MO001/4/2018

Theoretical Computer Science I
COS1501/XOS1501

Semesters 1 and 2

School of Computing

IMPORTANT INFORMATION

This contains important information about your module.

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1 Introduction

Dear student

As noted in Tutorial Letter 101, this is a blended module, and therefore much of your module material is available on myUnisa. However, in order to support you in your learning process, we also provide this study material in printed format.

Below you will find all the material that is available on the COS1501 site on myUnisa.

2 Home page

COS1501 – Theoretical Computer Science I

Dear students

Welcome to the myUnisa module site for COS1501. Your lecturers for this module are mentioned in Tutorial Letter 301, which is provided on myUnisa.

This module focuses on discrete mathematics with the specific focus areas being the basic ideas of set theory, relations, functions and logic, and we show you how to interpret definitions and apply rigorous proofs. We trust that you will find this module interesting.

There is no prescribed book for this module, but a study guide is provided. The learning units in this letter will guide you through the learning matter.

Statistics from previous years show that most students who have mastered the required study material and attempted to do all the assignments, have passed the module. So, please, do likewise.

Assessment

You will be assessed in various ways in this module.

- Three assignments should be submitted and it is important to attempt to do all of them. Assignment 01 is compulsory to obtain examination admission.
- Self-assessment questions for each assignment should also be attempted. Answers to all self-assessment questions will be provided.
- One two-hour, written examination is compulsory.

A work schedule is provided in Tutorial Letter 101. This schedule gives you an indication of the pace required so that you will be able to submit your assignments on time and are left with enough time for examination preparation.

General information

This module site is dedicated to support your learning for this module. Please make a regular habit of checking this site.

- **Home page**: This is the page you are currently on, and you will always start on this opening page of the COS1501 myUnisa site.
- **Learning units**: This is where you are guided through the required work for this semester, including what has to be studied, extra notes and assignments.
- Use the **Discussion Forums** to interact with other students about assignments, study groups, self-tests and the exam. Please post discussions in an appropriate forum and topic discussion line and do not use this forum for idle chat. We will be monitoring the discussion periodically and will contribute when necessary. You will be assigned to an e-tutor early in the semester. You can also communicate and post questions on the assigned site of your e-tutor. If your questions are not addressed, contact one of the lecturers via e-mail or telephone.

- **Assessment Info**: Use this page to submit assignments for the semester. You can also see your assignment marks here.
- **Announcements** are added when there is an issue that we want to tell you about or to provide useful information, so please keep checking these. The contents of the announcement will be e-mailed to your Unisa myLife account – make sure that you check this regularly too.
- Go to **Official Study Material** to find PDF versions of the study guide, tutorial letters and past examination papers.
- The **Additional Resources** option is where you will find other relevant documents such as Tutorial Letter 101 with the work schedule and assignments, assignment solutions and discussions, among other documents. You will often be referred to **Additional Resources** in the **Learning Units**.
- The **Schedule** will remind you of important current events and information such as assignment and examination dates.

Contacting us

You can contact us, or other module lecturers, by using the following resources.

- Tutorial Letter 301 provides important information about the School of Computing (SOC), including all the lecturers' contact information.
- You are welcome to visit the SOC website (http://osprey.unisa.ac.za) where you can view the lecturers' contact information and availability (http://osprey.unisa.ac.za/reg.htm). (Right click on the link and select <<Open in New Window>> to view the site properly.) Should a lecturer not be available, please phone the school's general number (011 670 9200) to leave a message.
- E-mail: COS1501-18-S1@unisa.ac.za for semester 1 or COS1501-18-S2@unisa.ac.za for semester 2.

3 Learning Unit 0: Orientation

Introduction

Welcome
Welcome to the learning units that will provide information and guidelines to your study material for the semester. Here you will find out about the study guide and the assignments that need to be done. It is important that you access the assessment and study plan page, which provides guidelines to enable you to cover all the study material in the allotted time (and submitting the assignments in on time). Also read Tutorial Letter 101 in detail.

This module focuses on discrete mathematics and its application in computer science. Concepts and skills needed for a theoretical understanding of computer science are introduced. If you feel a flash of fear when you hear the word “theoretical”, just remember that theory is much tidier than practice: one knows precisely what the meaning of the terms is and the problems always have solutions. Rigorous proofs are expected in this module.

This module does not involve practical work on a computer. However, there is a strong emphasis on practicing the skills acquired by doing the exercises provided in the study guide and the assignments. By completing all the assignments and the self-assessment questions and assignment, you are actually preparing for the examinations. It is very important to work consistently throughout the semester in order to master the contents of this module. It is also important to write down the solutions to activities, self-assessment exercises and assignments so that you get used to the correct mathematical format required for this module. You will have to write down the correct format in the examination.

This first learning unit (Learning Unit 0) covers more general topics that relate to the module. The rest of the learning units will guide you through the work that has to be studied as part of this module. You can use the table of contents in the learning units to move to the unit that you want to view.

Once you start moving around these pages, you can always return to the table of contents by clicking the link at the top or bottom centre of each page. Here you can also find links to the previous page and the next page in the sequence of pages in the learning units.

Purpose
On completing this module, you will be able to apply the fundamental knowledge and skills of discrete mathematics critically. The module forms part of the theoretical foundation of a computer science major. This background is relevant to computing fields such as relational databases, the development of provably correct programs, and the analysis of algorithms that will contribute to the development of computing in Southern Africa, Africa or globally. The module will support further studies and applications in the computing discipline.

This module forms part of the theory of a computer science major, supporting further studies and applications in the sector of computer programming, bioinformatics and linguistics. These concepts and skills contribute to the development of the computing field in southern Africa, Africa and globally.

Outcomes

Specific outcome 1: Construct logical arguments, using a variety of mathematical tools.

Specific outcome 2: Construct proofs in a clear and concise way using mathematical reasoning techniques.

Specific outcome 3: Demonstrate knowledge and understanding of the definitions, laws and operations of set theory.

Specific outcome 4: Synthesise and critically analyse relations, functions and binary sets that are represented as sets containing ordered pairs.

Specific outcome 5: Perform operations on vectors and matrices.

E-tutors

Unisa offers online tutorials (e-tutoring) to students registered for modules at NQF levels 5, 6 and 7; that is, qualifying first-year, second-year and third-year modules. Registered students will be allocated to an e-tutor and will then be able to interact with this tutor and an assigned group of students. You will receive an SMS informing you about your group, the name of your e-tutor and instructions on how to log onto myUnisa in order to receive further information on the e-tutoring process. Of course, you can still contact either of the module lecturers if you need to.

We strongly encourage you make full use of the e-tutoring facility. The e-tutor will facilitate the forum and can be contacted via e-mail. You can expand your understanding of the module content by attempting and discussing the additional exercises that will be provided on the forum. It is important to post your problems online – in this way, other students who have the same problems will also learn from the communications. The e-tutor will discuss old exam papers and make you aware of why students lose marks in the exam. The point of the e-tutor is to help you, and it would be a pity if you did not use this valuable resource.

Study guide

There is no prescribed book for this module. Instead, you are provided with a study guide that is also available as a PDF file on myUnisa under **Additional Resources**.

In the introductory learning unit of the study guide, information about this module and aspects on “how to study” is provided.

Number sets are introduced and then we move on to set theory, studying sets based on these number sets. We investigate elements of, and operations on, sets. Equality between sets is investigated by using Venn diagrams or by providing a formal proof. The concept of inclusion-exclusion is introduced. Certain kinds of sets turn out to be particularly useful, so they get special names, such as relation and function. Relations are investigated, examining their properties and testing for particular types of relation. We also study functions and their properties, as well as operations on functions. Binary operations are studied and then vectors and matrices are introduced. An introduction to logic is provided. Concepts from the field of logic, such as truth tables, connectives, quantification and predicates are introduced. Different mathematical proof strategies are implemented. Throughout, great emphasis is placed on sound reasoning and the ability to construct mathematical proofs.

Note: Tutorial Letter 102 provides solutions to all the self-assessment exercises that are provided in the study guide. This tutorial letter is also available as a PDF file on myUnisa under **Additional Resources**.

Recommended books
Should you wish to know more about a particular topic, you may consult any of the following books. (These books are not necessarily included in the study collection of the Unisa library.)

Ensley, DE & Crawley, JW. Discrete Mathematics: Mathematical Reasoning and Proof with Puzzles, Patterns and Games. John Wiley & Sons, Inc., 2006.
Visit http://www.gobookee.net/ensley-crawley-discrete-mathematics/

Grimaldi, RP. Discrete and Combinatorial Mathematics: An applied Introduction, 5th edition. Pearson Education, 2004.

Johnsonbaugh, R. Discrete Mathematics, 7th edition. Pearson Education Inc., 2009.

Labuschagne, WA. A User-friendly Introduction to Discrete Mathematics for Computer Science. Pretoria, UNISA, 1999.

Rosen, KH. Discrete Mathematics and its Application, 6th edition. McGraw-Hill, 2007.

Books online

You can also find books on the theory of computation in the online resources from the library.

- Go to oasis.unisa.ac.za
- Click on Library Links → Search for Information Resources → A-Z list of electronic resources → s → Safari Business and Tech Books Online
- In the search box at the top right of the page, search the entire site for “discrete mathematics”.
- You can then click on a book’s title, which should take you to the Table of Contents for that book. From here, you can click on the chapter you want to read.

Electronic resources

Computer-aided instruction tutorials

A computer-aided instruction (CAI) tutorial named “Relations” is provided on a CD that you should have received in your study package. It is not essential for you to work through these supplementary tutorials. However, they can help you to understand some study material better.

The minimum system requirements to run these tutorials is a Pentium II computer, preferably with 32MB RAM, a CD-ROM drive, a mouse and a 16-bit (High Colour) colour display, with at least Microsoft Windows 95.

How to run the tutorial:

- Insert the CD-ROM into your CD-ROM drive.
- Navigate to the CD drive in Windows Explorer.
- Double click on the executable file Relation.exe.

Note: If you have problems in running this tutorial (e.g. in Windows 7 an irritating pop-up is displayed repeatedly), you can copy the whole cos1501 directory to the hard disk of your computer, and then double click on the executable file Relations.exe.

To download the tutorial:

You can also download it from http://osprey.unisa.ac.za/TechnicalReports/cos1501/cos1501.zip if you want to:

- Go to the given web link.
- Save cos1501.zip to your computer.
- Double click on the saved cos1501.zip file.
- Choose *Extract all files* from top row of buttons on the opened page.
- Choose a destination for the extracted files.
- Click on the Extract Button.
- If asked whether you want to merge folders, you should select “No”.
- Double click on the cos1501 folder, double click on the Relations folder, and then double click on the Relations.exe icon (it looks like a ball with a red ribbon around it).
- You can now navigate through the tutorial.

We have tried these steps without experiencing any problems. Depending on your browser and operating system, there might be a slight variation in these steps. Ask someone experienced with computers to help you if you struggle.

Problems while running the tutorial:

- If the first screen is unattractive and spotty, your screen setting may be of a too low quality. Change your computer’s display settings to 16-bit or higher.
- The bottom part of the screen and left or right of the screen appear to be “missing”. Your computer is probably set at the incorrect resolution. This should be changed to at least 800x600.

- The picture on your screen appears in a window in the centre of the screen with open space around it. Your resolution setting is higher than what is required. It is not essential that you reset it. Reset it if you want a bigger picture on your screen. (See procedure above.)

Osprey web server
The School of Computing also uses the Osprey web server (http://osprey.unisa.ac.za/). The purpose of this server is to provide information about the school. It does not offer student administration services. There is also a discussion forum here that can be used.

Assessment

Assessment plan
You will be assessed in several ways in this module.

- There are three assignments to be submitted. You will find the assignment questions in Tutorial Letter 101.
  - All three assignments are multiple-choice question (MCQ) assignments.
  - The first assignment counts 20%, the second assignment counts 40% and the third assignment counts 40% to the semester mark.
  - You should do the assignments individually, not in groups.
  - The assignments should be submitted to myUnisa for marking.
  - Together, these three assignments count 20% towards the final module mark.
- Self-assessment questions are provided for Assignments 02 and 03.
  - These should not be submitted, but it is very important that you attempt these self-assessment questions in preparation of the examination.
- Example Assignments 01, 02 and 03 with solutions are provided.
  - These should give you a good indication of how to attempt your assignments, which should be submitted.
- There is one written examination.
  - This exam counts 80% towards the final module mark.
  - You must submit at least the first assignment by the due date to obtain admission to the examination.

Why do assignments? In the first place, we need to provide proof to the Department of Higher Education and Training that you are an active student. Therefore, it is compulsory to submit Assignment 01 by its due date. Furthermore, experience has shown that a student who does not work systematically during the semester is likely to give up and does not even attempt to write the examination.

Your final mark will be calculated as follows:
Semester mark (out of 100) × 20% + Examination mark (out of 100) × 80%

In order to pass this module, a final mark of at least 50% is required.

Example: The following example shows how the assessment system works, assuming that Assignments 01, 02 and 03 have been submitted.

Assignment	**Mark**	**× Weight**	**Contribution to semester mark**
01	80%	× 0.20	16%
02	80%	× 0.40	32%
03	80%	× 0.40	32%
		Semester mark	80%

The resulting semester mark is 80%.
Suppose you obtain 90% in the examination. The final mark will be calculated as follows:
(80 × 0.20)% + (90 × 0.80)% = (16 + 72)% = 88%.

Note: The semester mark will not contribute towards the results of students writing a supplementary examination.

Due dates
You will find a list of due dates for the assignments in Tutorial Letter 101.

Submission of assignments
Submit assignments **electronically** via myUnisa. For detailed information and requirements as far as assignments are concerned, refer to the *my Studies @ Unisa* brochure that you received with your study package.

The steps to be followed when you submit an assignment via myUnisa:

- Go to myUnisa at https://my.unisa.ac.za
- Log on with your student number and password.
- Choose the relevant module (COS1501) in the orange block.
- Click on **Assignments** in the menu on the left-hand side of the screen.
- Click on the assignment number of the assignment that you want to submit.
- Follow the instructions.

