ✓ or (n cannot be less than 1 and greater than 4 at the same time. We therefore discard this possibility.)

$n > 1$ and $n < 4$✓ ie $n > 0$✓ (n can therefore only have the values 2 and 3 which is indeed greater than zero).

Note : Please read the questions carefully and answer ALL the questions. You need to show that you can see that $n > 0$.

Question 6.2 (2 marks)

Statement: If $7x^3 - 4x + 8$ is even, then x is odd.

Converse statement: If x is odd✓, then $7x^3 - 4x + 8$ is even✓.

(Make sure you know the difference between the contrapositive and the converse).

Question 6.3 (5 marks)

Given: $x^2 + 2x$ is odd, then $x - 1$ is even.

Contrapositive: To prove: if $x - 1$ is odd then $x^2 + 2x$ is even ✓

NOTE: Always write down the contrapositive statement as above, and then start your proof from there.

You could have done this in two ways – we give both:

Method 1:

Suppose $x - 1$ is odd, ✓ ie $x - 1 = 2m + 1$ ✓1/2

ie $x = 2m + 2$✓1/2.

$= (2m + 2)^2 + 2\cdot(2m + 2)$ ✓

$= 4m^2 + 8m + 4 + 4m + 4$

$= 4m^2 + 12m + 8$✓
$= 2(2m^2 + 6m + 4)$ ✓ which is even. (2 x anything is always even).

Method 2:

Suppose $x – 1$ is odd, ✓ ie x must be even, ie $x = 2m$ ✓ (You have to include this step)

$= (2m)^2 + 2\cdot(2m)$ ✓

$= 4m^2 + 4m$ ✓
$= 2(2m^2 + 2m)$ ✓ which is even.

(Make sure you can do a direct proof, a proof by contradiction, and a contrapositive proof).

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