DSC1520 NOTES

WITH

SOLVED PAST QUESTIONS
Study Unit 1: Mathematical preliminaries
Chapter 1: Sections 1.1 – 1.6

1. Basics

• Numbers: different type of numbers – Natural, Real, etc. Also called constants

• Basic operations
  o + (add); \(2 + 3 = 5\)
  o – (subtract); \(3 – 2 = 1\)
  o \(\times\) (multiply); also \(\bullet\); \(3 \times 2 = 3 \cdot 2 = 6\)
  o \(\div\) (division) also / or fraction \((\frac{1}{2} = 1 \text{ divide by } 2)\);
    \[6 \div 3 = \frac{6}{3} = \frac{6}{3} = 2\]

Remember:
1 \(\times\) anything = anything
   \(1 \times 8 = 8\)
0 \(\times\) anything = 0
   \(0 \times 4 = 0\)
1 + anything = one more than anything
   \(1 + 345 = 346\)
0 + anything = anything
   \(0 + 34 = 34\)
anything \(\div\) 0 = not allowed
   \(12 \div 0 = \text{not allowed}\)
0 \(\div\) anything = 0
   \(0 \div 7 = 0\)
• Brackets ( ) : group operations together
  
  \[(3 + 4) - 3\]
  \[= 7 - 3\]
  \[= 4\]

• Order of operation: BODMAS
  Brackets; Of; Divide; Multiply; Add; Subtract

  \[40 - 4 \times (5 + 8) + 20\]
  \[= 40 - 4 \times (13) + 20\]
  \[= 40 - 52 + 20\]
  \[= 8\]

• Variables: used for unknown or generalisation of things: place holder: use alphabetic characters for example \(X\) or \(A\) or \(Y\). Can take on different values

  \[3x + 2y + 7g + x\]

  3x is known as a term with coefficient 3 and variable x

  Remember: the last term \(x\) has a coefficient value of 1 in front of it namely \(1x\)

  o Operations on variables or unknown:
    ▪ + and − : only if same variable, then + or − coefficients and variable stays the same

      \[3x + 4x + 3 = (3 + 4)x + 3 = 7x + 3\]
      \[5x - x - 6 = (5 - 1)x - 6 = 4x - 6\]
- x and ÷: only if same variable, then x and ÷ coefficient and unknowns

\[ 3a \times 4a = (3 \times 4) (a \times a) = 12a^2 \]

\[ 2x^2 \div x = \left(\frac{2}{1}\right) \frac{x^2}{x} = 2x \]

- Laws of operations

- Commutative law: order
  - \[ a + b = b + a \]
    \[ 3 + 4 = 4 + 3 = 7 \]
  - \[ a \times b = b \times a \]
    \[ 3 \times 4 = 4 \times 3 = 12 \]
  - \[ a - b \neq b - a \]
    \[ 4 - 3 = 1 \neq 3 - 4 = -1 \]
  - \[ a \div b \neq b \div a \]
    \[ 4 \div 2 = 2 \neq 2 \div 4 = 0.5 \]

- Associative law: ( )
  - \[ (a + b) + c = a + (b + c) \]
    \[ (3 + 4) + 2 = 7 + 2 = 9 \]
    \[ 3 + (4 + 2) = 3 + 6 = 9 \]
  - \[ (a \times b) \times c = a \times (b \times c) \]
    \[ (3 \times 4) \times 2 = 12 \times 2 = 24 \]
    \[ 3 \times (4 \times 2) = 3 \times 8 = 24 \]
  - \[ (a - b) - c \neq a - (b - c) \]
    \[ (3 - 4) - 2 = -1 - 2 = -3 \]
    \[ 3 - (4 - 2) = 3 - 2 = 1 \]
  - \[ (a \div b) \div c \neq a \div (b \div c) \]
    \[ (12 \div 2) \div 2 = 6 \div 2 = 3 \]
    \[ 12 \div (2 \div 2) = 12 \div 1 = 12 \]
- Distributive law (addition):
  - \( a \times (b + c) = ab + ac \)
    
    \[
    3 \times (4 + 2) = 3 \times 6 = 18 \\
    (3 \times 4) + (3 \times 2) = 12 + 6 = 18
    \]

- Exponent or Power: \((\text{something})^\text{power}\): short way of writing something multiplied over and over with itself.

  The **bottom** number: base
  The **top** number: exponent or power

  \[
  3 \times 3 = 3^2 \quad \text{base} = 3 \quad \text{power} = 2 \\
  15 \times 15 \times 15 \times 15 \times 15 = 15^5 \\
  Y^3 = Y \times Y \times Y
  \]

- Rules of exponents

  Let A and B be any two bases and \(x\) and \(y\) any two powers then

  1. \( A^x \times A^y = A^{x+y} \)
     
     \[
     2^2 \times 2^3 = 2^{2+3} = 2^5 \\
     2 \times 2 \times 2 \times 2 = 2^4
     \]

  2. \( A^x \div A^y = A^{x-y} \)
     
     \[
     2^4 \div 2^3 = 2^{4-3} = 2^1
     \]

  3. \( (A \times B)^x = A^x \times B^x \)

     \[
     (2 \times 3)^3 = 2^3 \times 3^3
     \]

  4. \( (A/B)^x = A^x / B^x \)

     \[
     (2 / 3)^3 = 2^3 / 3^3
     \]

  5. \( (A^x)^y = A^{xy} \)

     \[
     (2^3)^3 = 2^{3 \times 3} = 2^9
     \]
Remember: If $a$ is any number

1. $(a)^0 = 1$ but $0^0 = 0$ \[ 4^0 = 1 \]

2. $(a)^1 = a$

3. \[ \frac{1}{a^n} = a^{-n} \]

4. \[ \sqrt[n]{a} = a^{\frac{1}{n}} \]

5. $a^{2x+4} = a^{15}$ (base the same) \textbf{then} $2x + 4 = 15$

- Roots: Is the reverse of the power statement.
  \[ \sqrt{25} \] - what number must I multiply 2 times with itself to get an answer of 25 \textbf{=>} 5 because $5^2 = 25$
  \[ \sqrt[3]{8} \] - what number must I multiply 3 times with itself to get an answer of 8 \textbf{=>} 2 because $2^3 = 8$ etc.

- Simplify: write it another way

- Solve for $x$: Determine an answer for $x$

- Remember: When multiplying positive and negative numbers
  - $- \times - = +$
  - $- \times + = -$
  - $+ \times - = -$
  - $+ \times + = +$
2. Fractions

- Fraction is a part of a whole: like a slice of a pizza

\[
\text{fraction} = \frac{\text{number of slices}}{\text{number of slices in whole pizza}} = \frac{\text{numerator}}{\text{denominator (name of fraction)}}
\]

For example, \(\frac{1}{4}\) is one slice of a pizza consisting of 4 pieces.

- Can only add and subtract “same pizzas” if not convert to “same pizzas” – common denominator

\[
\frac{1}{4} + \frac{2}{4} = \frac{1+2}{4} = \frac{3}{4}
\]

\[
\frac{10}{20} + \frac{15}{20} = \frac{10+15}{20} = \frac{25}{20} = \frac{25\div20}{20} = \frac{1}{20}
\]

\[
\frac{2}{5} + \frac{1}{10} = \frac{20}{50} + \frac{5}{50} = \frac{25}{50} = \frac{1}{2} \quad \text{or} \quad \frac{2}{5} + \frac{1}{10} = \frac{4}{10} + \frac{1}{10} = \frac{5}{10} = \frac{1}{2}
\]

\[
\frac{1}{5} + \frac{1}{2} + \frac{2}{3} = \frac{6}{30} + \frac{15}{30} + \frac{20}{30} = \frac{41}{30} = 1\frac{11}{30}
\]
• If add or subtract whole and slices of pizzas: change whole pizzas to slices:

\[ \frac{b}{c} = a \times \frac{b}{c} = (a \times c) + b \]

\[
\begin{align*}
\frac{1}{4} + \frac{1}{2} &= \frac{1}{4} + \frac{3}{8} \\
\frac{2}{8} + \frac{12}{8} &= \frac{14}{8} = \frac{16}{8}
\end{align*}
\]

• Multiply: multiply the numbers across the top lines and multiply the numbers across the bottom lines

\[
\begin{align*}
\frac{1}{4} \times \frac{2}{3} &= \frac{1 \times 2}{4 \times 3} = \frac{2}{12}
\end{align*}
\]

• Divide by = multiply by inverse of fraction

\[
\begin{align*}
\frac{1}{4} \div \frac{2}{3} &= \frac{1}{4} \times \frac{3}{2} = \frac{3}{8}
\end{align*}
\]

No 1 of discussion class
Question 1

Simplify

\[
\frac{1}{6} - \frac{5}{6} \div \frac{2}{3} + \frac{1}{3} \times \frac{3}{4}
\]

Solution

\[
\frac{1}{6} - \frac{5}{6} \div \frac{2}{3} + \frac{1}{3} \times \frac{3}{4}
= \frac{1}{6} - \frac{5}{6} \times \frac{3}{2} + \frac{1}{3} \times \frac{3}{4}
= \frac{1}{6} - \frac{15}{12} + \frac{3}{12}
= \frac{2}{12} - \frac{15}{12} + \frac{3}{12}
= \frac{2 - 15 + 3}{12}
= \frac{-10}{12}
= -\frac{5}{6}
\]

Multiply fractions

Common denominator

Add and subtract fractions

Simplify by dividing nominator and denominator by 2
3. Solve equations in 1 variable

- Move values so that unknown is on its own on one side of equation by +, - x or ÷ both sides with same values

\[ 4x + 7 = 14 - 3x + 5 \]
\[ 4x + 7 + 3x = 14 - 3x + 3x + 5 \]
\[ 4x + 3x + 7 - 7 = 14 + 5 - 7 \]
\[ 7x = 12 \]
\[ 7x/7 = 12/7 \]
\[ x = 12/7 \]

4. Simple inequalities

- Equation if something = something
- Inequality something > or < or ≥ or ≤
- Use number line to demonstrate

\[ 3x + 20 > 14 - 16x \]
\[ 3x + 16x + 20 > 14 - 16x + 16x \]
\[ 19x + 20 - 20 > 14 - 20 \]
\[ 19x > -6 \]
\[ x > -6/19 \]

- When solving for \( x \) remember > and < change if multiply or divide by (−) value.

\[ -2x > 4x + 4 \]
\[ -2x - 4x > 4x - 4x + 4 \]
\[ -6x > 4 \]
\[ -6x/-6 < 4/-6 \]
\[ x < -4/6 \]
Question 2

Solve for \(x\) in

\[
-2x + \frac{5}{6} + \frac{x}{2} \geq -2x - 4\left(\frac{-x - 1}{3} - \frac{1}{4}\right)
\]

Solution

\[
-2x + \frac{5}{6} + \frac{x}{2} \geq -2x - 4\left(\frac{-x}{3} - \frac{5}{4}\right)
\]

Change fraction

\[
-2x + \frac{5}{6} + \frac{x}{2} \geq -2x + \left(\frac{4x}{3} + \frac{20}{4}\right)
\]

Multiply 4 into ( )

\[
-2x + \frac{5}{6} + \frac{x}{2} \geq -2x + \frac{4x}{3} + 5 \quad \text{Remember } \frac{20}{4} = 5
\]

\[
-2x + 2x + \frac{x}{2} - \frac{4x}{3} \geq 5 - \frac{5}{6}
\]

Move all the same terms to one side

\[
\frac{x - 4x}{2} \geq \frac{5}{6}
\]

\[
3x - 8x \geq 30 - 5
\]

Common denominator

\[
\frac{-5x}{6} \geq \frac{25}{6}
\]

Multiply both sides by 6

\[
-\frac{5x}{6} \times \frac{6}{1} \geq \frac{25}{6} \times \frac{6}{1}
\]

\[
-5x \geq 25 \quad \text{Divide both sides by } -5
\]

\[
\frac{-5x}{-5} \leq \frac{25}{-5}
\]

Inequality sign changes because we divide by a negative number

\[
x \leq -5
\]
5. Calculating percentages

- Pizza with 100 slices
  \[ \frac{\text{something}}{100} \]
- \% = fraction \[ \frac{70}{100} \]
- \% always of something: 25\% of 75: of means multiply

15\% of students at a university are male. How many male students are there in a total of 500 students?

\[ \frac{15}{100} \times 500 = 75 \]

The university expects a 10\% increase in the number of students for the next year. How many students do they expect in total?

New total = previous number + 10\% of previous number

\[ = 500 + (10\% \times 500) \]
\[ = 500 + (\frac{10}{100} \times 500) \]
\[ = 500 + (50) \]
\[ = 550 \]

% increase or decrease: value + (% increase of value)

\[ = 500 \times (0.1 + 1) = 500 \times 1.1 = 550 \]
Study Unit 2 : Linear functions
Chapter 2 : Sections 2.1 – 2.4 and 2.6

1. Function
   • Humans = relationships
   • Function = mathematical form of a relationship
     Temperature and number of ice cream sold
   • Independent variable – if variable : x
   • Dependent variable – then variable : y
   • Function of x : \( f(x) = y \) \( f(x) = 2x + 3 \) or \( y = 2x + 3 \)
   • Relationship of x and y : ordered pair \((x;y)\)
     If temp is 20° then number of ice creams sold is 400
     If temp is 30° then number of ice creams sold is 600
     \((x_1 ; y_1) = (20 ; 400)\) and \((x_2 ; y_2) = (30 ; 600)\)
   • To graph relationship use Cartesian plane
     • 2 number lines : x-axis : horizontal
       y-axis : vertical
       intersection : origin

![Graph of linear function](image-url)
2. Linear function

- Relationship between 2 variables is linear and graph is a straight line

\[ y = mx + c \quad \text{or} \quad y = ax + b \]

with \( y \) and \( x \) variables and \( m \) and \( c \) values and \((x_1;y_1)\) and \((x_2;y_2)\) are 2 points on the line.

- \( m = \) slope or how steep is line or how does \( y \)-values change if \( x \) values change
  \[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

- \( c = \) \( y \)-intercept : cut \( y \)-axis : where \( x = 0 \)

\[ y = mx + c \]

\[ y \]-intercept = \( c \)
\[ x = 0 \]
\[ (0;c) \]
Other lines:

\[ y = -mx + c \]

- \( m = \text{slope} = \text{negative} \)
- \( y \)-intercept = \( c \)
- \( x = 0 \)
- \( x \)-intercept
- \( y = 0 \)

- If two lines are parallel they have the same slope
• How to determine equation of line:
  
  o Need 2 points on line \((x_1;y_1)\) and \((x_2;y_2)\)

  1. Calculate \(m = \frac{y_2 - y_1}{x_2 - x_1}\)

  2. Substitute \(m\) and any one of the 2 points into function \(y = mx + c\) to determine \(c\)

No 3a of discussion class

• How to draw a line:

  o Need two points on line or

  o Equation of line

  1. Calculate any 2 points on line by choosing a x-value and calculate the y-value

No 3b of discussion class
Question 3a

Find the equation of the line passing through the points (1; 20) and (5; 60).

Solution

\[ y = mx + c. \]

Let \((x_1 ; y_1) = (1 ; 20)\) and \((x_2 ; y_2) = (5 ; 60)\)

The slope \(m\) is

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{60 - 20}{5 - 1} = \frac{40}{4} = 10
\]

Therefore \(y = 10x + c.\)

Substitute any one of the points into the equation of the line to determine \(c\). Let’s choose the point \((1 ; 20)\). Then

\[
y = 10x + c \\
20 = 10 \times 1 + c \\
20 = 10 + c \\
-c = 10 - 20 \\
-c = -10 \\
c = 10
\]

The equation of the line is \(y = 10x + 10\).
Question 3 b

Draw the graph of the line $y = 10x + 10$.

Solution

Need two points to draw line:

Choose any x or y value and calculate y or x:

Choose $x = 0$ then $y = 10(0) + 10$

$y = 10 \rightarrow$ point 1 = (0 ; 10)

Choose $y = 0$ then $0 = 10x + 10$

$-10x = 10$

$x = 10/-10$

$x = -1 \rightarrow$ point 2 = (-1 ; 0)
How to determine a slope, y-intercept and x-intercept if given the equation of a line:

for example $3x + 4y - 8 = 4$ or $y = 4x + 20$

1. Write it in the format $y = mx + c$

2. Compare with standard form => slope is $m$, y-intercept is $c$

3. To calculate the x-intercept make $y = 0$ and solve for $x$

1. Write in format $y = mx + c$
   $3x + 4y - 8 = 4$
   
   $4y = 4 + 8 - 3x$
   $4y = 12 - 3x$
   $y = \frac{12}{4} - \frac{3}{4}x$
   $y = 3 - \frac{3}{4}x$

2. slope = $m = -\frac{3}{4}$  y-intercept = $c = 3$

   x-intercept is where $y = 0$ but $y = 3 - \frac{3}{4}x$

   $0 = 3 - \frac{3}{4}x$
   $\frac{3}{4}x = 3$
   $x = 3 \times \frac{4}{3}$
   $x = \frac{12}{3}$
   $x = 4$
3. Application in economics

Relationship between price $P$ and quantity $Q$ of a product

- **Demand function**
  - If the price of a product $\uparrow$ then the demand $\downarrow$
  - $P = a - bQ$ with
    - $a =$ y-intercept (c)
    - $b =$ slope = negative

- **Supply function**
  - If the price of a product $\uparrow$ then the supply $\uparrow$
  - $P = c + dQ$ with
    - $c =$ y-intercept
    - $d =$ slope = positive
Cost function

- **Fixed cost**
- **Cost**
  - Variable cost dependent on quantity $Q$
  - $TC = FC + VC \times Q \Rightarrow y = c + mx$

A supermarket’s fixed cost is R5000 per month and the salary per employee is R2000 per month. What is the supermarket’s linear cost function if the number of employees is $Q$?

Cost = 5000 + 2000Q

- **Revenue**
  - What you earn
  - $R = \text{Price} \times \text{Quantity}$
  - $R = P \times Q \Rightarrow px$

- **Profit**
  - Revenue – cost

- **Depreciation**
  - A R200 000 car depreciates linearly to R40 000 in 8 years’ time. Derive a linear equation for the value of the car after $x$ years with $0 \leq x \leq 8$. 
Let \( y = \text{value} \) and \( x = \text{time or years} \)

\[ y = mx + c \]

Need two points on graph:

**Given \((8; 40\,000)\) and \((0; 200\,000)\)**

\[
\text{Now } m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{40000 - 200000}{8 - 0} = \frac{-160000}{8} = -20000
\]

\[ y = -20\,000x + c \]

Take any one of two points: Say point 2

\[ 200\,000 = -20\,000(0) + c \]

\[ c = 200\,000 \]

Depreciation: \( y = -20\,000x + 200\,000 \)
• Elasticity
  • Important in economics
  • Think what happens with an elastic band: if you apply little pressure the band expand a little bit and if you apply a lot of pressure the band expand a lot.
  • How sensitive demand is for price change
  • If the price P and % change in price goes up or down what will happen to the % change in quantity
  • Ratio of % change
  • \[ \varepsilon = \frac{\text{% change in demand}}{\text{% change in price}} \]
  • Price elasticity of demand or supply
    o Point : At a point
    o Arc : Over an interval

1. Price elasticity of demand
  • Point \((P_0;Q_0)\)
    Demand : \(P = a - bQ\)
    \[ \varepsilon_d = -\frac{1}{b} \cdot \frac{P_0}{Q_0} \]
  • In terms of \(P\)
    \[ \varepsilon_d = -\frac{1}{b} \cdot \frac{P}{Q} = \frac{P}{P-a} \]

Discussion class  4a + 4b
**Question 4a**

If the demand function is \( P = 80 - 2Q \), where \( P \) and \( Q \) are the price and quantity respectively, determine the expression for price elasticity of demand if the price \( P = 20 \).

