Tutorial Letter 202/2/2014

Quantitative Modelling
DSC1520

Second Semester

Department of Decision Sciences

Important Information:
This tutorial letter contains the solutions for Assignment 02.
Dear student

By now you should have worked through chapters 1 to 3 of the textbook and completed your first and second assignments. As the assignments contain questions from old examination papers, you are in fact already preparing for the examination. Do as many examples as possible - the more examples you work through, the better you will be able to recognise a problem and solve it.

Remember, help is just a phone call or an e-mail away. Please contact me if you need any help with the third assignment. My contact details are as follows:

Office: Hazelwood Campus, Room 4-37, Unisa
Tel: +27 12 433 4602
E-mail: mabemgv@unisa.ac.za

I wish you everything of the best with your preparation for the third assignment.

Ms Victoria Mabe-Madisa
ASSIGNMENT 02: SOLUTIONS

Question 1
To determine the point of intersection of two lines we need to determine a point \((x; y)\) so that the \(x\) and \(y\) values satisfy the equations of both lines. We need to solve the two equations simultaneously. There are different methods you can use to solve the set of equations.

(a) **Elimination method:**

**Step 1:** Eliminate one variable, say \(x\), by adding or subtracting one equation or multiple of an equation from another equation:

Let

\[
2x + 3y = 5 \quad (1)
\]

and

\[
20x - 9y = 21 \quad (2)
\]

be the equations of the two lines.

Now 3 times equation (1) plus equation (2) will eliminate \(y\). But \(3 \times \text{equation } (1)\) is:

\[
6x + 9y = 15 \quad (3)
\]

Now equation \(3 \times \text{equation } (1)\) plus equation (2) is

\[
\begin{align*}
6x + 9y &= 15 \\
20x - 9y &= 21
\end{align*}
\]

\[
26x = 36
\]

Now solve for \(x\):

\[
x = \frac{36}{26} = \frac{18}{13}
\]

**Step 2:** Solve for \(y\). Substitute the value of \(x\) into any one of the equations and solve for \(y\).

Substituting the value of \(x = \frac{18}{13}\) into, say, equation (1):

\[
2 \left( \frac{18}{13} \right) + 3y = 5
\]

\[
3y = 5 - \frac{36}{13}
\]

\[
y = \left( 5 - \frac{36}{13} \right) \times \frac{1}{3}
\]

\[
y = \frac{29}{39}
\]

The two lines intersect in the point \((x; y) = \left( \frac{18}{13}; \frac{29}{39} \right)\).
(b) **Substitution method:**

**Step 1:** Change one of the equations so that a variable is the subject of the equation.

Say $x$ in equation (1):

$$2x + 3y = 5 \quad (1)$$

follows

$$x = \frac{5}{2} - \frac{3}{2}y \quad (3)$$

**Step 2:** Substitute the value of $x$ (equation (3)) into the unchanged equation (2) and solve for $y$. Substituting $x = \frac{5}{2} - \frac{3}{2}y$ into $20x - 9y = 21$:

$$20 \left( \frac{5}{2} - \frac{3}{2}y \right) - 9y = 21$$
$$\frac{100}{2} - \frac{60}{2}y - 9y = 21$$
$$50 - 30y - 9y = 21$$
$$-39y = -29$$
$$y = \frac{-29}{-39}$$
$$y = \frac{29}{39}$$

**Step 3:** Substitute the calculated value of the variable in step 2 into any of the given equations and calculate the value of the other variable. Substitute $y = \frac{29}{39}$ into equation (1) or equation (2). Let’s say we choose equation (1):

$$2x + 3y = 5$$

Substituting $y = \frac{29}{39}$:

$$2x + 3 \left( \frac{29}{39} \right) = 5$$

$$2x + \left( \frac{29}{13} \right) = 5$$

$$2x = 5 - \frac{29}{13}$$
$$2x = \frac{65 - 29}{13}$$
$$x = \left( \frac{36}{13} \right) \div 2$$
$$x = \frac{36}{13} \times \frac{1}{2}$$
$$x = \frac{18}{13}$$

The two lines intersect in the point $(x ; y) = \left( \frac{18}{13} ; \frac{29}{39} \right)$. 
Question 2

We need to solve the following system of equations:

\[
\begin{align*}
  x - 2y + 3z &= -11 \quad (1) \\
  2x - z &= 8 \quad (2) \\
  3y + z &= 10 \quad (3)
\end{align*}
\]

**Step 1:** Determine two equations with the same two unknowns (variables) by adding or subtracting two of the three equations at a time.