Note that administrative assignment enquiries (e.g. whether or not the university has received your assignment or the date on which an assignment was returned to you) should be addressed as stipulated in the *my Studies @ Unisa* brochure and not to the academic department.

Plagiarism of assignments
When your assignment has been marked, a percentage will be awarded. This is an indication of how correct your answers were, as well as the quality of your assignment. Copying other students' solutions, or the official solutions from a previous or current semester, is plagiarism, which is a punishable offence that may lead to expulsion from the university. At the very least, you will receive 0% for a plagiarised assignment.

Study plan

The study plan is available in Tutorial Letter 101 and as a PDF document in the **Additional Resources** option of the COS1501 website on myUnisa. You can use this plan to guide your studies through the semester. By following this guide, you should ensure that you get all the assignments done by their due dates and you will be able to cover the whole syllabus.

Examination

Use your *my Studies @ Unisa* brochure for general examination guidelines and examination preparation guidelines.

Owing to regulatory requirements, in order to be considered for examination admission a student must submit at least Assignment 01.

The examination consists of one two-hour, written examination paper, which will be in English only. You will be examined on the content of the study guide, all tutorial letters and this MO.

The maximum mark for each question is indicated in brackets next to the question. Read all the questions carefully and then answer them in any order. Remember to number the questions correctly.

Most of the examination questions will closely resemble the exercises in the study guide (solutions to these questions are provided in Tutorial Letter 102) and assignment questions.

Keep in mind that you had the study guide and the myUnisa learning units open next to you when you attempted the assignments. They will not be there when you write the examination paper! Therefore, when you have reached the point in your preparation for the examination where you consider yourself to have mastered the material, we suggest that you try to do past examination papers **without** consulting books or study material.

Note: We include an example examination paper with solutions in this MO. There are other downloadable examination papers available on myUnisa as part of the official study material. The model solutions for these papers will not be made available. You are welcome to ask questions on specific topics that you do not understand.

We further recommend that you do not attempt to memorise anything that you do not understand during your preparation for the examination. Instead, you should take the time to understand the study material thoroughly in order to be able to apply it.

The examination paper will test you on **study units 3 to 10** of the study guide, and the material in **all** the tutorial matter. **Study units 1 to 2** of the study guide provide background material.

Bear in mind that the assignment questions do not cover all the work that we test in the examination paper. You have to prepare **all the work** prescribed above.

It will be expected of you to write down the answers to all the examination questions and provide proofs where required.

How to prepare:
Cover all the study units in the study guide thoroughly and **test yourself** by doing the activities and exercises before you look at the solutions in Tutorial Letter 102.

Work through the 2009 examination paper and the extra examples, with the solutions provided in this MO. It is very important that you also do all the self-assessment questions provided in this MO. Make sure that you understand the model solutions to **all assignment questions** provided in the tutorial letters.

Take note of the **structure and notation of solutions** provided in all tutorial matter. For example, when a proof is required and connectives such as '**iff**' or '**if…then**' are left out, the proof is not convincing. Also, **symbols** (e.g. '$\cap$') should be used as **connectives** for **sets** (e.g. $Y \cap W$), and **words** (e.g. 'and') should be used as **connectives** in **sentences** (e.g. $x \in Y$ **and** $x \in W$). Some proofs should start with the word '**assume**' and then logic reasoning should follow. These and other notation issues are mentioned in the hints provided in the assignments.

The examination, and the supplementary examination that follows in the examination period of the following semester, will have a structure and format similar to the 2009 examination paper, but the order of the sections will be in line with the order in which the concepts appear in the study guide. Other previous years' papers are also available on myUnisa.

Additional advice:

Venn diagrams: Draw your diagrams in stages as described on page 51 of the study guide. Also remember to draw the sets within the context of a universal set, name the sets and provide subscripts for the diagrams.

Relations and functions: You need to **apply** the definitions, not merely give them. For example, when you want to prove that a relation is functional and you only write the statement 'for every *x* there is only one *y*', you will receive no marks since it is neither a properly formulated definition nor a proof. **Relate each answer to the actual definition of the specific relation or function** given in the question.

When solving problems, do not forget the useful **shortcut notations** that help you to express an English sentence in precise mathematical notation. For example, an even number can be expressed as $2k$, an odd number as $2k + 1$, a multiple of three as $3k$ and so on, and two consecutive numbers could be k and $k + 1$, with $k \in \mathbb{Z}$.

Mathematical proofs and counterexamples: If you are required to provide a mathematical proof and you give an example instead of a general proof, you will receive no marks for that answer. In other words, **never** attempt to prove that something is true by using an example. However, you **should use a counterexample** to prove that something is not true. If you are required to give a **counterexample** and you attempt to give some mathematical proof instead of a **counterexample**, you will receive no marks for that answer.

Note: Read all the hints provided with assignments in Tutorial Letter 101. These hints will help you to avoid making general mistakes in the examination.

We wish you many fruitful hours of examination preparation and success when you write the examination!

Activity

One of the things that myUnisa does not have is the ability for you to create an online profile for yourself. This means that other students (and even your lecturers) know very little about you.

Please take the time to go the discussion forum, and then to the introductions forum. In a new topic, please introduce yourself to the group.

- Tell us who you are: your name and age (if you want to)
- Maybe tell us a little about your background: where are you from, in which city do you currently live, do you have a full-time job, and so on.
- Also, read some of the introductions and respond to the introductions of the other students.

Progressing through the learning units

Each one of the learning units that follows this introductory section is broken down into three parts:

- Study material: Here you will find out which study unit of the study guide you are expected to study in the learning unit to meet the assessment outcomes. (Note that **study units** are provided in the **study guide,** whereas **learning units** are provided in this MO.) An indication of the time span in which you should cover the material, if you are to keep up to date with your studies, is provided.
- In the learning units, we provide as activities some additional exercises and example assignments that you should attempt. We urge you never to look at a solution before you have spent some time grappling with the problem. Remember that there is no effortless way to acquire worthwhile skills.

We suggest that you tackle each learning unit as follows: Follow the guidelines in the introduction study unit of the study guide to cover the relevant material in the study guide and Tutorial Letter 102. Then go through the corresponding notes in the learning unit carefully and do the activities as you meet them. Do not tackle a learning unit before you have mastered the previous one.

The learning units that deal with the assignments will be structured differently. In these units, information regarding the scope of the assignment and the assignment questions and solutions are provided.

4 Learning units 1 and 2 – Numbers systems: $\mathbb{Z}^+$, $\mathbb{Z}^{\geq}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$

Study Material

Study guide
You should cover study units 1 and 2 in the study guide.

Time allocated
You will need one week to master these learning units. You can find the time frame for the study programme in Tutorial Letter 101.

Notes

Background
Some background material to number systems is proved in the study units to recap some material that you probably covered in school. Different classifications of types of numbers and their characteristics are discussed, and how these numbers can be re-written by scientists to make advanced mathematics possible.

Number systems are introduced in study units 1 and 2 in the study guide. Key questions are provided, which you should be able to answer when you have covered these study units.

Applications
You should be familiar with the properties of the different number systems $\mathbb{Z}^+$ (the set of positive integers), $\mathbb{Z}^{\geq}$ (the set of non-negative integers), $\mathbb{Z}$ (the set of integers), $\mathbb{Q}$ (the set of rational numbers) and $\mathbb{R}$ (the set of real numbers) and be able to apply the different laws and definitions relevant to each number system. You should also be able to factorise expressions.

Activities

Do the activities provided in study units 1 and 2 to consolidate your knowledge of the work in this learning unit.

Study unit 1

Activity 1-11:

1. *Factorising:* If we need to factorise an expression of the form $x^2 + ax + b$ or ($x^2 – ax – b$ or $x^2 + ax – b$ or $x^2 – ax + b$) we need to find some c and some d, such that a = c + d and b = (c)(d) and $(x + c)(x + d) = x^2 + ax + b$.

(a) $x^2 + 6x + 9 = x \cdot x + (3x + 3x) + (3)(3)$
$= (x + 3)(x + 3)$

(b) $x^2 – x – 2 = x \cdot x + (x – 2x) + (1)(-2)$
$= (x + 1)(x – 2)$

(c) $x^2 – 5x + 6 = x \cdot x – 2x –3x + (-2)(-3)$
$= (x – 2)(x – 3)$

(d) $x^2 + 4x – 12 = x \cdot x – 2x + 6x + (-2)(6)$

$= (x - 2)(x + 6)$

2. *Solve $x^2 - 4x + 4 = 0$ by factorising.*

Well, at school we learnt that:
the "+" in front of the last term means that our factorised version
$x^2 - 4x + 4$ will be either of the form $(x + ?)(x + ?)$ or of the form $(x - ?)(x - ?)$,
and that the "−" in front of the middle term tells us that our factorised version must be of the form
$(x - ?)(x - ?)$.

Experimenting with numbers that, when multiplied, give us 4, we soon find that our factorised version has to be $(x - 2)(x - 2)$, or $(x - 2)^2$ if you prefer.
Our equation $x^2 - 4x + 4 = 0$ may therefore be rewritten as $(x - 2)^2 = 0$.
By Property 9, at least one of the factors on the left-hand side must be zero, and both factors are
$(x - 2)$, so we get that $x - 2 = 0$
ie that $x = 2$.

3. *Complete the square to solve $x^2 - 4x = 12$.*

If	$x^2 - 4x$	=	12	
then	$x^2 - 4x + 4 - 4$	=	12	(by Property 8, since $4 - 4 = 0$)
ie	$x^2 - 4x + 4$	=	12 + 4	(by Property 6 with $k = 4$)
ie	$(x - 2)^2$	=	16	(factorise)
ie	$x - 2 = 4$ or $x - 2$	=	−4	(taking square roots)
ie	$x = 6$ or x	=	−2	(using Property 6 again, with $k = 2$).

4. *Is 21 a prime number?*

No. Refer to the definition of prime numbers on p 16.
The numbers 3 and 7 are factors of 21 ($3 \times 7 = 21$).

5. *What is the value of 5! (5 factorial)?*

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$.

Study unit 2

Activity 2-8

1. *Define the words "even" and "odd" for positive integers.*

Definitions:

- An integer n is *even* if n is a multiple of 2.

(We can say a positive integer n is even if $n = 2k$ for some positive integer k. You can think of even positive integers as numbers n of the form $n = 2k$, where k is some positive integer.)

- An integer n is *odd* if n is not even.

(Using the general form of an even positive integer, we can now say that n is odd if $n = 2k + 1$ for some positive integer k. You can think of odd positive integers as numbers n of the form $n = 2k + 1$, where k is some positive integer.)

2. *Is it the case that $m + (n \cdot k) = (m + n)(m + k)$ for all positive integers m, n and k?*

Substitute a few values and see whether the idea is plausible.
Take m = 1, n = 2, and k = 3, then the left-hand side is
$m + (n \cdot k) = 1 + (2 \cdot 3) = 7$
while the right-hand side becomes
$(m + n)(m + k) = (1 + 2) \cdot (1 + 3) = 3 \cdot 4 = 12$.
This is a counterexample to show that it is not the case that $m + (n \cdot k) = (m + n)(m + k)$ for all positive integers m, n and k.

3. *Are there any even prime numbers besides 2?*
No. Any even prime number other than 2 would have three factors: 1, because 1 is a factor of every number; 2, because the number we are talking about is supposedly even; and the number itself, because a number is always a factor of itself. However, primes cannot have so many factors, which means that 2 is the only even prime number.

4. *If m and n are even positive integers, is m + n even?*
If m and n are even positive integers, then each is a multiple of 2, in other words

$m = 2k$ for some $k \in \mathbb{Z}^+$.
and $n = 2j$ for some $j \in \mathbb{Z}^+$.
So
$$m + n = 2k + 2j = 2(k + j)$$
which means m + n is also even.

5. *If m and n are odd positive integers, is m·n odd?*
If m and n are odd positive integers, then both m and n can be written in the following general form:

$m = 2k + 1$ for some k in $\mathbb{Z}^+$.
and $n = 2j + 1$ for some j in $\mathbb{Z}^+$.
So
$$m \cdot n = (2k + 1)(2j + 1) = 4kj + 2k + 2j + 1 = 2(2kj + k + j) + 1$$
which means that m·n is odd.

An additional exercise:
If m and n are prime, is m + n and m − n prime?
No, not usually.
It can occasionally happen that m + n is also prime: take m = 3 and n = 2 then m + n = 5.
But for other values m + n may not be prime: take m = 3 and n = 7, for instance. 3 + 7 = 10, which is not a prime number.

What about the difference between two prime numbers? The difference m − n will sometimes be prime and sometimes not. E.g. 5 – 3 = 2, which is a prime number, but 23 – 3 = 20, which is not prime.

5 Learning unit 3 – Sets

Study Material

Study guide
You should cover study unit 3 in the study guide. Where the study unit refers to the CAI tutorial, you should also work through the relevant part in the CAI tutorial. The theory, examples and exercises will help you

understand the concepts. If you have not received your CAI tutorial with your study material, please see the *Orientation* section for downloading instructions.

Time allocated
You will need one week to master this learning unit.

Notes

Background
The previous study units covered different number systems that you will come across in the remaining units in this study guide. In this study unit, you are introduced to set theory. Set theory includes topics involving sets. It is essential for your computing studies that you understand set theory concepts because sets can be considered as fundamental building blocks for mathematics.

Applications
You should be familiar with all the notations and properties of sets. We use set notation to define different kinds of sets. In this study unit, we build new sets from old ones. The Venn diagrams in the next study unit will visually help you to understand the definitions in this study unit.

Activities

Do the activities provided in study unit 3 to consolidate your knowledge of the work in this learning unit.

Study unit 3

Activity 3-3

1. *In each of the following cases, describe the set more concisely, firstly, using list notation and then using set-builder notation.*

(a) list notation: {0, 2, 4, 6, 8}.
set-builder notation: $\{x \in \mathbb{Z}^{\geq} \mid x$ is an even non-negative integer and $x < 10\}$ (property description)

(b) The roster method: $\{-11, -9, -7, -5, -3, -1\}$.
There is more than one way (we give only two) to describe this set using set-builder notation:
$\{x \in \mathbb{Z} \mid x$ is an odd negative integer and $x > -13\}$ (property description)
or, if you prefer:
$\{y \in \mathbb{Z} \mid y$ is odd and $-13 < y < 0\}$ (property description)

(c) The roster method: $\{\ \} = \emptyset$
(because there does not exist an integer that is, at the same time, positive and less than 1).
In set-builder notation:
One possibility is $\{x < 1 \mid x \in \mathbb{Z}^{+}\}$.

