Tutorial Letter 202/1/2012

QUANTITATIVE MODELLING

Semester 1

Department of Decision Sciences

This tutorial letter contains solutions for Assignment 2 and 3
Dear student

You have completed the three assignments for the course and all that is left to pass this module is the examination. It is now time to start your revision for the examination. Try to work through all the assignments (first and second semester), evaluation exercises, the discussion class problems and the previous examination paper when you prepare for the examination. The questions in the May/June examination paper are similar to the problems in the above mentioned. You are also welcome to try the second semester’s assignments. The slides of the discussion classes are available on myUnisa, under the section “Announcements”. Remember, practice makes perfect! The more examples you work through the more you will be able to recognise a problem and know how to solve it.

Remember help is just a phone call or e-mail away. Please contact me if you need any help with the second assignment. My contact details and contact hours are as follows:

08:00 to 13:30 (Mondays to Fridays) (appointments and telephone)
13:30 to 16:00 (Mondays to Thursdays) (telephone only)

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Lastly, I wish you everything of the best with your preparation for the last hurdle, the examination.

Ms Victoria Mabe-Madisa
ASSIGNMENT 2: SOLUTIONS

Question 1

To determine the point of intersection of two lines we need to determine a point \((x;y)\) so that the \(x\) and \(y\) values satisfy the equations of both lines. We need to solve the two equations simultaneously. There are different methods you can use to solve the set of equations.

(a) **Elimination method:**

**Step 1:** Eliminate one variable, say \(x\), by adding or subtracting one equation or multiple of an equation from another equation:

Let \(2x + 3y = 5\) \hspace{1cm} (1)

and \(20x - 9y = 21\) \hspace{1cm} (2)

be the equations of the two lines.

Now 3 times equation (1) plus equation (2) will eliminate \(y\). But \(3 \times \text{equation (1)}\) is:

\[
6x + 9y = 15 \hspace{1cm} (3)
\]

Now equation \(3 \times \text{equation (1)}\) plus equation (2) is

\[
\begin{align*}
6x + 9y &= 15 \\
20x - 9y &= 21 \\
26x &= 36
\end{align*}
\]

Now solve for \(x\):

\[
x = \frac{36}{26} = \frac{18}{13}
\]
Step 2: Solve for \( y \). Substitute the value of \( x \) into any one of the equations and solve for \( y \).

Substituting the value of \( x = \frac{18}{13} \) into, say, equation (1):

\[
2\left(\frac{18}{13}\right) + 3y = 5
\]
\[
3y = 5 - \frac{36}{13}
\]
\[
y = \left(5 - \frac{36}{13}\right) \times \frac{1}{3}
\]
\[
y = \frac{29}{39}
\]

The two lines intersect in the point \( (x, y) = \left(\frac{18}{13}, \frac{29}{39}\right) \)

(b) Substitution method:

Step 1: Change one of the equations so that a variable is the subject of the equation.

Say \( x \) in equation (1):

From \( 2x + 3y = 5 \) (1)

follows \( x = \frac{5}{2} - \frac{3}{2}y \) (3)

Step 2: Substitute the value of \( x \) (equation (3)) into the unchanged equation (2) and solve for \( y \).

Substituting \( x = \frac{5}{2} - \frac{3}{2}y \) into \( 20x - 9y = 21 \):

\[
20\left(\frac{5}{2} - \frac{3}{2}y\right) - 9y = 21
\]
\[
100 - 60y - 9y = 21
\]
\[
50 - 69y = 21
\]
\[
-69y = -29
\]
\[
y = \frac{29}{39}
\]
\[
y = \frac{29}{39}
\]
Step 3: Substitute the calculated value of the variable in step 2 into any of the given equations and calculate the value of the other variable. Substitute $y = \frac{29}{39}$ into equation (1) or equation (2). Let’s say we choose equation (1): $2x + 3y = 5$

\[
2x + 3\left(\frac{29}{39}\right) = 5
\]
\[
2x + \left(\frac{29}{13}\right) = 5
\]
\[
2x = 5 - \frac{29}{13}
\]
\[
2x = \frac{65 - 29}{13}
\]
\[
x = \frac{36}{13} + 2
\]
\[
x = \frac{36}{13} \times \frac{1}{2}
\]
\[
x = \frac{18}{13}
\]

The two lines intersect in the point $(x; y) = \left(\frac{18}{13}; \frac{29}{39}\right)$.