**Solution**

Now

\[
\varepsilon_d = - \frac{1}{b} \cdot \frac{P}{Q}
\]

Given \( P = 80 - 2Q \) and \( P = 20 \).

Comparing \( P = 80 - 2Q \) with \( P = a - bQ \) \( \rightarrow a = 80 \) and \( b = 2 \).

To determine the value of \( Q \) we substitute \( P = 20 \) into the equation and solve for \( Q \)

\[
\begin{align*}
20 &= 80 - 2Q \\
20 - 80 &= -2Q \\
-60 &= -2Q \\
Q &= \frac{-60}{-2} \\
Q &= 30
\end{align*}
\]

Now
At $P = 20$ a 1% increase (decrease) in price will cause a 0.33% decrease (increase) in the quantity demanded.

**Question 4 b**

If the demand function is $P = 80 - 2Q$, where $P$ and $Q$ are the price and quantity respectively, determine the expression for price elasticity of demand in terms of $P$ only.

**Solution**

Now demand in terms of $P$

$$\varepsilon_d = \frac{P}{P - a}$$

Given $P = 80 - 2Q$ and $P = 20$.

Comparing $P = 80 - 2Q$ with $P = a - bQ$, $a = 80$ and $b = 2$.

Thus $$\varepsilon_d = \frac{P}{P - 80}.$$
• Arc

Over an interval
Use the average P and Q at beginning and end of interval.

\[ P_1 \to P_2 \text{ and } Q_1 \to Q_2 \]

\[ \varepsilon_d = \frac{1}{b} \cdot \frac{P_1 + P_2}{Q_1 + Q_2} \]

**Discussion class 5**

**Question 5**

Given the demand function \( P = 60 - 0,2Q \) where \( P \) and \( Q \) is the price and quantity respectively, calculate the arc price elasticity of demand when the price decreases from R50 to R40.

**Solution**

Arc elasticity of demand \( \varepsilon_d = \frac{1}{b} \cdot \frac{P_1 + P_2}{Q_1 + Q_2} \)

Given function \( P = 60 - 0,2Q \), with \( a = 60 \) and \( b = 0,2 \)

Given \( P_1 = 50 \) and \( P_2 = 40 \).

Need to determine \( Q_1 \) and \( Q_2 \).

\[
\begin{align*}
P & = 60 - 0,2Q \\
0,2Q & = 60 - P \\
Q & = 300 - 5P
\end{align*}
\]
Determine $Q_1$ and $Q_2$ by substituting $P_1 = 50$ and $P_2 = 40$ into the equation. Thus

If $P_1 = 50$ then $Q_1 = 300 - 5 \times 50 = 50$
If $P_2 = 40$ then $Q_2 = 300 - 5 \times 40 = 100$

Therefore

\[
\text{Elasticity of demand} = -\frac{1}{b} \times \frac{P_1 + P_2}{Q_1 + Q_2} = -\frac{1}{0,2} \times \frac{50 + 40}{50 + 100} = -\frac{1}{0,2} \times \frac{90}{150} = -\frac{90}{30} = -3
\]

2. Price elasticity of supply

\[
\text{Demand : } P = c + dQ \\
\varepsilon_s = \frac{1}{d} \cdot \frac{P_0}{Q_0}
\]
1. Two equations in two unknowns

- Algebraically

**Method 1: Elimination**

**Step 1:** Eliminate 1 variable

- $-$, $+$ one equation or multiple of equation from other equation

- Indication of size of multiple -> number in front of variable

**Step 2:** Solve variable 1

**Step 3:** Substitute value of variable 1 back into any one of equations and solve variable 2

**Discussion class example 6(a)**
Question 6a

Solve the following set of linear equations by using the elimination method:

\[
\begin{align*}
y + 2x &= 3 & \text{eq(1)} \\
y - x &= 2 & \text{eq(2)}
\end{align*}
\]

Solution

Step 1: Eliminate 1 variable – say y

Subtract eq2 from eq1 and solve x

\[
\begin{align*}
y + 2x &= 3 \\
-(y - x) &= 2
\end{align*}
\]

\[
\begin{align*}
0 + 3x &= 1 \\
x &= \frac{1}{3}
\end{align*}
\]

Step 2: Substitute value of x back into any one of equations and solve y

\[
y + 2x = 3
\]

\[
y + 2 \left( \frac{1}{3} \right) = 3
\]

\[
y = 3 - \frac{2}{3}
\]

\[
y = 2 \frac{1}{3}
\]
Method 2: Substitution

**Step 1:** Change one of the equation so that any variables is the subject of the equation – eq3

**Step 2:** Substitute eq3 into the unchanged equation and solve first variable

**Step 3:** Substitute answer step2 into any equation and solve the second variable

Discussion class example 6(b)

Question 6b

Solve the following set of linear equations by using the substitution method:

\[ y + 2x = 3 \] – eq(1)
\[ y - x = 2 \] – eq(2)

Solution

Step 1: Make 1 variable subject of an equation

Say y in eq1:

\[ y + 2x = 3 \]
\[ y = 3 - 2x \] – eq3

Step 2: Substitute the value of y into other equation

Substitute eq3 into eq2:
Substitute $y = 3 - 2x$ into $y - x = 2$

$$(3 - 2x) - x = 2$$

$$-3x = 2 - 3$$

$$-3x = -1$$

$$x = \frac{1}{3}$$

Step 3: Substitute value of variable into any equation

Substitute $x = \frac{1}{3}$ into eq(1) or eq(2) – choose eq(2)

$$y - x = 2$$

$$y - \frac{1}{3} = 2$$

$$y = 2 \frac{1}{3}$$

- Graphically

1. Draw 2 equations – solution intersect

Discussion class example 6(c)
Question 6c

Solve the following set of linear equations graphically

\[ y + 2x = 3 \quad \text{– eq}(1) \]
\[ y - x = 2 \quad \text{– eq}(2) \]

Solution

Draw 2 lines \(\rightarrow\) intersecting = solution

Need 2 points to draw a line:

Eq1: If \( x = 0 \) then \( y + 2(0) = 3 \) or \( y = 3 \) \(\rightarrow\) \((0 ; 3)\)

If \( y = 0 \) then \( 0 + 2x = 3 \) or \( x = 3/2 \) \(\rightarrow\) \((3/2 ; 0)\)

Eq2: If \( x = 0 \) then \( y - (0) = 2 \) or \( y = 2 \) \(\rightarrow\) \((0 ; 2)\)

If \( y = 0 \) then \( 0 - x = 2 \) or \( x = -2 \) \(\rightarrow\) \((-2 ; 0)\)

Graph points and draw lines

2. Solutions

1. 1 Unique – lines intersect
2. No Solution – lines parallel
3. Infinity – lines on top of each other
3. Three equations in three unknowns

Eliminate 1 variable of 3 variables then use method as above. Determine 2 eq’s with the same 2 unknowns then use method as above.

**Step 1:** Write all the equation in the same format – variables one side and values right hand side

**Step 2:** Eliminate 1 of the 3 variables by + or – one equation from another or one multiply by a value for example:
- \( eq1 + eq2 = eq4 \) or 
- \( eq1 - (2 \times eq2) = eq4 \)

**Step 3:** Do the same for any other 2 equations \( \rightarrow eq5 \)

**Step 4:** Now you have 2 equations with the same two variables namely eq4 and eq5 solve as previously explained

**Note:**
- If one equation has just two variables make one of the variables the subject of the equation \( \rightarrow eq4 \)
- Add or subtract other two eq’s that has 3 unknowns \( \rightarrow \) new equation with 2 variables \( \rightarrow eq5 \).
- Substitute eq4 into the eq5 and solve your first variable. Etc.

**Discussion class example 7**
Question 7

Solve the following set of equations

\[ x - y + z = 0 \]  \hspace{0.5cm} (1)  
\[ 2y - 2z = 2 \]  \hspace{0.5cm} (2)  
\[ -x + 2y + 2z = 29 \]  \hspace{0.5cm} (3)

Solution

Step 1: Get 2 eq’s with the same 2 unknowns

- eq(2) : already 2 variables
- Add eq(1) and eq(3):
  \[ x - y + z = 0 \]
  \[ -x + 2y + 2z = 29 \]
  \[ \underline{0 + y + 3z = 29} \]  \hspace{0.5cm} (4)

Step 2: Solve 2eq with 2 unknowns – any method

- Substitution : Make \( y \) the subject of eq4:
  \[ y = 29 - 3z \]  \hspace{0.5cm} (5)

- Substitute eq5 into eq2 and solve \( z \):
\[2y - 2z = 2 \quad (2)\]
\[2(29 - 3z) - 2z = 2\]
\[58 - 6z - 2z = 2\]
\[-8z = 2 - 58\]
\[-8z = -56\]
\[z = \frac{-56}{-8}\]
\[z = 7\]

Step 3: Substitute \(z = 7\) into eq2
\[2y - 2z = 2\]
\[2y - 2(7) = 2\]
\[2y = 2 + 14\]
\[2y = 16\]
\[y = 8\]

Step 4: Substitute \(z = 7\) and \(y = 8\) into eq1
\[x - y + z = 0\]
\[x - 8 + 7 = 0\]
\[x = 1\]

Solution: \(x = 1; y = 8\) and \(z = 7\)
2. Applications of simultaneous equations in business

a. Equilibrium market

market equilibrium:

quantity demanded = quantity supplied

price customers willing pay = price producers accept

\[ Q_d = Q_s \text{ or } P_d = P_s \]

• Algebraically:

Solve the demand function and supply function simultaneously. If demand function: \( P_d = a - bQ_d \) and supply: \( P_s = c + dQ_s \) then

\[ a - bQ_d = c + dQ_s \]

• Graphically:

Intersection of 2 functions

Discussion class example 8 (a) and (b)
Question 8

In a market we have the following:

Demand function: \( Q = 50 - 0,1P \)

Supply function: \( Q = -10 + 0,1P \)

where \( P \) and \( Q \) are the price and quantity respectively.

(a) Calculate the equilibrium price and quantity.

(b) Draw the two functions, and label the equilibrium point.

(c) Calculate the consumer surplus at equilibrium.

(d) Calculate the producer surplus at equilibrium.

Solution

(a) Equilibrium is the price and quantity where the demand and supply functions are equal. Thus determine \( Q_d = Q_s \), or

\[
50 - 0.1P = -10 + 0.1P
\]

\[
-0.2P = -60
\]

\[
P = \frac{-60}{-0.2} = 300
\]
To calculate the quantity at equilibrium we substitute the value of $P$ into the demand or supply function and calculate $Q$. Say we use the demand function then

$$Q = 50 - 0.1(300)$$

$$Q = 20$$

The equilibrium price is equal to 300 and the quantity to 20.

(b)

![Equilibrium price and quantity graph](image)

(c) and (d) see later
b. Break-even analysis

Do not make a profit or a loss

- Profit = 0
- Profit = revenue – cost = 0
- Total revenue = total cost

Solve:

- Algebraically: solve simultaneous equations
  
  Revenue = cost

- Graphically: where cost and revenue functions intercepts

**Discussion class example 9**
Question 9

A company manufactures and sells $x$ toy hand held radios per week. The weekly cost are given by

$$c(x) = 5000 + 2x$$

How many radios should they manufacture to break–even if a radio sells for R202?

Solution

Break-even: Revenue = cost

Revenue = price x quantity = $202x$ and Cost = $5000 + 2x$

$$202x = 5000 + 2x$$

$$202x - 2x = 5000$$

$$200x = 5000$$

$$x = 25$$

OR

Break-even: profit = 0

Profit = Revenue – cost

$$Profit = 202x - (5000 + 2x) = 0$$

$$200x - 5000 = 0$$

$$200x = 5000$$

$$x = 25$$
c. Producer and consumer surplus

• Consumer surplus

The consumer surplus for demand is the difference between

1. the amount the consumer is willing to pay for successive units ($Q = 0$ to $Q = Q_0$) of a product

   and

2. the amount that the consumer actually paid for $Q_0$ units of the product at a market price of $P_0$ per unit.

$$CS = \text{Amount willing to pay} - \text{Amount actually paid}$$

Example:

Calculate the consumer surplus for the demand function

$$P = 50 - 4Q$$

when the market price is $P = 10$. 


Now:

**CS = Amount willing to pay – Amount actually paid**

- First we calculate the amount actually paid:

  The number of items the consumer will purchase at a price $P = 10$ is:

  \[
  P = 50 - 4Q \\
  10 = 50 - 4Q \\
  10 - 50 = -4Q \\
  -40 = -4Q \\
  \frac{-40}{-4} = Q \\
  10 = Q \\
  Q = 10.
  \]

  The consumer will buy a quantity of 10 items if the price is R10 per item.

  Therefore the consumer will actually spend in total

  \[ P \times Q = 10 \times 10 = R100. \quad – (B) \]

  Now graph the demand function and the price and quantity as below:
Now the area $B$ of the rectangle $0P_0E_0Q_0$ with area = length $\times$ breadth under the given demand function $P = 50 - 4Q$ is equal to $10 \times 10 = 100$ which is the same as what the consumer will spend.

Thus graphically the Area $B$ = amount actually spend
Now the consumer will pay R100 for 10 units. But what will he be willing to pay if the product is scarcer, say only 5 units?

Then

\[ P = 50 - 4Q = 50 - (4 \times 5) = R30 \text{ per unit.} \]

And if the \( Q = 2 \), then

\[ P = 50 - 4Q = 50 - (4 \times 2) = R42 \text{ per unit.} \]

Graph 2

Thus the total amount which the consumer is thus willing to pay for the first 10 items = the area C under the demand function between \( P = 0 \) and \( P = 10 \).
Area \((C)\) = area triangle \((A)\) plus area square \((B)\)

\[
\text{Area } (C) = \left( \frac{1}{2} \times \text{base} \times \text{height} \right) + \left( \text{length} \times \text{breadth} \right).
\]

\[
\begin{align*}
\text{Area } C &= \left[ \frac{1}{2} \times 10 \times (50 - 10) \right] + (10 \times 10) \\
&= \left[ \frac{1}{2} \times 10 \times 40 \right] + 100 \\
&= \frac{400}{2} + 100 \\
&= 200 + 100 \\
&= 300
\end{align*}
\]

The total amount which the consumer is willing to pay for the first 10 units is thus R300

Now the consumer surplus is defined as:

\[
\text{CS} = \text{amount willing to pay} - \text{amount actual pay}
\]

\[
= 300 - 100
\]

\[
= 200.
\]

Alternatively to summarise:

\[
\text{CS} = \text{amount willing to pay} - \text{amount actual pay}
\]

\[
\text{CS} = \text{area } (C) \text{ in graph } 1 - \text{area } (B) \text{ in graph } 2
\]

\[
= (\text{area of } △ + \text{area of } □) - \text{area of } □
\]

\[
= (A) + (B) - (B) = (A)
\]
area of triangle

= \frac{1}{2} \times \text{height} \times \text{base}

= \frac{1}{2} \times (50 - 10) \times (10)

= 200 \text{ (same as calculated earlier).}

In general:

If you need to determine the demand surplus for a demand function of $P = a - bQ$ then the consumer surplus can be calculated by calculating an area of the triangle $P_0E_0a$ which is equal to

\[
\frac{1}{2} \times \text{height} \times \text{base}
\]

\[
= \frac{1}{2} \times (a - P_0) \times (Q_0 - 0)
\]

\[
= \frac{1}{2} \times (a - P_0) \times (Q_0)
\]

with

- $P_0$ the value given to you as the market price
- $Q_0$ is the value of the demand function if $P = P_0$
  (Substitute $P_0$ into the demand function and calculate $Q_0$)
- $a$ is the $y$-intercept of the demand function
  $P = a - bQ$ also known as the value of $P$ if $Q = 0$, or the place where the demand function intercepts the $y$-axis.
Method:

1. Draw a rough graph of the demand function
2. Calculate $Q_0$ if $P_0$ is given and
3. Get the value of $a$ from demand function
4. Calculate the area or $CS = \frac{1}{2} \times (a - P_0) \times (Q_0)$

Discussion class example 10
Question 10

Calculate the consumer surplus for the demand function \( P = 60 - 4Q \) when the market price is \( P = 12 \).

Solution

- **Calculate Q if P = 12**
  \[
  P = 60 - 4Q \\
  12 = 60 - 4Q \\
  4Q = 60 - 12 \\
  4Q = 48 \\
  Q = 12
  \]

- **Draw a rough sketch of graph**

  ![Graph](image)

- **Calculate the consumer surplus**

  \[
  CS = \frac{1}{2} \times \text{base} \times \text{height} \\
  = \frac{1}{2} \times (12) \times (60 - 12) \\
  = 288
  \]
Producer surplus

PS = revenue producer receives at $Q_0$ – revenue producer willing to accept between 0 and $Q_0$

Producer surplus = 

$$= \frac{1}{2} \times Q_0 \times (P_0 - b)$$

Discussion class example 8(c) and (d)
Question 8(c)

At equilibrium $Q$ is equal to 20 and $P$ equal to 300. Therefore the consumer surplus is the area of the shaded triangle below:

\[ CS = \frac{1}{2} \times \text{base} \times \text{height} \]
\[ = \frac{1}{2} \times 300 \times (50 - 20) \]
\[ = 4500 \]
(d) The producer surplus is the area of the shaded triangle below:

\[ PS = \frac{1}{2} \times \text{base} \times \text{height} \]

\[ = \frac{1}{2} \times 300 \times (20 - (-10)) \]

\[ = \frac{1}{2} \times 300 \times 30 \]

\[ = 4500 \]
4. Linear inequalities

(a) Graphics of linear inequality – study guide C.2

Example: $2x + y \leq 120$

**Step 1:** Change $\geq$ or $\leq$ or $>$ or $<$ to $=$ and draw the graph of the line.

**Step 2:** Determine region inequality true and colour area

- Substitute a point on either side into inequality and see which point makes the inequality true

(b) Solving a system of inequalities – study guide C.3

Draw each inequality as above (step 1 and step 2)

Select area true for whole system – area simultaneously coloured – feasible region

**Discussion class example 11 (a) + (b)**

See (c) later
Question 11

(a) Draw the lines representing the following constraints:

\[ 2x + y \leq 10 \quad (1) \]
\[ x + 2y \leq 140 \quad (2) \]
\[ x + y \leq 80 \quad (3) \]
\[ x_1, x_2 \geq 0 \quad (4) \]

(b) Show the feasible region.

(c) Determine the maximum value of

\[ P = 20x + 30y \]

subject to the constraints above.

Solution

Step 1:
Change the inequality sign (\( \geq \) or \( \leq \) or \( > \) or \( < \)) to an equal sign (=) and graph of the line.

