Tutorial letter 201/2/2018

Quantitative Modelling 1
DSC1520

Semester 2

Department of Decision Sciences

Solutions to Assignment 1
Dear Student

This tutorial letter contains the solutions to the first compulsory assignment. Please contact me if you have any questions or need any help with the next assignment.

Kind regards

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Question 1

\[ C = 3q + z \]

Let \((q_1; C_1) = (10; 40)\) and \((q_2; C_2) = (20; 70)\)

The slope \(m\) is

\[ m = \frac{C_2 - C_1}{q_2 - q_1} = \frac{70 - 40}{20 - 10} = \frac{30}{10} = 3 \]

Substituting point \((10 ; 40)\) in equation \(C = 3q + z\)

\[ 40 = 3 \times 10 + z \]
\[ z = 10. \]

The equation of the line is \(C = 3q + 10\) and the cost of manufacturing 35 items is

\[ y = 3(35) + 10 \]
\[ = 115 \]

[Option 1]

Question 2

The slope \(m\) is

\[ m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{2 - 1}{4 - 3} \]
\[ = 1 \]
\[ = -\frac{3}{5} \]

giving \(y = -\frac{3}{5}x + c\).

To determine the \(y\)-intercept \(c\) of the line we use one of the given points \((x_1; y_1)\) and \((x_2; y_2)\), say \((3 ; 1)\) to find

\[ 1 = -\frac{9}{5} + c \]

which results in \(c = \frac{14}{5} \)

The equation of the line is therefore \(y = -\frac{3}{5}x + \frac{14}{5} \).

[Option 1]
Question 3

According to the textbook, the price elasticity of demand is

\[ \varepsilon_d = -\frac{1}{b} \cdot \frac{P}{Q}, \]

with \( a \) and \( b \) the values of the demand function

\[ P = a - bQ. \]

The given demand function is

\[ P = 70 - 0.5Q. \]

It is therefore clear that \( a = 70 \) and \( b = 0.5 \).

The demand function is \( P = 70 - 0.5Q \). To find \( Q \) in terms of \( P \) we need to write the demand function with \( Q \) as the subject, resulting in

\[ Q = \frac{P - 70}{-0.5}. \]

We can now substitute \( b \) and \( Q \) into the formula for elasticity of demand to find

\[ \varepsilon_d = -\frac{1}{0.5} \times \frac{P}{P - 70} \]
\[ = -1 \times \frac{P}{P - 70} \times \frac{-0.5}{1} \]
\[ = -\frac{1}{2 \times \frac{P}{P - 70}} \]

Or

From the demand function

\[ P = a - bQ, \]

we find that \(-bQ = P - a\).

Therefore, price elasticity of demand

\[ \varepsilon_d = -\frac{1}{b} \times \frac{P}{Q} \]
\[ = \frac{P}{-bQ} \]
\[ = \frac{P}{P - a}. \]

For our demand function with \( a = 70 \),

\[ \varepsilon_d = \frac{P}{P - 70}. \]

(See page 89, equation 2.14)
Question 4

\[ x - 2y + 3z = -11 \quad (1) \]
\[ 2x - z = 8 \quad (2) \]
\[ 3y + z = 10 \quad (3) \]
\[ z = 2x - 8 \quad (4) \]
\[ y = \frac{10 - z}{3} \quad (5) \]

Substitute equation (4) into equation (5):
\[ y = \frac{10 - (2x - 8)}{3} = \frac{18 - 2x}{3} \quad (6) \]

Substitute equation (2) and equation (6) into equation (1):
\[ x - \frac{2}{3}(18 - 2x) + 3(2x - 8) = -11 \]
\[ x = 3 \]

\[ \frac{25}{3} x = 25 \]

Substitute \( x = 3 \) into equation (4) and equation (6):
\[ z = 2 \times 3 - 8 = -2 \]

and
\[ y = \frac{18 - 2 \times 3}{3} = \frac{18 - 6}{3} = 4 \]

Therefore \( x = 3; y = 4 \) and \( z = -2 \). The sum is 5.