Question 2

We need to solve the following system of equations:

\[
x + 2y - z = 5 \quad (1)
\]
\[
2x - y + z = 2 \quad (2)
\]
\[
y + z = 2 \quad (3)
\]

Step 1: Determine two equations with two unknowns (variables) by adding or subtracting two of the three equations at a time:
Now equation (3) is already an equation with only two variables namely \( y \) and \( z \). To determine another equation with two variables namely \( y \) and \( z \) we must eliminate \( x \) by subtracting equation (2) from two times equation (1):

\[
\begin{align*}
2x + 4y - 2z &= 10 \\ -2x + y + z &= 2 \\
\hline
0 + 5y - 3z &= 8
\end{align*}
\]

Now we have 2 equations with two unknowns namely:

\[
\begin{align*}
5y - 3z &= 8 \\
y + z &= 2
\end{align*}
\]

**Step 2**: Next we solve two equations with two unknowns, using the substitution method or elimination method. Let’s use the substitution method:

- **Step 1**: Make \( y \) the subject of equation (3):
  \[
y = 2 - z
\]
  \( (5) \)

- **Step 2**: Substitute equation (5) into equation (4) and solve for \( z \):
  \[
  5y - 3z = 8 \\
  5(2 - z) - 3z = 8 \\
  10 - 5z - 3z = 8 \\
  -8z = 8 - 10 \\
  -8z = -2 \\
  z = \frac{-2}{-8} = 0,25
  \]

- **Step 3**: Substitute \( z = 0,25 \) into equation (3) and solve for \( y \):
  \[
  y + z = 2 \\
  y + 0,25 = 2 \\
  y = 2 - 0,25 \\
  y = 1,75
  \]
- Step 4: Substitute $z = 0,25$ and $y = 1,75$ into equation (1) and solve for $x$:

$$x + 2y - z = 5 \tag{1}$$

$$\begin{align*}
  x + 2(1,75) - 0,25 &= 5 \\
  x + 3,5 - 0,25 &= 5 \\
  x + 3,25 &= 5 \\
  x &= 5 - 3,25 \\
  x &= 1,75
\end{align*}$$

Therefore the solution of the set of equations is $x = 1,75; y = 1,75$ and $z = 0,25$.

**Question 3**

At the break-even point no profit or loss is made or revenue = cost.

The cost to produce a number $x$ of a toy is given as $C = 2 700 + 25x$. The total revenue from selling $x$ toys is the price $\times$ quantity $= 45x$.

Therefore at the break-even point:

Revenue = cost

$$45x = 2 700 + 25x$$

Next we solve for $x$ to determine the number of toys. Make $x$ the subject of the equation:

$$\begin{align*}
  45x - 25x &= 2 700 \\
  20x &= 2 700 \\
  x &= 135.
\end{align*}$$

135 toys should be produced to break even.


**Question 4**

Equilibrium is the price and quantity where the demand and supply functions are equal. As the given demand and supply functions are linear lines it means the point where the lines of the demand and supply function intersect.

To determine the point of intersection of two lines we need to determine a point \((P; Q)\) so that the \(P\) and \(Q\) values satisfy the equations of both lines. We need to solve the two equations simultaneously. There are different methods you can use to solve the set of equations.

(a) **Elimination method:**

**Step 1:** Eliminate one variable, say \(P\), by adding or subtracting one equation or multiple of an equation from another equation:

Let 

\[
Q = 50 - 0.1P 
\]  

(1)

and 

\[
Q = -10 + 0.1P
\]  

(2)

be the equations of the two lines.

Now equation (1) plus equation (2) will eliminate \(P\).

\[
\begin{align*}
Q &= 50 - 0.1P \\
Q &= -10 + 0.1P \\
\hline
2Q &= 40 
\end{align*}
\]

Now solve for \(Q\):

\[
\begin{align*}
Q &= \frac{40}{2} \\
Q &= 20
\end{align*}
\]

**Step 2:** Solve for \(P\). Substitute the value of \(Q\) into any one of the equations and solve for \(P\).

Substituting the value of \(Q = 20\) into, say, equation (1):

\[
\begin{align*}
20 &= 50 - 0.1P \\
-30 &= -0.1P \\
30 &= P \\
0.1 &= P \\
300 &= P
\end{align*}
\]

The two lines intersect in the point \((P; Q) = (300 ; 20)\).
(b) **Substitution method:**

**Step 1:** Change one of the equations so that a variable is the subject of the equation.

Say $Q$ in equation (1): $Q$ is already the subject of the equation.

\[ Q = 50 - 0,1P \]  

(1)

**Step 2:** Substitute the value of $Q$ (equation (1)) into the unchanged equation (2) and solve for $P$. Now

\[ Q = -10 + 0,1P \]
\[ 50 - 0,1P = -10 + 0,1P \]
\[ -0,1P - 0,1P = -10 - 50 \]
\[ -0,2P = -60 \]
\[ P = \frac{-60}{-0,2} \]
\[ P = 300 \]

**Step 3:** Substitute the calculated value of the variable in step 2 into any equation and calculate the value of the other variable. Substitute $P = 300$ into equation (1) or equation (2). Let’s say we choose equation (1):

\[ Q = 50 - 0,1P \]
\[ = 50 - 0,1(300) \]
\[ = 50 - 30 \]
\[ = 20 \]

The two lines intersect in the point $(P; Q) = (300; 20)$.