(d) Because of the nature of real numbers, ie between any two real numbers a and b one can always find another real number, namely $(a + b)/2$, it is not really possible to represent this set using the roster method. In set-builder notation: $\{x \in \mathbb{R} \mid x > 2\}$.

2. *In each of the following cases, give an unambiguous description in English.*

(a) $\{-1, 0, 1\}$:
One possibility is to speak of the set having −1, 0 and 1 as its only elements.

Another is to speak of the set of all integers greater than −2 and less than 2.

(b) $\{ x \in \mathbb{R} \mid 0 < x < 1 \}$: The set of all real numbers greater than 0 and less than 1.

(c) {0}:
Again we can give many descriptions of this set in English:
- The set having 0 as its only element.
- The set of all non-negative integers less than 1.
- The set of all integers greater than -1 and less than 1.
- The set of all integers that are simultaneously not positive and not negative.

(d) $\{\mathbb{Z}\}$: The set containing the set of integers, $\mathbb{Z}$, as its only member.

Does it bother you to have a set like $\mathbb{Z}$ as an element inside another set? Remember that the purpose of a set is just to group together the things we are interested in, and these things may well be sets themselves. If we are interested in the number sets, we may group together as elements not just $\mathbb{Z}$, but also $\mathbb{Z}^+$, $\mathbb{Z}^{\geq}$, $\mathbb{Q}$ and $\mathbb{R}$ to form the set $\{\mathbb{Z}, \mathbb{Z}^+, \mathbb{Z}^{\geq}, \mathbb{Q}, \mathbb{R}\}$, and so on.

Activity 3-6

1. Let U = {1, 2, 3, 4, 5}, A = {1, 2, 3} and B = {3, 4, 5}. Determine the required sets.

(a) $A \cup B = \{1, 2, 3, 4, 5\} = B \cup A$ (1, 2, 3, 4 and 5 are elements of A or B or both.)

(b) $A \cap B = \{3\} = B \cap A$ (3 is an element of both A and B)

(c) $A - B = \{1, 2\}$ (1 and 2 are elements of A but not of B)
$B - A = \{4, 5\}$ (4 and 5 are elements of B but not of A)

(d) $A + B = \{1, 2, 4, 5\} = B + A$ (1, 2, 4 and 5 are elements that belong either to A or to B but not to both)

2. *Let U = {a, e, i, o, u}, A = {i, o, u} and B = {a, e, o, u}. Determine the following sets:*

(a) $A' = \{a, e\}$ (a and e are elements of U but not of A)
$(A')' = \{i, o, u\}$
$= A$

(b) $B' = \{i\}$
$(B')' = \{a, e, o, u\}$

(c) $A \cup B = \{a, e, i, o, u\}$
$(A \cup B)' = \emptyset$

(d) $A' \cap B' = \{a, e\} \cap \{i\}$
$= \emptyset$ (We can also refer to this set merely as $A' \cap B'$.)

(e) $A \cap B = \{o, u\}$
$(A \cap B)' = \{a, e, i\}$

(f) $A' \cup B' = \{a, e\} \cup \{i\}$
$= \{a, e, i\}$ (Can also be called $A' \cup B'$.)

(g) $A - B = \{i\}$
$B - A = \{a, e\}$

(h) $A \cap B' = \{i, o, u\} \cap \{i\}$
$= \{i\}$ (or $A \cap B'$)
$B \cap A' = \{a, e, o, u\} \cap \{a, e\}$
$= \{a,e\}$ (or $B \cap A'$)

(i) $A + B = \{i, o, u\} + \{a, e, o, u\}$
$= \{a, e, i\}$
$B + A = \{a, e, i\}$

3. *Let $U = \{1, 2, 3, 4, 5\}$, $A = \{3\}$ and $B = \{\{3\}, 4, 5\}$. Determine $\mathcal{P}$ (A) and $\mathcal{P}$ (B).*

***Note**: If the cardinality of some finite set C is n (ie $|C| = n \geq 0$), then a total of 2^n subsets of C can be formed, so $|\mathcal{P}(C)| = 2^n$. In the case of $B = \{\{3\}, 4, 5\}$, B has 3 elements namely {3}, 4 and 5, so the power set of B, namely $\mathcal{P}(B)$ has $2^3 = 8$ elements.*

$\mathcal{P}$ (A) = $\boldsymbol{\{}$ $\emptyset$, {3} $\boldsymbol{\}}$ (All the subsets of A = {3} are elements of $\mathcal{P}$ (A).)

$\mathcal{P}$ (B) = $\boldsymbol{\{}$ $\emptyset$, {{3}}, {4}, {5}, {{3}, 4}, {{3}, 5}, {4, 5}, {{3}, 4, 5} $\boldsymbol{\}}$

Subsets of **B** are *members* of **$\mathcal{P}$ (B).** We determine two *members* of **$\mathcal{P}$ (B)**:

{3}, 4 and 5 are *members* of **B** = **{**{3}, 4, 5**}**.
We form subsets of B:
Keep the outside brackets of B, then throw away the members 4 and 5 of B, then we are left with the *subset* **{**{3}**}** of B, which is then a *member* of **$\mathcal{P}$ (B)**. (Note that {3} is **not** a member of $\mathcal{P}$ (B).)
Keep the outside brackets of B, then throw away the members {3} and 4 of B, then we are left with the *subset* **{**5**}** of B, which is then a member of $\mathcal{P}$ (B). All the subsets of B are the members of $\mathcal{P}$ (B).

4. *Let $U = \{a, e, i, o, u\}$, $A = \{i, o, u\}$ and $B = \{a, e, o, u\}$. Determine the following sets:*

(a) $\mathcal{P}$ (A) = $\boldsymbol{\{}$ $\emptyset$, {i}, {o}, {u}, {i, o}, {i, u}, {o, u}, {i, o, u} $\boldsymbol{\}}$

$\mathcal{P}$ (B) = $\boldsymbol{\{}$ $\emptyset$, {a}, {e}, {o}, {u}, {a, e}, {a, o}, {a, u},
{e, o}, {e, u}, {o, u}, {a, e, o}, {a, o, u},
{a, e, u}, {e, o, u}, {a, e, o, u} $\boldsymbol{\}}$

(b) $\mathcal{P}$ ($A \cap B$) = $\mathcal{P}$ ({o, u}) = $\boldsymbol{\{}$ $\emptyset$, {o}, {u}, {o, u} $\boldsymbol{\}}$

$\mathcal{P}$ (A) ∩ $\mathcal{P}$ (B) = **{** ∅, {o}, {u}, {o, u} **}**

(c) $\mathcal{P}$ (A′) = $\mathcal{P}$ ({a, e}) = **{** ∅, {a}, {e}, {a, e} **}**

In order to be able to determine ($\mathcal{P}$ (A))′, we need to determine $\mathcal{P}$ (U) first.

$\mathcal{P}$ (U) = **{** ∅, {a}, {e}, {i}, {o}, {u},
{a, e}, {a, i}, {a, o}, {a, u}, {e, i},
{e, o}, {e, u}, {i, o}, {i, u}, {o, u},
{a, e, i}, {a, i, o}, {a, o, u}, {a, i, u},
{a, e, o}, {a, e, u}, {e, i, o}, {e, o, u},
{e, i, u}, {i, o, u}, {a, e, i, o}, {a, e, i, u},
{a, e, o, u}, {a, i, o, u}, {e, i, o, u},
{a, e, i, o, u} **}**

($\mathcal{P}$ (A))′ = **{** {a}, {e}, {a, e}, {a, i}, {a, o}, {a, u},
{e, i}, {e, o}, {e, u}, {a, e, i}, {a, i, o},
{a, o, u}, {a, i, u}, {a, e, o}, {a, e, u},
{e, i, o}, {e, o, u}, {e, i, u}, {a, e, i, o},
{a, e, i, u}, {a, e, o, u}, {a, i, o, u},
{e, i, o, u}, {a, e, i, o, u} **}**

(d) $\mathcal{P}$ (A) ∪ $\mathcal{P}$ (B)
= **{** ∅, {i}, {o}, {u}, {a}, {e}, {i, o}, {i, u}, {o, u}, {a, e}, {a, o}, {a, u},
{e, o}, {e, u}, {i, o, u}, {a, e, o}, {a, o, u}, {a, e, u}, {e, o, u}, {a, e, o, u} **}**

$\mathcal{P}$ (A ∪ B) = $\mathcal{P}$ ({a, e, i, o, u})
= $\mathcal{P}$ (U)

6 Assignment 1

Assignment 1 scope

What is covered?
This assignment is a multiple-choice assignment and covers study units 1 to 3 of the study guide.

Assignment submission
This assignment should be submitted either electronically via myUnisa (the preferred route), or by filling in a mark-reading sheet and submitting it via one of the regional centres or the post office.

Time allocated
You will need one week to complete this assignment.

Due date
See Tutorial Letter 101 for the due date and unique assignment number for this assignment.

PDF version

You can get a PDF version of this assignment question in **Additional Resources**.

Assignment 1 questions

You can find the questions in Tutorial Letter 101 or download them from the **Additional Resources** page.

Assignment 1 solutions

After the closing date, a discussion of the assignment will be posted on the **Additional Resources** page. You will be informed of this via an announcement on myUnisa.

Sample Assignment 1 questions and solutions

In Tutorial Letter 102, available on the **Additional Resources** page, you will find a sample assignment 1 that you can work through before attempting Assignment 1.

7 Learning unit 4 – Proofs involving sets

Study Material

Study guide
You need to refer to study unit 4 in the study guide. Where the study unit refers to the CAI tutorial, you should also work through the relevant part in the CAI tutorial. The theory, examples and exercises will help you understand the concepts. If you have not received your CAI tutorial with your study material, please see the *Orientation* section for downloading instructions.

Time allocated
You will need one week to cover this learning unit.

Notes

Background
We can draw pictures of sets, and we call these Venn diagrams. Venn diagrams are tools we can use in different problem scenarios involving sets. In this study unit, the definitions provided in the previous study unit can be applied in proofs.

Applications
We can use Venn diagrams to get an indication of whether two sets are equal or not. If two sets are not equal, a counterexample can be found to prove that the sets are not equal, otherwise a formal proof can be provided to show that the sets are equal. The inclusion-exclusion principle can be applied by using Venn diagrams to solve counting problems.

Activities

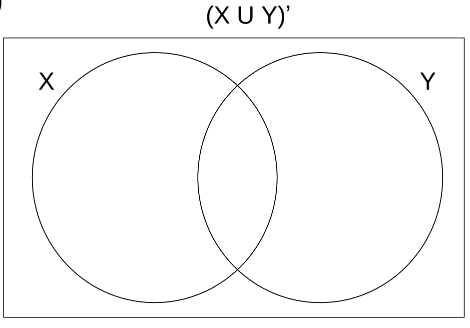
Do the activities provided in study unit 4 to consolidate your knowledge of the work in this learning unit.

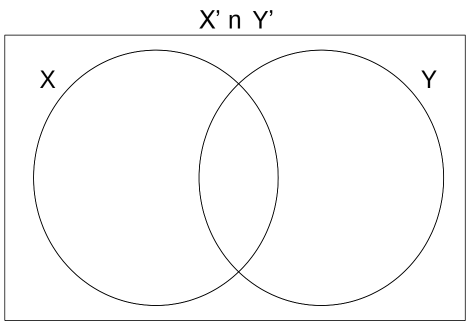
Study unit 4

Activity 4-4

1. *Draw the following diagrams:*

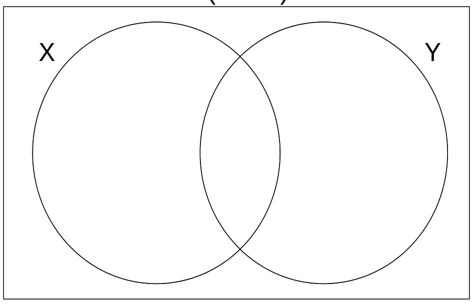
1(a) $(X \cup Y)'$ (First, draw a Venn diagram for $X \cup Y$.) 1(b) $X' \cap Y'$ (First, draw Venn diagrams for X' and Y'.)