Now equations (2) and (3) are already equations in two variables. But the variables in equation (2) are \( x \) and \( z \), and in equation (3) they are \( y \) and \( z \). The equations have to have the same variables. To determine another equation with two variables we can subtract equation (2) from 2 times equation (1):

Now 2 times equation (1) is \( 2x - 4y + 6z = -22 \).

Two times equation (1) minus equation (2):

\[
\begin{align*}
  2x - 4y + 6z &= -22 \\
  -(2x - z = 8) & \quad \text{or} \quad 2x - 4y + 6z &= -22 \\
  -4y + 7z &= -30 \quad (4)
\end{align*}
\]

Thus equation (3): \( 3y + z = 10 \) and equation (4): \( -4y + 7z = -30 \) are two equations with the same two variables namely \( y \) and \( z \).

**Step 2:** Next we solve two equations with the same two unknowns, using any method as described in question 1. Say we use the substitution method:

Make \( z \) the subject of equation (3) and substitute into equation (4) and solve for \( y \):

Now \( z = 10 - 3y \).

Substitute the value of \( z \), namely \( z = 10 - 3y \), into equation (4) and solve for \( y \):

\[
\begin{align*}
  -4y + 7z &= -30 \\
  -4y + 7(10 - 3y) &= -30 \\
  -4y + 70 - 21y &= -30 \\
  -25y &= -30 - 70 \\
  -25y &= -100 \\
  y &= \frac{-100}{-25} \\
  y &= 4
\end{align*}
\]
**Step 3:** Substitute $y = 4$ into equation (3) and solve for $z$:

\[
3y + z = 10 \\
3(4) + z = 10 \\
12 + z = 10 \\
z = 10 - 12 \\
z = -2
\]

**Step 4:** Substitute $y = 4$ and $z = -2$ into equation (1) or (2) and solve for $x$. Say we use equation (2):

\[
2x - z = 8 \\
2x = 8 + z \\
2x = 8 + (-2) \\
2x = 6 \\
x = \frac{6}{2} \\
x = 3
\]

Therefore $x = 3$; $y = 4$ and $z = -2$.

**Question 3**

Equilibrium is the price and quantity where the demand and supply functions are equal. It means the point where the lines of the demand and supply function intersect.

Therefore we need to determine the value of $P$ and $Q$ for which $P_d = P_s$ or $Q_d = Q_s$. Now given is $P_d = 100 - 0,5Q$ and $P_s = 10 + 0,5Q$. Thus

\[
P_d = P_s \implies 100 - 0,5Q = 10 + 0,5Q \\
-0,5Q - 0,5Q = 10 - 100 \\
-Q = -90 \\
Q = \frac{-90}{-1} \\
Q = 90.
\]

To calculate the price at equilibrium, we substitute the value of $Q$ into the demand or supply function and calculate $P$. Say we use the demand function, then

\[
P = 100 - 0,5(90) \\
P = 100 - 45 \\
P = 55.
\]

The equilibrium price is equal to 55 and the quantity is 90.
**Question 4**

Breakeven is when no profit is made or when revenue is equal to cost.

Revenue or Income is defined as price times quantity or
\[ R = p \times q \text{ or } p \times x. \]

Now given is quantity as \( x \) and price as R3,10. Thus
\[
\begin{align*}
\text{Revenue} &= R(x) = p \times x \\
&= 3,10 \times x \\
&= 3,10x.
\end{align*}
\]

The revenue function is equal to \( R(x) = 3,10x \).

The cost function is given as \( c(x) = 300 + 0.92x \).

Thus at breakeven:
\[
\begin{align*}
R(x) &= c(x) \\
3,10x &= 300 + 0.92x.
\end{align*}
\]

Now we need to solve for \( x \):
\[
\begin{align*}
3,10x - 0.92x &= 300 \\
2.18x &= 300 \\
x &= \frac{300}{2.18} \\
x &\approx 137.61468 \\
x &\approx 138
\end{align*}
\]

Approximately 138 items must be sold to breakeven.

**Question 5**

We need to graphically represent \( y \geq 1 - \frac{1}{3}x \). To draw a linear inequality we first change the inequality sign (\( \geq \) or \( \leq \) or \( > \) or \( < \)) to an equal sign (\( = \)) and draw the graph of the line. But we need two points to draw a line. Choose the two points where the lines cut through the \( x \)-axis (\( x \)-axis intercept, thus \( y = 0 \)) and \( y \)-axis (\( y \)-axis intercept, \( x = 0 \)). Calculate \((0; y)\) and \((x; 0)\) and draw a line through the two points.

