Tutorial letter 202/2/2012

QUANTITATIVE MODELLING

DSC1520

Semester 2

Department Decision Sciences

Assignment 2: Solutions
Dear student

I hope at this stage you should have worked through almost two thirds of the work prescribed for this module and completed your second assignment. As the assignments contain questions from old examination papers you are already in a way preparing for the examination. Practice makes perfect! Continue doing as many examples as possible. The more examples you work through the more you will be able to recognise a problem and know how to solve it.

Remember help is just a phone call or e-mail away. Please contact me if you need any help with the second assignment. My contact details and contact hours are as follows:

08:00 to 13:30 (Mondays to Fridays) (appointments and telephone)
13:30 to 16:00 (Mondays to Thursdays) (telephone only)

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Lastly, I wish you everything of the best with your preparation for the third assignment.

Victoria Mabe-Madisa
ASSIGNMENT 2: SOLUTIONS

Question 1

To determine the point of intersection of two lines we need to determine a point \((x;y)\) so that the \(x\) and \(y\) values satisfy the equations of both lines. Thus we need to solve the two equations simultaneously. There are different methods you can use to solve the set of equations.

(a) **Elimination method:**

**Step 1:** Eliminate one variable, say \(x\), by adding or subtracting one equation or multiple of an equation from another equation:

Let \(x + 2y = 5\) \hspace{1cm} (1)

and \(2x - 3y = -4\) \hspace{1cm} (2)

Now equation (2) minus 2 times equation (1) will eliminate \(x\). But \(2 \times \text{equation (1)}\) is:

\[
2x + 4y = 10 \hspace{1cm} (3)
\]

Now equation (2) minus \(2 \times \text{equation (1)}\) or equation (2) – equation (3) is

\[
\begin{align*}
2x - 3y &= -4 \\
-2x - 4y &= 10 \\
-7y &= -14
\end{align*}
\]

Now solve for \(y\):

\[
y = \frac{-14}{-7} = 2
\]

**Step 2:** Solve for \(x\). Substitute the value of \(y\) into any one of the equations and solve for \(x\). Substitute the value of \(y = 2\) into, say, equation (1):

\[
x + 2(2) = 5 \\
x + 4 = 5 \\
x = 5 - 4 \\
x = 1
\]

The two lines intersect in the point \((x ; y) = (1 ; 2)\).
(b) **Substitution method:**

**Step 1:** Change one of the equations so that a variable is the subject of the equation. 
Say \(x\) in equation (1):

\[
\text{Let } x + 2y = 5 \quad \text{(1)} \\
\text{then } x = -2y + 5 \quad \text{(3)}
\]

**Step 2:** Substitute the value of \(x\) (equation (3)) into the unchanged equation (2) and solve for \(x\). Substitute \(x = -2y + 5\) into \(2x - 3y = -4\):

\[
2x - 3y = -4 \\
2(-2y + 5) - 3y = -4 \\
-4y + 10 - 3y = -4 \\
-7y = -4 - 10 \\
-7y = -14 \\
y = \frac{-14}{-7} \\
y = 2
\]

**Step 3:** Substitute the calculated value of the variable in step 2 into any equation and calculate the value of the other variable. Substitute \(y = 2\) into equation (1) or equation (2). Let’s say we choose equation (1):

\[
x + 2(2) = 5 \\
x + 4 = 5 \\
x = 5 - 4 \\
x = 1
\]

The two lines intersect in the point \((x ; y) = (1 ; 2)\).

**Question 2**

We need to solve the following system of equations:

\[
x - 2y + 3z = -11 \quad \text{(1)} \\
2x - z = 8 \quad \text{(2)} \\
3y + z = 10 \quad \text{(3)}
\]

**Step 1:** Determine two equations with the **same** two unknowns (variables) by adding or subtracting two of the three equations at a time.
Now equations (2) and (3) are already equations in two variables. But the variables in equation (2) are \( x \) and \( z \), and in equation (3) they are \( y \) and \( z \). The equations have to have the same variables. To determine another equation with two variables we can subtract equation (2) from 2 times equation (1):

Now 2 times equation (1) is \( 2x - 4y + 6z = -22 \).

Two times equation (1) minus equation (2):

\[
\begin{align*}
2x - 4y + 6z &= -22 \\
-2x + z &= 8 \\
\hline
-4y + 7z &= -30
\end{align*}
\]

or

\[
\begin{align*}
2x - 4y + 6z &= -22 \\
-4y + 7z &= -30
\end{align*}
\]

Thus equation (3): \( 3y + z = 10 \) and equation (4): \( -4y + 7z = -30 \) are two equations with the same two variables namely \( y \) and \( z \).