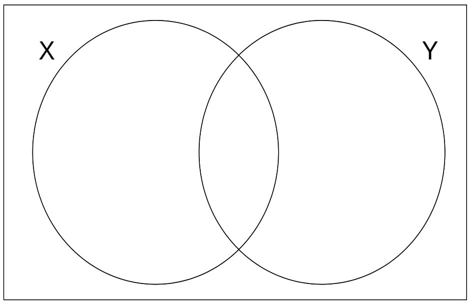


1(c) $(X \cap Y)'$ (First, draw a Venn diagram for $X \cap Y$.) 1(d) $X' \cup Y'$

(X n Y)'



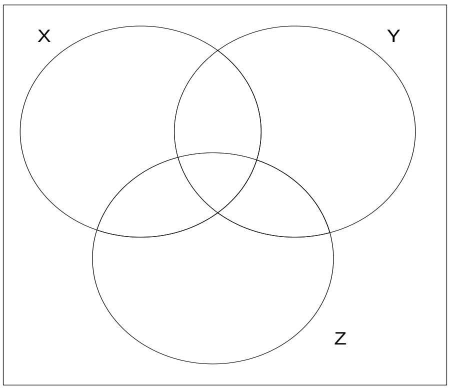
X' U Y'



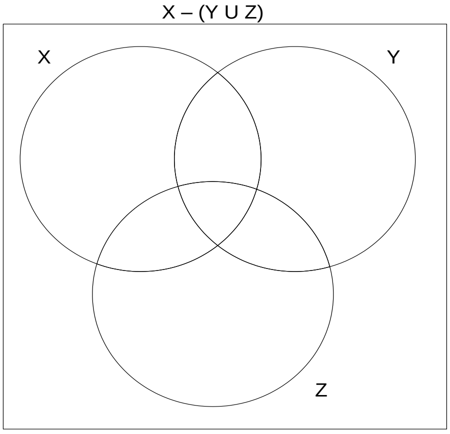
2. *Draw the following Venn diagrams:*

(a) **Note**: Draw the universal set for each diagram; draw all three sets in each diagram; name all three sets; provide a subscript for each diagram; colour in only the area relevant to the subscript.
$X - (Y \cup Z)$:

X

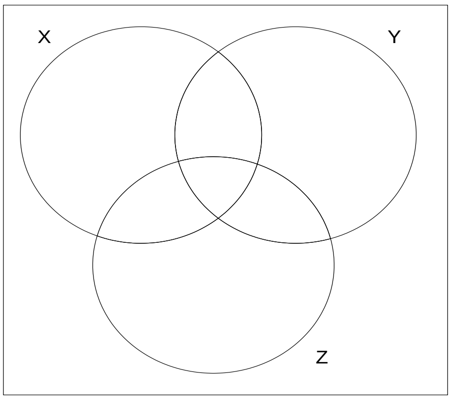


X – (Y U Z)

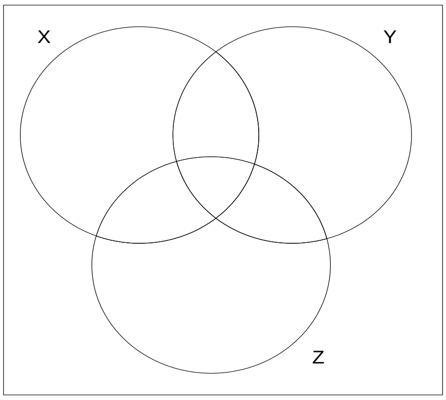


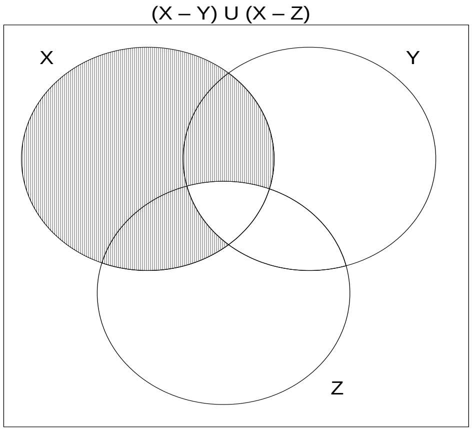
2(b) $(X-Y) \cup (X-Z)$

X – Y

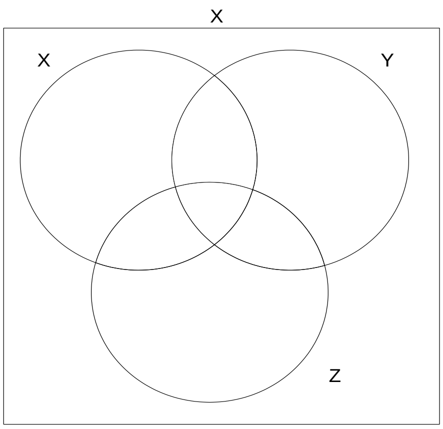


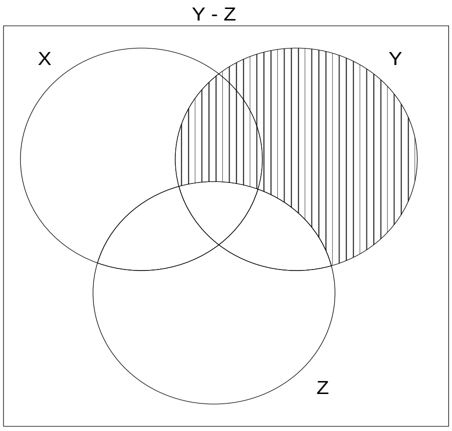
X – Z

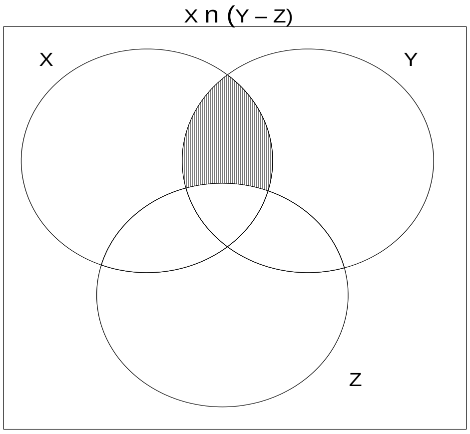




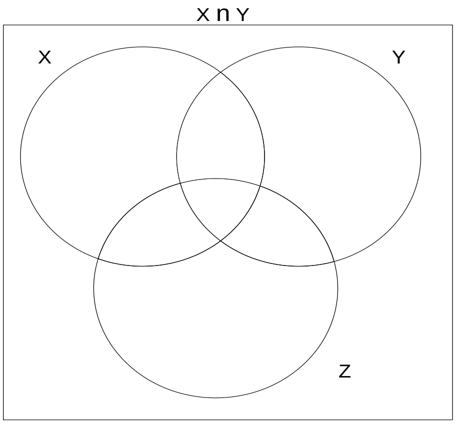
2(c) X ∩ (Y − Z)

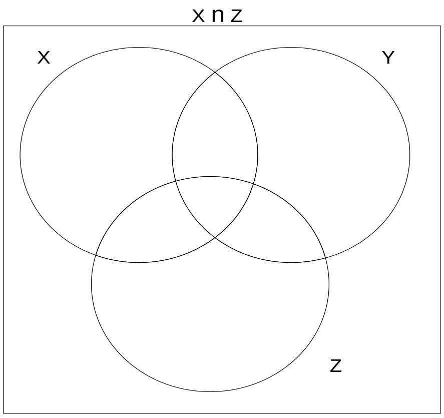




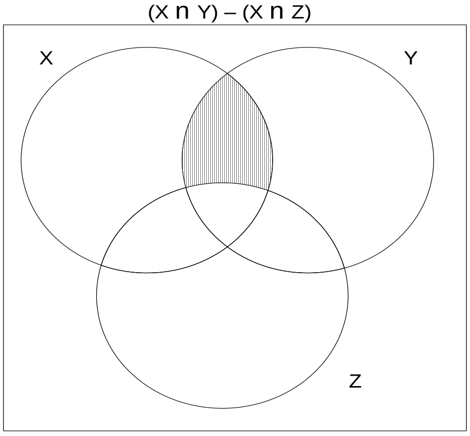


2(d) (X ∩ Y) − (X ∩ Z)



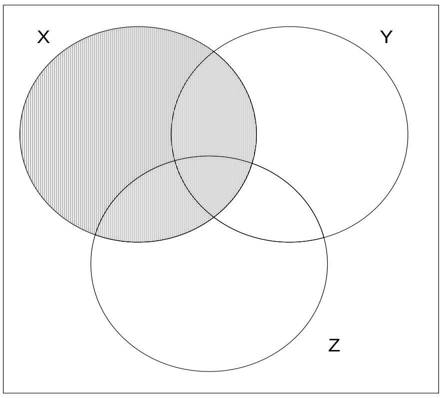


(X n Y) – (X n Z)

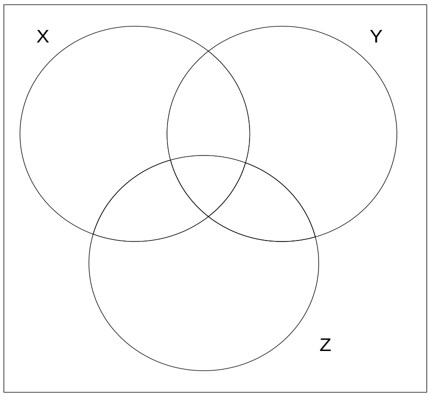


2(e) X ∩ (Y + Z)

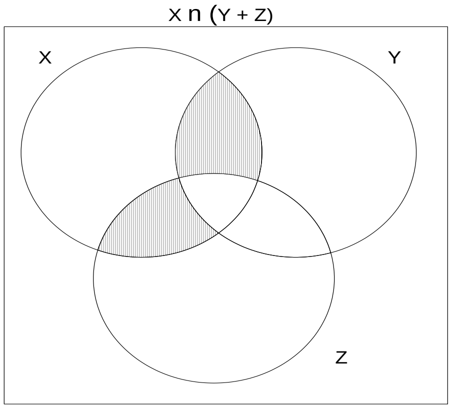
X



Y + Z

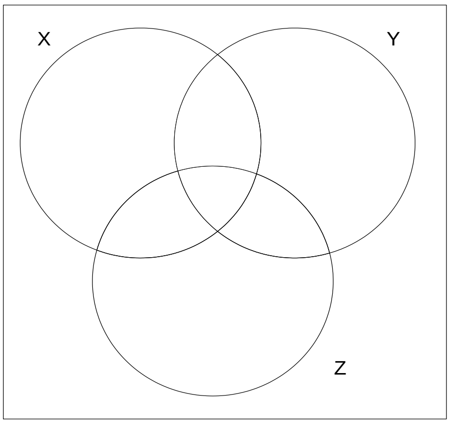


X n (Y + Z)

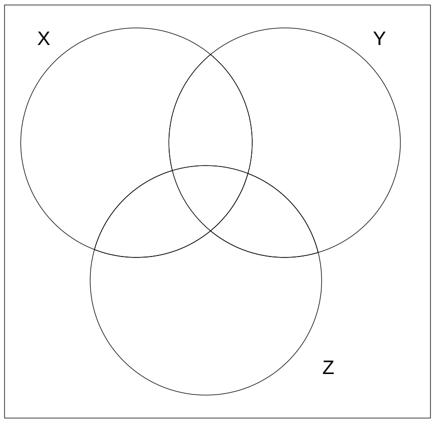


2(f) (X ∩ Y) + (X ∩ Z)

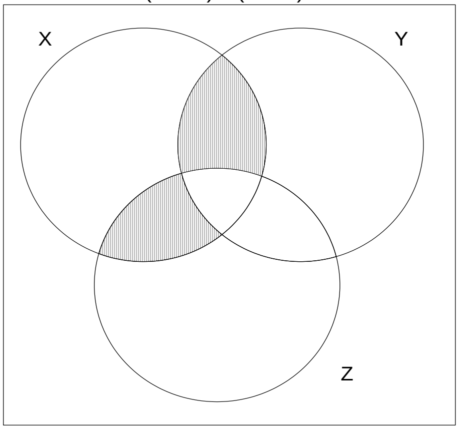
X n Y



X n Z



(X n Y) + (X n Z)



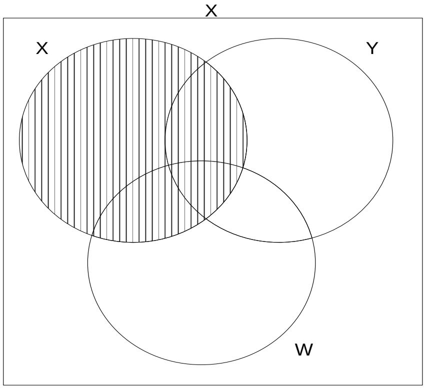
Activity 4-5

Use Venn diagrams to determine whether or not the given equations hold.

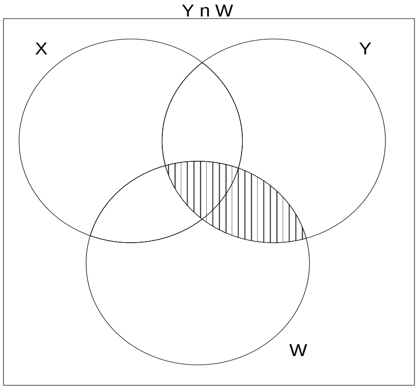
1. (a) Is $X - (Y \cap W) = (X - Y) \cup (X - W)$?

Left-hand side: $X - (Y \cap W)$

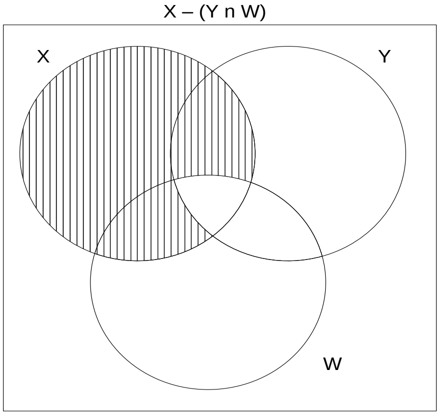
X



Y n W

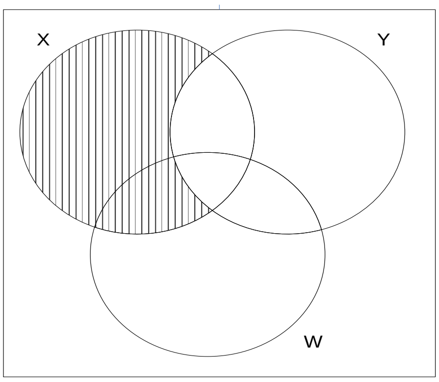


X – (Y n W)

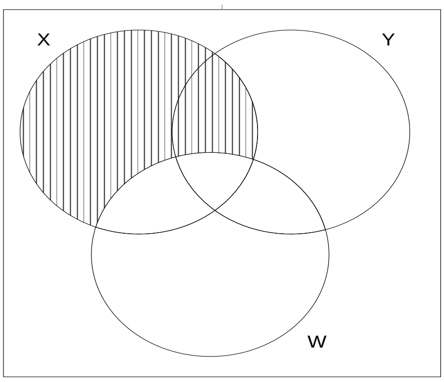


Right-hand side: $(X - Y) \cup (X - W)$

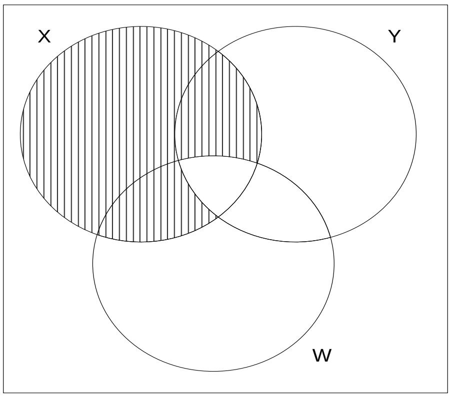
X - Y



X - W



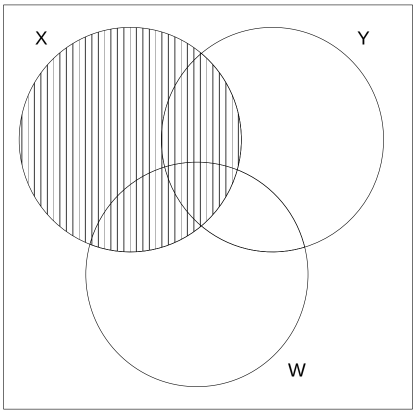
(X – Y) U (X – W)



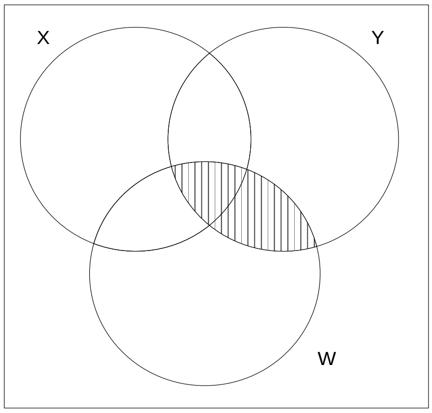
(b) Is $X \cap (Y \cap W) = (X \cap Y) \cap W$?