**Step 1**: Change the \( \geq \) sign to a \( = \) sign. Choose the values of \( x \) and \( y \) randomly or use the points \((0; y)\) and \((x; 0)\) as below, sketch the two points and draw the line of the graph through the two points.

Let \( x = 0 \), then \( y \) is equal to
\[
\begin{align*}
y &\geq 1 - \frac{1}{3}x, \\
y &\geq 1 - \frac{1}{3}(0) \\
y &= 1.
\end{align*}
\]
Let \( y = 0 \), then \( x \) is equal to

\[
\begin{align*}
y &\geq 1 - \frac{1}{3}x. \\
0 & = 1 - \frac{1}{3}x. \\
-1 & = -\frac{1}{3}x \\
-1 \times -3 & = x \\
x & = 3.
\end{align*}
\]

The two points calculated are \((0; 1)\) and \((3; 0)\).

**Step 2:** Determine the feasible region for the inequality by substituting a point on either side of the drawn line into the equation of the inequality. The inequality region is the area where the selected point makes the inequality true. Shade the feasible area as follows:

Select points \((0; 0)\) to the left of the line and \((1; 1)\) to the right of the line.

Now the point \((0; 0)\) to the left of the line makes the inequality

\[
\begin{align*}
y &\geq 1 - \frac{1}{3}x \\
0 &\geq 1 - \frac{1}{3}(0) \\
0 &\geq 1
\end{align*}
\]

false. Therefore the feasible region is to the right of the line.
Question 6
When the market price per unit is $P = 30$, the consumer will purchase

\[
P = 48 - 0.2Q \\
30 = 48 - 0.2Q \\
30 - 48 = -0.2Q \\
-18 = -0.2Q \\
Q = \frac{-18}{-0.2} \\
\Rightarrow 90 \text{ units.}
\]

- Draw a rough sketch of the graph.

Thus, the total amount that the consumer is willing to pay for the first 90 items = the area $C$ (i.e. area A plus area B) under the demand function between $P = 0$ and $P = 48$.

Area ($C$) = area triangle ($A$) plus area square ($B$)
Area (C) = \( \frac{1}{2} \times \text{base} \times \text{height} \) + \( \text{length} \times \text{width} \).

\[
\text{Area } C = \left[ \frac{1}{2} \times 90 \times (48 - 30) \right] + 30 \times 90 \\
= \left[ \frac{1}{2} \times 90 \times 18 \right] + 2700 \\
= \frac{1620}{2} + 2700 \\
= 810 + 2700 \\
= 3510.
\]

The total amount that the consumer is willing to pay for the first 90 units is, R3510.

Now the amount that the consumer actually pays is the area of the square (B).

\[
\text{Area of the square } (B) \\
= \text{length} \times \text{width} \\
= 30 \times 90 \\
= 2700.
\]

The amount that the consumer actually pays is R2700.

Now the consumer surplus is defined as:

\[
\text{CS} = \text{amount willing to pay} - \text{amount actual spent} \\
= 3510 - 2700 \\
= 810.
\]

**Question 7**

Let

\[
x_1 + x_2 \leq 13 \quad (1) \\
2x_1 - x_2 \leq 8 \quad (2) \\
-2x_1 + 3x_2 \leq 12 \quad (3) \\
x_1, x_2 \geq 0 \quad (4)
\]

In this case we are working with \( x_1 \) and \( x_2 \) which might feel odd to you. Up to now we have worked with \( x \) and \( y \). Now remember \( x \) and \( y \) are just placeholders for numbers. We can use any character to denote placeholders. In this case we are just using \( x_1 \) and \( x_2 \) instead of \( x \) and \( y \).

**Step 1:**

To draw a linear inequality we first change the inequality sign (\( \geq \) or \( \leq \) or \( > \) or \( < \)) to an equal sign (=) and draw the graph of the line. But we need two points to draw a line. Choose the two points where the lines cut through the \( x_1 \)-axis (\( x_1 \)-axis intercept, thus \( x_2 = 0 \)) and \( x_2 \)-axis (\( x_2 \)-axis intercept, \( x_1 = 0 \)). Calculate \((0; x_2)\) and \((x_1; 0)\) and draw a line through the two points. See the table on the next page for calculations.
Step 2:
Determine the feasible region for each inequality by substituting a point on either side of the drawn line into the equation of the inequality. The inequality region is the area in which the selected point, makes the inequality true. See the table below for calculations.

