DSC1520 ASSIGNMENT 3
POSSIBLE SOLUTIONS

Question 1
Find the derivative of the function: \( G(x) = x(x^2 - 4\sqrt{x} + 4) \)

Replace \( \sqrt{x} \) with \( x^{\frac{1}{2}} \), expand the brackets and simplify before differentiating

\[
G(x) = x(x^2 - 4x^{\frac{1}{2}} + 4)
\]

\[
G(x) = x^3 - 4x^{\frac{3}{2}} + 4x
\]

Apply the “Power Rule” of differentiation.

If \( G(x) = x^n \) then \( G'(x) = nx^{n-1} \)

Also, if \( G(x) = ax^n \) then \( G'(x) = anx^{n-1} \)

Note: The derivative of \( ax \) is \( a \) and the derivative of a constant term \( c \) is \( 0 \).

\[
G'(x) = 3x^{3-1} - 4 \left(\frac{3}{2}\right) x^{\frac{3}{2}-1} + 4
\]

\[
G'(x) = 3x^2 - 6x^{\frac{1}{2}} + 4
\]

Replace \( x^{\frac{1}{2}} \) with \( \sqrt{x} \)

\[
G'(x) = 3x^2 - 6\sqrt{x} + 4
\]
Question 2

Differentiate the function

\[ f(x) = \frac{x^2 + 6}{2x + 5} \]

Apply the “Quotient rule” of differentiation.

Let \( f(x) = \frac{u}{v} \)

\( u = x^2 + 6; \ du = 2x \) and \( v = 2x + 5; \ dv = 2 \)

\[ f'(x) = \frac{vdu - udv}{v^2} \]

\[ f'(x) = \frac{(2x + 5)2x - (x^2 + 6)2}{(2x + 5)^2} \]

\[ f'(x) = \frac{4x^2 + 10x - 2x^2 - 12}{(2x + 5)^2} \]

\[ f'(x) = \frac{4x^2 - 2x^2 + 10x - 12}{(2x + 5)^2} \]

\[ f'(x) = \frac{2x^2 + 10x - 12}{(2x + 5)^2} \]

Factor out 2, the common factor on the numerator

\[ f'(x) = \frac{2(x^2 + 5x - 6)}{(2x + 5)^2} \]

Now, factorize the bracket on the numerator

\[ f'(x) = \frac{2(x + 6)(x - 1)}{(2x + 5)^2} \]
Question 3

Find the derivative of the function

\[ P(x) = x^5e^{3x} + \frac{x + 1}{x} \]

Apply the “Product rule” on \( x^5e^{3x} \) and “Quotient rule” on \( \frac{x+1}{x} \)

Let \( P(x) = f(x) + g(x) \) where

\[ f(x) = x^5e^{3x} \text{ and } g(x) = \frac{x+1}{x} \]

\[ P'(x) = f'(x) + g'(x) \]

\[ f(x) = uv \text{ where} \]

\[ u = x^5; \ du = 5x^4 \text{ and } v = e^{3x}; \ dv = 3e^{3x} \]

\[ f'(x) = u dv + v du \quad \text{[Product rule]} \]

\[ f'(x) = x^5 \cdot 3e^{3x} + e^{3x} \cdot 5x^4 \]

\[ f'(x) = 3x^5e^{3x} + 5x^4e^{3x} = e^{3x}(3x^5 + 5x^4) \]

\[ g(x) = \frac{u}{v} \text{ where} \]

\[ u = x + 1; \ du = 1 \text{ and } v = x; \ dv = 1 \]

\[ g'(x) = \frac{v du - udv}{v^2} \]

\[ g'(x) = \frac{x(1) - (x + 1)1}{x^2} = \frac{x - x - 1}{x^2} = -\frac{1}{x^2} \]

\[ \therefore P'(x) = e^{3x}(3x^5 + 5x^4) - \frac{1}{x^2} \]
Question 4

Find the derivative of

$$\frac{d}{dx}(\ln x + 4x^{-2}) = \frac{1}{x} + 4(-2)x^{-2-1} = \frac{1}{x} - 8x^{-3}$$

Question 5

Evaluate

$$\int x^2(1 + \frac{1}{x^2})dx$$

Expand the bracket, simplify and apply the “Power rule” of integration.