Left-hand side: $X \cap (Y \cap W)$

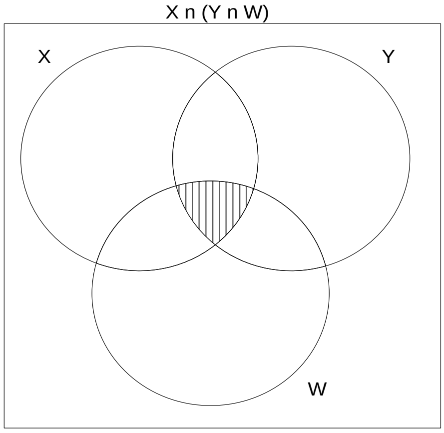
X



Y n W

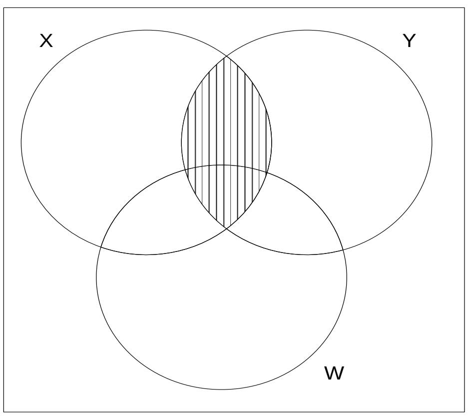


X n (Y n W)

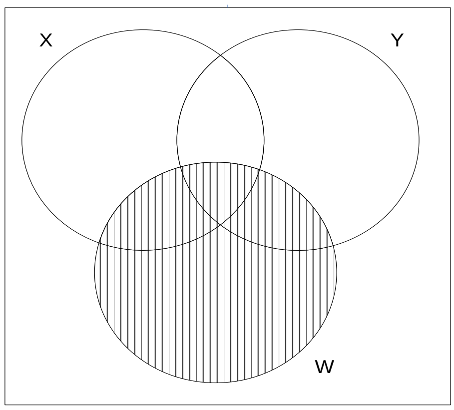


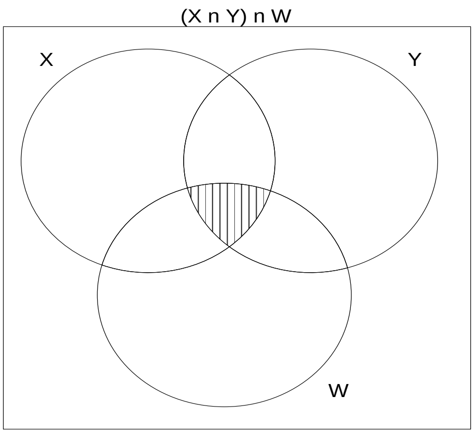
Right-hand side: $(X \cap Y) \cap W$

X n Y



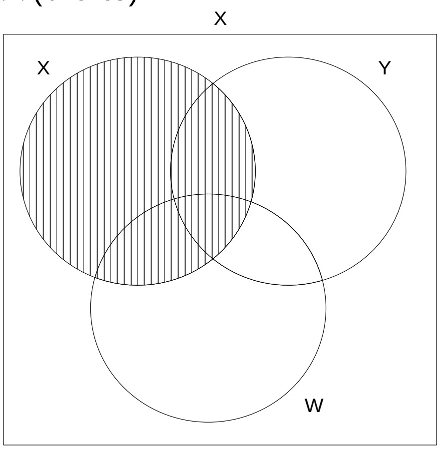
W

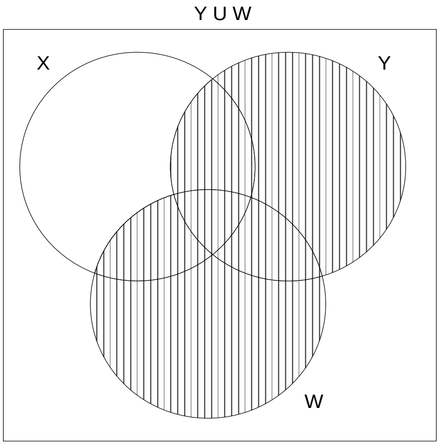


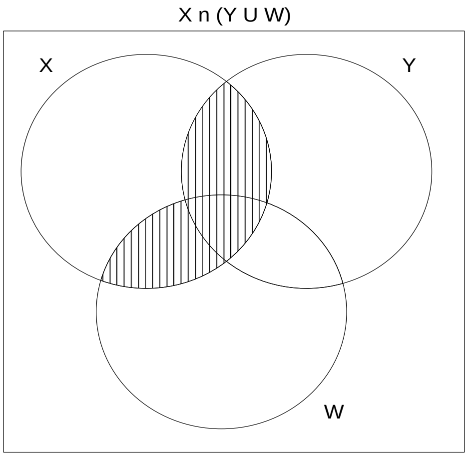


(c) Is X ∩ (Y ∪ W) = (X ∩ Y) ∪ (X ∩ W)?

Left-hand side: X ∩ (Y ∪ W)

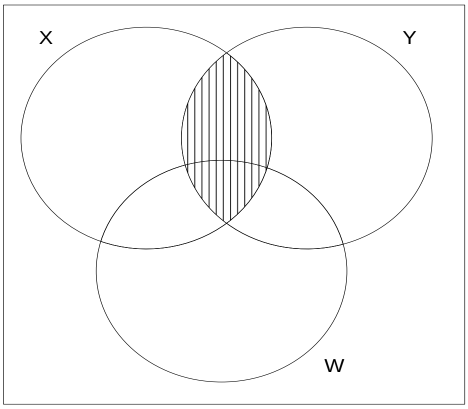




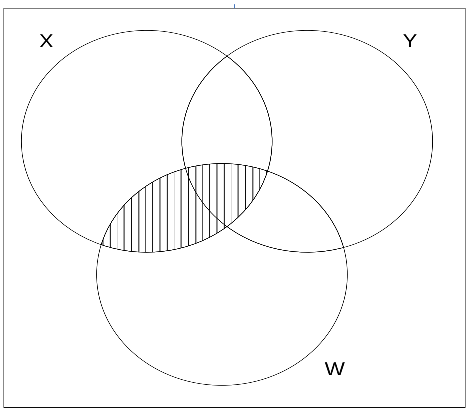


Right-hand side: $(X \cap Y) \cup (X \cap W)$

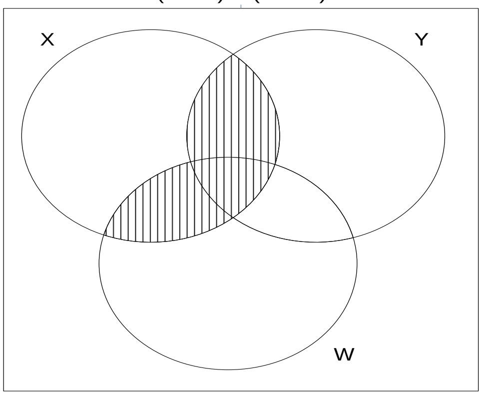
X n Y



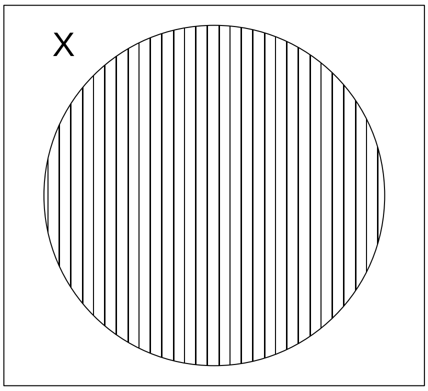
X n W



(X n Y) U (X n W)



(d) The Venn diagrams for both $(X')'$ and X look like this:



Activity 4-6

Using if and only if statements, write out a proof in words for each of the following identities, where X, Y and W are arbitrary subsets of a universal set U:

1(a) $(X')' = X$

It seems we should be able to produce a cast-iron proof.

$x \in (X')'$
iff $x \notin X'$
iff $x \in X$.
Therefore $(X')' = X$.

(b) $X - (Y \cap W) = (X - Y) \cup (X - W)$

$x \in X - (Y \cap W)$

iff $x \in X$ and $x \notin (Y \cap W)$
iff $x \in X$ and ($x \in Y'$ or $x \in W'$)
iff ($x \in X$ and $x \in Y'$) or ($x \in X$ and $x \in W'$)
iff $x \in (X - Y)$ or $x \in (X - W)$
iff $x \in (X - Y) \cup (X - W)$

Thus $X - (Y \cap W) = (X - Y) \cup (X - W)$ for all subsets X, Y and W of U.

(c) $X \cap (Y \cap W) = (X \cap Y) \cap W$

$x \in X \cap (Y \cap W)$
iff $x \in X$ and $x \in (Y \cap W)$
iff $x \in X$ and ($x \in Y$ and $x \in W$)
iff ($x \in X$ and $x \in Y$) and $x \in W$
iff $x \in (X \cap Y)$ and $x \in W$
iff $x \in (X \cap Y) \cap W$.

We can conclude that $X \cap (Y \cap W) = (X \cap Y) \cap W$ for all subsets X, Y and W of U.

(d) $X \cap (Y \cup W) = (X \cap Y) \cup (X \cap W)$

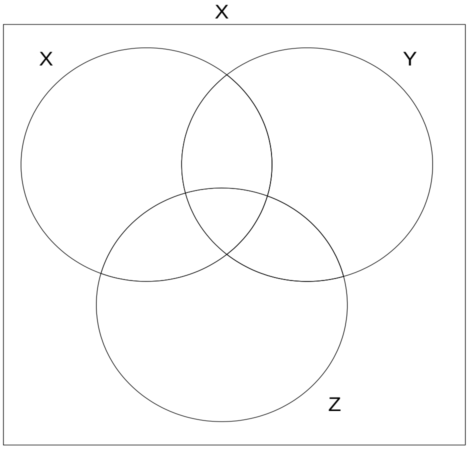
$x \in X \cap (Y \cup W)$
iff $x \in X$ and $x \in (Y \cup W)$
iff $x \in X$ and ($x \in Y$ or $x \in W$)
iff ($x \in X$ and $x \in Y$) or ($x \in X$ and $x \in W$)
iff ($x \in X \cap Y$) or ($x \in X \cap W$)
iff $x \in (X \cap Y) \cup (X \cap W)$.
Thus $X \cap (Y \cup W) = (X \cap Y) \cup (X \cap W)$.

Activity 4-8

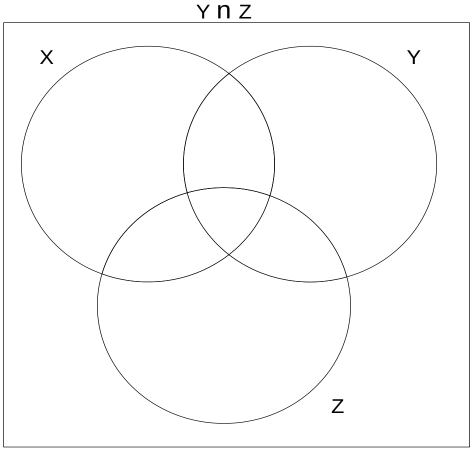
1. *Is it the case that for all X, Y, Z $\subseteq$ U, $X + (Y \cap Z) = (X + Y) \cap (X + Z)$?*

Determine the Venn diagram for $X + (Y \cap Z)$:

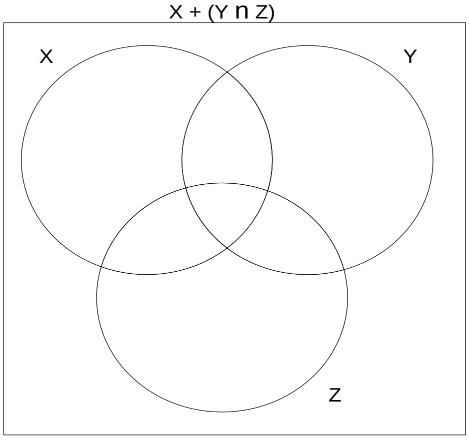
X



Y n Z

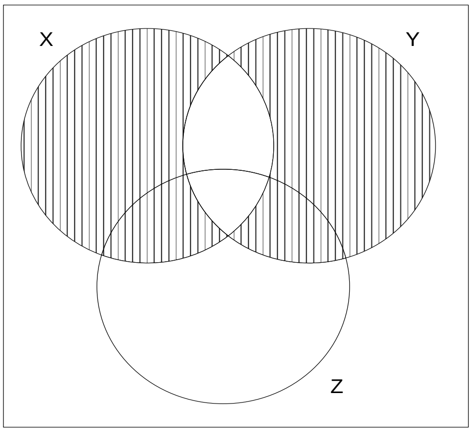


X + (Y n Z)

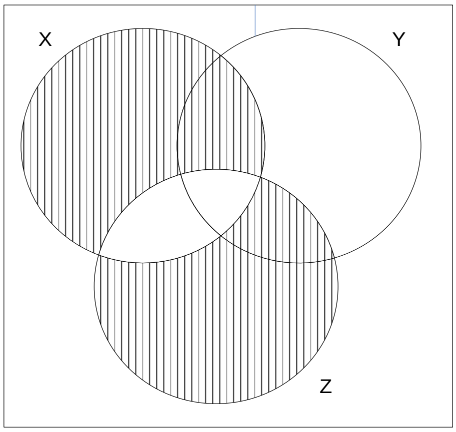


Determine the Venn diagram for (X + Y) $\cap$ (X + Z):

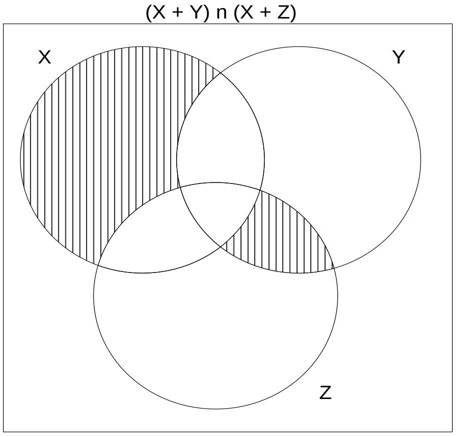
X + Y



X + Z



(X + Y) n (X + Z)



From the Venn diagrams it is clear that it is *not* the case that for all X, Y, Z $\subseteq$ U,
X + (Y $\cap$ Z) = (X + Y) $\cap$ (X + Z).