<table>
<thead>
<tr>
<th>Inequality</th>
<th>Step 1</th>
<th>Step 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_1 + x_2 \leq 13$</td>
<td>$x_1 + x_2 = 13$ Point: (0; 13)</td>
<td>Select points (0,0) below the line and (13;13) above the line. Above: $13 + 13 \leq 13$ – False Below: $0 + 0 \leq 13$ – True Area below the line is true.</td>
</tr>
<tr>
<td>$2x_1 - x_2 \leq 8$</td>
<td>$2x_1 - x_2 = 8$ Point (0; −8)</td>
<td>Select points (0,0) above the line and (13;13) below the line. Below: $2(13) - 13 \leq 8$ – False Above: $2(0) - 0 \leq 8$ – True Area above the line is true.</td>
</tr>
<tr>
<td>$-2x_1 + 3x_2 \leq 12$</td>
<td>$-2x_1 + 3x_2 = 12$ Point (0; 4)</td>
<td>Select points (0,0) below the line and (0;13) above the line. Above: $-2(0) + 3(13) \leq 12$ – False Below: $-2(0) + 3(0) \leq 12$ – True Area below the line is true.</td>
</tr>
<tr>
<td>$x_1, x_2 \geq 0$</td>
<td>$x_2 = -6$ Point (−6; 0)</td>
<td>Area above the $x_1$-axis and to the right of the $x_2$-axis is true.</td>
</tr>
</tbody>
</table>

Step 3:
The feasible region is the region where all the inequality regions are true at the same time.
**Question 8**

We need to determine the maximum value of the function \( P = 20x + 30y \) subject to the given constraints. To do this we determine all the corner points of the feasible region of the constraints and substitute them into the objective function (function you want to maximise or minimise) to determine the maximum or minimum value of the function. The method can be summarised as follows:

**Step 1:** Determine the coordinates of all corners of the feasible region by

- determining the point where the two lines intersect - solving two equations with two unknowns 
  or
- read the coordinates of the intersection point from the graph.

**Step 2:** Substitute the corner points into the objective function and calculate the value of the objective function.

**Step 3:** Choose the corner point that results in the highest (maximisation) or the lowest (minimisation) objective function value.

**Step 1:**

The corner points of the feasible region in the graph below are the points A, B, C, D and the origin is \((0 ; 0)\).

To determine the coordinates of the corners of the feasible region we can

- read the coordinates of the intersection points from the graph or
- determine the point where the lines intersect by solving two equations with two unknowns using the substitution or elimination methods.
**Point A:**

Point A is the point where line (2) cuts the y-axis. These coordinates of the point can be read from the graph as (0; 70).

**Point B:**

Point B is the point where lines (2) and (3) intersect. These coordinates of the point can be read from the graph as (20; 60) or can be calculated by solving two equations with two unknowns by using for example the method of substitution.

We need to solve the following two equations simultaneously:

\[ x + 2y = 140 \quad (2) \quad \text{and} \quad x + y = 80 \quad (3) \]

First we make \( x \) the subject of equation (2) by subtracting \( 2y \) from each side of equation (2):

\[
\begin{align*}
x + 2y - 2y &= 140 - 2y \\
x &= 140 - 2y
\end{align*}
\]

Substitute the value of \( x \) namely \( x = 140 - 2y \) into equation (3) and solve for \( y \):

\[
\begin{align*}
x + y &= 80 \quad (3) \\
(140 - 2y) + y &= 80 \\
-2y &= 80 - 140 \\
-y &= -60 \\
y &= 60
\end{align*}
\]

Substitute the value of \( y = 60 \) into equation (3) and solve for \( x \):

\[
\begin{align*}
x + y &= 80 \\
x + 60 &= 80 \\
x &= 80 - 60 \\
x &= 20
\end{align*}
\]

The coordinates of point B are (20; 60).

**Point C:**

Point C is the point where line (1) and (3) intersect.