$$\int x^2 \left(1 + \frac{1}{x^2}\right)dx = \int x^2 + \frac{x^2}{x^2}dx = \int (x^2 + 1)dx$$

Power rule of integration

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + c$$

Note: $$\int a dx = ax + c$$, where a is any constant term.

$$\int (x^2 + 1)dx = \frac{x^{2+1}}{2+1} + x + c = \frac{x^3}{3} + x + c = \frac{1}{3}x^3 + x + c$$
Question 6

Evaluate the following definite integral:

\[ \int_{-2}^{2} (x^2 - 3) \, dx \]

\[ \int_{-2}^{2} (x^2 - 3) \, dx = \left[ \frac{x^{2+1}}{2+1} - 3x \right]_{-2}^{2} = \left[ \frac{x^3}{3} - 3x \right]_{-2}^{2} \]

Note: There is no constant of integration, c in a definite integral.

\[ = \left[ \frac{(2)^3}{3} - 3(2) \right] - \left[ \frac{(-2)^3}{3} - 3(-2) \right] \]

\[ = \left[ \frac{8}{3} - 6 \right] - \left[ -\frac{8}{3} + 6 \right] \]

\[ = \left[ -\frac{10}{3} \right] - \left[ \frac{10}{3} \right] \]

\[ = -\frac{20}{3} \]

\[ = -6 \frac{2}{3} \]
Question 7

Evaluate the following integral:

\[ \int \sqrt{9x - 5} \, dx. \]

Replace the root sign with an exponent of \( \frac{1}{2} \)

\[ \int \sqrt{9x - 5} \, dx = \int (9x - 5)^{\frac{1}{2}} \, dx. \]

Apply the “u” substitution or use standard integrals

The “u” substitution method.

Let \( u = 9x - 5; \frac{du}{dx} = 9; du = 9 \, dx; \frac{du}{9} = dx \)

Now, in the original integral replace \( 9x - 5 \) with \( u \).

Also, replace \( dx \) with \( \frac{du}{9} \).

\[ \int (9x - 5)^{\frac{1}{2}} \, dx = \int u^{\frac{1}{2}} \cdot \frac{du}{9} \]

\[ = \frac{1}{9} \int u^{\frac{1}{2}} \, du = \frac{1}{9} \cdot \frac{u^{\frac{1}{2}+1}}{\frac{1}{2}+1} + c = \frac{1}{9} \cdot \frac{u^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{27}u^{\frac{3}{2}} + c \]

But \( u = 9x - 5 \)

Replace \( u \) with \( 9x - 5 \)

Also, replace the exponent of \( \frac{3}{2} \) with the equivalent root

\[ \int \sqrt{9x - 5} \, dx = \frac{2}{27} \sqrt{(9x - 5)^3} + c \]
Question 8

Evaluate the following integral:

\[ \int \frac{x^2 + 4}{x^3} \, dx. \]

Express as separate fractions and simplify.

\[
= \int \left( \frac{x^2}{x^3} + \frac{4}{x^3} \right) \, dx \\
= \int \left( \frac{1}{x} + 4x^{-3} \right) \, dx \\
= \ln x + 4 \left( \frac{x^{-3+1}}{-3+1} \right) + c \\
= \ln x + \frac{4x^{-2}}{-2} + c \\
= \ln x - 2x^{-2} + c
\]
### Table of derivatives including some standard derivatives

<table>
<thead>
<tr>
<th>Function: $f(x)$</th>
<th>Derivative: $f'(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>0</td>
</tr>
<tr>
<td>$ax$</td>
<td>$a$</td>
</tr>
<tr>
<td>$x^n$</td>
<td>$nx^{n-1}$</td>
</tr>
<tr>
<td>$ax^n$</td>
<td>$anx^{n-1}$</td>
</tr>
<tr>
<td>$[f(x)]^n$</td>
<td>$nf'(x)[f(x)]^{n-1}$</td>
</tr>
<tr>
<td>$e^x$</td>
<td>$e^x$</td>
</tr>
<tr>
<td>$e^{g(x)}$</td>
<td>$g'(x)e^{g(x)}$</td>
</tr>
<tr>
<td>$\ln x$</td>
<td>$\frac{1}{x}$</td>
</tr>
<tr>
<td>$\ln g(x)$</td>
<td>$\frac{g'(x)}{g(x)}$</td>
</tr>
</tbody>
</table>
Power Rule

\[ f(x) = ax^n; \quad f'(x) = anx^{n-1} \]