2. *Find examples of sets A and B such that $\mathcal{P}$ (A $\cup$ B) is not a subset of $\mathcal{P}$ (A) $\cup$ $\mathcal{P}$ (B).*

This means we need to find a counterexample to show that it is not the case that $\mathcal{P}$ (A $\cup$ B) is a subset of $\mathcal{P}$ (A) $\cup$ $\mathcal{P}$ (B) for all sets A and B of U.

Let a universal set be U = {1, 2 } and let A = {1} and B = {2}.

$\mathcal{P}$ (A) $= \{ \emptyset, \{1\} \}$

$\mathcal{P}$ (B) $= \{ \emptyset, \{2\} \}$

$\mathcal{P}$ (A) $\cup$ $\mathcal{P}$ (B) $= \{ \emptyset, \{1\}, \{2\}\}$

$\mathcal{P}$ (A $\cup$ B) $= \mathcal{P}$ ({1, 2})

$= \{ \emptyset, \{1\}, \{2\}, \{1, 2\}\}$

3. *Is it the case that for all X, Y $\subseteq$ U, $\mathcal{P}$ (X) $\cap$ $\mathcal{P}$ (Y) = $\mathcal{P}$ (X $\cap$ Y)? Justify your answer*.

S $\in \mathcal{P}$ (X) $\cap$ $\mathcal{P}$ (Y)

iff S $\in \mathcal{P}$ (X) and S $\in \mathcal{P}$ (Y)

iff S $\subseteq$ X and S $\subseteq$ Y
iff the elements of S all belong to X and all belong to Y
iff the elements of S all belong to X $\cap$ Y
iff S $\subseteq$ X $\cap$ Y
iff S $\in \mathcal{P}$ (X $\cap$ Y).

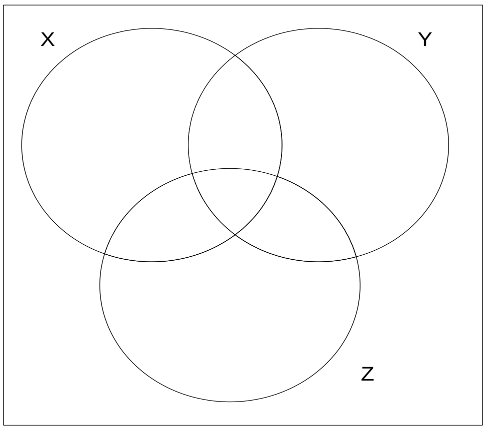
This proves that $\mathcal{P}$ (X) $\cap$ $\mathcal{P}$ (Y) = $\mathcal{P}$ (X $\cap$ Y).

4. *Use Venn diagrams to investigate whether or not, for all sets X, Y, Z $\subseteq$ U,*
X − (Y − Z) = (X − Y) − Z. If the statement appears to hold, give a proof; if not, give a counterexample.

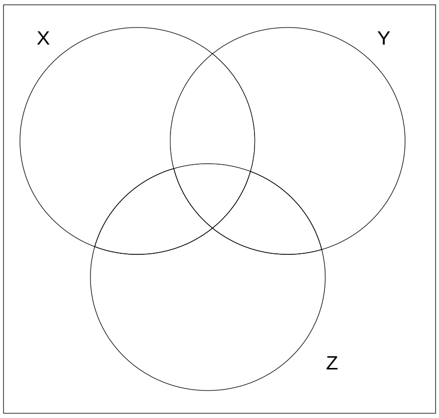
We draw the Venn diagrams as follows:

Left-hand side:

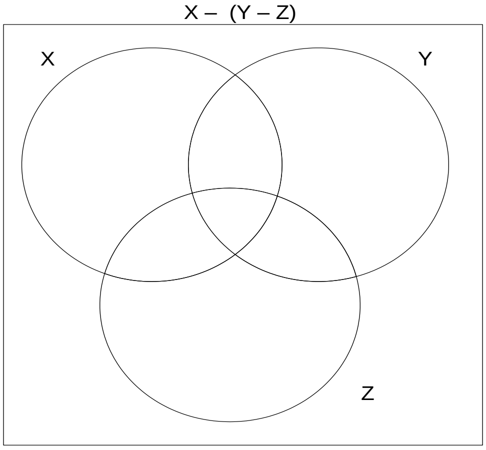
X



Y – Z

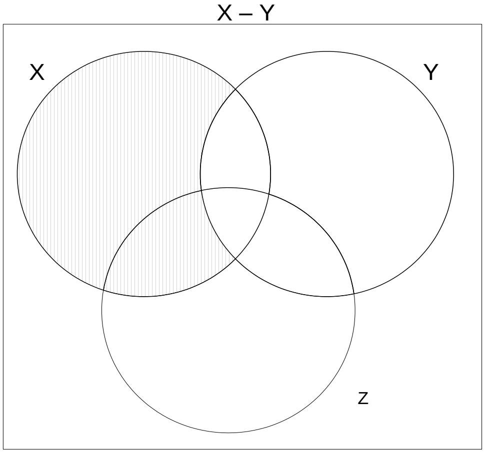


X – (Y – Z)

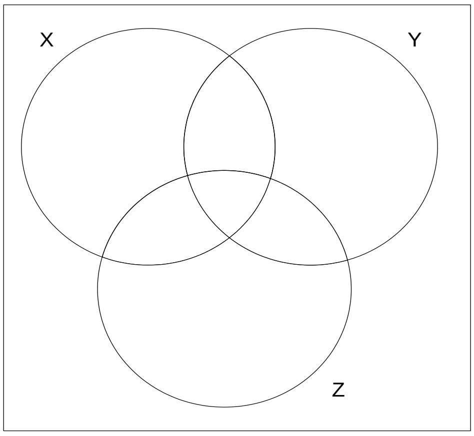


Right-hand side:

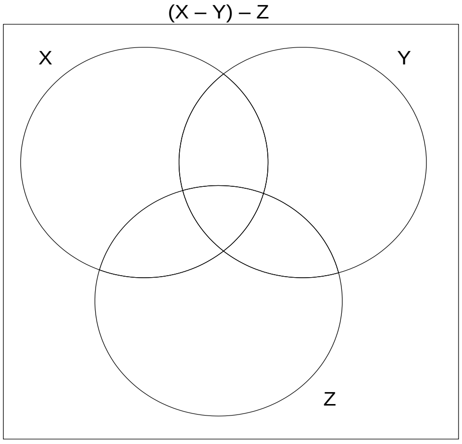
X – Y



Z



(X – Y) – Z



The shaded areas representing X − (Y − Z) and (X − Y) − Z differ, and therefore it seems as if the claim that

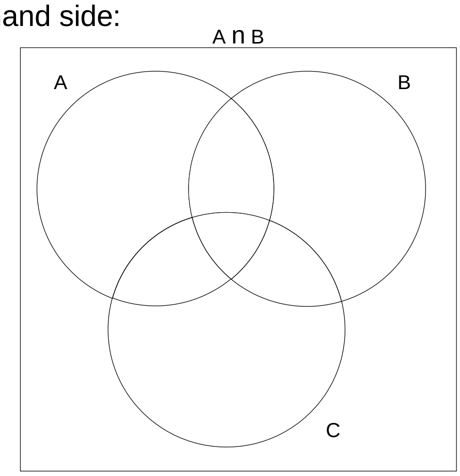
X − (Y − Z) = (X − Y) − Z is not always true. We have to give a counterexample with specific values for X, Y and Z. The two final Venn diagrams differ in the region $X \cap Z$, so we choose, for example, $1 \in X$ and $1 \in Z$.

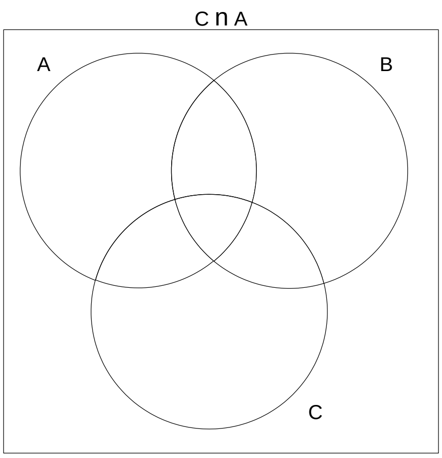
Let X = {1, 2}, Y = {2, 3}, and Z = {1, 3},
then X − (Y − Z) = {1, 2} - {2} = {1}.
On the other hand, (X - Y) - Z = {1} - {1,3} = { }.

So, in this case, X − (Y−Z) ≠ (X−Y) − Z.

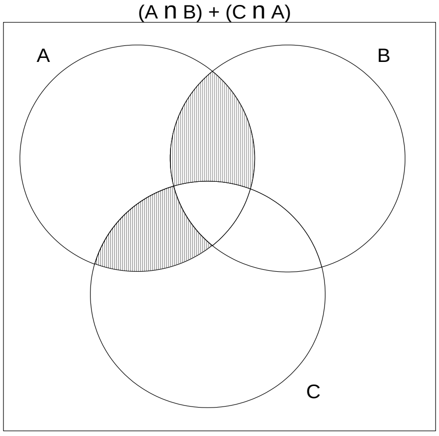
5. *Use Venn diagrams to investigate whether or not, for all subsets A, B and C of U,*
$(A \cap B) + (C \cap A) = (A \cap B') \cup (B - C)$. If the statement appears to hold, give a proof; if not, give a counterexample.

We draw the Venn diagrams as follows:
Left-hand side:



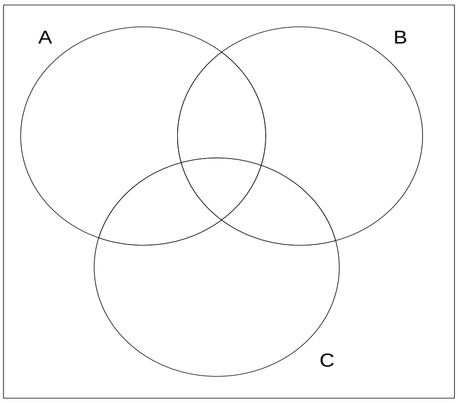


$(A \cap B) + (C \cap A)$

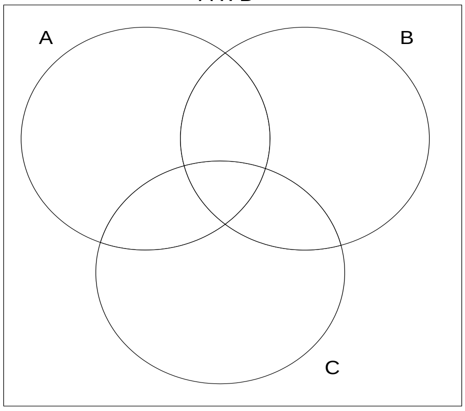


Right-hand side:

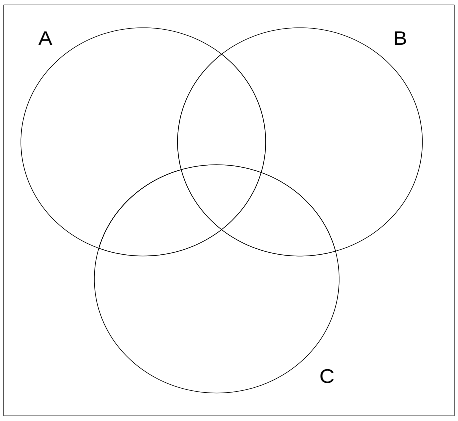
B'



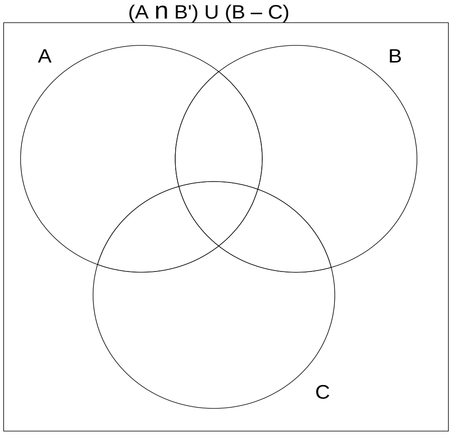
$A \cap B'$



$B - C$



$(A \cap B') \cup (B - C)$



It appears that the expression is not an identity, so we need a counterexample. That is, we want a concrete example of sets that show that the left-hand side is different from the right-hand side.

Counterexample:

The final diagrams differ in some areas of B. We choose the element 2 that resides in set B only.

Let $A = \{1\}$, $B = \{1, 2\}$, $C = \{1, 3\}$ and $U = \{1, 2, 3\}$ with U as the universal set.

Determine B':

$B' = U - B = \{3\}$

Now $(A \cap B) + (C \cap A)$ (Determine which members reside in either $A \cap B$ or $C \cap A$, but not in both $A \cap B$ and $C \cap A$.)