**Question 5**

Consumer surplus is the monetary value of the benefit that accrues to consumers from the matching of supply and demand in the market. The consumer surplus is the difference between the amount the consumer is willing to spend for successive units of a product from

$Q = 0$ to $Q = Q_0$ and the amount that the consumer actually spent on $Q_0$ units of the product at a market price of $P_0$ per unit:

\[ CS = \text{Amount willing to pay} - \text{Amount actually paid} \]
If you need to determine the demand surplus for a linear demand function of \( P = a - bQ \) then the consumer surplus can be calculated by calculating an area of the triangle \( P_0Q_0a \) which is equal to

\[
\frac{1}{2} \times \text{height} \times \text{base} = \frac{1}{2} \times (a - P_0) \times (Q_0 - 0) = \frac{1}{2} \times (a - P_0) \times (Q_0)
\]

with

- \( P_0 \) the value given to you as the market price,
- \( Q_0 \) the value of the demand function if \( P \) equals the given market price (substitute \( P_0 \) into the demand function and calculate \( Q_0 \)), and
- \( a \) the \( y \)-intercept of the demand function \( P = a - bQ \) also known as the value of \( P \) if \( Q = 0 \), or the point where the demand function intercepts the \( y \)-axis.

In general we can summarise the steps of determining the consumer surplus as follows:

**Method:**

1. Calculate \( Q_0 \) if \( P_0 \) is given.
2. Draw a rough graph of the demand function.
3. Read the value of \( a \) from the demand function – the \( y \)-intercept of the demand function.
4. Calculate the area of \( CS = \frac{1}{2} \times (a - P_0) \times (Q_0) \).
**Step 1:** First we need to determine $Q$ from the demand function if $P = 30$ or when the market per unit is $P = 30$, the consumer will purchase

\[
\begin{align*}
P &= 48 - 0.2Q \\
30 &= 48 - 0.2Q \\
30 - 48 &= -0.2Q \\
-18 &= -0.2Q \\
Q &= \frac{-18}{-0.2} \\
Q &= 90 \text{ units}
\end{align*}
\]

**Step 2:** Next we draw a rough sketch of the demand function:

![Demand function diagram](image)

**Step 3:** Now the consumer surplus is the area of the shaded triangle of the rough sketch of step 2:

\[
\text{Area } A = \left[\frac{1}{2} \times 90 \times (48 - 30)\right]
= \left[\frac{1}{2} \times 90 \times 18\right]
= \frac{1620}{2}
= 810
\]

The consumer surplus is equal to 810 if the price $P$ is equal to 30.
Question 6

We need to graphically represent $y \geq 3 - 3x$. To draw a linear inequality we first change the inequality sign ($\geq$ or $\leq$ or $>$ or $<$) to an equal sign ($=$) and draw the graph of the line. But we need two points to draw a line. Choose the two points where the lines cut through the $x$-axis ($x$-axis intercept, thus $y = 0$) and $y$-axis ($y$-axis intercept, $x = 0$). Calculate $(0; y)$ and $(x; 0)$ and draw a line through the two points.

**Step 1:** Change the $\geq$ sign to $=$ sign. Choose the values of $x$ and $y$ randomly or use the points $(0 ; y)$ and $(x ; 0)$ as below, sketch the two points and draw the line of the graph through the two points:

Let $x = 0$ then $y$ is equal to

$$y = 3 - 3x$$
$$y = 3 - 3(0)$$
$$y = 3$$

Let $y = 0$ then $x$ is equal to

$$y = 3 - 3x$$
$$0 = 3 - 3x$$
$$3x = 3$$
$$x = 1$$

The two points calculated are $(0;3)$ and $(1;0)$.
**Step 2:** Determine the feasible region for the inequality by substituting a point on either side of the drawn line into the equation of the inequality. The inequality region is the area which selected point, made the inequality true. Shade the feasible area:

Select points (0;0) to left of the line and (1;1) to the right of the line.

Now the point (0;0) on the left of the line makes the inequality
\[ y \geq 3 - 3x \]
\[ 0 \geq 3 - 3(0) \]
\[ 0 \geq 3 \]
false. So the feasible region can’t be to the left of the line it is to the right of the line.