The coordinates of the point can be read from the graph as (40; 40) or you can calculate it by solving two equations with two unknowns by using for example the method of substitution:

We need to solve the following two equations simultaneously:

\[ 2x + y = 120 \quad (1) \quad \text{and} \quad x + y = 80 \quad (3) \]

First we make \( y \) the subject of equation (1) by subtracting \( 2x \) from each side of equation (1):

\[
\begin{align*}
2x + y - 2x &= 120 - 2x \\
y &= 120 - 2x \quad (4)
\end{align*}
\]
Substitute the value of \( y \) of equation (4) into equation (3) and solve for \( x \):

\[
\begin{align*}
x + y &= 80 \quad (3) \\
x + (120 - 2x) &= 80 \\
x + 120 - 2x &= 80 \\
-x &= 80 - 120 \\
-x &= -40 \\
x &= 40
\end{align*}
\]

Substitute the value of \( x = 40 \) into equation (3) and solve for \( y \):

\[
\begin{align*}
x + y &= 80 \\
40 + y &= 80 \\
y &= 80 - 40 \\
y &= 40
\end{align*}
\]

The coordinates of Point C are (40 ; 40).

**Point D:**

Point D is the point where line (1) cuts the \( x \)-axis. This coordinate of the point can be read from the graph as (60 ; 0).

**Step 2:**

Substitute the corner points of the feasible region into the objective function and determine the value of the objective function for each corner point:

<table>
<thead>
<tr>
<th>Corner points of feasible region</th>
<th>Value of ( P = 20x + 30y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A : x = 0 ; y = 70 )</td>
<td>( P = 20(0) + 30(70) = 2100 )</td>
</tr>
<tr>
<td>( B : x = 20 ; y = 60 )</td>
<td>( P = 20(20) + 30(60) = 2200 ) ← Maximum</td>
</tr>
<tr>
<td>( C : x = 40 ; y = 40 )</td>
<td>( P = 20(40) + 30(40) = 2000 )</td>
</tr>
<tr>
<td>( D : x = 60 ; y = 0 )</td>
<td>( P = 20(60) + 30(0) = 1200 )</td>
</tr>
<tr>
<td>Origin: ( x = 0 ; y = 0 )</td>
<td>( P = 20(0) + 30(0) = 0 )</td>
</tr>
</tbody>
</table>

**Step 3:**

Choose the corner point resulting in the highest (maximisation) objective function value.

Maximum of \( P \) is at point B where \( x = 20 \), \( y = 60 \) and \( P = 2200 \).
Question 9

It is given that \( x \) and \( y \) are the number of units of cake 1 and cake 2, respectively. To help us with the formulation of the problem, we summarise the information given in a table with the headings: resources (items with restrictions), the variables \( (x \text{ and } y) \) and capacity (amount or number of resources available).

<table>
<thead>
<tr>
<th>Resources</th>
<th>( x )</th>
<th>( y )</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flour</td>
<td>1,8</td>
<td>0,75</td>
<td>18kg</td>
</tr>
<tr>
<td>Eggs</td>
<td>3</td>
<td>2</td>
<td>36</td>
</tr>
<tr>
<td>Sugar</td>
<td>0,4</td>
<td>0,6</td>
<td>10kg</td>
</tr>
<tr>
<td>Number of units of cakes</td>
<td></td>
<td></td>
<td>Never negative</td>
</tr>
</tbody>
</table>

Using the table, the following constraints can be defined:

\[
\begin{align*}
1,8x + 0,75y & \leq 18 \text{ (flour)} \\
3x + 2y & \leq 36 \text{ (eggs)} \\
0,4x + 0,6y & \leq 10 \text{ (sugar)} \\
x, y & \geq 0
\end{align*}
\]

Question 10

It is given that \( x \) and \( y \) are the number of two-person and four-person boats, respectively.

To help us with the formulation of the problem, we summarise the information given in a table with the headings: resources (items with restrictions), the variables \( (x \text{ and } y) \) and capacity (amount or number of resources available).

<table>
<thead>
<tr>
<th>Resources</th>
<th>( x )</th>
<th>( y )</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cutting</td>
<td>0,9</td>
<td>1,8</td>
<td>864</td>
</tr>
<tr>
<td>Assembly</td>
<td>0,8</td>
<td>1,2</td>
<td>672</td>
</tr>
<tr>
<td>Profit</td>
<td>2,500</td>
<td>4,000</td>
<td>15</td>
</tr>
<tr>
<td>Number of boats</td>
<td></td>
<td></td>
<td>Never negative</td>
</tr>
</tbody>
</table>

Using the table, the following constraints can be defined:

\[
\begin{align*}
0,9x + 1,8y & \leq 864 \\
0,8x + 1,2y & \leq 672 \\
x, y & \geq 0
\end{align*}
\]