$= \{1\} + \{1\}$

$= \{ \}$

while $(A \cap B') \cup (B - C)$ (Determine which members reside in $A \cap B'$ or $B - C$.)
$= \{ \} \cup \{2\}$
$= \{2\}$.

In this example it is not the case that $(A \cap B) + (C \cap A) = (A \cap B') \cup (B - C)$.

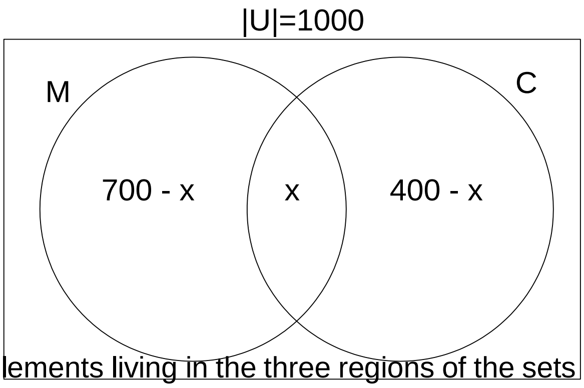
Activity 4-10

1. *Suppose that of 1000 first-year students, 700 take mathematics, 400 take computer science and 800 take mathematics or computer science.*

(a) *How many take mathematics and computer science?*
Let U be the set of first-year students, M the set of those taking mathematics, and C the set of those taking computer science. Then

$|U| = 1000$,
$|M| = 700$,
$|C| = 400$,
$|M \cup C| = 800$ and
$|M \cap C| = x$ (We do not know how many take mathematics and computer science.)

By using this information, we can fill in the number of elements that reside in each region of the two sets, starting with the region in the middle that has x elements, and then $|M - C| = 700 - x$ and $|C - M| = 400 - x$.



We add the number of elements living in the three regions of the sets M and C, and since the total number of elements that reside in these regions is $|M \cup C| = 800$, we can determine x.
$|M - C| + x + |C - M| = 800$
ie $700 - x + x + 400 - x = 800$
ie $x = 300$, ie 300 students take mathematics and computer science.

(b) *How many students take mathematics but not computer science?*
$|M - C| = 700 - x = 700 - 300 = 400$.

(c) *How many students do not take any of the two subjects?*
There are 1000 students and 800 take mathematics or computer science, so
$|(M \cup C)'| = |U| - |M \cup C| = 1000 - 800 = 200$ do not take any of the two subjects.

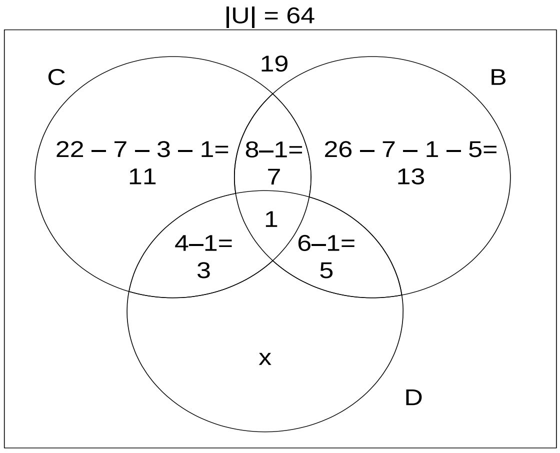
2. *A builder has a team of 64 construction workers. Of these, 45 are trained in the use of heavy machinery, ie cranes, bulldozers and backhoes. A total of 22 can operate cranes, 26 can operate backhoes, 4 can operate cranes and bulldozers, 6 can operate backhoes and bulldozers, 8 can*

operate cranes and backhoes, and 1 can operate all three kinds of machines. How many can operate bulldozers?

First, we set out the available information neatly. Let U be the set of all workers in the team. Let C be the set of those who can operate cranes, B those who can operate backhoes and D those who can operate dozers. Then

$|U| = 64$, $|C| = 22$, $|B| = 26$,
$|D| = x$, the unknown we want to solve, ie those who operate bulldozers
$|C \cap D| = 4$,
$|B \cap D| = 6$,
$|C \cap B| = 8$,
$|C \cap D \cap B| = 1$,
$|C \cup D \cup B| = 45$ and
$|(C \cup D \cup B)'| = 64 - 45 = 19$.

Now we can fill in the various regions. We initially fill in x for the value of $|D - (B \cup C)|$.



$|C \cup B \cup D| = 45 = 11 + 7 + 1 + 3 + 5 + 13 + x$
ie $45 = 40 + x$
ie $x = 5$, ie 5 workers can operate dozers only.
Thus $3 + 1 + 5 + 5 = 14$ workers can operate bulldozers.

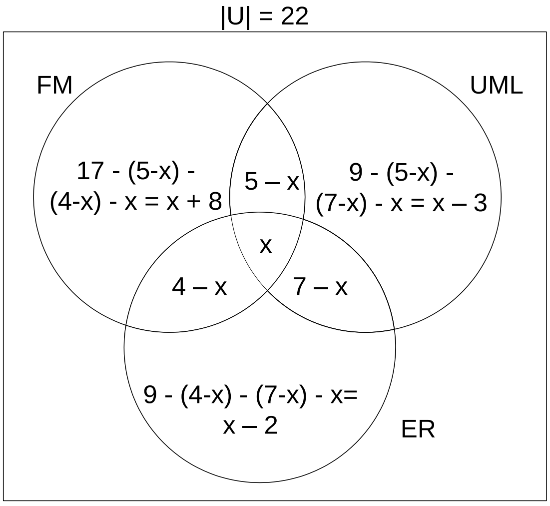
3. *A large software company employs 22 software engineers for the design of systems. Of these engineers, 17 are well versed in the secrets of a formal method (FM), 9 can use the Unified Modelling Language (UML), and 9 are familiar with the use of entity-relationship (ER) diagrams. If 5 engineers can use both an FM and UML, 4 can use both an FM and ER diagrams and 7 can use both UML and ER diagrams, answer the following:*

(a) How many engineers can use all 3 techniques, namely an FM, UML and ER diagrams?
Let U denote the number of software engineers. Let FM be the engineers who are well versed in the secrets of a formal method, let UML be those who can use the UML and ER those who are familiar with the use of entity-relationship diagrams. Then
$|U| = 22$,
$|FM| = 17$,
$|UML| = 9$,

$|ER| = 9$,
$|FM \cap UML| = 5$,
$|FM \cap ER| = 4$,
$|UML \cap ER| = 7$ and
$|FM \cap UML \cap ER| = x$, ie the number of engineers who can use all 3 techniques.

Start by filling x into the intersection of the three circles. Then fill in the intersections of each pair of circles, eg 5 engineers can use both FM and UML, so insert 5 – x in the remaining overlap between FM users and UML users, and so on ...

Now fill in the outstanding figures for each individual technique by subtracting the numbers already inside a particular circle from the total users who use that technique, eg, in total, 9 use UML, so fill in: 9 – (5–x) – x – (7–x) = x–3 in the open region of the UML circle, and so on …



Now let us solve x:
$(x + 8) + (5 - x) + (x - 3) + x + (4 - x) + (7 - x) + (x - 2) = 22$
ie $x + 19 = 22$
ie $x = 3$
Therefore, 3 engineers can use all three techniques.

(b) *How many engineers can use UML only?*
$x - 3 = 3 - 3 = 0$, ie no engineer uses UML only.

Activity 4-11

Prove the given sets equal.

1. $\{y \in \mathbb{Z}^+ \mid y \text{ is an even prime number}\} = \{u \in \mathbb{Z}^+ \mid u^2 = 4\}$

Proof:
If $x \in \{y \in \mathbb{Z}^+ \mid y \text{ is an even prime number}\}$
then $x \in \mathbb{Z}^+$ and x is an even prime number
ie $x \in \mathbb{Z}^+$ and $x = 2$ (since 2 is the only even prime number)
ie $x \in \{u \in \mathbb{Z}^+ \mid u^2 = 4\}$.

Conversely, if $x \in \{u \in \mathbb{Z}^+ \mid u^2 = 4\}$
then $x \in \mathbb{Z}^+$ and $x = 2$ (since $2 \in \mathbb{Z}^+$) (this excludes -2)
ie $x \in \mathbb{Z}^+$ and x is an even prime
ie $x \in \{y \in \mathbb{Z}^+ \mid y \text{ is an even prime}\}$.

2. $\mathcal{P}(\{0,1\}) = \{\emptyset\} \cup \{\{0\}\} \cup \{\{1\}\} \cup \{\{0,1\}\}$

Proof:
$S \in \mathcal{P}(\{0,1\})$
iff $S \in \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$
iff $S = \emptyset$ or $S = \{0\}$ or $S = \{1\}$ or $S = \{0,1\}$
iff $S \in \{\emptyset\}$ or $S \in \{\{0\}\}$ or $S \in \{\{1\}\}$ or $S \in \{\{0,1\}\}$
iff $S \in \{\emptyset\} \cup \{\{0\}\} \cup \{\{1\}\} \cup \{\{0,1\}\}$.

Note: The following rules have to be applied when doing exercises 3 – 5.

Let *a* and *b* be two factors, then consider the following options:

(i) If $ab < 0$, i.e ab is a negative number,
then *a* is a negative number and *b* is a positive number (since a minus times a plus gives a minus)
OR
a is a positive number and *b* is a negative number (since a plus times a minus gives a minus)

(ii) If $ab > 0$, i.e ab is a positive number,
then *a* is a negative number and *b* is a negative number (since a minus times a minus gives a plus)
OR
a is a positive number and *b* is a positive number (since a plus times a plus gives a plus)

3. $\{x \in \mathbb{R} \mid x^2 + 6x + 5 < 0\} = \{x \in \mathbb{R} \mid -5 < x < -1\}$

Proof:
$y \in \{x \in \mathbb{R}: x^2 + 6x + 5 < 0\}$
iff $y \in \mathbb{R}$ and $y^2 + 6y + 5 < 0$
iff $y \in \mathbb{R}$ and $(y + 1)(y + 5) < 0$
iff $y \in \mathbb{R}$ and either ($y + 1 > 0$ and $y + 5 < 0$) or ($y + 1 < 0$ and $y + 5 > 0$)
iff $y \in \mathbb{R}$ and either ($y > -1$ and $y < -5$) or ($y < -1$ and $y > -5$)
iff $y \in \mathbb{R}$ and $-5 < y < -1$ (since there is no real number simultaneously less than -5 and greater than -1)
iff $y \in \{x \in \mathbb{R} \mid -5 < x < -1\}$

4. $\{x \in \mathbb{Z} \mid x^2 - 5x + 4 < 0\} = \{x \in \mathbb{Z}^+ \mid x \text{ is a prime factor of } 6\}$

Proof:
$y \in \{x \in \mathbb{Z} \mid x^2 - 5x + 4 < 0\}$
iff $y \in \mathbb{Z}$ and $y^2 - 5y + 4 < 0$

iff $y \in \mathbb{Z}$ and $(y - 1)(y - 4) < 0$
iff $y \in \mathbb{Z}$ and either $(y - 1 < 0$ and $y - 4 > 0)$ or $(y - 1 > 0$ and $y - 4 < 0)$
iff $y \in \mathbb{Z}$ and either $(y < 1$ and $y > 4)$ or $(y > 1$ and $y < 4)$
iff $y \in \mathbb{Z}$ and $1 < y < 4$
iff $y \in \mathbb{Z}$ and $y \in \{2, 3\}$
iff $y \in \mathbb{Z}^+$ and $y \in \{2, 3\}$ (since 2 and 3 are positive integers.)
iff $y \in \{x \in \mathbb{Z}^+ \mid x$ is a prime factor of 6$\}$

5. $\{x \in \mathbb{R} \mid x^2 + x - 2 > 0\} = \{x \in \mathbb{R} \mid x < -2 \text{ or } x > 1\}$
Note: There is a mistake in the exercise given in the study guide.

Proof:
$y \in \{x \in \mathbb{R} \mid x^2 + x - 2 > 0\}$
iff $y \in \mathbb{R}$ and $y^2 + y - 2 > 0$
iff $y \in \mathbb{R}$ and $(y - 1)(y + 2) > 0$
iff $y \in \mathbb{R}$ and either $(y - 1 < 0$ and $y + 2 < 0)$ or $(y - 1 > 0$ and $y + 2 > 0)$
iff $y \in \mathbb{R}$ and either $(y < 1$ and $y < -2)$ or $(y > 1$ and $y > -2)$
iff $y \in \mathbb{R}$ either $y < -2$ or $y > 1$
iff $y \in \{x \in \mathbb{R} \mid x < -2 \text{ or } x > 1\}$

Activity 4-12

1. *Determine whether or not for $V, W, Z \subseteq U$, if $V \subseteq W$, then $V \cup Z \subseteq W \cup Z$ and $V \cap Z \subseteq W \cap Z$. Provide either a proof or a counterexample, whichever is appropriate.*

Let us first try to prove that if $V \subseteq W$ then $V \cup Z \subseteq W \cup Z$.

Suppose $V \subseteq W$.
Let $x \in V \cup Z$,
then $x \in V$ or $x \in Z$
ie $x \in W$ or $x \in Z$ ($V \subseteq W$, so if $x \in V$ then $x \in W$)
ie $x \in W \cup Z$.
Therefore, $V \cup Z \subseteq W \cup Z$.

Let us now consider whether it is the case that if $V \subseteq W$, then $V \cap Z \subseteq W \cap Z$.
Suppose $V \subseteq W$.
Let $x \in V \cap Z$,
then $x \in V$ and $x \in Z$
ie $x \in W$ and $x \in Z$ (because $V \subseteq W$)
ie $x \in W \cap Z$.
We can conclude that if $V \subseteq W$ then $V \cap Z \subseteq W \cap Z$.

2. *Is it the case that, for all subsets $X, Y, W \subseteq U$, if $X = Y$ and $Y = W$, then $X = W$, and if $X \subset Y$ and $Y \subset W$, then $X \subset W$? Justify your answer.*

In general, what does '$A \subset B$' mean? It means that A is a subset of (but not equal to) B, ie A is a proper subset of B.
First, we have to attempt to prove that if X = Y and Y = W, then X = W.
If X = Y and Y = W then we know that X has exactly the same elements as Y and that Y has exactly the same elements as W.
Therefore, X and W contain exactly the same elements and, hence, X = W.