Step 2: Next we solve two equations with the same two unknowns, using any method as described in question (3). Say we use the substitution method:

Make \( z \) the subject of equation (3) and substitute into equation (4) and solve for \( y \):

Now \( z = 10 - 3y \):

Substitute the value of \( z \), namely \( z = 10 - 3y \), into equation (4) and solve for \( y \):

\[
\begin{align*}
-4y + 7z &= -30 \\
-4y + 7(10 - 3y) &= -30 \\
-4y + 70 - 21y &= -30 \\
-25y &= -30 - 70 \\
-25y &= -100 \\
y &= -100 / -25 \\
y &= 4
\end{align*}
\]

Step 3: Substitute \( y = 4 \) into equation (3) and solve for \( z \):

\[
\begin{align*}
3y + z &= 10 \\
3(4) + z &= 10 \\
12 + z &= 10 \\
z &= 10 - 12 \\
z &= -2
\end{align*}
\]
Step 4: Substitute \( y = 4 \) and \( z = -2 \) into equation (1) or (2) and solve for \( x \). Say we use equation (2):

\[
2x - z = 8 \\
2x - (-2) = 8 \\
2x + 2 = 8 \\
2x = 8 - 2 \\
2x = 6 \\
x = 6 / 2 \\
x = 3
\]

Therefore \( x = 3 \), \( y = 4 \) and \( z = -2 \).

Or alternatively,

\[
x - 2y + 3z = -11 \quad \text{(1)} \\
2x - z = 8 \quad \text{(2)} \\
3y + z = 10 \quad \text{(3)}
\]

Make \( z \) the subject of equation (2) and \( y \) the subject of equation (3):

\[
z = 2x - 8 \quad \text{(4)} \\
y = \frac{10 - z}{3} \quad \text{(5)}
\]

Substitute equation (4) into equation (5):

\[
y = \frac{10 - (2x - 8)}{3} = \frac{18 - 2x}{3} \quad \text{(6)}
\]

Substitute equation (2) and equation (6) into equation (1):

\[
x - \frac{2}{3}(18 - 2x) + 3(2x - 8) = -11 \\
x - 12 + \frac{4}{3}x + 6x - 24 = -11 \\
\frac{3 + 4 + 18}{3}x = -11 + 12 + 24 \\
\frac{25}{3}x = 25 \\
x = 3
\]
Substitute $x = 3$ into equation (4) and equation (6):

$$z = 2 \times 3 - 8 = -2$$

and

$$y = \frac{18 - 2 \times 3}{3} = \frac{18 - 6}{3} = 4$$

Therefore $x = 3$, $y = 4$ and $z = -2$.

**Question 3**

Equilibrium is the price and quantity where the demand and supply functions are equal. It means the point where the lines of the demand and supply function intersect.

Therefore we need to determine the value of $P$ and $Q$ for which $P_d = P_s$ or $Q_d = Q_s$. Now given is $P_d = 100 - 0.5Q$ and $P_s = 10 + 0.5Q$. Thus

$$P_d = P_s$$

$$100 - 0.5Q = 10 + 0.5Q$$

$$-0.5Q - 0.5Q = 10 - 100$$

$$-Q = -90$$

$$Q = 90$$

To calculate the price at equilibrium, we substitute the value of $Q$ into the demand or supply function and calculate $P$. Say we use the demand function, then

$$P = 100 - 0.5(90)$$

$$P = 100 - 45$$

$$P = 55$$

The equilibrium price is equal to 55 and the quantity is 90.

**Question 4**

Break-even is when no profit is made or when revenue is equal to cost. Revenue or Income is defined as price times quantity or

$$R = p \times q \text{ or } p \times x.$$
Now given is quantity as $x$ and price is given as R3,10. Thus

$$\text{Revenue} = R(x) = p \times x$$

$$= 3,10 \times x$$

$$= 3,10x$$

The revenue function is thus equals to $R(x) = 3,10x$ and the cost function is given as $c(x) = 300 + 0,92x$.

Thus at breakeven:

$$R(x) = c(x)$$

$$3,10x = 300 + 0,92x.$$ 

Now we need to solve for $x$:

$$3,10x - 0,92x = 300$$

$$2,18x = 300$$

$$x = \frac{300}{2,18}$$

$$x \approx 137,61468$$

$$x \approx 138.$$ 

Approximately 138 items must be sold to breakeven.