Step 2:
Determine the feasible region – substitute a point on either side of the equality
<table>
<thead>
<tr>
<th>Inequality</th>
<th>y-axis intercept</th>
<th>x-axis intercept</th>
<th>Inequality region</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2x + y \leq 120)</td>
<td>(2x + y = 120)</td>
<td>(2x + 0 = 120)</td>
<td>Select points ((0,0)) below the line</td>
</tr>
<tr>
<td>((1))</td>
<td>(2(0) + y = 120)</td>
<td>(x = \frac{120}{2} = 60)</td>
<td>(2(0) + 0 \leq 120 ) – True</td>
</tr>
<tr>
<td></td>
<td>(y = 120)</td>
<td></td>
<td>Area below the line</td>
</tr>
<tr>
<td></td>
<td>Point: ((0 ; 120))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x + 2y \leq 140)</td>
<td>(x + 2y = 140)</td>
<td>(x + 2(0) = 140)</td>
<td>Select points ((0,0)) to left of the line</td>
</tr>
<tr>
<td>((2))</td>
<td>(0 + 2y = 140)</td>
<td>(x = 140)</td>
<td>(0 + 2(0) \leq 140 ) – True</td>
</tr>
<tr>
<td></td>
<td>(y = \frac{140}{2} = 70)</td>
<td></td>
<td>Area to the left of the line</td>
</tr>
<tr>
<td></td>
<td>Point ((0 ; 70))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x + y \leq 80)</td>
<td>(x + y = 80)</td>
<td>(x + 0 = 80)</td>
<td>Select points ((0,0)) below the line</td>
</tr>
<tr>
<td>((3))</td>
<td>(0 + y = 80)</td>
<td>(x = 80)</td>
<td>(0 + 0 \leq 80 ) – True</td>
</tr>
<tr>
<td></td>
<td>(y = 80)</td>
<td></td>
<td>Area below the line</td>
</tr>
<tr>
<td></td>
<td>Point ((0 ; 80))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(x, y \geq 0)</td>
<td></td>
<td></td>
<td>Area above the x-axis and to the right of the y-axis</td>
</tr>
</tbody>
</table>
Step 3: Draw graph and show feasible region

(c) See later
Application of system of linear inequalities in business

1. Linear programming

- Real life – find “best” value under certain conditions – optimisation

- Need to find maximum (profit) or minimum (cost) subject to certain constrains for example resources: labour or materials.

- LP is the problem of maximising or minimising a linear function (profit or cost) called the **objective function** subject to linear **constraints** expressed as inequalities or equations.

- **Formulating a LP problem:**
  - Define the decision variables for example $x$ and $y$
  - summarise the information given in a table with the headings:
    - resources (things that have restrictions on),
    - the variables ($x$ and $y$) and
    - capacity (amount or number available of the resources).

- **Discussion class example 12 and 13**
Question 12

Giapetto’s Woodcarving manufactures two types of wooden toys: soldiers and trains.

- A soldier sells for R27 and uses R10 worth of raw materials.
- A train sells for R21 and uses R9 worth of raw materials.
- The manufacturer of wooden soldiers and trains requires two types of skilled labour: carpentry and finishing.
- A soldier requires 2 hours of finishing and 1 hour of carpentry
- A train requires 1 hour of finishing and 1 hour of carpentry.
- Each week, only 100 finishing hours and 80 carpentry hours are available at most.
- The demand for trains is unlimited, but at most 40 soldiers are bought each week.
- Giapetto’s has a weekly budget of R10 000 for the raw material.

If \( x \) is the number of soldier toys made per week and \( y \) is the number of train toys made per week, formulate the linear constraints that describe Giapetto’s situation and write down the revenue function (objective function) if his objective is to maximise his revenue.
Solution

First we define the variables. Let $x$ be the number of soldier toys manufactured per week and $y$ the number of train toys manufactured per week. To help us with the formulation we summarise the information given in a table with the headings: resources (things that have restrictions on), the variables ($x$ and $y$) and capacity (amount or number available of the resources).

<table>
<thead>
<tr>
<th>Resource</th>
<th>$x$</th>
<th>$y$</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carpentry</td>
<td>1</td>
<td>1</td>
<td>80</td>
</tr>
<tr>
<td>Finishing</td>
<td>2</td>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>Raw materials budget</td>
<td>10</td>
<td>9</td>
<td>10 000</td>
</tr>
<tr>
<td>Sales / Revenue</td>
<td>27</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>Maximum per week</td>
<td>40</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Number of toys</td>
<td></td>
<td></td>
<td>Always positive</td>
</tr>
</tbody>
</table>
Using the table the following constraints can be defined:

\[
\begin{align*}
  x + y & \leq 80 & \text{Carpentry} \\
  2x + y & \leq 100 & \text{Finishing} \\
  10x + 9y & \leq 10000 & \text{Budgetary constraint raw materials} \\
  x & \leq 40 & \text{Maximum demand per week} \\
  x, y & \geq 0 & \text{Non-negativity}
\end{align*}
\]

As he would like to maximise his revenue and as revenue is equal to quantity times the price or sales, the objective function can be written as \(27x + 21y\).
• Solving a LP graphically:

Many methods example simplex method and graphics

Use graphical method in DSC1520.

**Step 1**: Draw all the inequalities

**Step 2**: Determine the feasible region

**Step 3**: Determine coordinates of all corners of feasible region by substitution or read from graph

**Step 4**: Substitute corner points into objective function

**Step 5**: Choose corner point that result in highest/maximisation or lowest/minimisation objective function value.

**Discussion class example 11 (c)**
Question 11(c)

Determine all the corner points of the feasible region and substitute them into the objective function (function you want to maximise or minimise) and determine the maximum value.

<table>
<thead>
<tr>
<th>Corner points of feasible region</th>
<th>Value of $P = 20x + 30y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: $x = 60; y = 0$</td>
<td>$P = 20(60) + 30(0) = 1200$</td>
</tr>
<tr>
<td>B: $x = 0; y = 70$</td>
<td>$P = 20(0) + 30(70) = 2100$</td>
</tr>
<tr>
<td>C: $x = 20; y = 60$</td>
<td>$P = 20(20) + 30(60) = 2200 \rightarrow \text{Maximum}$</td>
</tr>
<tr>
<td>D: $x = 40; y = 40$</td>
<td>$P = 20(40) + 30(40) = 2000$</td>
</tr>
<tr>
<td>Origin: $x = 0; y = 0$</td>
<td>$P = 20(0) + 30(0) = 0$</td>
</tr>
</tbody>
</table>

Maximum $P$ is at point C where $x = 20; y = 60$ and $P = 2200$. 
Study Unit 4 : Non-Linear functions

Chapter 4 : Sections 4.1– 4.4

Types of non-linear functions:

• Polynomials
  • Linear function : \( y = mx + c \) – chapter 2
  • Quadratic function : \( y = ax^2 + bx + c \)
  • Cubic function : \( y = ax^3 + bx^2 + cx + d \)
  • \( n \)-th order polynomials : \( y = ax^n + bx^{n-1} + cx^{n-2} + \ldots \) + constant

• Exponential functions
  \( y = a^x \): general format (a = constant)
  \( y = 10^x \): scientific format
  \( y = e^x \): natural format

• Logarithm functions
  \( y = \log_a x \)
  \( y = \log_{10} x = \log x \)
  \( y = \log_e x = \ln x \)

• Hyperbolic functions \( y = \frac{a}{bx + c} \)
1. Quadratic function

\[ y = ax^2 + bx + c \]

- polynomial of **degree** 2 – highest power of \( x \)
- \( c \) : **y-axis intercept** : cuts \( y \)-axis
  - point where \( x = 0 \) : \((0 ; c)\)
- \( a \) – **shape** of the function
  - \( a > 0 \) : \( a \) positive : smiling face
  - \( a < 0 \) : \( a \) negative : sad face
- **vertex** : turning point : function maximum or minimum
  - point \((x;y)\) with \( x = \frac{-b}{2a} \) with \( a \), \( b \) and \( c \) the coefficients of the standard quadratic function \( y = ax^2 + bx + c \)
  - \( y \) the function value if \( x = \frac{-b}{2a} \) (substitute the answer for \( x \) into the function and solve \( y \))
- **Roots** or \( x \)-intercept : cuts \( x \)-axis
  - make \( y = 0 \) and solve \( x \) by using the quadratic formula
  
  \[
  x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
  \]

  with \( a \), \( b \) and \( c \) the values in the standard quadratic equation \( 0 = ax + bx + c \)
Example:

Vertex: Turning point
Maximum

\( \left( \frac{-b}{2a} ; \; f \left( \frac{-b}{2a} \right) \right) \)

y-intercept: \( x = 0 \)

(0; c)

Root: \( y = 0 \)

\[ x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \]

Root: \( y = 0 \)

\[ x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \]
Example:

Find the coordinates of the vertex of the graph \( y = 4x^2 - x - 3 \).

- **Vertex** = extreme point = turning point = max or min
- \( a > 0 \) thus graph = smiling face => min exists
- \( x \) coordinate of minimum is \( x = \frac{-b}{2a} \)

Comparing \( y = 4x^2 - x - 3 \) with \( y = ax^2 + bx + c \Rightarrow a = 4, b = -1, c = -3 \).

Thus

\[
x = \frac{-b}{2a} = \frac{-(-1)}{2(4)} = \frac{1}{8} = 0.125
\]

- \( y \) value if \( x = 0.125 \) is

\[
y = 4x^2 - x - 3
\]

\[
y = 4(0.125)^2 - 0.125 - 3
\]

\[
y = -3.0625
\]

Coordinates of the vertex of the graph \( y = 4x^2 - x - 3 \) are

\[(0.125; -3.0625)\]
Study Unit 4 : Non-Linear functions

Chapter 4 : Sections 4.1– 4.4

Types of non-linear functions:

- **Polynomials**
  - Linear function : \( y = mx + c \) – chapter 2
  - Quadratic function : \( y = ax^2 + bx + c \)
  - Cubic function : \( y = ax^3 + bx^2 + cx + d \)
  - \( n \)-th order polynomials : \( y = ax^n + bx^{n-1} + cx^{n-2} + \ldots + \) constant

- **Exponential functions**
  \[ y = a^x : \text{general format (}a = \text{constant)} \]
  \[ y = 10^x : \text{scientific format} \]
  \[ y = e^x : \text{natural format} \]

- **Logarithm functions**
  \[ y = \log_a x \]
  \[ y = \log_{10} x = \log x \]
  \[ y = \log_e x = \ln x \]

- **Hyperbolic functions**
  \[ y = \frac{a}{bx + c} \]
1. Quadratic function

\[ y = ax^2 + bx + c \]

- polynomial of **degree** 2 – highest power of \( x \)

- **\( c \): y-axis intercept** : cuts y-axis
  - point where \( x = 0 \) : \( (0 ; c) \)

- **\( a \) – shape** of the function
  - \( a > 0 \) : \( a \) positive : smiling face
  - \( a < 0 \) : \( a \) negative : sad face

- **vertex** : turning point : function maximum or minimum
  - point \((x;y)\) with \( x = \frac{-b}{2a} \) and \( y \) the function value if
    \[ x = \frac{-b}{2a} \] (substitute the answer for \( x \) into the function and solve \( y \))

- **Roots** or \( x \)-intercept : cuts \( x \)-axis
  - make \( y = 0 \) and solve \( x \) by
    - factorisation or quadratic formula
      \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
Example:

- **Vertex**: Turning point
- **Maximum**

- **Root**: \( y = 0 \)
- **Root**: \( y = 0 \)

- **y-intercept**: \( x = 0 \)

- **c** (0; c)

- **\( x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \)**
- **\( x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \)**
Example:

Find the coordinates of the vertex of the graph \( y = 4x^2 - x - 3 \).

- Vertex = extreme point = turning point = max or min
- \( a > 0 \) thus graph = smiling face => min exists
- \( x \) coordinate of minimum is \( x = \frac{-b}{2a} \)

Comparing \( y = 4x^2 - x - 3 \) with \( y = ax^2 + bx + c \) => \( a = 4 \), \( b = -1 \), \( c = -3 \). Thus

\[
x = \frac{-b}{2a} = \frac{-(\text{-}1)}{2(4)} = \frac{1}{8} = 0.125
\]

- \( y \) value if \( x = 0.125 \) is

\[
y = 4x^2 - x - 3 = 4(0.125)^2 - 0.125 - 3 = -3.0625
\]

Coordinates of the vertex of the graph \( y = 4x^2 - x - 3 \) are \((0.125; -3.0625)\)
Graph: \( y = ax^2 + bx + c \)

- Choose random \( x \) values and substitute into function to calculate \( y \). Draw \((x \; y)\) coordinate points and graph function.

Or

- Calculate the
  - Roots or \( x \)-intercept
  - \( y \)-intercept: \((0; c)\)
  - turning point or vertex

and draw coordinate points and graph of function.

Application:

- supply and demand; break-even etc.
- Maximum or minimum

Discussion class example 14
Question 14

The demand function for a commodity is \( Q = 6000 - 30P \). Fixed costs are R72 000 and the variable costs are R60 per additional unit produced.

(a) Write down the equation of total revenue and total costs in terms of \( P \).

(b) Determine the profit function in terms of \( P \).

(c) Determine the price at which profit is a maximum, and hence calculate the maximum profit.

(d) What is the maximum quantity produced?

(e) What is the price and quantity at the break-even point(s)?
Solution

(a) Given are the quantity demanded as \( Q = 6000 - 30P \), the fixed costs of R72 000 and the variable costs per unit of R60. Now

\[
TR = PQ = P(6000 - 30P) = 6000P - 30P^2
\]

Total cost = Fixed Cost + Variable Cost
\[
TC = 72000 + 60Q = 72000 + 60(6000 - 30P) = 72000 + 360000 - 1800P = 432000 - 1800P
\]

(b) Profit is total revenue minus total cost. Thus

\[
\text{Profit} = TR - TC = 6000P - 30P^2 - (432000 - 1800P) = -30P^2 + 7800P - 432000
\]

(c) The profit function derived in (b) is a quadratic function with \( a = -30; b = 7800; \) and \( c = -432000. \)
As \( a < 0 \) the shape of the function looks like a “sad face” and the function thus has a maximum at the function’s turning point or vertex \((P; Q)\).

The price \( P \) at the turning point or where the profit is a maximum is

\[
P = -\frac{b}{2a} = -\frac{7800}{2 \times -30} = \frac{-7800}{-60} = 130
\]

and thus the maximum profit

\[
\text{Profit} = -30(130)^2 + 7800(130) - 432000 = 75000.
\]

(d) The maximum quantity produced at the maximum price of R130 calculated in (c) is

\[
Q = 6000 - 30(130) = 2100.
\]

(e) At break-even the profit is equal to zero. Thus

\[
\text{Profit} = -30P^2 + 7800 - 432000 = 0.
\]
As the profit function is a quadratic function we use the quadratic formula with 
\( a = -30 \) and \( b = 7800 \) and \( c = -432\,000 \) to solve \( P \). Thus

\[
P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

\[
= \frac{-7800 \pm \sqrt{(7800)^2 - 4(-30)(-432000)}}{2 \times -30}
\]

\[
= \frac{-7800 \pm \sqrt{9000000}}{60}
\]

\[
= \frac{-7800 \pm 3000}{60}
\]

\[
= \frac{-4800}{60} \text{ or } \frac{-10800}{-60}
\]

\[
= 80 \text{ or } 180
\]

Now if \( P = 80 \) then \( Q = 6000 - 30(80) = 3600 \) and if \( P = 180 \) then \( Q = 6000 - 30(180) = 600 \).

Thus the two break-even points are where the price is R80 and the quantity 3 600, and where the price is R180 and the quantity 600.
2. **Cubic function**

\[ y = ax^3 + bx^2 + cx + d \]

- polynomial of **degree** 3 – highest power of \( x \)
- 1 or 3 roots
- 0 or 2 turning points

**Example:**

![Cubic function graph](image)

**Graph:** Choose random \( x \) values and substitute into function to calculate \( y \). Draw \((x; y)\) coordinate points and graph function

**Application:**
- supply and demand; break-even etc.
- Maximum or minimum - Differentiation
3. **Exponential functions**

\[ y = a^x : \text{general format (} a = \text{constant} \) \]

\[ y = 10^x : \text{scientific format} \]

\[ y = e^x : \text{natural format} \]

- \( a \) = base = constant
- \( x \) = index or power = variable
- \( e \) = unending number = 2.7182818….

**Example**

![Graph of exponential function]

**Properties:**

- Continuously pass through point \((0 ; 1)\)
- If power > 0 and \(a > 0\), curve increases : growth curve
- If power < 0 and \(a > 0\), curve decrease ; decay curve
- If \(a > 0\) curve above x-axis
- If \(a < 0\) curve below x-axis
Graph: Choose random $x$ values and substitute into function to calculate $y$. Draw $(x; y)$ coordinate points and graph function.

Rules: Let $A$ and $B$ be any two bases and $x$ and $y$ any two powers then

1. $A^x \times A^y = A^{x+y}$
   \[2^2 \times 2^3 = 2^{2+3} = 2^5\]

2. $A^x \div A^y = A^{x-y}$
   \[2^4 \div 2^3 = 2^{4-3} = 2^1\]

3. $(A \times B)^x = A^x \times B^x$
   \[(2 \times 3)^3 = 2^3 \times 3^3\]

4. $(A/B)^x = A^x / B^x$
   \[(2 / 3)^3 = 2^3 / 3^3\]

5. $(A^x)^y = A^{x \times y}$
   \[(2^3)^3 = 2^{3 \times 3} = 2^9\]

Note: If $a$ is any number

1. $(a)^0 = 1$ but $0^0 = 0$
   \[4^0 = 1\]

2. $(a)^1 = a$

3. $\frac{1}{a^n} = a^{-n}$
   \[2/x^4 = 2x^{-4}\]

4. $\sqrt[n]{a} = a^{\frac{1}{n}}$
   \[\sqrt{24} = 24^{\frac{1}{2}}\]

5. $a^{2x+4} = a^{15}$ (base the same) then $2x + 4 = 15$

Discussion class example 15

Application:

Discussion class example 16a
Question 15

Simplify the following expression

\[
\left(\frac{4L^2}{L^{-2}}\right)^2
\]

Solution

\[
\left(\frac{4L^2}{L^{-2}}\right)^2 = (4L^2 \times L^2)^2 \quad \text{since} \quad \frac{1}{a^b} = a^{-b}
\]

\[
= (4L^{2+2})^2 \quad \text{since} \quad a^b \times a^c = a^{b+c}
\]

\[
= (4L^4)^2
\]

\[
= 4^2 L^{4\times2} \quad \text{since} \quad (a^a)^b = a^{a\times b}
\]

\[
= 16L^8
\]
Question 16

An investment in a bank is said to grow according to the following formula:

\[ P(t) = \frac{6000}{1 + 29e^{-0.4t}} \]

where \( t \) is time in years and \( P \) is the amount (principle plus interest).

(a) What is the initial amount invested?
(b) Determine algebraically the time in years when the amount will be R4 000.