Product Rule

- Used to differentiate a “product” of two different functions.

\[ f(x) = uv: \text{ where } u \text{ and } v \text{ are both functions of } x. \]

\[ f'(x) = udv + vdu: \text{ where } du \text{ and } dv \text{ are the derivatives of } u \text{ and } v \text{ with respect to } x, \text{ respectively}. \]

Quotient Rule

- Used to differentiate a “quotient” or a fraction of two functions.

\[ f(x) = \frac{u}{v}: \text{ where } u \text{ and } v \text{ are both functions of } x. \]

\[ f'(x) = \frac{vdu - udv}{v^2} \]

du and dv are the derivatives of u and v with respect to x, respectively.

The Chain Rule

- Used to differentiate a function of a function or a multiple of these.
- By making the necessary substitutions, a chain of derivatives is used to compute the derivative of the particular function, for example,

If \( y = f(u) \text{ where } u = f(v) \text{ and } v = f(w) \) then

\[ y'(w) = \frac{dy}{dw} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dw} \]

Notice how the du and dv terms will disappear, by cancelling each other out, to yield the desired derivative, \( \frac{dy}{dw} \).
# Table of integrals including standard integrals

<table>
<thead>
<tr>
<th>Integral</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \int 0 , dx )</td>
<td>( c )</td>
</tr>
<tr>
<td>( \int a , dx )</td>
<td>( ax + c )</td>
</tr>
<tr>
<td>( \int x^n , dx )</td>
<td>( \frac{x^{n+1}}{n+1} + c )</td>
</tr>
<tr>
<td>( \int ax^n , dx )</td>
<td>( \frac{ax^{n+1}}{n+1} + c )</td>
</tr>
<tr>
<td>( \int f'(x)[f(x)]^n , dx )</td>
<td>( \frac{[f(x)]^{n+1}}{n+1} + c )</td>
</tr>
<tr>
<td>( \int e^x , dx )</td>
<td>( e^x + c )</td>
</tr>
<tr>
<td>( \int f'(x)e^{f(x)} , dx )</td>
<td>( e^{f(x)} + c )</td>
</tr>
<tr>
<td>( \int \frac{1}{x} , dx )</td>
<td>( \ln x + c )</td>
</tr>
<tr>
<td>( \int \frac{f'(x)}{f(x)} , dx )</td>
<td>( \ln f(x) + c )</td>
</tr>
</tbody>
</table>

## Definite integrals

\[
\int_{a}^{b} f(x) \, dx = F(b) - F(a)
\]
Question 9

What is the value of maximum revenue if total revenue is given by

\[ R(x) = -\frac{1}{5}x^2 + 30x + 81 \]

where \( x \) is the quantity?

Maximum revenue occurs when \( R'(x) = 0 \)

where \( R'(x) \) is the derivative of \( R(x) \), the total revenue.

\[ R'(x) = -\frac{1}{5}(2)x^{2-1} + 30 = -\frac{2}{5}x + 30 \]

But \( R'(x) = 0 \) at maximum revenue.

\[-\frac{2}{5}x + 30 = 0 \]

\[ 30 = \frac{2}{5}x \]

\[ \frac{5}{2} \times 30 = \frac{5}{2} \times \frac{2}{5}x \]

\[ 75 = x \]

Thus the maximum revenue is given by substituting 75 for \( x \) in the total revenue function.

\[ R(x) = -\frac{1}{5}(75)^2 + 30(75) + 81 = 1206 \]

OR Since the total revenue function is a quadratic function, the maximum revenue occurs at the turning point where \( x = -\frac{b}{2a} \) where \( a = -\frac{1}{5} \) and \( b = 30 \)

At maximum revenue

\[ x = -\frac{b}{2a} = -\frac{30}{2(-\frac{1}{5})} = 75 \]
Question 10

Total revenue is given by

\[ TR = 2x^5 - \frac{1}{2}x^2 + 10x + 15, \]

where \( x \) is the number of units sold. What is the marginal revenue when five units are sold?