Or alternatively, if we use the point (1;1) on the right of the line makes the inequality
\[ y \geq 3 - 3x \]
\[ 1 \geq 3 - 3(1) \]
\[ 1 \geq 0 \]
true. So the feasible region is to the right of the line.
Question 7

Let

\[
\begin{align*}
  x_1 + x_2 &\leq 13 \\
  2x_1 - x_2 &\leq 8 \\
  -2x_1 + 3x_2 &\leq 12 \\
  x_1, x_2 &\geq 0
\end{align*}
\]

In this case we are working with \(x_1\) and \(x_2\) which might feel funny to you. Up to now we have worked with \(x\) and \(y\). Now remember \(x\) and \(y\) are just placeholders for numbers. We can use any character to denote placeholders. In this case we are just using \(x_1\) and \(x_2\) in stead of \(x\) and \(y\).

**Step 1:** To draw a linear inequality we first change the inequality sign (\(\geq\) or \(\leq\) or \(>\) or \(<\)) to an equal sign (\(=\)) and draw the graph of the line. But we need two points to draw a line. Choose the two points where the lines cut through the \(x_1\)-axis (\(x_1\)-axis intercept, thus \(x_2=0\)) and \(x_2\)-axis (\(x_2\)-axis intercept, \(x_1 = 0\)). Calculate \((0 ; x_2)\) and \((x_1 ; 0)\) and draw a line through the two points. See the table below for calculations.

**Step 2:** Determine the feasible region for each inequality by substituting a point on either side of the drawn line into the equation of the inequality. The inequality region is the area which selected point, made the inequality true. See table below for calculations.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>(x)-axis intercept</th>
<th>(y)-axis intercept</th>
<th>Inequality region</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Point ((x ; y)) if (x_1 = 0)</td>
<td>Point ((x ; y)) if (x_2 = 0)</td>
<td></td>
</tr>
<tr>
<td>(x_1 + x_2 \leq 13)</td>
<td>(x_1 + x_2 = 13)</td>
<td>(x_1 + x_2 = 13)</td>
<td>Select points ((0,0)) below the line and ((13;13)) above the line.</td>
</tr>
<tr>
<td></td>
<td>(0 + x_2 = 13)</td>
<td>(x_1 + 0 = 13)</td>
<td>Above: (13 + 13 \leq 13) – False</td>
</tr>
<tr>
<td></td>
<td>(x_2 = 13)</td>
<td>(x_1 = 13)</td>
<td>Below: (0 + 0 \leq 1) – True</td>
</tr>
<tr>
<td></td>
<td>Point ((0 ; 13))</td>
<td>Point ((13 ; 0))</td>
<td>Area below the line is true.</td>
</tr>
<tr>
<td>(2x_1 - x_2 \leq 8)</td>
<td>(2x_1 - x_2 = 8)</td>
<td>(2x_1 - x_2 = 8)</td>
<td>Select points ((0,0)) to left of the line and ((13;13)) to the right of the line.</td>
</tr>
<tr>
<td></td>
<td>(2(0) - x_2 = 8)</td>
<td>(2x_1 + 0 = 8)</td>
<td>Right: (2(13) - 13 \leq 8) – False</td>
</tr>
<tr>
<td></td>
<td>(x_2 = -8)</td>
<td>(x_1 = 4)</td>
<td>Left: (2(0) - 0 \leq 8) – True</td>
</tr>
<tr>
<td></td>
<td>Point ((0 ; -8))</td>
<td>Point ((4 ; 0))</td>
<td></td>
</tr>
<tr>
<td>$-2x_1 + 3x_2 \leq 1$</td>
<td>$-2x_1 + 3x_2 = 12$</td>
<td>$-2x_1 + 3x_2 = 12$</td>
<td>Area to the left of the line is true.</td>
</tr>
<tr>
<td>---------------------</td>
<td>---------------------</td>
<td>---------------------</td>
<td>--------------------------------------</td>
</tr>
<tr>
<td>$-2(0) + 3x_2 = 12$</td>
<td>$x_2 = 4$</td>
<td>$-2x_1 + 3(0) = 12$</td>
<td>Select points (0;0) below the line</td>
</tr>
<tr>
<td></td>
<td>Point (0 ; 4)</td>
<td>$x_2 = -6$</td>
<td>and (0;13) above the line.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Above: $-2(0) + 3(13) \leq 12$ – False</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Below: $-2(0) + 3(0) \leq 12$ – True</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>Area below the line is true.</td>
</tr>
</tbody>
</table>

**Step 3:** The feasible region is the region where all the inequality regions are true at the same time.
Question 8

To determine the maximum value of the function \( P = 20x + 30y \) subject to the given constraints we determine all the corner points of the feasible region of the constraints and substitute them into the function you want to maximise or minimise (objective function) and determine the maximum value. The method can be summarised as follows:

**Step 1:** Determine the coordinates of all corners of the feasible region by
- determining the point where the two lines intersect - solving two equations with two unknowns or
- read the coordinates of the interception point from the graph.