**Question 5**

The consumer surplus for demand is the difference between the amount the consumer is willing to spend for successive units of a product from $Q = 0$ to $Q = Q_0$ and the amount that the consumer actually spent on $Q_0$ units of the product at a market price of $P_0$ per unit.

$$CS = \text{Amount willing to pay} - \text{Amount actually paid}$$

If you need to determine the demand surplus for a demand function of $P = a - bQ$ then the consumer surplus can be calculated by calculating an area of the triangle $P_0Q_0a$ which is equal to

$$\frac{1}{2} \times \text{height} \times \text{base} = \frac{1}{2} \times (a - Q_0) \times (P_0 - 0) = \frac{1}{2} \times (a - Q_0) \times (P_0)$$

with

- $P_0$ the value given to you as the market price,
- $Q_0$ the value of the demand function if $P$ equals the given market price (substitute $P_0$ into the demand function and calculate $Q_0$), and
- $a$ the $y$-intercept of the demand function $P = a - bQ$ also known as the value of $P$ if $Q = 0$, or the point where the demand function intercepts the $y$-axis.
In general we can summarise the steps of determining the consumer surplus as follows:

**Method:**
1. Calculate $Q_0$ if $P_0$ is given.
2. Draw a rough graph of the demand function.
3. Read the value of $a$ from the demand function – the $y$-intercept of the demand function.
4. Calculate the area of $CS = \frac{1}{2} \times (a - P_0) \times (Q_0)$.

**Step 1:** First we need to determine $Q$ from the demand function $P = 50 - 4Q$ if $P = 10$. Therefore

\[
P = 50 - 4Q \\
10 = 50 - 4Q \\
4Q = 50 - 10 \\
4Q = 40 \\
Q = 10
\]

**Step 2:** Next we draw a rough sketch of the demand function:
Step 3: Now the consumer surplus is the area of the shaded triangle of the rough sketch of step 2:

\[ CS = \frac{1}{2} \times \text{base} \times \text{height} \]
\[ = \frac{1}{2} \times 10 \times (50 - 10) \]
\[ = \frac{1}{2} \times 10 \times 40 \]
\[ = 200 \]

The consumer surplus is equal to 200 if the price \( P \) is equal to 10.

Question 6

We need to graphically represent \( y \leq 4 - 2x \). To draw a linear inequality we first change the inequality sign (\( \geq \) or \( \leq \) or \( > \) or \( < \)) to an equal sign (=) and draw the graph of the line. But we need two points to draw a line. Choose the two points where the lines cut through the \( x \)-axis (\( x \)-axis intercept, thus \( y = 0 \)) and \( y \)-axis (\( y \)-axis intercept, \( x = 0 \)). Calculate \((0 ; y)\) and \((x ; 0)\) and draw a line through the two points.

Step 1: Change the \( \leq \) sign to an = sign. Choose the values of \( x \) and \( y \) randomly or use the points \((0 ; y)\) and \((x ; 0)\) as below, sketch the two points and draw the line of the graph through the two points:

Let \( x = 0 \) the \( y \) is equal to
\[ y = 4 - 2x \]
\[ y = 4 - 2(0) \]
\[ y = 4 \]

Let \( y = 0 \) then \( x \) is equal to
\[ y = 4 - 2x \]
\[ 0 = 4 - 2x \]
\[ -4 = -2x \]
\[ \frac{-4}{-2} = x \]
\[ x = 2 \]

The two point calculated are \((0 ; 4)\) and \((2 ; 0)\).
Step 2: Determine the feasible region for the inequality by substituting a point on either side of the drawn line into the equation of the inequality. The inequality region is the area where the selected point, make the inequality true. Shade the feasible area: Select points $(0; 0)$ to the left of the line and $(3; 3)$ to the right of the line.

Now the point $(0; 0)$ on the left side of the line makes the inequality

\[
y \leq 4 - 2x
\]

\[
0 \leq 4 - 2(0)
\]

\[
0 \leq 4
\]

false. So the feasible region is to the left of the line. Or alternatively, if we use the point $(3; 3)$ on the right of the line makes the inequality

\[
y \leq 4 - 2x
\]

\[
3 \leq 4 - 2(3)
\]

\[
3 \leq -2
\]

false. The feasible region is to the left of the line.
Question 7

Step 1: To graph a linear inequality we first change the inequality sign (≥ or ≤ or > or <) to an equal sign (=) and draw the graph of the line. But we need two points to draw a line. Choose the two points where the lines cut the x-axis (x-axis intercept, thus \( y = 0 \)) and y-axis (y-axis intercept, \( x = 0 \)). Calculate \((0; y)\) and \((x; 0)\) and draw a line through the two points. See the table below for the calculations.