Solution

(a) Initial means \( t = 0 \)

\[ P = \frac{6000}{1 + 29e^{-0.4 \times 0}} \]

Using your calculator's \( e^x \) key

\[ P = \frac{6000}{30} = 200. \]
(b) If \( P = 4000 \) then

\[
4000 = \frac{6000}{1 + 29e^{-0.4t}}
\]

\[
1 + 29e^{-0.4t} = \frac{6000}{4000}
\]

\[
1 + 29e^{-0.4t} = \frac{3}{2}
\]

Divide numerator and denominator by 2000

\[
29e^{-0.4t} = \frac{3}{2} - 1
\]

\[
29e^{-0.4t} = \frac{1}{2}
\]

\[
e^{-0.4t} = \frac{1}{2} \div \frac{29}{1}
\]

\[
e^{-0.4t} = \frac{1}{2} \times \frac{1}{29}
\]

\[
e^{-0.4t} = \frac{1}{58}
\]
\[ \ln(e^{-0.4t}) = \ln\left(\frac{1}{58}\right) \]

Take \ln on both sides

\[ -0.4t \ln e = \ln\left(\frac{1}{58}\right) \]

\[ \ln a^b = b \ln a \]

\[ t = \frac{\ln\left(\frac{1}{58}\right)}{-0.4} \quad \text{Since } \ln e = 1 \]

\[ t = 10.15110753 \quad \text{Using your calculator, rounded to 8 decimal places} \]

\[ t = 10.2 \text{ years} \quad \text{Rounded to one decimal place} \]
4. **Logarithmic functions**

\[ y = \log_a x \]

\[ y = \log_{10} x = \log x \]

\[ y = \log_e x = \ln x \]

- Logs is the power of a number: \( \log 10 = 1; \log 100 = 2 \)
- \( \log_{\text{base}} \text{number} = \text{power} \) same as \( \text{Number} = \text{base}^{\text{power}} \)
  
  \[ \log_2 8 = 3 \quad \Rightarrow \quad 8 = 2^3 \]

**Rules:**

1. \( \log (u \times v) = \log u + \log v \)
2. \( \log (u \div v) = \log u - \log v \)
3. \( \log u^j = j \log u \)
4. \( \log_a x = \log x / \log a = \ln x / \ln a \)

**Discussion class example 17 + 18**

**Application:**

**Discussion class example 16(b)**
Question 17

Evaluate \( \frac{\log_{3} 12.34}{\ln \sqrt{12.34}} \)

Solution

Since \( \log_{a} b = \frac{\ln b}{\ln a} \) we can write

\[
\frac{\log_{3} 12.34}{\ln \sqrt{12.34}} = \frac{\ln 12.34}{\ln 3} \times \frac{1}{\ln \sqrt{12.34}}
\]

Using your calculator, rounded to 3 decimal places

\[= 1.820\]
**Question 18**

Solve for $Q$ if $\log Q - \log \left( \frac{Q}{Q+1} \right) = 0.8$

**Solution**

$$\log(Q) - \log \left( \frac{Q}{Q+1} \right) = 0.8$$

$$\log \left( \frac{Q}{Q+1} \right) = 0.8$$

$$\log_a \frac{a}{b} = \log a - \log b$$

$$\log \left( Q \times \frac{Q+1}{Q} \right) = 0.8$$

$$\log(Q + 1) = 0.8$$

$$Q + 1 = 10^{0.8}$$

$\log_a b = c$ can be written as $a^c = b$

$$Q = 10^{0.8} - 1$$

Using your calculator, rounded to 9 decimal places

$$Q = 5.309573445$$

$$Q = 5.31$$  Rounded to 2 decimal places
Study Unit 5 : Calculus

Chapter 6: Sections 6.1, 6.2.1, 6.3.1
Chapter 8: Section 8.1, 8.2 and 8.5

• In Business world the study of change important

Example: change in the sales of a company; change in the value of the rand; change in the value of shares; change in the interest rate etc.

• Equally important is the rate at which these changes take place.

Example: If the sales of a company increased by R2 000 000,00, it is important to know whether this change occurred over one year, two years or ten years.

• Rate of change: changes over time, change in costs for different production quantities in a production process, etc.

• Change consists of two components: size and direction.

Let’s look at linear function : y = mx + c.

• Slope (m) is the change in y which corresponds to change of one unit in the value of x: \[ \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1}. \]
• Slope => indication of rate of change = constant
• Value of the slope => size of the change
• Sign of the slope => direction of the change.
  • positive sign – an increase;
  • negative sign – a decrease.

Let’s look at a non-linear function, for example a quadratic function

The size and direction are not constant, but change continuously.

Use the mathematical technique of Differentiation to determine the rate of change.
1. Differentiation

- Slope of a curve = change in $y$ / change in $x$ = rate of curve change
- Use differentiation to get slope at a given point
- $\frac{dy}{dx} = f'(x)$ derivative of $y$ with respect to $x$
- Pronounce this as ”dee-y-dee-x.”
- Use rules of differentiation to obtain the derivative
- Many rules of differentiation – look at just 1 namely

**Power Rule:**

If $f(x) = x^n$, then $f'(x)$ or $\frac{d}{dx}(x^n) = nx^{n-1}$ for $n \neq 0$

For example

- $n$ integer and positive
  \[ \frac{d}{dx}(x^4) = 4x^{4-1} = 4x^3 \]

- $n$ integer and negative
  \[ \frac{d}{dx}(x^{-3}) = -3x^{-3-1} = -3x^{-4} = \frac{3}{x^4} \]
• n is a fraction and positive

\[
\frac{d}{dx} \left( x^\frac{1}{2} \right) = \frac{1}{2} x^{\frac{1}{2} - 1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \frac{1}{x^{\frac{1}{2}}} = \frac{1}{2\sqrt{x}}
\]

• n is a fraction and negative

\[
\frac{d}{dx} \left( x^{-\frac{1}{2}} \right) = -\frac{1}{2} x^{-\frac{1}{2} - 1} = -\frac{1}{2} x^{-\frac{3}{2}} = -\frac{1}{2} \frac{1}{x^{\frac{3}{2}}} = -\frac{1}{2\sqrt{x^3}}
\]

**Note:**

1. The derivative of any constant term say \(a\), that is a term which consists of a number only, is zero:

\[
\frac{da}{dx} = 0, \text{ where } a \text{ is a constant.}
\]

**Example:** \(f(x) = 4\) then \(f'(x) = 0\).
2. \[ \frac{d}{dx} [a f(x)] = a f'(x). \]

Example: \( f(x) = 7x^5 \) then \( f'(x) = 7 \times 5x^4 = 35x^4. \)

3. If \( f(x) = g(x) + h(x) \), then \( f'(x) = g'(x) + h'(x). \)

Example: \( f(x) = 7x^5 + 2x^3 \) then \( f'(x) = 35x^4 + 6x^2 \)

Example: \( f(x) = 4 + 8x^2 \) then \( f'(x) = 0 + 16x \)

Steps:

1. First we need to simplify the given expression so that we can use the basic rule of differentiation.

2. Secondly we differentiate the new expression using the basic rule \( \frac{d}{dx} x^n = nx^{n-1} \) where \( n \neq 0 \) of differentiation

For example:

1. \[ \frac{d}{dx} \left( \frac{1}{x^3} \right) = \frac{d}{dx} (x^{-3}) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4} \]

2. \[ \frac{d}{dx} \sqrt{x} = \frac{d}{dx} (x^{\frac{1}{2}}) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2} \left( \frac{1}{2} x^{-\frac{1}{2}} \right) = \frac{1}{2\sqrt{x}} \]

Discussion class example 19
Question 19

Differentiate the following expression

\[
\frac{x^3 - 4x^2 + 4x}{x - 2}
\]

Solution

• First we need to simplify the given expression so that we can use the basic rule of differentiation.

\[
\frac{x^3 - 4x^2 + 4x}{x - 2} = \frac{x(x^2 - 4x + 4)}{x - 2}
\]

\[
= \frac{x(x - 2)(x - 2)}{x - 2}
\]

\[
= x(x - 2)
\]

\[
= x^2 - 2x
\]

• Next we can differentiate the new expression using the basic rule \( \frac{d}{dx} x^n = nx^{n-1} \) where \( n \neq 0 \). Therefore

\[
\frac{d}{dx} \left( \frac{x^3 - 4x^2 + 4x}{x - 2} \right) = \frac{d}{dx} (x^2 - 2x)
\]

\[
= 2x^{2-1} - 2x^{1-1}
\]

\[
= 2x - 2
\]
• Application:
  - Minimum or maximum, vertex, turning point => slope = 0 => dy/dx = 0

  Discussion class example 14 (c) using differentiation

  - Also called rate of change because slope is rate of change
  - Slope of a tangent line at a given point – derivative at that point.
  - Marginal analysis
    - Marginal revenue: \( MR = dTR/dQ \)
    - Marginal cost: \( MC = dTC/dQ \)

  Discussion class example 20
Question 20
What is the marginal cost when $Q = 10$ if the total cost is given by:

$$TC = 20Q^4 - 30Q^2 + 300Q + 200$$

Solution
The marginal cost function is the differentiated total cost function. Thus by differentiating the total cost function we can determine the marginal cost function. Now if the total cost function is

$$TC = 20Q^4 - 30Q^2 + 300Q + 200$$

then the marginal cost function is

$$MC = \frac{dTC}{dQ} = 80Q^3 - 60Q + 300.$$

Now the marginal cost function’s value when $Q$ is equal 10 is

$$MC = 80(10)^3 - 60(10) + 300 = 80000 - 600 + 300 = 79700.$$
2. Integration

- Is the reverse of differentiation

\[ \frac{d(y)}{dx} \rightarrow y \rightarrow \int d(y) \]

- Indefinite integral: different rules

Steps:

- Simplify function before you integrate – write it so that you can apply the integration rule for example

  \( \int (ax + b) = \int ax + \int b \)

- \( \int \frac{1}{\sqrt{x}} = \int x^{\frac{1}{2}} \)

- Apply basic integration rule

\[ \int x^n dx = \frac{x^{n+1}}{n+1} + c \text{ where } n \neq -1 \]

\[ \int a \ d x = \frac{ax^{0+1}}{1} + c = ax + c \text{ where a is a constant} \]

Discussion class example 21 and 22

- Hint: test your answer: differentiate answer, must be equal to function integrated.
Question 21

Evaluate the following

\[ \int (x^2 + 2x + 3)\,dx \]

Solution

To integrate the function we make use of the basic rule of integration namely

\[ \int x^n = \frac{x^{n+1}}{n+1} + c \]

when \( n \neq -1 \). Therefore:

\[ \int (x^2 + 2x + 3)\,dx = \int x^2\,dx + \int 2x\,dx + \int 3\,dx \]

\[ = \frac{x^{2+1}}{2+1} + \frac{2x^{1+1}}{1+1} + \frac{3x^{0+1}}{0+1} + c \]

\[ = \frac{x^3}{3} + \frac{2x^2}{2} + \frac{3x}{1} + c \]

\[ = \frac{x^3}{3} + x^2 + 3x + c \]
Question 22

Determine \( \int \frac{Q+1}{\sqrt{Q}} \, dQ \)

Solution

First simplify the function to be integrated:

\[
\frac{Q+1}{\sqrt{Q}} = \frac{(Q+1)}{Q^{\frac{1}{2}}}
\]

\[
= (Q+1)Q^{-\frac{1}{2}}
\]

\[
= Q^2 + Q^{-\frac{1}{2}}
\]

Integrate the function using rule \( \int x^n = \frac{x^{n+1}}{n+1} + c \) when \( n \neq -1 \)

\[
\int \left( Q^2 + Q^{-\frac{1}{2}} \right) \, dQ = \int Q^2 \, dQ + \int Q^{-\frac{1}{2}} \, dQ
\]

\[
= \frac{Q^{3}}{3} + \frac{Q^{-\frac{3}{2}}}{1} + c
\]

\[
= \sqrt{Q^3} \cdot \frac{2}{3} + \sqrt{Q} \cdot \frac{2}{1} + c
\]

\[
= \frac{2\sqrt{Q^3}}{3} + 2\sqrt{Q} + c
\]
Definite integral: area under a given curve between two points $a$ and $b$:

$$\int_{a}^{b} x^n \, dx = \frac{x^{n+1}}{n+1} (x = b) - \frac{x^{n+1}}{n+1} (x = a)$$

- **Steps:**

  1. Simplify the function
  2. Integrate the function by applying the basic rule of integration
  3. Calculate the value of the integrated function at the value $a$ – substitute the values $a$ into the integrated function – answer 1
  4. Calculate the value of the integrated function at the value $b$ – substitute the values $b$ into the integrated function – answer 2
  5. Subtract answer 2 from answer 1

**Discussion class example 23**
Question 23

Evaluate

\[ \int_{-1}^{1} (z + 1) \, dz \]

Solution

Integrate the function using the basic rule \( \int x^n = \frac{x^{n+1}}{n+1} + c \) when \( n \neq -1 \) and substitute the values into the integrated function:

\[
\int_{-1}^{1} (z + 1) \, dz = \frac{1}{2} z^{1+1} + z \bigg|_{-1}^{1} \\
= (\frac{z^2}{2} + z) \bigg|_{-1}^{1} \\
= (1^2 + 1) - \left( \frac{(-1)^2}{2} + (-1) \right) \\
= 1\frac{1}{2} - (\frac{1}{2} - 1) \\
= 1\frac{1}{2} - (-\frac{1}{2}) \\
= 1\frac{1}{2} + \frac{1}{2} = 2
\]
2. Discussion classes: Questions and solutions

The slides of my notes of the discussion classes can be found on myUnisa.

Question 1
Calculate
\[
\frac{1}{6} - \frac{5}{6} + \frac{2}{3} + \frac{1}{3} \times \frac{3}{4}
\]

Solution
\[
\frac{1}{6} - \frac{5}{6} + \frac{2}{3} + \frac{1}{3} \times \frac{3}{4} = \frac{1}{6} - \frac{5}{6} + \frac{2}{3} \times \frac{3}{4} = \frac{1}{6} - \frac{5}{6} \times \frac{3}{4} + \frac{1}{3} \times \frac{3}{4}
\]
\[
= \frac{1}{6} - \frac{15}{12} + \frac{3}{12}
\]
\[
= \frac{2 - 15}{12} + \frac{3}{12}
\]
\[
= \frac{-13 + 3}{12}
\]
\[
= \frac{-10}{12}
\]
\[
= \frac{-5}{6}
\]

Question 2
Solve for \(x\) in

\[-2x + \frac{5}{6} + \frac{x}{2} \geq -2x - 4 \left( \frac{-x}{3} - \frac{1}{4} \right)\]
Solution

\[-2x + \frac{5}{6} + \frac{x}{2} \geq -2x - 4 \left( -\frac{x}{3} - 1 \frac{1}{4} \right)\]

\[-2x + \frac{5}{6} + \frac{x}{2} \geq -2x - 4 \left( -\frac{x}{3} - \frac{5}{4} \right)\]  
change fraction

\[-2x + \frac{5}{6} + \frac{x}{2} \geq -2x + \left( \frac{4x}{3} + \frac{20}{4} \right)\]  
multiply 4 into ( )

\[-2x + \frac{5}{6} + \frac{x}{2} \geq -2x + \frac{4x}{3} + 5\]  
remember \(\frac{20}{4} = 5\)

\[-2x + 2x + \frac{x}{2} \geq \frac{5x}{3} - \frac{5}{6}\]  
move all similar terms to one side

\[\frac{x}{2} - \frac{4x}{3} \geq 5 - \frac{5}{6}\]

\[\frac{3x - 8x}{6} \geq \frac{30}{6} - \frac{5}{6}\]  
common denominator

\[-\frac{5x}{6} \geq \frac{25}{6}\]

\[-\frac{5x}{6} \times \frac{6}{1} \geq \frac{25}{6} \times \frac{6}{1}\]

\[-5x \geq 25\]

\[-\frac{5x}{-5} \leq \frac{25}{-5}\]  
inequality sign changes because we divide by a negative number

\[x \leq -5\]

Question 3

(a) Find the equation of the line passing through the points (1; 20) and (5; 60).

(b) Draw the graph of the line \(y = 10x + 10\).

Solution

(a) Let \((x_1 ; y_1) = (1 ; 20)\) and \((x_2 ; y_2) = (5 ; 60)\). We need to determine the slope \(m\) and the \(y\) intercept \(c\) of the line \(y = mx + c\).

Now the slope \(m\) is defined by

\[m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{60 - 20}{5 - 1} = \frac{40}{4} = 10\]

Therefore \(y = 10x + c\).
Now both \((x_1 ; y_1)\) and \((x_2 ; y_2)\) lie on the line. We can thus substitute any one of the points into the equation of the line to determine \(c\). Let’s choose the point \((1; 20)\). Then

\[
\begin{align*}
y &= 10x + c \\
20 &= 10 \times 1 + c \\
20 &= 10 + c \\
-c &= 10 - 20 \\
-c &= -10 \\
c &= 10
\end{align*}
\]

The equation of the line is \(y = 10x + 10\).

(b) Given the line \(y = 10x + 10\), we need two points to draw a line. Select any \(x\) or \(y\) value and calculate the value of the point.

Say we choose \(x = 0\), then \(y = 10(0) + 10 = 10\). Therefore point 1 = \((0; 10)\).

Choose \(y = 0\) then

\[
\begin{align*}
0 &= 10x + 10 \\
-10x &= 10 \\
x &= 10 / (-10) \\
x &= -1. \text{ Therefore point 2 = } (-1; 0).
\end{align*}
\]

Please note that you can use any \(x\) and/or \(y\) value to calculate the two points. Normally \(x = 0\) and \(y = 0\) are used to simplify the calculation.

Next we plot the two calculated points of the line and draw the line:
Question 4
If the demand function is \( P = 80 - 2Q \), where \( P \) and \( Q \) are the price and quantity, respectively, determine the expression for price elasticity of demand
(a) if the price \( P = 20 \).
(b) in terms of \( P \) only.

Solution
(a) The demand function is given as \( P = 80 - 2Q \).

Now the price elasticity of demand is
\[
\varepsilon_d = -\frac{1}{b} \cdot \frac{P}{Q}
\]
with \( a \) and \( b \) the values of the standard demand function \( P = a - bQ \).

To determine the price elasticity of demand, we need to determine the values of \( b \), \( Q \) and \( P \). Given \( P = 80 - 2Q \) and \( P = 20 \). Comparing \( P = 80 - 2Q \) with \( P = a - bQ \), we can conclude that \( a = 80 \) and \( b = 2 \).

Now \( a \), \( b \) and \( P \) are known. All we need to calculate is the value of \( Q \). To determine the value of \( Q \), we substitute \( P = 20 \) into the equation of the demand function and solve for \( Q \):
\[
\begin{align*}
20 &= 80 - 2Q \\
20 - 80 &= -2Q \\
-60 &= -2Q \\
Q &= 30
\end{align*}
\]

Now
\[
\varepsilon_d = -\frac{1}{b} \cdot \frac{P}{Q}
\]
\[
= -\frac{1}{2} \times \frac{20}{30}
\]
\[
= -\frac{1}{3}
\]
\[
= -0.33 \text{ (rounded to 2 decimal places)}
\]

At \( P = 20 \) a 1% increase (decrease) in price will cause a 0.33% decrease (increase) in the quantity demanded.
(b) The expression for the price elasticity of demand is \( \varepsilon_d = -\frac{1}{b} \times \frac{P}{Q} \) with \( a \) and \( b \) the values of the standard demand function \( P = a - bQ \).

To determine the price elasticity of demand we thus need to determine the values of \( b \), \( Q \) and \( P \). It is given that \( P = 80 - 2Q \) and question asked in terms of \( P \) thus \( P = P \). Comparing \( P = 80 - 2Q \) with \( P = a - bQ \), we can say that \( a = 80 \) and \( b = 2 \). At this stage \( Q \) is unknown.

The demand function now denotes the relationship between the price \( P \) and the demand \( Q \). If given \( P \), we can derive \( Q \) by substituting \( P \) into the demand function and we can then solve for \( Q \).