**Step 2:** Substitute the corner points into the objective function.

**Step 3:** Choose the corner point which results in the highest (maximisation) or the lowest (minimisation) objective function value.

**Step 1:** The corner points of the feasible region are the points A, B, C, D and the origin (0;0).

![Diagram](image.png)

**Point A:**
The point where the line (3) cuts the \( y \)-axis: (0;4)

**Point B:**
The point where line (3) and (1) intersect. This coordinates of the point can be read from the graph as \( \left( \frac{1}{2}; 4 \right) \) or you can calculate it by solving two equations with two unknowns by using for example if we use the method of substitution:

Let \( 2x + y = -5 \) be equation (1) and \( y = 4 \) equation (3)

Substitute equation (3) into the equation (1):
\[2x + 4 = 5\]
\[2x = 1\]
\[x = \frac{1}{2}\]

Point B is the point \(\left(\frac{1}{2}; 4\right)\)

**Point C:**

The point where line (1) and (2) intersect. This coordinates of the point can be read from the graph as \((2;1)\) or you can calculate it by solving two equations with two unknowns by using for example if we use the method of substitution:

Let \(2x + y = 5\) be equation (1) and \(x = 2\) equation (2). Substituting the value of \(x\) of equation (2) into equation (1):

\[2(2) + y = 5\]
\[y = 5 - 4\]
\[y = 1\]

The point C is \((2; 1)\)

**Point D:**

The point where the line (2) cuts the \(x\)-axis: \((2; 0)\).

**Step 2:** Substitute the corner points of the feasible region into the objective function and determine the value of the objective function for each corner point:

<table>
<thead>
<tr>
<th>Corner points of feasible region</th>
<th>Value of (P = 20x + 30y)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: (x = 0;\ y = 4)</td>
<td>(P = 20(0) + 30(4) = 120)</td>
</tr>
<tr>
<td>B: (x = \frac{1}{2};\ y = 4)</td>
<td>(P = 20\left(\frac{1}{2}\right) + 30(4) = 130)(--\text{Maximum})</td>
</tr>
<tr>
<td>C: (x = 2;\ y = 1)</td>
<td>(P = 20(2) + 30(1) = 70)</td>
</tr>
<tr>
<td>D: (x = 2;\ y = 0)</td>
<td>(P = 20(2) + 30(0) = 40)</td>
</tr>
<tr>
<td>Origin: (x = 0;\ y = 0)</td>
<td>(P = 20(0) + 30(0) = 0)</td>
</tr>
</tbody>
</table>
**Step 3:** Choose the corner point which results in the highest (maximisation) objective function value.

Maximum of $P$ is at point B where $x = \frac{1}{2}$, $y = 4$ and $P = 130$.

**Question 9**

First we define the decision variables. Let $x$ be the number of Product A manufactured and $y$ the number of Product B manufactured, respectively.

To help us with the formulation, we summarise the information given in a table with the headings: resources (items with restrictions and that is used in the manufacturing process), the variables ($x$ and $y$) and capacity (amount or number of the resources available).

<table>
<thead>
<tr>
<th>Resources</th>
<th>$x$ Number of Product A</th>
<th>$y$ Number of Product B</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Processing</td>
<td>30 minutes 0,5 hours</td>
<td>12 minutes 0,2 hours</td>
<td>4 hours 240 minutes</td>
</tr>
<tr>
<td>Assembly</td>
<td>18 minutes 0,3 hours</td>
<td>72 minutes 1,2 hours</td>
<td>6 hours 360 minutes</td>
</tr>
<tr>
<td>Packaging</td>
<td>24 minutes 0,4 hours</td>
<td>48 minutes 0,8 hours</td>
<td>4,8 hours 288 minutes</td>
</tr>
<tr>
<td>Number of Product A ($x$) and Product B ($y$)</td>
<td></td>
<td></td>
<td>Never negative</td>
</tr>
</tbody>
</table>

Now the units of the resources and capacity must be the same. Using the table, the following constraints can be defined if we use **minutes** as units:

$$30x + 12y \leq 240$$
$$18x + 72y \leq 360$$
$$24x + 48y \leq 288$$

$$x, y \geq 0$$

Using the table, the following constraints can be defined if we use **hours** as units:

$$0,5x + 0,2y \leq 4$$
$$0,3x + 1,2y \leq 6$$
$$0,4x + 0,8y \leq 4,8$$

$$x, y \geq 0$$
Question 10

First we define the decision variables. Let \( x \) and \( y \) be the number of pairs of boots and jackets manufactured respectively.

Secondly to help us with the formulation, we summarise the information in a table with the headings:

- resources (items with restrictions),
- the variables (\( x \) and \( y \)) and
- capacity (amount or number of the resources available).