Step 2: Determine the feasible region for each inequality by substituting a point on either side of this line into the equation of the inequality. The inequality region is the area where the selected point makes the inequality true. The calculations are given below:

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Step 1</th>
<th>Step 2</th>
</tr>
</thead>
</table>
| \( y \geq 5 - 2.5x \) | \( y \geq 5 - 2.5 \)  
\( y = 5 - 2.5(0) \)  
\( y = 5 \)  
Point : \((0; 5)\) | \( y \geq 5 - 2.5 \)  
\( 0 = 5 - 2.5 \)  
\( 2.5x = 5 \)  
\( x = \frac{5}{2.5} \)  
\( x = 2 \)  
Point : \((2; 0)\) | Select the point \((0;0)\) below the line \( y \geq 5 - 2.5x \). Thus \( 0 \geq 5 - 2.5(0) \)  
\( 0 \geq 5 \) false.  
As the point \((0;0)\) lies below the line \( y \geq 5 - 2.5x \) the area below the line \( y \geq 5 - 2.5x \) is false. Thus the area that makes the inequality true lies above the line \( y \geq 5 - 2.5x \). See area 1 in the graph in step 3. |
| \( y \leq 3 - x \) | \( y \leq 3 - x \)  
\( y = 3 - x \)  
\( y = 3 - 0 \)  
\( y = 3 \)  
Point \((0; 3)\) | \( y \leq 3 - x \)  
\( 0 = 3 - x \)  
\( x = 3 \)  
Point \((3; 0)\) | Select the point \((0;0)\) below the line \( y \leq 3 - x \). Thus \( 0 \leq 3 - 0 \)  
\( 0 \leq 3 \) true.  
As the point \((0;0)\) lies below the line \( y \leq 3 - x \) the area below the line \( y \leq 3 - x \) is true. Thus the area that makes the inequality true lies below the line \( y \leq 3 - x \). See area 2 in the graph in step 3. |
| \( x \geq 0 \) | | Area above the x-axis |
| \( y \geq 0 \) | | Area to the right of the y-axis |
**Step 3:** The feasible region (grey shaded area) is the one where all the inequalities are true simultaneously.

![Graph showing feasible region](image)

**Question 8**

It is given that $x$ and $y$ are the number of litres of chocolate and vanilla ice creams, respectively. To help us with the formulation of the problem, we summarise the information given in a table with the headings: resources (items with restrictions), the variables ($x$ and $y$) and capacity (amount or number of resources available).

<table>
<thead>
<tr>
<th>Resource</th>
<th>$x^*$</th>
<th>$y^*$</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Milk</td>
<td>0.4</td>
<td>0.5</td>
<td>1000</td>
</tr>
<tr>
<td>Sugar</td>
<td>0.15</td>
<td>0.25</td>
<td>400</td>
</tr>
<tr>
<td>Number of ice creams</td>
<td></td>
<td></td>
<td>Never negative</td>
</tr>
</tbody>
</table>

Using the table, the following constraints can be defined:

\[
0.4x + 0.5y \geq 1000 \\
0.15x + 0.25y \leq 400 \\
x, y \geq 0
\]

**Question 9**

It is given that $x$ and $y$ are the number of two-person and four-person boats, respectively. To help us with the formulation of the problem, we summarise the information given in a table with the headings: resources (items with restrictions), the variables ($x$ and $y$) and capacity (amount or number of resources available).
<table>
<thead>
<tr>
<th>Resource</th>
<th>Two-person</th>
<th>Four-person</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutting</td>
<td>0.9</td>
<td>1.8</td>
<td>864</td>
</tr>
<tr>
<td>Assembly</td>
<td>0.8</td>
<td>1.2</td>
<td>672</td>
</tr>
<tr>
<td>Profit</td>
<td>2500</td>
<td>4000</td>
<td></td>
</tr>
<tr>
<td>Number of boats</td>
<td></td>
<td></td>
<td>Never negative</td>
</tr>
</tbody>
</table>

Using the table, the following constraints can be defined:

\[
0.9x + 1.8y \leq 864
\]

\[
0.8x + 1.2y \leq 672
\]

\[x, y \geq 0\]

**Question 10**

We need to determine the maximum value of the function \( P = 20x + 30y \) subject to the given constraints. To do this we determine all the corner points of the feasible region of the constraints and substitute them into the objective function (function you want to maximise or minimise) to determine the maximum or minimum value of the function. The method can be summarised as follows:

**Step 1:** Determine the coordinates of all corners of the feasible region by
- determining the point where the two lines intersect - solving two equations with two unknowns or
- read the coordinates of the intersection point from the graph.