[Option 4]

Question 5

\[ -2x + \frac{5}{6} + \frac{x}{2} \geq -2x - 4 \left( -\frac{x}{3} - \frac{1}{4} \right) \]
\[ -2x + \frac{5}{6} + \frac{x}{2} \geq -2x + \left( \frac{4x}{3} + \frac{20}{4} \right) \]
\[ \frac{x}{2} - \frac{4x}{3} \geq 5 - \frac{5}{6} \]
\[ \frac{3x - 8x}{6} \geq \frac{30 - 5}{6} \]
\[ \frac{-5x}{6} \geq \frac{25}{6} \]
\[ x \leq -5 \]

[Option 1]
Question 6

The arc price elasticity of a demand function \( P = a - bQ \) between two prices \( P_1 \) and \( P_2 \) is

\[
\text{arc elasticity of demand} = -\frac{1}{b} \times \frac{P_1 + P_2}{Q_1 + Q_2}
\]

with \( b \) the slope of the demand function and \( P_1, P_2 \) and \( Q_1, Q_2 \) the price and quantity demanded.

\[
\begin{align*}
P &= 70 - 0.5Q \\
0.5Q &= 70 - P \\
Q &= 140 - 2P
\end{align*}
\]

We can now determine \( Q_1 \) and \( Q_2 \) by substituting \( P_1 = 7 \) and \( P_2 = 5 \) into the equation.

Therefore if \( P_1 = 7 \) then \( Q_1 = 140 - 2 \times 7 = 126 \)

and if \( P_2 = 5 \) then \( Q_2 = 140 - 2 \times 5 = 130 \).

\[
\text{elasticity of demand} = -\frac{1}{b} \times \frac{P_1 + P_2}{Q_1 + Q_2}
\]

\[
= -\frac{1}{0.5} \times \frac{7 + 5}{126 + 130} = -\frac{1}{0.5} \times \frac{12}{256} = -\frac{12}{128} = -0.09375 \approx -0.094; \text{ inelastic.}
\]

[Option 5]

Question 7

The price elasticity of demand measures the responsiveness (sensitivity) of quantity demanded to changes in the good’s own price i.e The sensitivity of quantity demanded to change in price. [Option 2]
Question 9

\[ 9x + 2.5y \geq 150 \]
\[ 3x + y \leq 100 \]
\[ x, y \geq 0. \]

[Option 2]

Question 10

\[ PS = \frac{1}{2} \times \text{base} \times \text{height} \]
\[ = \frac{1}{2} \times 20 \times (90 - 50) \]
\[ = \frac{800}{2} \]
\[ = 400. \]

[Option 3]

Question 11

We need to determine the maximum value of the function

\[ P = 20x + 30y \]

subject to the given constraints.

The corner points of the feasible region in the graph below are the points A, B, C, D and the origin (0;0).

Point A:

Point A is the point where line (2) cuts the y-axis. This coordinates of the point can be read from the graph as (0;70).

Point B:
Point B is the point where line (2) and (3) intersect. This coordinates of the point can be read from the graph as (20;60) or you can find them by calculation.

\[
\begin{align*}
  x + 2y &= 140 \quad (2) \\
  x + y &= 80 \quad (3)
\end{align*}
\]

\[
\begin{align*}
  x + 2y - 2y &= 140 - 2y \\
  x &= 140 - 2y
\end{align*}
\]

Substitute the value of \( x \) namely \( x = 140 - 2y \) into equation (3) and solve \( y \):

\[
\begin{align*}
  x + y &= 80 \quad (3) \\
  (140 - 2y) + y &= 80 \\
  -y &= 80 - 140 \\
  -y &= -60 \\
  y &= 60
\end{align*}
\]

Substitute the value of \( y = 60 \) into equation (3) and solve \( x \):

\[
\begin{align*}
  x + y &= 80 \\
  x + 60 &= 80 \\
  x &= 80 - 60 \\
  &= 20
\end{align*}
\]

The coordinates of Point B are (20;60).

**Point C:**

Point C is the point where line (1) and (3) intersect. The coordinates of the point can be read from the graph as (40;40) or you can find them by calculation.

\[
\begin{align*}
  2x + y &= 120 \quad (1) \\
  x + y &= 80 \quad (3) \\
  2x + y - 2x &= 120 - 2x \\
  y &= 120 - 2x \quad (4)
\end{align*}
\]

Substituting the value of \( y \) of equation (4) into equation (3) and solve for \( x \):

\[
\begin{align*}
  x + y &= 80 \quad (3) \\
  x + (120 - 2x) &= 80 \\
  -x &= 80 - 120 \\
  &= -40 \\
  &= 40.
\end{align*}
\]

Substitute the value of \( x = 40 \) into equation (3) and solve \( y \):

\[
\begin{align*}
  x + y &= 80 \\
  10 + y &= 80 \\
  y &= 80 - 40 \\
  &= 40
\end{align*}
\]
The coordinates of Point C are (40; 40).