<table>
<thead>
<tr>
<th>Resources</th>
<th>( x ) Number of Product A boots</th>
<th>( y ) Number of Product B jackets</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Making</td>
<td>4 hours</td>
<td>2 hours</td>
<td>800 hours</td>
</tr>
<tr>
<td>Finishing</td>
<td>3 hours</td>
<td>4 hours</td>
<td>1200 hours</td>
</tr>
<tr>
<td>Number of boots</td>
<td></td>
<td></td>
<td>150</td>
</tr>
<tr>
<td>Number of Product A (x) and Product B (y)</td>
<td></td>
<td></td>
<td>Never negative</td>
</tr>
</tbody>
</table>

Using the table, the following constraints can be defined if we use **hours** as units:

\[
4x + 2y \leq 800 \\
3x + 4y \leq 1200 \\
x \geq 150 \\
x, y \geq 0
\]
ASSIGNMENT 3: SOLUTIONS

Question 1

Total revenue is given by

\[ R(x) = -\frac{1}{5}x^2 + 30x + 81 \]

We need to determine the maximum value of the revenue \( R(x) \). We first have to calculate the \( x \)-value of the maximum value and then substitute it in the revenue equation to calculate the maximum revenue. There are different ways of determining the coordinates of the maximum or minimum point of a quadratic function.

Method 1:

The \( x \)-coordinate of the turning point, vertex or extreme point can be calculated by using the formula

\[ x = \frac{-b}{2a} \]

with \( a \), \( b \), and \( c \) the coefficients in the standard quadratic function \( y = ax^2 + bx + c \).

It is given that \( R(x) = -\frac{1}{5}x^2 + 30x + 81 \). Comparing it with the standard quadratic function \( y = ax^2 + bx + c \), we can see that \( a = -\frac{1}{5} \), \( b = 30 \) and \( c = 81 \) for the given function. Therefore the \( x \)-value of the maximum revenue function is at

\[ x = \frac{-b}{2a} = \frac{-30}{2 \times -\frac{1}{5}} = \frac{-30}{-\frac{2}{5}} = \frac{30 \times 5}{2} = \frac{150}{2} = 75 \]
To calculate the maximum revenue value, we substitute the calculated $x$-value into the given equation namely $R(x) = -\frac{1}{5}x^2 + 30x + 81$ and calculate the maximum revenue value. Therefore

$$R(x) = -\frac{1}{5}x^2 + 30x + 81$$
$$= -\frac{1}{5}(75)^2 + 30(75) + 81$$
$$= 1206$$

The function has a maximum value in the point (75; 1206) that is where the maximum revenue is equal to 1206.

**Method 2:**

You can also make use of the method of differentiation, as discussed in Chapter 6, to determine the maximum or minimum value or vertex of a function.

The maximum or minimum value of a function is the point where the differentiated function is equal to zero or $\frac{dy}{dx} = 0$. The function $R(x) = -\frac{1}{5}x^2 + 30x + 81$ was given.

Differentiating the function $R(x)$, using the basic rule of differentiation namely $\frac{d}{dx}x^n = nx^{n-1}$ with $n \neq 0$, we get

$$R(x) = -\frac{1}{5}x^2 + 30x + 81$$
$$\frac{d}{dx} R(x) = -\frac{1}{5}(2)x^{2-1} + 30x^{1-1} + 0$$

because $\frac{d}{dx}a = 0$ if $a$ is a constant

$$\frac{d}{dx} R(x) = -\frac{2}{5}x^1 + 30x^0$$

but $x^0 = 1$

$$\frac{d}{dx} R(x) = -\frac{2}{5}x + 30$$
But the maximum or minimum occurs when \( \frac{dy}{dx} = 0 \). Therefore

\[
\frac{d}{dx} \left( R(x) \right) = 0 \\
- \frac{2}{5} x + 30 = 0 \\
- \frac{2}{5} x = -30 \\
x = -30 \times - \frac{5}{2} \\
x = \frac{150}{2} \\
x = 75
\]

To calculate the maximum revenue value, we substitute the calculated \( x \)-value into the given equation namely \( R(x) = - \frac{1}{5} x^2 + 30x + 81 \) and calculate the maximum revenue value. Therefore

\[
R(x) = - \frac{1}{5} x^2 + 30x + 81 \\
= - \frac{1}{5} (75)^2 + 30(75) + 81 \\
= 1206.
\]

The function has a maximum value in the point \((75; 1206)\), that is where the maximum revenue is equal to 1 206.

[Option 2]

**Question 2**

The roots or solutions of a function are the coordinates of the point where the function, if drawn, cuts the \( x \)-axis. We therefore need to determine the value of \( x \) in the point where the function is equal to zero:

\[
y = 0 \quad \text{or} \quad 5x^2 - 6x + 1 = 0.
\]

We make use of the quadratic formula

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
with $a$, $b$, and $c$ the values of the coefficients in the function $0 = ax^2 + bx + c$, to determine the roots.