To determine the value of \( Q \) we need to change the equation of the demand function so that \( Q \) is the subject of the equation. That means we write \( Q \) in terms of \( P \). Now

\[
P = 80 - 2Q \\
P - 80 = -2Q \\
\frac{P - 80}{-2} = Q.
\]

As we have determined the values of \( b \), \( P \) and \( Q \) we can now substitute them into the formula for the price elasticity of demand:

\[
\varepsilon_d = -\frac{1}{2} \times \frac{P}{P - 80} \\
= -\frac{1}{2} \times \frac{P}{P - 80} \times \frac{-2}{1} \\
= \frac{P}{P - 80}
\]

**Or alternatively**

You can use the given formula of price elasticity of demand in terms of \( P \) of a demand function in the form \( P = a - bQ \) as given in the textbook on page 78, equation 2.14 (Edition 2) and page 89, equation 2.14 (Edition 3).

\[
\varepsilon_d = \frac{P}{P - a}
\]

Now \( a = 80 \) (intercept on the \( y \)-axis of the demand function) \( \varepsilon_d = \frac{P}{P - 80} \).
Question 5
Given the demand function $P = 60 - 0.2Q$ where $P$ and $Q$ are the price and quantity respectively, calculate the arc price elasticity of demand when the price decreases from R50 to R40.

Solution
The arc price elasticity of a demand function $P = a - bQ$ between two prices $P_1$ and $P_2$ is

$$\text{arc elasticity of demand} = -\frac{1}{b} \times \frac{P_1 + P_2}{Q_1 + Q_2}$$

with $b$ the slope of the demand function and $P_1$, $P_2$ and $Q_1$, $Q_2$ the price and quantity demanded.

Now for the given function $P = 60 - 0.2Q$, we can derive that $a = 60$ and $b = 0.2$. It is given that $P_1 = 50$ and $P_2 = 40$. All we need to determine are $Q_1$ and $Q_2$. By making $Q$ the subject of the equation, we can rewrite the equation $P = 60 - 0.2Q$ as

$$P = 60 - 0.2Q$$
$$0.2Q = 60 - P$$
$$Q = 300 - 5P.$$ We can now determine $Q_1$ and $Q_2$ by substituting $P_1 = 50$ and $P_2 = 40$ into the equation.

Therefore if $P_1 = 50$ then $Q_1 = 300 - 5 \times 50 = 50$
and if $P_2 = 40$ then $Q_2 = 300 - 5 \times 40 = 100$.

Therefore

$$\text{elasticity of demand} = -\frac{1}{0.2} \times \frac{50 + 40}{50 + 100}$$
$$= -\frac{1}{0.2} \times \frac{90}{150}$$
$$= -\frac{90}{30}$$
$$= -3$$
**Question 6**

Solve the following set of linear equations:

\[ y + 2x = 3 \quad (1) \]
\[ y - x = 2 \quad (2) \]

by using the

(a) elimination method.
(b) substitution method.
(c) graphical method.

**Solution**

(a) **Elimination method**:

Step 1: Eliminate one variable, say \( y \), by adding or subtracting one equation or multiple of an equation from another equation:

Subtract equation (2) from equation (1) and solve for \( x \):

\[ y + 2x = 3 \quad (1) \]
\[ -(y - x = 2) \quad (2) \]
\[ 0 + 3x = 1 \]
\[ x = \frac{1}{3} \]

Step 2: Solve for \( y \). Substitute value of \( x \) back into any one of equations and solve \( y \).

Substitute the value of \( x \) into say equation (1):

\[ y + 2x = 3 \quad (1) \]
\[ y + 2 \left( \frac{1}{3} \right) = 3 \]
\[ y = 3 - \frac{2}{3} \]
\[ y = 2 \frac{1}{3} \]
(b) **Substitution method:**

Step 1: Change one of the equations so that a variable is the subject of the equation, say $y$ in equation (1):

$$y + 2x = 3$$

$$y = 3 - 2x \quad (3)$$

Step 2: Substitute the value of $y$ (equation (3)) into the unchanged equation (2) and solve for $x$. Substitute $y = 3 - 2x$ into $y - x = 2$:

$$y - x = 2$$

$$(3 - 2x) - x = 2$$

$$-3x = 2 - 3$$

$$-3x = -1$$

$$x = \frac{1}{3}$$

Step 3: Substitute the calculated value of the variable into any equation and calculate the value of the other variable. Substitute $x = \frac{1}{3}$ into equation (1) or equation (2). Let’s say we choose equation (2):

$$y - x = 2 \quad (2)$$

$$y - \frac{1}{3} = 2$$

$$y = 2 + \frac{1}{3}$$

$$y = \frac{7}{3}$$

(c) **Graphical method:**

To solve the two equations simultaneously by using the graphical method, we first draw the graphs of the two lines and secondly determine the point where the two lines intersect.

To draw a line we need two points:
Equation (1): $y + 2x = 3$

Let’s choose $x = 0$, then $y + 2(0) = 3$ or $y = 3$. Therefore one point is $(0 ; 3)$.

Next we choose $y = 0$. If $y = 0$ then $0 + 2x = 3$ or $x = \frac{3}{2}$. Therefore a second point is $(3/2 ; 0)$.

Equation (2): $y - x = 2$

If $x = 0$ then $y - (0) = 2$ or $y = 2$. Thus the first point is $(0; 2)$.

If $y = 0$ then $0 - x = 2$ or $x = -2$. Thus a second point is $(-2; 0)$

Please note that you can use any $x$ or $y$ values to calculate the two points. Normally $x = 0$ and $y = 0$ are used to simplify the calculation.

Next we plot the two calculated points of the lines and draw the lines. The solution is the point where the two lines intercept, namely the point $(\frac{1}{3}; 2\frac{1}{3})$. 

[Graph showing two lines intersecting at a point marked as $(\frac{1}{3}; 2\frac{1}{3})$.]
Question 7
Solve the following sets of equations:
(a)  
\[
\begin{align*}
  x - y + z &= 0 \quad (1) \\
  2y - 2z &= 2 \quad (2) \\
  -x + 2y + 2z &= 29 \quad (3)
\end{align*}
\]

(b)  
\[
\begin{align*}
  x - 2y + 3z &= -11 \quad (1) \\
  2x - z &= 8 \quad (2) \\
  3y + z &= 10 \quad (3)
\end{align*}
\]

Solution
(a) We need to solve the following system of equations:
\[
\begin{align*}
  x - y + z &= 0 \quad (1) \\
  2y - 2z &= 2 \quad (2) \\
  -x + 2y + 2z &= 29 \quad (3)
\end{align*}
\]

Step 1: Determine two equations with two unknowns (variables) by adding or subtracting two of the three equations at a time:

Now equation (2) is already an equation with only two variables. To determine another equation with just two variables we add equation (1) and equation (3):
\[
\begin{align*}
  x - y + z &= 0 \quad (1) \\
  -x + 2y + 2z &= 29 \quad (3) \\
  \hline
  0 + y + 3z &= 29 \quad (4)
\end{align*}
\]

Step 2: Next we solve two equations with two unknowns, using any method described previously. Let’s use the substitution method:
- Step 1: Make \( y \) the subject of equation (4):
  \[ y = 29 - 3z \quad (5) \]
• Step 2: Substitute equation (5) into equation (2) and solve for \( z \):

\[
2y - 2z = 2 \tag{2}
\]

\[
2(2z - 3z) - 2z = 2
\]

\[
58 - 6z - 2z = 2
\]

\[
-8z = 2 - 58
\]

\[
-8z = -56
\]

\[
z = \frac{-56}{-8} = 7
\]

• Step 3: Substitute \( z = 7 \) into equation (2) and solve for \( y \):

\[
2y - 2z = 2 \tag{2}
\]

\[
2y - 2(7) = 2
\]

\[
2y = 2 + 14
\]

\[
2y = 16
\]

\[
y = 8
\]

• Step 4: Substitute \( z = 7 \) and \( y = 8 \) into equation (1) and solve for \( x \):

\[
x - y + z = 0 \tag{1}
\]

\[
x - 8 + 7 = 0
\]

\[
x = 1
\]

Therefore the solution of the set of equations is \( x = 1, y = 8 \) and \( z = 7 \).

(b) We need to solve the following system of equations:

\[
x - 2y + 3z = -11 \tag{1}
\]

\[
2x - z = 8 \tag{2}
\]

\[
3y + z = 10 \tag{3}
\]
Make $z$ the subject of equation (2) and $y$ the subject of equation (3):

\begin{align*}
z &= 2x - 8 \quad (4) \\
y &= \frac{10 - z}{3} \quad (5)
\end{align*}

Substitute equation (4) into equation (5):

\begin{align*}
y &= \frac{10 - (2x - 8)}{3} = \frac{18 - 2x}{3} \quad (6)
\end{align*}

Substitute equation (2) and equation (6) into equation (1):

\begin{align*}
x - \frac{2}{3} (18 - 2x) + 3(2x - 8) &= -11 \\
x - 12 + \frac{4}{3} x + 6x - 24 &= -11 \\
\frac{3 + 4 + 18}{3} x &= -11 + 12 + 24 \\
\frac{25}{3} x &= 25 \\
x &= 3
\end{align*}

Substitute $x = 3$ into equation (4) and equation (6):

\begin{align*}
z &= 2 \times 3 - 8 = -2
\end{align*}

and

\begin{align*}
y &= \frac{18 - 2 \times 3}{3} = \frac{18 - 6}{3} = 4
\end{align*}

Therefore $x = 3$, $y = 4$ and $z = -2$. 
Question 8

In a market we have the following:

Demand function: \( Q = 50 - 0,1P \)
Supply function: \( Q = -10 + 0,1P \)

where \( P \) and \( Q \) are the price and quantity respectively.

(a) Calculate the equilibrium price and quantity.
(b) Draw the two functions, and label the equilibrium point.

Solution

(a) Equilibrium is the price and quantity where the demand and supply functions are equal. Therefore determine \( Q_d = Q_s \) or

\[
\begin{align*}
50 - 0,1P &= -10 + 0,1P \\
-0,1P - 0,1P &= -10 - 50 \\
-0,2P &= -60 \\
P &= \frac{60}{-0,2} \\
P &= 300
\end{align*}
\]

To calculate the quantity at equilibrium, we substitute the value of \( P \) into the demand or supply function and calculate \( Q \). Say we use the demand function, then

\[
\begin{align*}
Q &= 50 - 0,1(300) \\
Q &= 20
\end{align*}
\]

The equilibrium price is equal to 300 and the quantity is 20.
Question 9

A company manufactures and sells \( x \) hand-held toy radios per week. The weekly costs are given by

\[
c(x) = 5000 + 2x
\]

What is the company’s break-even point if a radio sells for R202?

Solution

The break-even point of a company occurs when the company does not make a profit or a loss, meaning break-even is where the revenue of the company is equal to the cost. Therefore

\[
\text{revenue} = \text{cost}.
\]

Now revenue = price \( \times \) quantity. Therefore

\[
\text{revenue} = 202x.
\]

Cost is given by

\[
c(x) = 5000 + 2x.
\]
Therefore break-even is when
\[ \text{revenue} = \text{cost} \]
\[ 202x = 5000 + 2x \]
\[ 202x - 2x = 5000 \]
\[ 200x = 5000 \]
\[ x = 25 \]

Or alternatively

At break-even the profit is equal to zero. Therefore
\[ \text{profit} = 0 \]
\[ \text{profit} = \text{revenue} - \text{cost} \]
\[ \text{profit} = 202x - (5000 + 2x) = 0 \]
\[ 200x - 5000 = 0 \]
\[ 200x = 5000 \]
\[ x = 25 \]

Thus the company breaks even when they manufacture 25 hand-held toy radios.

**Question 10**

Calculate the consumer surplus for the demand function \( P = 60 - 4Q \) when the market price is \( P = 12 \).

**Solution**

Consumer surplus is the monetary value of the benefit that accrues to consumers from the matching of supply and demand in the market. The consumer surplus is the difference between the amount the consumer is willing to spend for successive units of a product from \( Q = 0 \) to \( Q = Q_0 \) and the amount that the consumer actually spent on \( Q_0 \) units of the product at a market price of \( P_0 \) per unit:

\[ CS = \text{Amount willing to pay} - \text{Amount actually paid} \]
If you need to determine the demand surplus for a linear demand function of \( P = a - bQ \) then the consumer surplus can be calculated by calculating an area of the triangle \( P_0Q_0a \) which is equal to

\[
\frac{1}{2} \times \text{height} \times \text{base} = \frac{1}{2} \times (a - Q_0) \times (P_0 - 0) = \frac{1}{2} \times (a - Q_0) \times (P_0)
\]

with

- \( P_0 \) the value given to you as the market price,
- \( Q_0 \) the value of the demand function if \( P \) equals the given market price (substitute \( P_0 \) into the demand function and calculate \( Q_0 \)), and
- \( a \) the \( y \)-intercept of the demand function \( P = a - bQ \) also known as the value of \( P \) if \( Q = 0 \), or the point where the demand function intercepts the \( y \)-axis.

In general we can summarise the steps of determining the consumer surplus as follows:

**Method:**

1. Calculate \( Q_0 \) if \( P_0 \) is given.
2. Draw a rough graph of the demand function.
3. Read the value of \( a \) from the demand function – the \( y \)-intercept of the demand function.
4. Calculate the area of \( CS = \frac{1}{2} \times (a - P_0) \times (Q_0) \).
First we need to determine $Q$ from the demand function if $P = 12$. Therefore

\[ P = 60 - 4Q \]
\[ 12 = 60 - 4Q \]
\[ 4Q = 60 - 12 \]
\[ 4Q = 48 \]
\[ Q = 12 \]

Next we draw a rough sketch of the demand function:

Now the consumer surplus is the area of the shaded triangle:

\[ CS = \frac{1}{2} \times \text{base} \times \text{height} \]
\[ = \frac{1}{2} \times 12 \times (60 - 12) \]
\[ = 288 \]

**Question 11**

(a) Graph the lines representing the following constraints:

\[ 2x + y \leq 120 \quad (1) \]
\[ x + 2y \leq 140 \quad (2) \]
\[ x + y \leq 80 \quad (3) \]
\[ x_1, x_2 \geq 0 \quad (4) \]

(b) Show the feasible region.

(c) Determine the maximum value of $P = 20x + 30y$ subject to the constraints above.
Solution
(a) and (b)

Step 1:
To graph a linear inequality we first change the inequality sign (≥ or ≤ or > or <) to an equal sign (=) and draw the graph of the line. But we need two points to draw a line. Choose the two points where the lines cut the x-axis (x-axis intercept, thus y = 0) and y-axis (y-axis intercept, x = 0). Calculate (0 ; y) and (x ; 0) and draw a line through the two points. See the table below for the calculations.

Step 2:
Determine the feasible region for each inequality by substituting a point on either side of this line into the equation of the inequality. The inequality region is the area where the selected point makes the inequality true. The calculations are given below:

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Step 1</th>
<th>Step 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>2x + y ≤ 120</td>
<td>x = 0</td>
<td>Select the point (0;0) below the line 2(0) + 0 ≤ 120 – True Area below the line</td>
</tr>
<tr>
<td>x + 2y ≤ 140</td>
<td>x = 0</td>
<td>Select the point (0;0) to the left of the line 0 + 2(0) ≤ 140 – True Area to the left of the line</td>
</tr>
<tr>
<td>x + y ≤ 80</td>
<td>x = 0</td>
<td>Select the point (0;0) below the line 0 + 0 ≤ 80 – True Area below the line</td>
</tr>
<tr>
<td>x, y ≥ 0</td>
<td></td>
<td>Area above the x-axis and to the right of the y-axis</td>
</tr>
</tbody>
</table>
Step 3:
The feasible region is the one where all the inequalities are true simultaneously.

(c) Determine all the corner points of the feasible region and substitute them into the objective function (function you want to maximise or minimise) and determine the maximum value.

Step 1: Determine the coordinates of all corners of the feasible region by solving two equations with two unknowns (substitution) or read them from the graph.

Step 2: Substitute the corner points into the objective function.

Step 3: Choose the corner point which results in the highest (maximisation) or the lowest (minimisation) objective function value.

<table>
<thead>
<tr>
<th>Corner points of feasible region</th>
<th>Value of $P = 20x + 30y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: $x = 60; y = 0$</td>
<td>$P = 20(60) + 30(0) = 1200$</td>
</tr>
<tr>
<td>B: $x = 0; y = 70$</td>
<td>$P = 20(0) + 30(70) = 2100$</td>
</tr>
<tr>
<td>C: $x = 20; y = 60$</td>
<td>$P = 20(20) + 30(60) = 2200\leftarrow\text{Maximum}$</td>
</tr>
<tr>
<td>D: $x = 40; y = 40$</td>
<td>$P = 20(40) + 30(40) = 2000$</td>
</tr>
<tr>
<td>Origin: $x = 0; y = 0$</td>
<td>$P = 20(0) + 30(0) = 0$</td>
</tr>
</tbody>
</table>

Maximum of $P$ is at point C where $x = 20, y = 60$ and $P = 2200$. 
Question 12

Giapetto woodcarving manufactures two types of wooden toys: soldiers and trains.
- A soldier sells for R27 and uses R10 worth of raw materials.
- A train sells for R21 and uses R9 worth of raw materials.
- The manufacturer of wooden soldiers and trains requires two types of skilled labour: carpentry and finishing.
- A soldier requires 2 hours of finishing and 1 hour of carpentry.
- A train requires 1 hour of finishing and 1 hour of carpentry.
- Each week, at most 100 finishing hours and 80 carpentry hours are available.
- The demand for trains is unlimited, but at most 40 soldiers are bought each week.
- Giapetto’s has a weekly budget of R10 000 for the raw materials.

If \( x \) is the number of toy soldiers made per week and \( y \) is the number of trains made per week, formulate the linear constraints that describe Giapetto’s situation and write down the revenue function (objective function) if his objective is to maximise his revenue.

Solution

We first define the variables. Let \( x \) be the number of toy soldiers manufactured per week and \( y \) the number of trains manufactured per week. To help us with the formulation, we summarise the information given in a table with the headings: resources (items on which there are restrictions), the variables (\( x \) and \( y \)) and capacity (amount or number of the resources available).

<table>
<thead>
<tr>
<th>Resource</th>
<th>( x )</th>
<th>( y )</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soldier</td>
<td>Soldier</td>
<td>Train</td>
<td>Capacity</td>
</tr>
<tr>
<td>Carpentry</td>
<td>1</td>
<td>1</td>
<td>80</td>
</tr>
<tr>
<td>Finishing</td>
<td>2</td>
<td>1</td>
<td>100</td>
</tr>
<tr>
<td>Raw materials budget</td>
<td>10</td>
<td>9</td>
<td>10 000</td>
</tr>
<tr>
<td>Maximum per week</td>
<td>40</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>Sales / Revenue</td>
<td>27</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>Number of toys</td>
<td>Never negative</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Using the table, the following constraints can be defined:

\[
\begin{align*}
    x + y & \leq 80 & \text{Carpentry} \\
    2x + y & \leq 100 & \text{Finishing} \\
    10x + 9y & \leq 10000 & \text{Budgetary constraint on raw materials} \\
    x & \leq 40 & \text{Maximum demand per week} \\
    x, y & \geq 0 & \text{Non-negative}
\end{align*}
\]

As Giapetto would like to maximise his revenue, and as revenue is equal to quantity times the price or sales, the objective function can be written as \(27x + 21y\).

**Question 13**

A manufacturing plant makes two types of inflatable boats: a two-person boat and a four-person boat. Each two-person boat requires 0.9 labour hours from the cutting department and 0.8 labour hours from the assembly department. Each four-person boat requires 1.8 labour hours from the cutting department and 1.2 labour hours from the assembly department. The maximum hours available for the cutting and assembly departments are 864 and 672 respectively. The company makes a profit of R2 500 on a two-person boat and R4 000 on a four-person boat. Let \(x\) and \(y\) be the number of two-person boats and four-person boats made respectively. Formulate the constraints and objective function of the linear programming problem if the company would like to maximise its profit.