**Point D:**

Point D is the point where line (1) cuts the x-axis. This coordinate of the point can be read from the graph as (60; 0).

Substitute the corner points of the feasible region into the objective function and determine the value of the objective function for each corner point:

<table>
<thead>
<tr>
<th>Corner points of feasible region</th>
<th>Value of $P = 20x + 30y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A : x = 0 ; y = 70$</td>
<td>$P = 20(0) + 30(70) = 2100$</td>
</tr>
<tr>
<td>$B : x = 20 ; y = 60$</td>
<td>$P = 20(20) + 30(60) = 2200$ ← Maximum</td>
</tr>
<tr>
<td>$C : x = 40 ; y = 40$</td>
<td>$P = 20(40) + 30(40) = 2000$</td>
</tr>
<tr>
<td>$D : x = 60 ; y = 0$</td>
<td>$P = 20(60) + 30(0) = 1200$</td>
</tr>
</tbody>
</table>

Maximum of $P$ is at point B where $x = 20$, $y = 60$ and $P = 2200$.

[Option 4]

**Question 12**

Equilibrium is the price and quantity where the demand and supply functions are equal.

\[
P_d = P_s
\]

\[
100 - 0.5Q = 10 + 0.5Q
\]

\[
-0.5Q - 0.5Q = 10 - 100
\]

\[
Q = \frac{-90}{-1}
\]

\[
= 90.
\]

To calculate the price at equilibrium, we substitute the value of $Q$ into the demand or supply function and calculate $P$.

\[
P = 100 - 0.5(90)
\]

\[
P = 100 - 45
\]

\[
P = 55
\]

The equilibrium price is equal to 55 and the quantity is 90. [Option 1]
Question 13

Revenue or Income is defined as price times quantity or \( R = p \times q \) or \( p \times x \). Now given is quantity as \( x \) and price is given as \( p(x) = 5 - \frac{x}{1000} \). Thus

\[
\text{Revenue} = p \times x \\
= (5 - \frac{x}{1000}) \times x \\
= 5x - \frac{x^2}{1000} \\
= 5x - \frac{1}{1000}x^2 \\
= 5x - 0.001x^2.
\]

[Option 3]

Question 14

Let \( x \) = number produced

Total cost = variable cost + fixed cost

\[
TC = 4x + 64 \\
= 4(200) + 64 \\
= 800 + 64 \\
= 864
\]

[Option 2]

Question 15

We need to find the consumer surplus for demand function

\( P = 60 - 4Q \)

when the market price \( P = 16 \).

From the textbook page 128, we know that the consumer surplus is calculated as

\( \text{CS} = \text{Amount willing to pay} - \text{Amount actually paid} \).

This is determined by calculating the area of the triangle \( P_0E_0a \) in the following graph which is given to

\[
\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2} \times Q_0 (a - P_0)
\]

with

\[ P_0 \text{ the market price}, \]
\[ Q_0 \text{ the number of units demanded at price } P_0, \text{ and} \]
\[ a \text{ the } y\text{-intercept of the demand function } P = a - bQ. \]
In general we can summarise the steps of determining the consumer surplus as follows:

1. Calculate $Q_0$ if $P_0$ is given.
2. Draw a rough graph of the demand function.
3. Read the value of $a$ from the demand function, that is the $y$-intercept of the demand function.
4. Calculate the area of $CS = \frac{1}{2} \times Q_0 (a - P_0)$.

First we need to determine $Q$ from the demand function $P = 60 - 4Q$ if $P = 16$. That is

$$16 = 60 - 4Q$$

giving

$$Q = 11.$$  

**Draw** the demand function by using the point $(11; 16)$ found before and $(0; a)(0; 60)$. 
The value of $a$ is found by substituting $Q = 0$ into the demand function, that is
\[ P = 50 - bQ = 50. \]

The consumer surplus is the area of the shaded triangle in the sketch, that is
\[
CS = \frac{1}{2} \times \text{base} \times \text{height} \\
= \frac{1}{2} \times 11 \times (60 - 16) \\
= \frac{442}{2} = 221.
\]

The consumer surplus is equal to 221 if the price $P$ is equal to 16.

[Option 1]