Comparing the given equation $5x^2 - 6x + 1 = 0$ with the general form $ax^2 + bx + c = 0$, we conclude that $a = 5$, $b = -6$ and $c = 1$. Substituting $a$, $b$ and $c$ into the formula gives

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(5)(1)}}{2(5)}$$

$$x = \frac{6 \pm \sqrt{36 - 20}}{10}$$

$$x = \frac{6 \pm \sqrt{16}}{10}$$

$$x = \frac{6 \pm 4}{10}$$

$$x = \frac{6 + 4}{10} \quad \text{or} \quad \frac{6 - 4}{10}$$

$$x = \frac{10}{10} \quad \text{or} \quad \frac{2}{10}$$

$$x = 1 \quad \text{or} \quad 0,2.$$  

The roots of the function $5x^2 - 6x + 1 = 0$ are 1 and 0,2.

**Question 3**

Simplify the following expression

$$\left(\frac{\sqrt{L}}{L^2}\right)^2 = (\sqrt{L} \times L^{-2})^2$$

since $\frac{1}{a^b} = a^{-b}$

$$= (L^{\frac{1}{2}} \times L^{-2})^2$$

since $\sqrt{a} = a^{\frac{1}{2}}$

$$= (L \times L^4)$$

since $(a^{b})^{a} = a^{b \times a}$

$$= L^{5}$$

Or alternatively,
\[
\left( \frac{\sqrt{L}}{L^2} \right)^2 = \left( \frac{L^3}{L^4} \right)^2 \\
= \left( \frac{L^{\frac{3}{2}}}{L^{2\times\frac{1}{2}}} \right) \text{ since } (a^b)^a = a^{b\times a} \\
= \left( \frac{L}{L^4} \right) \\
= (L \times L^4) \text{ since } \frac{1}{a^b} = a^{-b} \\
= L^5
\]

[Option 4]

**Question 4**

Your calculator’s log key is log to the base 10. In the question asked we need to determine a log to the base 5. To use our calculator we thus need to change the base of 5 to a base of 10 or \(e\). The change of base logarithm, rule 4 in the textbook (page 172 (Edition 2) and page 190 (Edition 3)), states that

\[
\log_b(N) = \frac{\ln N}{\ln b} \quad \text{or} \quad \log_b(N) = \frac{\log N}{\log b}.
\]

Therefore

\[
\text{Log}_{18} \left( \frac{34.8}{1091.7} \right) \\
= \frac{\text{Log} \left( \frac{34.8}{1091.7} \right)}{\text{Log} 18} \quad \sin ce \ a^{-b} = a^{\frac{1}{b}} \\
= \frac{\text{Log} 0.0318}{\text{Log} 18} \\
= -1.1922
\]

Or alternatively,
\[
\log_{18} \left( \frac{34.8}{1091.7} \right)
\]

\[
\ln \left( \frac{34.8}{1091.7} \right) = \frac{\ln x}{\ln a}
\]

\[
= \frac{\ln 0.0318}{\ln 18}
\]

\[
= -1.1922
\]

**Question 5**

We need to determine how many months \((t)\), after the new employee has started working at the factory, will he/she be able to assemble 40 units of the product. Substituting the value of \(Q = 40\) into the equation we calculate the value of \(t\):

\[
Q(t) = 50 - 30e^{-0.05t}
\]

\[
40 = 50 - 30e^{-0.05t}
\]

\[
40 - 50 = -30e^{-0.05t}
\]

\[
-10 = -30e^{-0.05t}
\]

\[
-10 \times \frac{1}{-30} = -30e^{-0.05t} \times \frac{1}{-30}
\]

\[
\frac{1}{3} = e^{-0.05t}
\]

\[
\ln \frac{1}{3} = \ln e^{-0.05t}
\]

Take \(\ln\) on both sides

\[
\ln \frac{1}{3} = -0.05t \times \ln e
\]

because \(\ln a^b = b \ln a\)

\[
\ln \frac{1}{3} = -0.05t
\]

since \(\ln e = 1\)

\[
\frac{\ln \frac{1}{3}}{-0.05} = -0.05t
\]

\[
\frac{\ln \frac{1}{3}}{-0.05} = t
\]

\[
t = 21.97225
\]

\[
t \approx 22
\]

[Option 4]
Question 6

To determine the definite integral we first use the basic rule, namely \( \int x^n = \frac{x^{n+1}}{n+1} + c \) when \( n \neq -1 \), to integrate the function:

\[
\int_{-1}^{2} (6 - 4x) \, dx = \left( \frac{6x^{0+1}}{0+1} + \frac{4x^{1+1}}{1+1} \right) \bigg|_{-1}^{2} \\
= \left( \frac{6x^{1}}{1} + \frac{4x^{2}}{2} \right) \bigg|_{-1}^{2} \\
= (6x + 2x^2) \bigg|_{-1}^{2}
\]

remember \((6 - 4x) = (6x^0 - 4x)\) as \( x^0 = 1 \)