**Solution**

First we define the variables. Let \(x\) and \(y\) be the number of two-person boats and four-person boats made respectively. To help us with the formulation, we summarise the information given in a table with the headings: resources (items with restrictions), the variables (\(x\) and \(y\)) and capacity (amount or number of the resources available).
<table>
<thead>
<tr>
<th>Resource</th>
<th>( x )</th>
<th>( y )</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutting</td>
<td>0,9</td>
<td>1,8</td>
<td>864</td>
</tr>
<tr>
<td>Assembly</td>
<td>0,8</td>
<td>1,2</td>
<td>672</td>
</tr>
<tr>
<td>Profit</td>
<td>2500</td>
<td>4000</td>
<td></td>
</tr>
<tr>
<td>Number of boats</td>
<td></td>
<td></td>
<td>Never negative</td>
</tr>
</tbody>
</table>

Using the table, the following constraints can be defined:

\[
0,9x + 1,8y \leq 864 \\
0,8x + 1,2y \leq 672 \\
x, y \geq 0
\]

As the company would like to maximise their profit, the objective function can be written as

\[
2500x + 4000y.
\]

**Question 14**

The demand function for a commodity is \( Q = 6000 - 30P \). Fixed costs are R72 000 and the variable costs are R60 per additional unit produced.

(a) Write down the equation of total revenue and total costs in terms of \( P \).

(b) Determine the profit function in terms of \( P \).

(c) Determine the price at which profit is a maximum, and hence calculate the maximum profit.

(d) What is the maximum quantity produced?

(e) What is the price and quantity at the break-even point(s)?
Solution

(a) The quantity demanded is \( Q = 6000 - 30P \), the fixed costs of R72 000 and the variable costs per unit of R60 are given. Now

\[
\begin{align*}
\text{Total revenue} &= \text{Price} \times \text{Quantity} \\
TR &= PQ \\
TR &= P(6000 - 30P) \\
TR &= 6000P - 30P^2
\end{align*}
\]

\[
\begin{align*}
\text{Total cost} &= \text{Fixed cost} + \text{Variable cost} \\
TC &= 72000 + 60Q \\
TC &= 72000 + 60(6000 - 30P) \\
TC &= 72000 + 360000 - 1800P \\
TC &= 432000 - 1800P
\end{align*}
\]

(b) Profit is total revenue minus total cost. Thus

\[
\begin{align*}
\text{Profit} &= TR - TC \\
&= 6000P - 30P^2 - (432000 - 1800P) \\
&= -30P^2 + 7800P - 432000
\end{align*}
\]

(c) The profit function derived in (b) is a quadratic function with 
\( a = -30, b = 7800, c = -432000 \).

As \( a < 0 \) the shape of the function looks like a “sad face” and the function thus has a maximum at the function’s turning point or vertex \((P; Q)\).

The price \( P \) at the turning point, or where the profit is a maximum, is

\[
P = \frac{-b}{2a} = -\frac{7800}{2 \times -30} = \frac{-7800}{-60} = 130
\]

and thus the maximum profit

\[
\text{Profit} = -30(130)^2 + 7800(130) - 432000 = 75000.
\]
(d) The maximum quantity produced at the maximum price of R130 calculated in (c) is 
\[ Q = 6000 - 30(130) = 2100. \]

(e) At break-even the profit is equal to zero. Thus
\[ \text{Profit} = -30P^2 + 7800 - 432000 = 0. \]

As the profit function is a quadratic function we use the quadratic formula with 
\[ a = -30 \text{ and } b = 7800 \] and \[ c = -432000 \] to solve \( P \). Thus
\[
P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
= \frac{-7800 \pm \sqrt{(7800)^2 - 4(-30)(-432000)}}{2 \times -30}
\]
\[
= \frac{-7800 \pm \sqrt{9000000}}{-60}
\]
\[
= \frac{-7800 \pm 3000}{-60}
\]
\[
= \frac{-4800 \text{ or } -10800}{-60}
\]
\[
= 80 \text{ or } 180
\]

Now if \( P = 80 \) then \( Q = 6000 - 30(80) = 3600 \) and if \( P = 180 \) then \( Q = 6000 - 30(180) = 600 \).

Thus the two break-even points are where the price is R80 and the quantity 3 600, and where the price is R180 and the quantity 600.
Question 15

Simplify the following expression:

\[
\left( \frac{4L^2}{L^2} \right)^2
\]

Solution

\[
\left( \frac{4L^2}{L^2} \right)^2 = (4L^2 \times L^2)^2 \quad \text{since} \quad \frac{1}{a^b} = a^{-b}
\]

\[
= (4L^{2+2})^2 \quad \text{since} \quad a^b \times a^c = a^{b+c}
\]

\[
= (4L^4)^2
\]

\[
= 4^2 L^{4+2} \quad \text{since} \quad (a^b)^c = a^{ab}
\]

\[
= 16L^6
\]

Question 16

An investment in a bank is said to grow according to the following formula:

\[
P(t) = \frac{6000}{1 + 29e^{-0.4t}}
\]

where \( t \) is the time in years and \( P \) is the amount (principle plus interest).

(a) What is the initial amount invested?

(b) Determine algebraically the time in years when the amount will be R4 000.

Solution

(a) Initial means \( t = 0 \).

\[
P = \frac{6000}{1 + 29e^{-0.4 \times 0}} = \frac{6000}{30} = 200.
\]
(b) If $P = 4000$ then

$$4000 = \frac{6000}{1 + 29e^{-0.4t}}$$

$$1 + 29e^{-0.4t} = \frac{6000}{4000}$$

$$1 + 29e^{-0.4t} = \frac{3}{2}$$

divide nominator and denominator by 2000

$$29e^{-0.4t} = \frac{3}{2} - 1$$

$$29e^{-0.4t} = \frac{1}{2}$$

$$e^{-0.4t} = \frac{1}{2} \times \frac{29}{1}$$

$$e^{-0.4t} = \frac{1}{2} \times \frac{1}{29}$$

$$e^{-0.4t} = \frac{1}{58}$$

$$\ln(e^{-0.4t}) = \ln\left(\frac{1}{58}\right)$$

take ln on both sides

$$-0.4t \ln e = \ln\left(\frac{1}{58}\right)$$

$$\ln a^b = b \ln a$$

$$t = \frac{\ln\left(\frac{1}{58}\right)}{-0.4}$$

$$t = 10.15110753$$

using your calculator, rounded to 8 decimal places

$$t = 10.2 \text{ years}$$

rounded to one decimal place
Question 17

Evaluate \( \frac{\log_{12.34} 12.34}{\ln \sqrt{12.34}} \).

Solution

\[
\frac{\log_{12.34} 12.34}{\ln \sqrt{12.34}} = \frac{\ln 12.34}{\ln 3} \times \frac{1}{\ln \sqrt{12.34}}
\]

since \( \log_b a = \frac{\ln a}{\ln b} \)

\[= \frac{1.820}{\ln 3} \times \frac{1}{\ln \sqrt{12.34}} \]

using your calculator, rounded to 3 decimal places

Question 18

Solve for \( Q \) if \( \log Q - \log \left( \frac{Q}{Q+1} \right) = 0.8 \)

Solution

\[
\log(Q) - \log \left( \frac{Q}{Q+1} \right) = 0.8
\]

\[
\log \left( \frac{Q}{Q+1} \right) = 0.8
\]

\[
\log \frac{a}{b} = \log a - \log b
\]

\[
\log \left( Q \times \frac{Q+1}{Q} \right) = 0.8
\]

\[
\log(Q+1) = 0.8
\]

\[
Q + 1 = 10^{0.8}
\]

\[
Q = 10^{0.8} - 1
\]

\[
Q = 5.309573445
\]

using your calculator, rounded to 9 decimal places

\[
Q = 5.31
\]

rounded to 2 decimal places
Question 19

Differentiate the following expression:

\[ \frac{x^3 - 4x^2 + 4x}{x - 2} \]

Solution

First we need to simplify the given expression so that we can use the basic rule of differentiation.

\[
\frac{x^3 - 4x^2 + 4x}{x - 2} = \frac{x(x^2 - 4x + 4)}{x - 2} = \frac{x(x-2)(x-2)}{x - 2} = x(x-2) = x^2 - 2x
\]

Next we can differentiate the new expression using the basic rule \( \frac{d}{dx}x^n = nx^{n-1} \) where \( n \neq 0 \).

Therefore

\[
\frac{d}{dx} \left( \frac{x^3 - 4x^2 + 4x}{x - 2} \right) = \frac{d}{dx}(x^2 - 2x) = 2x^{2-1} - 2x^{1-1} = 2x - 2
\]

Question 20

What is the marginal cost when \( Q = 10 \) if the total cost is given by

\[ TC = 20Q^4 - 30Q^2 + 300Q + 200? \]
Solution
The marginal cost function is the differentiated total cost function. Thus by differentiating the total cost function we can determine the marginal cost function. Now if the total cost function is

\[ TC = 20Q^4 - 30Q^2 + 300Q + 200 \]

then the marginal cost function is

\[ MC = \frac{dTC}{dQ} = 80Q^3 - 60Q + 300. \]

Now the value of the marginal cost function when \( Q \) is equal 10, is

\[ MC(10) = 80(10)^3 - 60(10) + 300 = 80000 - 600 + 300 = 79700. \]

Question 21
Evaluate the following:

\[ \int (x^2 + 2x + 3)\,dx \]

Solution
To integrate the function we make use of the basic rule of integration, namely

\[ \int x^n = \frac{x^{n+1}}{n+1} + c \text{ when } n \neq -1. \]

Therefore

\[ \int (x^2 + 2x + 3)\,dx = \int x^2\,dx + \int 2x\,dx + \int 3\,dx \]

\[ = \frac{x^{2+1}}{2+1} + \frac{2x^{1+1}}{1+1} + \frac{3x^{0+1}}{0+1} + c \]

\[ = \frac{x^3}{3} + \frac{2x^2}{2} + \frac{3x}{1} + c \]

\[ = \frac{x^3}{3} + x^2 + 3x + c \]
Question 22

Determine \( \int \frac{Q^{1/2} + 2}{\sqrt{Q}} \, dQ \)

Solution

First simplify the function to be integrated:

\[
\frac{Q + 1}{\sqrt{Q}} = \frac{(Q+1)}{Q^{1/2}} = (Q+1)Q^{1/2} = Q^{1/2} + Q^{1/2}
\]

Now we can integrate the function using the basic integration rule \( \int x^n = \frac{x^{n+1}}{n+1} + c \) when \( n \neq -1 \).

\[
\int (Q^{1/2} + Q^{1/2}) \, dQ = \int Q^{1/2} \, dQ + \int Q^{1/2} \, dQ
\]

\[
= \frac{Q^{1/2+1}}{1/2+1} + \frac{Q^{1/2+1}}{1/2+1} + c
\]

\[
= \frac{Q^{3/2}}{3} + \frac{Q^{3/2}}{3} + c
\]

\[
= \sqrt{Q}x^{2/3} + \sqrt{Q}x^{2/3} + c
\]

\[
= \frac{2\sqrt{Q}}{3} + 2\sqrt{Q} + c
\]
Question 23

Evaluate

\[ \int_{-1}^{1} (z + 1) \, dz \]

Solution

First simplify the function to be integrated:

\[ \int_{-1}^{1} (z + 1) \, dz = \int_{-1}^{1} z \, dz + \int_{-1}^{1} 1 \, dz \]

Now we can integrate the function, using the basic rule \( \int x^n = \frac{x^{n+1}}{n+1} + c \) when \( n \neq -1 \), and then substitute the values between which the integral has to be calculated, into the integrated function:

\[ \int_{-1}^{1} z \, dz + \int_{-1}^{1} 1 \, dz = \frac{1}{2} z^{1+1} \bigg|_{-1}^{1} + z \bigg|_{-1}^{1} \]

\[ = \left( \frac{z^2}{2} + z \right) \bigg|_{-1}^{1} \]

\[ = \left( \frac{1^2}{2} + 1 \right) - \left( \frac{(-1)^2}{2} + (-1) \right) \]

\[ = \frac{1}{2} + \left( \frac{1}{2} \right) - \left( \frac{1}{2} + 1 \right) \]

\[ = \frac{1}{2} - \left( \frac{1}{2} \right) \]

\[ = \frac{1}{2} - \frac{1}{2} \]

\[ = \frac{1}{2} + \frac{1}{2} \]

\[ = 2 \]
3. Sample examination paper and solution

Examination paper:
The examination paper consists of two sections: Section A and Section B

SECTION A
Answer ALL the questions in this section on the mark-reading sheet supplied. Carefully follow the instructions for completing the mark-reading sheet.
Also pay attention to the following information. Suppose you are asked the following question:

$$3 + 2 \times -1 + 4 \div 2 =$$


The correct answer is [3]. Only one option (indicated as [1] [2] [3] [4] [5]) per question is correct. If you mark more than one option, you will not receive any marks for the question. If your answer is correct, you will receive 3 MARKS. Marks WILL NOT be subtracted for incorrect answers.

Section A consists of 20 questions and counts 60 marks. Hand in the completed mark-reading sheet with your answers for Section B. DO NOT STAPLE IT!

SECTION B
This section must be completed in the spaces provided below each question. Section B counts 40 marks.
Remember to include your MARK-READING SHEET.
SECTION A

Question 1
Find the slope of the line \( 0 = 6 + 3x - 2y \).

\[ \frac{2}{3} \quad \frac{3}{2} \quad 3 \quad 2 \quad \text{None of the above} \]

Question 2
\( \log_{3} \left( \frac{3}{\sqrt{3}} \right) \), to four decimal places, equals

\[ \begin{array}{l}
\text{[1]} \quad -0.0795. \\
\text{[2]} \quad 0.0795. \\
\text{[3]} \quad 2.0000. \\
\text{[4]} \quad 0.5000. \\
\text{[5]} \quad \text{none of the above.} \\
\end{array} \]

Question 3
Solve the inequality \( x^2 - 3x \geq 6 - 2x \)

\[ \begin{array}{l}
\text{[1]} \quad -2 \leq x \leq 3 \\
\text{[2]} \quad -6 \leq x \leq 1 \\
\text{[3]} \quad x \leq -2; x \geq 3 \\
\text{[4]} \quad x \leq -3; x \geq 2 \\
\text{[5]} \quad -3 \leq x \leq 2 \\
\end{array} \]
Question 4
Find the equation of the straight line passing through the points (4;2) and (2;4).

[1] \( y = -1x + 6 \)
[2] \( y = -1x \)
[3] \( y = 2x + 4 \)
[4] \( y = 1x + 2 \)
[5] None of the above

Question 5
Find the value of quantity \( Q \) for the demand function \( P = 60 - 4Q \) when the market price is \( P = 24 \).

[1] 8
[2] 9
[3] 10
[4] 11
[5] 12

Question 6
Calculate the consumer surplus for the demand function \( P = 60 - 4Q \) when the market price is \( P = 16 \).

[1] 242
[2] 484
[3] 88
[4] 32
[5] 352
Question 7

If the demand function is \( P = 90 - 0.05Q \), where \( P \) and \( Q \) are the price and quantity respectively, determine the expression for price elasticity of demand in terms of \( P \).

\[ \begin{align*}
[1] & \quad \frac{P}{P-90} \\
[2] & \quad \frac{P-90}{P} \\
[3] & \quad \frac{P}{P-1800} \\
[4] & \quad \frac{P-1800}{P} \\
[5] & \quad \text{None of the above}
\end{align*} \]

Question 8

The supply and demand functions are given by

\[
\begin{align*}
P &= 50 - 3Q \quad \text{(supply function)} \\
P &= 14 + 1.5Q \quad \text{(demand function)}
\end{align*}
\]

where \( P \) and \( Q \) are the price and quantity respectively. Calculate the level of excess supply if price \( P = 20 \).

\[ \begin{align*}
[1] & \quad 10 \\
[2] & \quad 4 \\
[3] & \quad 14 \\
[4] & \quad 6 \\
[5] & \quad \text{None of the above}
\end{align*} \]
Question 9
What is the value of maximum revenue if total revenue is given by
\[ R(x) = -\frac{1}{5}x^2 + 30x + 81 \]
where \( x \) is the quantity?

[1] 75
[2] 1206
[3] 152.65
[4] 81
[5] None of the above

Question 10
Solve the following system of linear equations:

\[
\begin{align*}
x + y + z &= 8 \\
x - 3y &= 0 \\
5y - z &= 10
\end{align*}
\]

[1] \( x = 6; y = 2; z = 0 \)
[2] \( x = 0; y = 6; z = 2 \)
[3] \( x = 2; y = 0; z = 6 \)
[4] \( x = -6; y = 2; z = 6 \)
[5] None of the above
Question 11
Determine the roots of $4x^2 + 3x - 1$.

1. $x = \frac{1}{4}; x = -1$
2. $x = \frac{1}{4}; x = 1$
3. $x = -\frac{1}{4}; x = 1$
4. $x = -\frac{1}{4}; x = -1$
5. None of the above

Question 12
If $y = 2^{-x}$, find $x$ if $y = 0.0625$.

1. $x = -2$
2. $x = 3$
3. $x = 4$
4. $x = 5$
5. None of the above

Question 13
Evaluate the following definite integral:

$$\int_{-2}^{2} (x^2 - 3)dx$$

1. $6 \frac{2}{3}$
2. $-6 \frac{2}{3}$
3. $3 \frac{1}{3}$
4. $-3 \frac{1}{3}$
5. None of the above
Question 14
Evaluate
\[ \int x^2 \left(1 + \frac{1}{x^2}\right) \, dx \]

[1] \(x^2 + x + c\)
[2] \(\frac{1}{3} x^3 + x + c\)
[3] \(x^2 + 1\)
[4] \(\frac{1}{2} x^2 + x + c\)
[5] None of the above

Question 15
Simplify
\[ \frac{d}{dx} \left[ \frac{x-x^2}{\sqrt{x}} \right] \]

[1] \(\frac{2}{3} \sqrt{x} + \frac{1}{2\sqrt{x}}\)
[2] \(-\frac{1}{2\sqrt{x}} - \frac{3}{2} \sqrt{x}\)
[3] \(\frac{3}{2} \sqrt{x} - \frac{1}{2\sqrt{x}}\)
[4] \(\frac{3}{2} \sqrt{x} + \frac{1}{2\sqrt{x}}\)
[5] None of the above
Question 16
The demand function of a firm is $Q = 150 - 0.5P$, where $P$ and $Q$ represent the quantity and price respectively. At what value of $Q$ is marginal revenue equal to zero?

[1] 150
[2] 75
[3] 113
[4] 0
[5] None of the above

Question 17
Given the demand function $P = 60 - 0.2Q$. What is the arc price elasticity of demand when the price decreases from R50 to R40?

[1] $-\frac{1}{3}$
[2] $\frac{1}{3}$
[3] $-3$
[4] 3
[5] None of the above
Question 18
Consider the market defined by the following functions:

demand function: \[ P = 60 - 0.6Q \]

supply function: \[ P = 20 + 0.2Q \]

where \( P \) and \( Q \) are the price and quantity respectively. Calculate the equilibrium price and quantity.