**Step 2:** Substitute the corner points into the objective function and calculate the value of the objective function.

**Step 3:** Choose the corner point which results in the highest (maximisation) or the lowest (minimisation) objective function value.

**Step 1:** The corner points of the feasible region in the graph below are the points A, B, C, D and the origin (0;0).
To determine the coordinates of the corners of the feasible region we can:

- read the coordinates of the intersection points from the graph or
- determine the point where the lines intersect by solving two equations with two unknowns using the substitution or elimination methods.

**Point A:**
Point A is the point where line (2) cuts the $y$-axis. This coordinates of the point can be read from the graph as $(0;70)$.

**Point B:**
Point B is the point where line (2) and (3) intersect. This coordinates of the point can be read from the graph as $(20;60)$ or you can calculate it by solving two equations with two unknowns by using for example if we use the method of substitution:

We need to solve the following two equations simultaneously:

\[
\begin{align*}
x + 2y &= 140 \quad (2) \\
x + y &= 80 \quad (3)
\end{align*}
\]

First we make $x$ the subject of equation (2) by subtracting $2y$ from each side of equation (2):

\[
\begin{align*}
x + 2y - 2y &= 140 - 2y \\
x &= 140 - 2y
\end{align*}
\]

Substitute the value of $x$ namely $x = 140 - 2y$ into equation (3) and solve $y$:

\[
\begin{align*}
x + y &= 80 \quad (3) \\
(140 - 2y) + y &= 80 \\
-2y &= 80 - 140 \\
y &= -60
\end{align*}
\]

Substitute the value of $y = 60$ into equation (3) and solve $x$:

\[
\begin{align*}
x + y &= 80 \\
x + 60 &= 80 \\
x &= 80 - 60 \\
x &= 20
\end{align*}
\]

The coordinates of Point B are $(20;60)$.

**Point C:**
Point C is the point where line (1) and (3) intersect.

The coordinates of the point can be read from the graph as $(40;40)$ or you can calculate it by solving two equations with two unknowns by using for example the method of substitution:
We need to solve the following two equations simultaneously:

\[ 2x + y = 120 \quad (1) \quad \text{and} \quad x + y = 80 \quad (3) \]

First we make \( y \) the subject of equation (1) by subtracting \( 2x \) from each side of equation (1):

\[
2x + y - 2x = 120 - 2x \\
y = 120 - 2x \quad (4)
\]

Substituting the value of \( y \) of equation (4) into equation (3) and solve for \( x \):

\[
x + (120 - 2x) = 80 \\
x + 120 - 2x = 80 \\
-x = 80 - 120 \\
-x = -40 \\
x = 40
\]

Substitute the value of \( x = 40 \) into equation (3) and solve \( y \):

\[
x + y = 80 \\
10 + y = 80 \\
y = 80 - 40 \\
y = 40
\]

The coordinates of Point C are (40;40).

Point D:
Point D is the point where line (1) cuts the \( x \)-axis. This coordinate of the point can be read from the graph as (60;0).

Step 2: Substitute the corner points of the feasible region into the objective function and determine the value of the objective function for each corner point:

<table>
<thead>
<tr>
<th>Corner points of feasible region</th>
<th>Value of ( P = 20x + 30y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: ( x = 0; y = 70 )</td>
<td>( P = 20(0) + 30(70) = 2100 )</td>
</tr>
<tr>
<td>B: ( x = 20; y = 60 )</td>
<td>( P = 20(20) + 30(60) = 2200 )</td>
</tr>
<tr>
<td>C: ( x = 40; y = 40 )</td>
<td>( P = 20(40) + 30(40) = 2000 )</td>
</tr>
<tr>
<td>D: ( x = 60; y = 0 )</td>
<td>( P = 20(60) + 30(0) = 1200 )</td>
</tr>
<tr>
<td>Origin: ( x = 0; y = 0 )</td>
<td>( P = 20(0) + 30(0) = 0 )</td>
</tr>
</tbody>
</table>

Step 3: Choose the corner point which results in the highest (maximisation) objective function value. Maximum of \( P \) is at point B where \( x = 20 \), \( y = 60 \) and \( P = 2200 \).