Secondly we substitute the given values between which we must integrate (2 and \(-1\) in this case) into the integrated function and subtract these two values thus \( F(x)\bigg|_{x=a}^{x=b} = F(a) - F(b) \):

\[
= (6(2) + 2(2)^2) - (6(-1) + 2(-1)^2) \\
= (12 - 8) - (-6 - 2) \\
= 4 - (-8) \\
= 4 + 8 \\
= 12
\]

[Option 3]
Question 7

To integrate the function, we make use of the basic rule of integration namely \( \int x^n = \frac{x^{n+1}}{n+1} + c \) when \( n \neq -1 \). But we first need to simplify the expression \( x(x+2) \) before we can use the rule. Therefore

\[
\int x(x^2 + 2)\,dx = \int (x^3 + 2x)\,dx = \int x^3\,dx + \int 2x\,dx = \frac{x^{3+1}}{3+1} + \frac{2x^{1+1}}{1+1} + c = \frac{1}{4}x^4 + \frac{2}{2}x^2 + c = \frac{1}{4}x^4 + x^2 + c
\]

[Option 1]

Question 8

The basic rule of differentiation states that \( \frac{d}{dx} x^n = nx^{n-1} \) when \( n \neq 0 \). To make use of this rule we first need to simplify the expression. We can simplify the expression by taking the \( x \) into the brackets and to write \( \sqrt{x} \) as \( x^{\frac{1}{2}} \) when changing from square root form to exponential form. Therefore

\[
y = x(x^2 - \sqrt{x})
\]

\[
y = (x^3 - x^{3/2})
\]

\[
y = x^3 - x^{\frac{3}{2}}
\]

because \( \sqrt{a} = a^{\frac{1}{2}} \)

\[
y = x^3 - x^{\frac{1}{2}}
\]

because \( a^b \times a^c = a^{b+c} \)

Now we differentiate the simplified expression, using the basic rule of differentiation namely \( \frac{d}{dx} x^n = nx^{n-1} \) when \( n \neq 0 \).
\[
\frac{d}{dx} \left[ x^3 - x^{\frac{3}{2}} \right] = 3x^2 - \frac{3}{2} x^{\frac{1}{2}} - 1
\]

\[
= 3x^2 - \frac{3}{2} x^{\frac{1}{2}}
\]

\[
= 3x^2 - \frac{3}{2} \sqrt{x}
\]

[Option 3]

**Question 9**

We need to determine the price if the marginal revenue is zero. The marginal revenue function is the differentiated revenue function. Thus we need to determine the revenue function first. It is given that price is \( P \) and demand is \( Q = 150 - 0.5P \).

Now Revenue is price \( \times \) demand that is \( R = P \times Q \)

Therefore substitute \( Q = 150 - 0.5P \) into the revenue formula:

\[
R = P \times (150 - 0.5P)
\]

\[
= 150P - 0.5P^2.
\]

To determine the marginal revenue we need to differentiate the revenue function. Thus the marginal revenue function (\( MR \)) is

\[
MR = \frac{dR}{dP} = \frac{d}{dP} (150P - 0.5P^2)
\]

\[
= 150 - (2 \times 0.5)P
\]

\[
= 150 - P.
\]

You are asked to find the value of \( P \) for which \( MR \) is equal to 0. Therefore, set

\[
MR = 0,
\]
that is

\[ 150 - P = 0 \]

\[ P = 150. \]

The value of \( P \) is equal to 150 if the marginal revenue is equal to 0.

[Option 1]

**Question 10**

The maximum or minimum value of a function is where the differentiated function is equal to zero or \( \frac{dy}{dx} = 0 \). Now given the function \( y = x^3 - 3x^2 \)

Now differentiating the function \( f(x) \) using the basic rule of differentiation namely \( \frac{d}{dx}x^n = nx^{n-1} \) with \( n \neq 0 \) we get:

\[
\begin{align*}
f(x) &= x^3 - 3x^2 \\
\frac{d}{dx}f(x) &= 3x^2 - (3\times2)x^{2-1} \\
\frac{d}{dx}f(x) &= 3x^2 - 6x
\end{align*}
\]

But the maximum or minimum occurs when \( \frac{dy}{dx} = 0 \). Thus

\[
\frac{d}{dx}f(x) = 3x^2 - 6x = 0
\]

\[ 3x^2 - 6x = 0 \]

\[ 3x(x - 2) = 0 \]

\[ x = 0 \text{ or } 2 \]

[Option 2]