[1] \( P = 300; Q = 20 \)
[2] \( P = 200; Q = 30 \)
[3] \( P = 20; Q = 300 \)
[4] \( P = 30; Q = 200 \)
[5] None of the above

Question 19
What is the point of intersection of the following lines?

\[ 2x + y - 5 = 0 \]
\[ 3x - 2y - 4 = 0 \]

[1] \( x = 3; y = 1 \)
[2] \( x = 1; y = 2 \)
[3] \( x = 2; y = 1 \)
[4] \( x = 1; y = 3 \)
[5] None of the above
Question 20

The graph of $y = -2x + x^2 - 3$ is represented by

[Diagram 1]
[Diagram 2]
[Diagram 3]
[Diagram 4]

[5] None of the above
SECTION B

Question 21

The monthly demand for a new line of computers, $t$ months after it has been introduced in the market, is given by

$$D(t) = 2000 - 1500e^{-0.05t} \text{ for } t > 0.$$ 

(a) Find demand two years after these computers were introduced.  
(b) Algebraically, determine the number of months after which demand will be 1 000 units.

Question 22

An electronics company manufactures radios and television sets. The time needed to manufacture a radio is 90 minutes and it takes 5 minutes to test the radio. The time needed to manufacture a television set is 150 minutes and it takes 15 minutes to test a television set. It costs R175 to make a radio and R850 to make a television set. The company has at most 95 hours of manufacturing time and at least 9 hours of testing time available. The production cost must not exceed R13 500.

Write down the inequalities that this production process must satisfy.

Question 23

ABC intends manufacturing and marketing a new product. It has been determined that the cost of producing the product, as a function of price, is given by

$$C(P) = 432\ 000 - 1800P.$$ 

The revenue generated when units are sold at price $P$ rand each is given by

$$R(P) = 6000P - 30P^2.$$ 

Plot the income and cost functions on the same graph. Indicate clearly on the graph, the break-even point(s) and profit area.
Question 24

(a) Graph the lines representing the following constraints:      (4)

1. $6x + 2y \leq 840$
2. $2x + y \leq 300$
3. $x + y \leq 250$
   \[ x, y \geq 0 \]

(b) Show the feasible region.                    (1)

(c) Determine the maximum value of $P = 120x + 95y$, subject to the constraints above.  (5)

[10]

Question 25

Let $f(x) = 3x^2 - x$. Find the equation of the line tangent to the graph $y = f(x)$ at $x = 1$.

[6]

[40]

Total: 100
Solutions

SECTION A

Question 1

In general the line \( y = mx + c \) has a slope of \( m \).

First we need to change the given function to the general format of a line, namely \( y = mx + c \). We need to change the equation so that \( y \) is the subject of the equation, i.e. writing it on its own on one side of the equation. Now we are given the line \( 0 = 6 + 3x - 2y \).

Therefore

\[
0 = 6 + 3x - 2y \\
2y = 6 + 3x \quad \text{add} \ 2y \ \text{on both sides} \\
y = \frac{6 + 3x}{2} \quad \text{divide both sides by} \ 2 \\
y = \frac{6}{2} + \frac{3x}{2} \\
y = 3 + \frac{3}{2}x
\]

The slope of the line \( 0 = 6 + 3x - 2y \) is equal to \( \frac{3}{2} \).

[Option 2]

Question 2

\[
\log_3 \left( \frac{3}{\sqrt{3}} \right) = \frac{\ln \left( \frac{3}{\sqrt{3}} \right)}{\ln 3} \\
= \frac{0.54931}{1.09861} \quad \text{using your calculator, rounded to 5 decimal places} \\
= 0.5000 \quad \text{rounded to 4 decimal places}
\]

[Option 4]
Question 3

First we write the inequality in the standard quadratic format \( y = ax^2 + bx + c \):

\[
x^2 - 3x \geq 6 - 2x \\
x^2 - 3x - 6 + 2x \geq 0 \\
x^2 - x - 6 \geq 0
\]

To determine the \( x \)-values for which the inequality holds, we use the quadratic formula to determine the solution or roots of the equation \( x^2 - x - 6 = 0 \) using the formula

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

with \( a, b, \) and \( c \) the values of the coefficients in the function \( ax^2 + bx + c = 0 \).

From the given function \( x^2 - x - 6 = 0 \), we can derive that \( a = 1, b = -1 \) and \( c = -6 \).

Substituting \( a, b \) and \( c \) into the formula gives

\[
x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-6)}}{2(1)} \\
x = \frac{+1 \pm \sqrt{1 + 24}}{2} \\
x = \frac{1 \pm \sqrt{25}}{2} \\
x = \frac{1 \pm 5}{2}
\]

\[
x = \frac{1 + 5}{2} \quad \text{or} \quad \frac{1 - 5}{2} \\
x = \frac{6}{2} \quad \text{or} \quad \frac{-4}{2} \\
x = 3 \quad \text{or} \quad -2.
\]
Graphing the solution of the equation $x^2 - x - 6 = 0$ on a number line:

Now to determine the area where the inequality $x^2 - x - 6 \geq 0$ is true, we substitute a point in the region smaller than $-2$, the region between $-2$ and $3$ and a value in the region greater than $3$, into the inequality. The inequality region is then the area in which the selected point makes the inequality true. You can choose any values in the different regions.

First we choose a value smaller than $-2$, for example $x = -3$. Now the left-hand side of the inequality $x^2 - x - 6 \geq 0$ is as follows:

\[
\text{LHS} = x^2 - x - 6 \\
= (-3)^2 - (-3) - 6 \\
= 9 + 3 - 6 \\
= 6
\]

The right-hand side: \( \text{RHS} = 0 \).

We need the \( \text{LHS} \geq \text{RHS} \) and the \( \text{LHS} \geq \text{RHS} \). The inequality $x^2 - x - 6 \geq 0$ is therefore true for values smaller than $-2$.

Next we choose a value between $-2$ and $3$ for example $x = 0$. Now the left-hand side of the inequality $x^2 - x - 6 \geq 0$ is

\[
\text{LHS} = x^2 - x - 6 \\
= (0)^2 - (0) - 6 \\
= -6.
\]

The right-hand side is \( \text{RHS} = 0 \).
We need the $\text{LHS} \geq \text{RHS}$ but the $\text{LHS} \leq \text{RHS}$. The inequality $x^2 - x - 6 \geq 0$ is therefore not true for values between $-2$ and 3.

Finally, we choose a value greater than 3, for example $x = 4$. Now the left-hand side of the inequality $x^2 - x - 6 \geq 0$ is

\[
\text{LHS} = x^2 - x - 6 = (4)^2 - (4) - 6 = 16 + 4 - 6 = 14
\]

The right-hand side is

\[
\text{RHS} = 0
\]

We need the $\text{LHS} \geq \text{RHS}$, and the $\text{LHS} \geq \text{RHS}$. The inequality $x^2 - x - 6 \geq 0$ is therefore true for values greater than 3.

The area of the inequality is therefore true if $x \leq -2$ and $x \geq 3$.

[Option 3]

**Question 4**

Let $(x_1 ; y_1) = (4 ; 2)$ and $(x_2 ; y_2) = (2 ; 4)$

We need to determine the slope $m$ and $y$-intercept $c$ of the line $y = mx + c$. Now the slope $m$ is

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{2 - 4} = \frac{2}{-2} = -1
\]

Therefore $y = -x + c$ or $y = -x + c$.

Now both $(x_1 ; y_1)$ and $(x_2 ; y_2)$ lie on the line. We can thus substitute any one of the points into the equation of the line to determine $c$. Let’s choose the point $(4 ; 2)$ then
\[ y = -1x + c \\
2 = -1 \times 4 + c \\
2 = -4 + c \\
2 + 4 = c \\
c = 6 \]

The equation of the line is \( y = -1x + 6 \).

**[Option 1]**

**Question 5**

We need to determine the value \( Q \) if \( P \) is equal to 24. Thus we substitute the value of \( P \) into the function and solve for \( Q \).

Therefore

\[
\begin{align*}
P &= 60 - 4Q \\
24 &= 60 - 4Q \\
4Q &= 60 - 24 \\
4Q &= 36 \\
Q &= 9
\end{align*}
\]

**[Option 2]**

**Question 6**

If you need to determine the demand surplus for a demand function of \( P = a - bQ \) then the consumer surplus can be calculated by calculating an area of the triangle \( P_0Q_0a \) which is equal to

\[
\frac{1}{2} \times \text{height} \times \text{base} = \frac{1}{2} \times (a - P_0) \times (Q_0 - 0) = \frac{1}{2} \times (a - P_0) \times (Q_0)
\]

with

- \( P_0 \) the value given as the market price,
- \( Q_0 \) the value of the demand function if \( P \) equals the given market price (Substitute \( P_0 \) into the demand function and calculate \( Q_0 \)), and
- \( a \) the \( y \)-intercept of the demand function \( P = a - bQ \), also known as the value of \( P \) if \( Q = 0 \), or the point where the demand function intercepts the \( y \)-axis.
In general we can summarise the steps of determining the consumer surplus as follows:

Method:
1. Calculate \( Q_0 \) if \( P_0 \) is given.
2. Draw a rough graph of the demand function.
3. Read the value of \( a \) from the demand function – the \( y \)-intercept of the demand function.
4. Calculate the area of \( CS = \frac{1}{2} \times (a - P_0) \times (Q_0) \).

First we calculate \( Q \) if \( P = 16 \). Therefore

\[
P = 60 - 4Q \\
16 = 60 - 4Q \\
4Q = 60 - 16 \\
4Q = 44 \\
Q = 11
\]

The consumer surplus is the area of the shaded triangle on the right:

\[
CS = \frac{1}{2} \times \text{base} \times \text{height} \\
= \frac{1}{2} \times 11 \times (60 - 16) \\
= \frac{1}{2} \times 11 \times 44 \\
= \frac{482}{2} \\
= 242
\]

[Option 1]
Question 7

The demand function is given as \( P = 90 - 0.05Q \). Now the price elasticity of demand is

\[
\varepsilon_d = -\frac{1}{b} \cdot \frac{P}{Q}
\]

with \( a \) and \( b \) the values of the standard demand function \( P = a - bQ \).

To determine the elasticity of demand, we need to determine the values of \( b, Q \) and \( P \). It is given that \( P = 90 - 0.05Q \) and question asked in terms of \( P \) thus \( P = P \). Comparing \( P = 90 - 0.05Q \) with \( P = a - bQ \) we can say that \( a = 90 \) and \( b = 0.05 \). Now \( a, b \) and \( P \) are known and \( Q \) is unknown at this stage.

The demand function denotes the relationship between the price \( P \) and the demand \( Q \). Therefore if \( P \) is given, we can derive \( Q \) by substituting \( P \) into the demand function and solving for \( Q \).

\( P = 90 - 0.05Q \). We need to change the equation so that \( Q \) is the subject of the equation.

That means we write \( Q \) in terms of \( P \) as asked. Now

\[
\begin{align*}
P &= 90 - 0.05Q \\
P - 90 &= -0.05Q \\
\frac{P - 90}{-0.05} &= Q \\
Q &= \frac{P - 90}{-0.05}.
\end{align*}
\]

As we have determined the values of \( b, P \) and \( Q \) we can now substitute them into the formula for the elasticity of demand:

\[
\varepsilon_d = -\frac{1}{0.05} \times \frac{P}{P - 90} \times \frac{P - 90}{P - 90} \times -0.05 \times 1
\]

\[
\varepsilon_d = \frac{P}{P - 90}
\]
Or alternatively,

you can use the given formula of elasticity of demand in terms of $P$ as given in the textbook 
on page 78, equation 2.14 (Edition 2) and page 89, equation 2.14 (Edition 3).

$$\varepsilon_d = \frac{P}{P-a}$$

Now $a = 90$ (intercept on the $y$-axis of the demand function)

$$\varepsilon_d = \frac{P}{P-90}$$

[Option 1]

**Question 8**

We need to determine the difference between the quantity supplied and the quantity 
demanded if the price is equal to 20. First we determine what the quantity supplied is if the 
price is 20. Thus we substitute the value $P = 20$ into the supply function and solve for the 
quantity supplied. Therefore

$$P = 50 - 3Q$$
$$20 = 50 - 3Q$$
$$3Q = 50 - 20$$
$$3Q = 30$$
$$Q = \frac{30}{3}$$
$$Q = 10$$
Now we determine what the quantity demanded is if the price is 20. Thus we substitute the value \( P = 20 \) into the demand function and solve for \( Q \). Therefore

\[
P = 14 + 1.5Q \\
20 = 14 + 1.5Q \\
-1.5Q = 14 - 20 \\
-1.5Q = -6 \\
Q = \frac{-6}{-1.5} \\
Q = 4
\]

The supplier supplied 10 units and only 4 units were demanded. Thus the supplier supplied \( 10 - 4 = 6 \) units more than were demanded.

[Option 4]

**Question 9**

The given total revenue function is a quadratic function. The minimum or maximum value of a quadratic function is where the quadratic function changes direction or turns, also called the vertex. The vertex is given by the coordinate pair \((x; y)\). There exists different methods to determine the maximum or minimum value of a quadratic function.

**Method 1:**

The \( x \)-coordinate of the vertex can be calculated by using the formula

\[
x = \frac{-b}{2a}
\]

with \( a, b, \) and \( c \) the coefficients in the standard quadratic function \( y = ax^2 + bx + c \).

It is given that \( R(x) = -\frac{1}{5}x^2 + 30x + 81 \). Comparing it with the standard form of the quadratic function \( y = ax^2 + bx + c \), we can see that \( a = -\frac{1}{5}, b = 30 \) and \( c = 81 \) for the given function. As the \( a \)-value is negative we can say that the graph of the function is in the form of a sad
face, thus a maximum extreme point exists for the function. Therefore the $x$-value of the extreme point or vertex is:

\[
x = \frac{-(30)}{2 \times -\frac{1}{5}} = \frac{-30}{\frac{2}{5}} = -\frac{30 \times 5}{2} = 150 \times \frac{2}{2} = 75
\]

To calculate the $y$-value of the vertex we substitute the calculated $x$-value into the given equation $R(x) = -\frac{1}{5}x^2 + 30x + 81$ and calculate $R(x)$. Therefore

\[
R(x) = -\frac{1}{5}x^2 + 30x + 81 = -\frac{1}{5}(75)^2 + 30(75) + 81 = 1206
\]

The function has a maximum value in the point $(75; 1206)$ or where the revenue is equal to 1206.

**Method 2:**

You can also make use of the method of differentiation, as discussed in Chapter 6, to determine the maximum or minimum value or vertex of a function.

The maximum or minimum value of a function is the point where the differentiated function is equal to zero or $\frac{dy}{dx} = 0$. The function $R(x) = -\frac{1}{5}x^2 + 30x + 81$ was given.

Differentiating the function $R(x)$, using the basic rule of differentiation namely $\frac{d}{dx}x^n = nx^{n-1}$ with $n \neq 0$, we get
\[
R(x) = -\frac{1}{5} x^2 + 30x + 81
\]

\[
\frac{d}{dx} R(x) = -\frac{1}{5} (2) x^{2-1} + 30 x^{1-1} + 0 \quad \text{because} \quad \frac{d}{dx} a = 0 \text{ if } a \text{ is a constant}
\]

\[
\frac{d}{dx} R(x) = -\frac{2}{5} x^1 + 30 x^0 \quad \text{but } x^0 = 1
\]

\[
\frac{d}{dx} R(x) = -\frac{2}{5} x + 30
\]

But the maximum or minimum occurs when \(\frac{dy}{dx} = 0\). Therefore

\[
\frac{d}{dx} R(x) = 0
\]

\[
-\frac{2}{5} x + 30 = 0
\]

\[
-\frac{2}{5} x = -30
\]

\[
x = -30 \times -\frac{5}{2}
\]

\[
x = \frac{150}{2} = 75
\]

To calculate the \(y\)-value of the extreme point or vertex we substitute the calculated \(x\)-value into the given equation \(R(x) = -\frac{1}{5} x^2 + 30x + 81\). Therefore

\[
R(x) = -\frac{1}{5} x^2 + 30x + 81
\]

\[
= -\frac{1}{5} (75)^2 + 30(75) + 81
\]

\[
= 1 206.
\]

The function has a maximum value in the point \((75; 1 206)\), that is where the revenue is equal to 1 206.
Method 3:
You can determine the minimum or maximum value of a quadratic function by using the symmetry of the quadratic function. Because the quadratic function is symmetrical, the vertex \((x; y)\) occurs halfway between the two roots of the quadratic function. Therefore you can determine the \(x\)-value of the vertex by calculate the roots and then determine the halfway mark \((\text{root 1 + root 2})/2\) and then substitute the \(x\)-value of the vertex into the quadratic function to determine the \(y\)-value of the vertex.

[Option 2]

Question 10

We need to solve the following system of equations:

\[
\begin{align*}
8 &= x + y + z \quad (1) \\
3 &= x - 3y \quad (2) \\
5 &= 5y - z \quad (3)
\end{align*}
\]

Make \(x\) the subject of equation (2) and \(z\) the subject of equation (3):

\[
\begin{align*}
x &= 3y \\ z &= -10 + 5y
\end{align*}
\]

Substitute equation (4) and equation (5) into equation (1):

\[
\begin{align*}
x + y + z &= 8 \\
(3y) + y + (-10 + 5y) &= 8 \\
9y &= 8 + 10 \\
9y &= 18 \\
y &= \frac{18}{9} \\
y &= 2
\end{align*}
\]

Substitute \(y = 2\) into equation (4) and equation (5):

\[
\begin{align*}
x &= 3y = 3 \times 2 = 6
\end{align*}
\]

and

\[
\begin{align*}
z &= -10 + 5y = -10 + 5(2) = -10 + 10 = 0.
\end{align*}
\]

Therefore \(x = 6; y = 2\) and \(z = 0\).  

[Option 1]
Question 11
The roots or solutions of a function can be found where the function, if drawn, intersects the x-axis. We therefore need to determine the value of \( x \) at the point(s) where the graph of the function intersects the x-axis, in other words where the function value is zero:

\[
y = 0 \quad \text{or} \quad 4x^2 + 3x - 1 = 0.
\]

We make use of the quadratic formula

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

with \( a, b, \) and \( c \) the values of the coefficients in the equation \( 0 = ax^2 + bx + c \) to determine the roots.

Comparing the given equation \( 4x^2 + 3x - 1 = 0 \) with the general format of \( 0 = ax^2 + bx + c \), we derive that \( a = 4; \ b = 3 \) and \( c = -1 \). Substituting \( a, b \) and \( c \) into the formula gives

\[
x = \frac{-3 \pm \sqrt{(3)^2 - 4(4)(-1)}}{2(4)}
\]

\[
x = \frac{-3 \pm \sqrt{9 + 16}}{8}
\]

\[
x = \frac{-3 \pm \sqrt{25}}{8}
\]

\[
x = \frac{-3 \pm 5}{8}
\]

\[
x = \frac{-3 + 5}{8} \quad \text{or} \quad \frac{-3 - 5}{8}
\]

\[
x = \frac{2}{8} \quad \text{or} \quad \frac{-8}{8}
\]

\[
x = \frac{1}{4} \quad \text{or} \quad -1.
\]

The roots of the function \( 4x^2 + 3x - 1 = 0 \) are \(-1\) and \( \frac{1}{4} \).  