Marginal revenue is the derivative of total revenue thus:

\[ MR = TR' = 2(5)x^{5-1} - \frac{1}{2}(2)x + 10 = 10x^4 - x + 10 \]

Given that \( x = 5 \) units,

\[ MR = 10(5)^4 - 5 + 10 = 6255 \]

Question 11

Suppose the total cost (in rand) of manufacturing radios is given by

\[ 2Q^3 - Q^2 + 80Q + 150 \]

where \( Q \) is the number of radios manufactured. What is the marginal cost if 10 radios are manufactured?

Marginal cost is the derivative of total cost.

\[ MC = TC' = 6Q^2 - 2Q + 80 \]

Given \( Q = 10 \)

\[ MC = 6(10)^2 - 2(10) + 80 = 660 \]

Therefore, the marginal cost if 10 radios are manufactured is R660.
Question 12

The annual revenue (in millions of rand) generated by a television company can be approximated by the function

\[ f(t) = 5.78 + 8.59 \ln t \]

where \( t \) is the number of years since the company started. The rate of change in revenue 15 years after the company started, is given by \( f'(t) \) at \( t = 15 \).

\[ f'(t) = 8.59 \left( \frac{1}{t} \right) = \frac{8.59}{t} \]

Given that \( t = 15 \) and revenue being in millions of rand, the rate of change in revenue is therefore

\[ f'(15) = \frac{8.59}{15} \times 1\,000\,000 \]

\[ = R572\,667 \text{ per annum.} \]
Question 13

The demand for seats at a mini soccer match is given by

\[ Q = 192 - P^2 \]

Where \( Q \) is the number of seats and \( P \) is the price per seat. Find the price elasticity of demand if seats cost R6 each. What does this value mean?

First, find \( Q \) when \( P = 6 \)

\[ Q = 192 - (6)^2 = 156 \]

Since the demand function is non-linear, the price elasticity of demand is given by

\[ \varepsilon_d = \frac{dQ}{dP} \times \frac{P}{Q} \]

\[ \frac{dQ}{dP} = -2P = -2(6) = -12 \]

\[ \therefore \varepsilon_d = -12 \left( \frac{6}{156} \right) \]

\[ \varepsilon_d = -0.46 \]

\[ |\varepsilon_d| = 0.46 < 1, \text{ therefore demand is inelastic.} \]
Question 14

Calculate the consumer surplus for the demand function

\[ P = \frac{40}{Q + 3} \]

When the market price is \( P = 10 \).

First we find \( Q \) when \( P = 10 = P_0 \)

\[
10 = \frac{40}{Q + 3}
\]

\[ 10(Q + 3) = 40 \]

\[ 10Q + 30 = 40 \]

\[ 10Q = 40 - 30 \]

\[ 10Q = 10 \]

\[ \frac{10Q}{10} = \frac{10}{10} \]

\[ Q = 1 = Q_0 \]

Consumer surplus for a non-linear demand function is given by

\[ \text{Consumer Surplus} = \int_0^{Q_0} P_d \, dQ - (P_0 \times Q_0) \]

\[ C.S. = \int_0^1 \frac{40}{Q + 3} \, dQ - (10 \times 1) \]

\[ C.S. = [40 \ln(Q + 3)]_0^1 - 10 \]

\[ C.S. = [40 \ln(1 + 3) - 40 \ln(0 + 3)] - 10 \]

\[ C.S. = [40 \ln 4 - 40 \ln 3] - 10 \]

\[ C.S. = 1.5 \]
Question 15

The marginal cost function for a good is given by

\[ MC = 2Q^2 - 1. \]

Find the total cost function if fixed costs are 300.

Since Marginal cost is the “derivative” of Total cost, it follows that:

Total cost is the “integral” of Marginal cost

\[ TC = \int MC \, dQ \]

\[ TC = \int (2Q^2 - 1) \, dQ \]

\[ TC = \frac{2(Q^2+1)}{2+1} - Q + c \]

\[ TC = \frac{2Q^3}{3} - Q + c \]

The constant term, c in the Total cost function represents Fixed costs.

\[ Total \ cost = \frac{2Q^3}{3} - Q + 300 \]

Refer any queries to 083 427 5621 or 081 215 3817

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