[Option 1]
Question 12

To solve for $x$ if $y = 2^{-x}$ and $y = 0.0625$ we substitute $y = 0.0625$ into the equation and solve for $x$. Therefore

$$0.0625 = 2^{-x}$$

Taking log or ln on both sides of the equation yields

$$\log 0.0625 = \log (2^{-x}) \quad \text{or} \quad \ln 0.0625 = \ln (2^{-x})$$

Apply rule 3 of logarithms ( $\ln a^b = b \ln a$):

$$\frac{\log 0.0625}{\log 2} = -x \quad \text{or} \quad \frac{\ln 0.0625}{\ln 2} = -x$$

$$-4.00 = -x \quad \text{or} \quad -4.00 = -x$$

$$x = 4.00 \quad \text{or} \quad x = 4.00 \quad \text{rounded to two decimals}$$

The value of $x$, if $y = 0.0625$, is equal to 4.00 (rounded to 2 decimal places). Actually it happens to be exactly 4 but one cannot say that for sure if a calculator was used.

[Option 3]
Question 13

To determine the definite integral we first use the basic rule, namely
\[ \int x^n = \frac{x^{n+1}}{n+1} + c \] when \( n \neq -1 \), to integrate the function and secondly we substitute the given values between which we must integrate (2 and –2 in this case) into the integrated function.

Therefore
\[
\int_{-2}^{2} (x^2 - 3) \, dx = \int_{-2}^{2} x^2 \, dx - \int_{-2}^{2} 3 \, dx
\]
\[
= \left[ \frac{x^3}{3} \right]_{-2}^{2} - 3 \left[ \frac{x}{1} \right]_{-2}^{2}
\]
\[
= \left[ \frac{2^3}{3} - \frac{(-2)^3}{3} \right] - [3(2) - 3(-2)]
\]
\[
= \left[ \frac{8}{3} + \frac{8}{3} \right] - [6 + 6]
\]
\[
= \frac{16}{3} - 12
\]
\[
= \frac{16 - 36}{3}
\]
\[
= -\frac{20}{3}
\]
\[
= -\frac{62}{3}
\]

[Option 2]

Question 14

First we simplify the expression before we integrate. Therefore
\[
x^2 \left( 1 + \frac{1}{x^2} \right) = x^2 + \frac{x^2}{x^2}
\]
\[
= x^2 + 1
\]

To determine the integral we use the basic rule, namely
\[ \int x^n = \frac{x^{n+1}}{n+1} + c \] when \( n \neq -1 \), to integrate the function
\[ \int x^2 \left(1 + \frac{1}{x^2}\right) \, dx = \int (x^2 + 1) \, dx \]

\[ = \frac{x^{2+1}}{2+1} + x + c \]

\[ = \frac{x^3}{3} + x + c \]

[Option 2]

**Question 15**

First we simplify the expression. We can write \( \sqrt{x} \) as \( x^{\frac{1}{2}} \) when changing from surd (square root) form to exponential form. Therefore

\[ \frac{x - x^2}{\sqrt{x}} = \frac{x - x^2}{x^{\frac{1}{2}}} \]

\[ = \frac{x^{1-\frac{1}{2}} - x^{2-\frac{1}{2}}}{x^{\frac{1}{2}}} \]

\[ = x^{\frac{1}{2}} - x^{\frac{3}{2}} \]

\[ \frac{a^b}{a^c} = a^{b-c} \]

Next we differentiate the simplified expression using the basic rule of differentiation, namely \( \frac{d}{dx} x^n = nx^{n-1} \) when \( n \neq 0 \):
Revenue is price × demand or \( R = P \times Q \).

It is given that price is \( P \) and demand is \( Q = 150 - 0.5P \).

Therefore substitute \( Q = 150 - 0.5P \) into the formula for \( R \):

\[
R = P \times (150 - 0.5P)
\]

\[
R = 150P - 0.5P^2
\]

To determine the marginal revenue we need to differentiate the ordinary revenue function. Thus the marginal revenue function \((MR)\) is

\[
MR = \frac{dR}{dP} = \frac{d}{dP}(150P - 0.5P^2)
\]

\[
MR = 150 - (2 \times 0.5)P
\]

\[
MR = 150 - P.
\]
You are asked at what value of \( Q \) is \( MR \) equal to 0. But \( MR \) contains only \( P \). We first need to solve for \( P \) and then substitute its value into the demand function to solve for \( Q \). (Remember the demand function denotes the relationship between \( P \) and \( Q \).) Therefore if \( MR = 0 \)

Then

\[
0 = 150 - P
\]

\[
P = 150.
\]

But we need to determine \( Q \). Now \( P \) and \( Q \) are related as \( Q = 150 - 0,5P \). Therefore

\[
Q = 150 - 0,5 \times (150)
\]

\[
Q = 150 - 75
\]

\[
Q = 75.
\]

The value of \( Q \) is equal to 75 if the marginal revenue is equal to 0.

[Option 2]

**Question 17**

The arc price elasticity of a demand function \( P = a - bQ \) between two prices \( P_1 \) and \( P_2 \) is

\[
\text{arc elasticity of demand} = \frac{1}{b} \times \frac{P_1 + P_2}{Q_1 + Q_2}
\]

with \( b \) the slope of the demand function and \( P_1, P_2 \) and \( Q_1, Q_2 \) the price and quantity demanded.

For the given function \( P = 60 - 0,2Q \), we see that \( a = 60 \) and \( b = 0,2 \) and it is given that \( P_1 = 50 \) and \( P_2 = 40 \). We only need to determine \( Q_1 \) and \( Q_2 \). We can rewrite the equation \( P = 60 - 0,2Q \), by making \( Q \) the subject of the equation, as
\[ P = 60 - 0.2Q \]
\[ 0.2Q = 60 - P \]
\[ Q = 300 - 5P. \]

We can now determine \( Q_1 \) and \( Q_2 \) by substituting \( P_1 = 50 \) and \( P_2 = 40 \) into the equation:

If \( P_1 = 50 \) then \( Q_1 = 300 - 5 \times 50 = 50 \)
and if \( P_2 = 40 \) then \( Q_2 = 300 - 5 \times 40 = 100 \).

Therefore

\[ \text{arc elasticity of demand} = -\frac{1}{b} \times \frac{P_1 + P_2}{Q_1 + Q_2} \]
\[ = -\frac{1}{0.2} \times \frac{50 + 40}{50 + 100} \]
\[ = -\frac{1}{0.2} \times \frac{90}{150} \]
\[ = -\frac{90}{30} \]
\[ = -3. \]

[Option 3]

**Question 18**

Equilibrium is the price and quantity where the demand and supply functions are equal. Thus

\( P_d = P_s \) or

\[ 60 - 0.6Q = 20 + 0.2Q \]
\[ -0.6Q - 0.2Q = 20 - 60 \]
\[ -0.8Q = -40 \]
\[ Q = \frac{-40}{-0.8} \]
\[ Q = 50 \]
To calculate the quantity at equilibrium, we substitute the value of $Q$ into the demand or supply function and calculate $P$. Say we use the demand function, then

\[ P = 60 - 0,6Q \]
\[ P = 60 - 0,6(50) \]
\[ P = 60 - 30 \]
\[ P = 30 \]

The equilibrium price is equal to 50 and the quantity to 30.

[Option 5]

**Question 19**

To determine the point of intersection of two lines we need to determine a point $(x ; y)$ so that the $x$ and $y$ values satisfy both equations of the lines. Thus we need to solve the two equations simultaneously.

Let \( 2x + y - 5 = 0 \) or \( 2x + y = 5 \) \( (1) \)
and \( 3x - 2y - 4 = 0 \) or \( 3x - 2y = 4 \) \( (2) \)

\[ 2 \times \text{equation (1)}: \quad 4x + 2y = 10 \quad (3) \]

Equation (2) + equation (3):
\[ 3x - 2y = 4 \]
\[ + (4x + 2y = 10) \]
\[ 7x = 14 \]

Now solve for $x$:
\[ x = \frac{14}{7} \]
\[ x = 2 \]

Substitute $x = 2$ into equation (1) or equation (2) and solve for $y$. If we, for example, substitute $x = 2$ into equation (1) we get
\[ 2(2) + y = 5 \]
\[ 4 + y = 5 \]
\[ y = 5 - 4 \]
\[ y = 1 \]
The two lines intersect in the point \((x ; y) = (2 ; 1)\).  

**Question 20**

Given the function \( y = -2x + x^2 - 3 \) or written in the format \( y = ax^2 + bx + c \), the function \( y = x^2 - 2x - 3 \) with \( a = 1 \), \( b = -2 \) and \( c = -3 \). To graph the function we need to determine the vertex, roots and \( y \)-intercept of the function:

The function has the shape of a “smiling face” as \( a > 0 \).

The \( y \)-intercept is the value \( c \) or the value \(-3\).

The vertex is the point

\[
\begin{align*}
    x &= \frac{-b}{2a} = \frac{-(-2)}{2(1)} = \frac{2}{2} = 1 \\
    y &= (1^2 - 2(1) - 3) = (1 - 2 - 3) = -4.
\end{align*}
\]

The roots, \( a = 1 \), \( b = -2 \) and \( c = 3 \), enable us to determine the value of the quadratic formula. Therefore

\[
\begin{align*}
    x &= \frac{2 \pm \sqrt{4 + 12}}{2} \\
    x &= \frac{2 \pm \sqrt{16}}{2} \\
    x &= \frac{2 \pm 4}{2} \\
    x &= \frac{2 + 4}{2} \quad \text{or} \quad \frac{2 - 4}{2} \\
    x &= \frac{6}{2} \quad \text{or} \quad \frac{-2}{2} \\
    x &= 3 \quad \text{or} \quad -1.
\end{align*}
\]
Hence the graph of the function $y = x^2 - 2x - 3$ is

\[ y \]

\[ x \]

\[ y = x^2 - 2x - 3 \]

---

**SECTION B**

**Question 21**

(a) We need to find the demand $D(t)$ after 2 years. But $t$ is defined in months. We therefore need to change 2 years to months. Now 2 years is equal to $2 \times 12 = 24$ months. If $t = 2$ years = 24 months then

\[
D = 2000 - 1500e^{-0.05 \times 24}
\]

\[
D = 2000 - 1500e^{-0.05 \times 24}
\]

\[
= 1548,208682 \quad \text{using your calculator, rounded to 6 decimal places}
\]

\[
= 1548 \quad \text{computers. rounded to an integer}
\]

(b) We need to determine the value of $t$ in months $D(t) = 1000$. If $D = 1000$, then
\[ 1000 = 2000 - 1500e^{-0.05t} \]
\[ 1500e^{-0.05t} = 2000 - 1000 = 1000 \]
\[ e^{-0.05t} = \frac{1000}{1500} = \frac{2}{3} \]

divide by 1 500

\[ \ln\left(e^{-0.05t}\right) = \ln\left(\frac{2}{3}\right) \]

take ln on both sides

\[ -0.05t \ln e = \ln\left(\frac{2}{3}\right) \]

since \( \ln e = 1 \)

\[ t = \frac{\ln\left(\frac{2}{3}\right)}{-0.05} \]

\[ t = 8,109302162 \]

using your calculator, 9 decimal places

\[ t = 8,11 \text{ months} \]

rounded to 2 decimal places

The demand will be equal to 100 units after 8,11 months (rounded to 2 decimal places).

**Question 22**

First we define the variables. Let \( x \) be the number of radios manufactured per week and \( y \) the number of television sets manufactured per week. To help us with the formulation we summarise the information given in a table with the headings: resources (items with restrictions), the variables (\( x \) and \( y \)) and capacity (amount or number of the resources available).

<table>
<thead>
<tr>
<th>Resources</th>
<th>Radio</th>
<th>Television</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manufacturing time (min)</td>
<td>90</td>
<td>150</td>
<td>( 95 \times 60 = 540 ) minutes</td>
</tr>
<tr>
<td>Testing time (min)</td>
<td>5</td>
<td>15</td>
<td>( 9 \times 60 = 540 ) minutes</td>
</tr>
<tr>
<td>Production cost</td>
<td>175</td>
<td>850</td>
<td>13 500</td>
</tr>
</tbody>
</table>

Using the table, we can formulate our linear program as follows:

Let \( x = \) number of radios

\( y = \) number of television sets
then

\[ 90x + 150y \leq 5700 \] \hspace{1cm} \text{manufacturing time}

\[ 5x + 15y \geq 540 \] \hspace{1cm} \text{testing time}

\[ 175x + 850y \leq 13500 \] \hspace{1cm} \text{production cost}

\[ x, y \geq 0 \] \hspace{1cm} \text{non-negative}

**Question 23**

The cost function is a linear function. We need two points to draw the line of the cost function. Choose the two points where the lines cut the x-axis (x-axis intercept, thus \( y = 0 \)) and y-axis (y-axis intercept, \( x = 0 \)). Calculate \( (0 ; y) \) and \( (x ; 0) \) and draw a line through the two points.

Therefore if \( P = 0 \) then

\[
C(P) = 432000 - 1800(0)
\]

\[
= 432000.
\]

This gives the point \((0 ; 432 000)\).

If \( C(P) = 0 \) then

\[
0 = 432000 - 1800(P)
\]

\[
1800P = 432000
\]

\[
P = \frac{432000}{1800}
\]

\[
P = 240.
\]

This gives the point \((240 ; 0)\).

The revenue function is a quadratic function. We need to determine the vertex, roots and y-intercept of the function to draw it. It is given that the function \( R(P) = 6000P - 30P^2 \) with the coefficients equal to \( a = -30 \), \( b = 6000 \) and \( c = 0 \).

The function has the shape of a “sad face” as \( a < 0 \).

The y-intercept is the value \( c \) which is 0.
The vertex is the point

\[ x = \frac{-b}{2a} = \frac{-(6000)}{2(-30)} = 100 \]

\[ y = 6000(100) - (30)(100)^2 = 300000. \]

The roots, \( a = -30, \ b = 6\ 000 \) and \( c = 0 \), enable us to determine the value of the quadratic formula. Therefore

\[ x = \frac{-(6000) \pm \sqrt{(6000)^2 - 4(0)(-30)}}{2(-30)} \]

\[ x = \frac{-6000 \pm \sqrt{(6000)^2}}{-60} \]

\[ x = \frac{-6000 \pm 6000}{-60} \]

\[ x = \frac{-6000 + 6000}{-60} \quad \text{or} \quad \frac{-6000 - 6000}{-60} \]

\[ x = 0 \quad \text{or} \quad 12000 \]

\[ x = 0 \quad \text{or} \quad -200. \]

The graphs of the two functions are shown in the figure below:
The break-even points are the points where the cost function is equal to the revenue function. These are the points A and B. Profit is revenue minus cost, therefore the profit area is where the revenue is greater than the cost (the revenue function lies above the cost function) and the loss is where the cost is greater than the revenue (the cost function lies above the revenue function).

**Question 24**

**Step 1:**
To graph a linear inequality, we first change the inequality sign (≥ or ≤ or > or <) to an equal sign (=) and draw the graph of the line. But we need two points to draw a line. Choose the two points where the lines cut the x-axis (x-axis intercept, thus \( y = 0 \)) and y-axis (y-axis intercept, \( x = 0 \)). Calculate \((0 ; y)\) and \((x ; 0)\) and draw a line through the two points. See the table below for the calculations.

**Step 2:**
Determine the feasible region for each inequality by substituting a point on either side of this line into the equation of the inequality. The inequality region is the area in which the selected point makes the inequality true. The calculations are given below:

<table>
<thead>
<tr>
<th>Inequality</th>
<th>( y )-axis intercept</th>
<th>( x )-axis intercept</th>
<th>Inequality region</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 6x + 2y \leq 840 )</td>
<td>( 6x + 2y = 840 )</td>
<td>( 6x + 2(0) = 840 )</td>
<td>Select point ( (0;0) ) below the line ( 6(0) + 2(0) \leq 840 ) – True</td>
</tr>
<tr>
<td></td>
<td>( 6(0) + 2y = 840 )</td>
<td>( x = 140 )</td>
<td>Area below the line</td>
</tr>
<tr>
<td></td>
<td>( y = 420 )</td>
<td>Point ((140; 0))</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Point ((0; 420))</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( 2x + y \leq 300 )</td>
<td>( 2x + y = 300 )</td>
<td>( 2x + (0) = 300 )</td>
<td>Select the point ( (0;0) ) to the left of line ( 2(0) + (0) \leq 300 ) – True</td>
</tr>
<tr>
<td></td>
<td>( 2(0) + y = 300 )</td>
<td>( x = 300 )</td>
<td>Area to the left of the line</td>
</tr>
<tr>
<td></td>
<td>( y = 300 )</td>
<td>Point ((300; 0))</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Point ((0; 300))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
\begin{align*}
  x + y &\leq 250 \\
  0 + y &\leq 250 \\
  y &\leq 250 \\
  \text{Point (0 ; 250)} \\
  x + 0 &\leq 250 \\
  x &\leq 250 \\
  \text{Point (250 ; 0)} \\
  x, y &\geq 0
\end{align*}

Select the point (0;0) below the line

\[0 + 0 \leq 250 \quad \text{– True}\]

Area below the line

\[x, y \geq 0\]

Area above the x-axis and to the right of the y-axis

\textbf{Step 3:}

The feasible region is the area where all the inequalities are true simultaneously.

\begin{figure}[h]
  \centering
  \includegraphics[width=\textwidth]{feasible_region.png}
  \caption{Feasible region}
\end{figure}

\textbf{Step 4:}

Determine all the corner points of the feasible region and substitute them into the objective function (function you want to maximise or minimise) and determine the maximum value.

\textbf{Step 1:} Determine the coordinates of all the corners of the feasible region by solving two equations with two unknowns (substitution) or read the values from the graph.

\textbf{Step 2:} Substitute the corner points into the objective function.

\textbf{Step 3:} Choose the corner point that results in the highest (maximisation) or lowest (minimisation) objective function value.
<table>
<thead>
<tr>
<th>Corner points of feasible region</th>
<th>Value of $P = 120x + 95y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: $x = 0; y = 250$</td>
<td>$P = 120(0) + 95(250) = 23,750$</td>
</tr>
<tr>
<td>B: $x = 50; y = 195$</td>
<td>$P = 120(50) + 95(195) = 24,525$ (maximum)</td>
</tr>
<tr>
<td>C: $x = 120; y = 60$</td>
<td>$P = 120(120) + 95(60) = 20,100$</td>
</tr>
<tr>
<td>D: $x = 140; y = 0$</td>
<td>$P = 120(140) + 95(0) = 16,800$</td>
</tr>
<tr>
<td>Origin: $x = 0; y = 0$</td>
<td>$P = 120(0) + 95(0) = 0$</td>
</tr>
</tbody>
</table>

The maximum of $P$ is at the point B where $x = 50, y = 195$ and $P = 24,525$.

**Question 25**

We need to determine the tangent line $y = mx + c$ to the graph of $f(x)$, where $x = 1$.

If $x = 1$ then the function value $f(x) = f(1) = 3(1)^2 - 1 = 2$. We therefore need to determine the tangent line at the point $(x; y) = (1; 2)$.

The slope $m$ of the tangent line can be determined by differentiation and is equal to

$$
\frac{df}{dx} = \frac{d(3x^2 - x)}{dx} = 6x - 1.
$$

The value of the slope in the point where $x = 1$, is therefore $m = 6(1) - 1 = 6(1) - 1 = 5$. Therefore the tangent line is equal to $y = 6x + c$.

As the tangent line runs through the point $(1; 2)$ we can use it to determine the value of $c$, namely

$$
y = 5x + c
$$

$$
2 = 5(1) + c
$$

$$
2 - 5 = c
$$

$$
c = -3.
$$

The equation of the tangent line is $y = 5x - 3$. 