Also, if $X \subset Y$ and $Y \subset W$ we can try to prove that $X \subset W$ as follows:
Suppose $X \subset Y$ and $Y \subset W$.
Let $x \in X$,
then $x \in Y$ (since $X \subset Y$)
ie $x \in W$ (because $Y \subset W$).
So $X \subseteq W$.
But Y has at least one element not in X (since $X \subset Y$)
and W has at least one element not in Y (since $Y \subset W$),
so W has at least two elements not in X
ie $X \neq W$,
so $X \subset W$.

What does this tell us about $\subseteq$? We now know that if $X \subseteq Y$ and $Y \subseteq W$, ie ($X \subset Y$ or X = Y) and ($Y \subset W$ or Y = W), then ($X \subset W$ or X = W) (from the above proofs)
ie $X \subseteq W$.
In other words, the fact that = and $\subset$ satisfy transitive laws for sets tells us that $\subseteq$ is also transitive.

3. *Is it the case that, for all subsets X of U, $X \cup \emptyset = X$? Justify your answer.*
We know that the set we obtain when we determine the union of two sets Y and Z contains all the elements of Y and all the elements of Z. So when we form the union of a subset X and $\emptyset$, the new set, $X \cup \emptyset$, contains all the elements of X and all the elements of $\emptyset$.

But since $\emptyset$ has no elements, $X \cup \emptyset$ will only contain the elements of X.
Therefore, $X \cup \emptyset = X$ for all subsets X of U.
In the style of our other proofs, we may also argue as follows:
Let $x \in X \cup \emptyset$,
then $x \in X$ or $x \in \emptyset$
ie $x \in X$ (because it is impossible for x to reside in $\emptyset$).
Thus, $X \cup \emptyset \subseteq X$.

Conversely, let $x \in X$,
then $x \in X$ or $x \in \emptyset$
ie $x \in X \cup \emptyset$.
Thus, $X \subseteq X \cup \emptyset$.
It follows that $X \cup \emptyset = X$.

4. *Is it the case that, for all subsets V and W of U, $V \cap W = \emptyset$ iff $V = \emptyset$ or $W = \emptyset$? Justify your answer.*
This claim has two parts. These are

if $V = \emptyset$ or $W = \emptyset$ then $V \cap W = \emptyset$, and
if $V \cap W = \emptyset$ then $V = \emptyset$ or $W = \emptyset$.
Both these parts must hold for the claim to be true.
Let us consider the first part. Suppose $V = \emptyset$ or $W = \emptyset$.
We know that, when the intersection of the two sets V and W is formed, the set $V \cap W$ contains the elements common to both V and W.
In this case, at least one of V or W is empty so there is no element common to V and W.
Therefore, $V \cap W$ is also empty.

Looking at the second part of the claim, we have to decide whether it is necessarily the case whenever $V \cap W = \emptyset$, it follows that $V = \emptyset$ or $W = \emptyset$
Well, if $V \cap W = \emptyset$ all we know is that V and W have no elements in common.
It is not necessarily the case that one of them is empty.

Consider the example $V = \{1, 2\}$ and $W = \{3, 4\}$.
It is clear that $V \cap W = \emptyset$ although neither V nor W is empty.
We can conclude that if $V = \emptyset$ or $W = \emptyset$ then $V \cap W = \emptyset$,
but if $V \cap W = \emptyset$, then it is not necessarily the case that $V=\emptyset$ or $W=\emptyset$.
Therefore, it is not the case that, for all subsets V and W, $V \cap W = \emptyset$ iff $V = \emptyset$ or $W = \emptyset$.

5. *Is it the case that, for every subset X of U, there exists a subset Y of U such that $X \cup Y = \emptyset$? Justify your answer.*

No. We give a counterexample.

We know that the set $X \cup Y$ contains all the elements of X as well as those of Y.
So if U is the set $\{1, 2\}$ and X is the subset $\{1\}$, then there is no subset Y of U such that $X \cup Y = \emptyset$.
To see this, note that there are just four possible values for Y,
namely, $Y = \emptyset$, $Y = \{1\}$, $Y = \{2\}$, and $Y = \{1, 2\}$.
In each case, $X \cup Y$ contains at least the element 1, so $X \cup Y \neq \emptyset$.

6. *Is it the case that for every subset X of U there is some subset Y such that $X \cap Y = U$? Justify your answer.*

No. We give a counterexample.

We know that the intersection $X \cap Y$ contains the elements that are common to X and Y.
So, if $X \cap Y = U$ it must be the case that X and Y have all the elements of U in common. It is not usually the case.
Take $U = \{1, 2\}$ and $X = \{1\}$.
Then there is no subset Y of U such that $X \cap Y = U$.
To see this, note that Y can have four possible values,
namely, $Y = \emptyset$, $Y = \{1\}$, $Y = \{2\}$, and $Y = \{1, 2\}$.
In none of these four cases does $X \cap Y$ contain the element 2.

7. *Using "if and only if" statements, determine the following*:

(a) *Is it the case that $X + Y = Y + X$ for all $X, Y \subseteq U$?*

$x \in X + Y$
iff $x \in X$ or $x \in Y$, but not both
iff $x \in Y$ or $x \in X$, but not both
iff $x \in Y + X$. We conclude that $X + Y = Y + X$.

(b) *Is it case that $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$ for all $X, Y, Z \subseteq U$?*

$x \in X \cap (Y + Z)$
iff $x \in X$ and $x \in (Y + Z)$
iff $x \in X$ and ($x \in Y$ or $x \in Z$), but not both
iff x is in X and in exactly one of Y or Z
iff either ($x \in X$ and $x \in Y$) or ($x \in X$ and $x \in Z$), but not both
iff $x \in X \cap Y$ or $x \in X \cap Z$, but not both
iff $x \in (X \cap Y) + (X \cap Z)$.
Therefore, $X \cap (Y + Z) = (X \cap Y) + (X \cap Z)$.

8 Learning unit 5 – Relations

Study Material

Study guide
You should cover study unit 5 in the study guide. Where the study unit refers to the CAI tutorial, you should also work through the relevant part in the CAI tutorial. The theory, examples and exercises will help you understand the concepts. If you have not received your CAI tutorial with your study material, please see the *Orientation* section for downloading instructions.

Time allocated
You will need one week to master this learning unit.

Notes

Background
Relations are special kinds of sets containing ordered pairs as elements. We consider the different definitions that define the properties of relations.

Applications
Definitions of relations can be applied to determine which properties the given relations have.

Activities

Do the activities provided in study unit 5 to consolidate your knowledge of the work in this learning unit.

Study unit 5

Activity 5-4

Suppose A = {1, 2, 3, 4}, B = {2, 5}, C = {3, 4, 7} and the universal set U = {1, 2, 3, 4, 5, 7}.
Write out the following Cartesian products by using list notation:

(a) $A \times B = \{(1, 2), (1, 5), (2, 2), (2, 5), (3, 2), (3, 5), (4, 2), (4, 5)\}$

(b) $B \times A = \{(2, 1), (2, 2), (2, 3), (2, 4), (5, 1), (5, 2), (5, 3), (5, 4)\}$

(c) $A \cup B = \{1, 2, 3, 4, 5\}$

$(A \cup B) \times C = \{(1, 3), (1, 4), (1, 7), (2, 3), (2, 4), (2, 7), (3, 3), (3, 4), (3, 7), (4, 3), (4, 4), (4, 7), (5, 3), (5, 4), (5, 7)\}$

(d) $A + B = \{1, 3, 4, 5\}$

$(A + B) \times B = \{(1, 2), (1, 5), (3, 2), (3, 5), (4, 2), (4, 5), (5, 2), (5, 5)\}$

Activity 5-8

1. *Let P and R be relations on A = { 1, 2, 3, {1}, {2} } given by*
P = { (1, {1}), (1, 2) } and
R = { (1, {1}), (1, 3), (2, {1}), (2, {2}), ({1}, 3), ({2}, {1}) }. Investigate the following:

(a) **Irreflexivity:**
Is it the case that for all $x \in A$, $(x, x) \notin R$?

Yes, R is irreflexive, since the first and second co-ordinates differ from each other in each ordered pair of R. We do not have all the following as elements of R: (1, 1), (2, 2), (3, 3), ({1}, {1}), ({2}, {2}).

(b) **Reflexivity:**
Is it the case that for all x $\in$ A, (x, x) $\in$ R?
No, R is not reflexive, we give a counterexample:
(1, 1) $\notin$ R.

(c) **Symmetry:**
If (x, y) $\in$ R, is it the case that (y, x) $\in$ R?
No, R is not symmetric. We give a counterexample:
(1, {1}) $\in$ R, but its mirror image, ({1}, 1) $\notin$ R.

(d) **Antisymmetry:**
If x $\neq$ y and (x, y) $\in$ R, is it the case that (y, x) $\notin$ R?
Yes, R is antisymmetric, since no member of R has its mirror image also living in R.
For each ordered pair (x, y) living in R, we have that (y, x) $\notin$ R.

(e) **Transitivity:**
If (x, y) $\in$ R and (y, z) $\in$ R, is it the case that (x, z) $\in$ R?
No, R is not transitive. We give a counterexample:
(2, {1}) $\in$ R, and ({1}, 3) $\in$ R, but (2, 3) $\notin$ R.

(f) **Trichotomy**:
Is it the case for all x, y $\in$ A, if x $\neq$ y then (x, y) $\in$ R or (y, x) $\in$ R?
No, R does not satisfy trichotomy. We give a counterexample:
(2, 3) $\in$ R and also (3, 2) $\in$ R.

This means that 2 $\neq$ 3, but we cannot compare the elements 2 and 3 of A in terms of R, because these 2 elements do not appear together in any ordered pair of R.

(g) **R ∘ R:** (x, w) $\in$ R ∘ R iff for some y there exist pairs (x, y) $\in$ R and (y, w) $\in$ R.
R = { (1, {1}), (1, 3), (2, {1}), (2, {2}), ({1}, 3), ({2}, {1}) }
We have that (1, {1}) $\in$ R and ({1}, 3) $\in$ R, hence (1, 3) $\in$ R ∘ R.

Also: (2, {1}) $\in$ R and ({1}, 3) $\in$ R, hence (2, 3) $\in$ R ∘ R.

(2, {2}) $\in$ R and ({2}, {1}) $\in$ R, hence (2, {1}) $\in$ R ∘ R.

({2}, {1}) $\in$ R and ({1}, 3) $\in$ R, hence ({2}, 3) $\in$ R ∘ R.

Thus, R ∘ R = { (1, 3), (2, 3), (2, {1}), ({2}, 3) }.

(h) **R ∘ P:** (x, w) $\in$ R ∘ P iff for some y there exist pairs (x, y) $\in$ P and (y, w) $\in$ R.

We have that (1, {1}) $\in$ P and ({1}, 3) $\in$ R, hence, (1, 3) $\in$ R∘P.

Also: (1, 2) $\in$ P and (2, {1}) $\in$ R, hence, (1, {1}) $\in$ R∘P.

(1, 2) $\in$ P and (2, {2}) $\in$ R, hence, (1, {2}) $\in$ R∘P.

Thus, R ∘ P = { (1, 3), (1, {1}), (1, {2}) }.

(i) **T:** T is a subset of R, so T is also a relation on A.
We have (a, B) ∈ T iff a ∈ B.
In each ordered pair in T, the first co-ordinate must be a member of the second co-ordinate.
We have T = { (1, {1}), (2, {2}) }.

2. *Let A = {a, b}. For each of the specifications given below, find suitable examples of relations on $\mathcal{P}$ (A).*

First of all, let us write down $\mathcal{P}$ (A):

$\mathcal{P}$ (A) = **{** ∅, {a}, {b}, {a,b} **}**,

and $\mathcal{P}$ (A) × $\mathcal{P}$ (A) = **{** (∅, ∅), (∅, {a}), (∅, {b}), (∅, {a,b}), ({a}, ∅), ({a}, {a}), ({a}, {b}), ({a}, {a,b}), ({b}, ∅), ({b}, {a}), ({b}, {b}), ({b}, {a,b}), ({a,b}, ∅), ({a,b}, {a}), ({a,b}, {b}), ({a,b},{a,b}) **}**.

(a) R is reflexive on $\mathcal{P}$ (A), symmetric and transitive:
Two examples of relations that satisfy this specification are **{** (∅, ∅), ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}) **}** and **{** (∅, ∅), ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}), (∅, {a}), ({a}, ∅) }.

(b) R is reflexive on $\mathcal{P}$ (A) and symmetric, but not transitive:
{ (∅, ∅), ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}), (∅, {a}), ({a}, ∅), ({a}, {a,b}), ({a,b}, {a}) **}**

This relation is not transitive because it contains both (∅, {a}) and ({a}, {a,b}) but not (∅, {a,b}). Another counterexample to transitivity is that both ({a,b}, {a}) and ({a}, ∅) belong to the relation, but not ({a,b}, ∅).

(c) R is reflexive on $\mathcal{P}$ (A), transitive, but is not symmetric and not antisymmetric:
{ (∅, ∅), ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}), (∅, {a}), (∅, {b}), ({a}, {b}), ({a,b}, ∅), ({a,b}, {a}), ({a,b}, {b}), ({b}, {a}) **}**

(d) R is simultaneously symmetric and antisymmetric:
{ (∅, ∅), ({a}, {a}), ({b}, {b}), ({a,b}, {a,b}) **}**

(e) R is irreflexive, antisymmetric, transitive:
{ (∅, {a}), (∅, {b}), (∅, {a,b}), ({a}, {b}), ({a}, {a,b}), ({b}, {a,b}).}

3. *Prove that if R is a relation on X, then R is transitive iff R ∘ R ⊆ R.*

First, we attempt to prove that if R is transitive, then we prove that R ∘ R ⊆ R.

Assume R is transitive.

Suppose (x, z) ∈ R ∘ R, then, according to the definition of composition, there exists some y ∈ X such that (x, y) ∈ R and (y, z) ∈ R.
Because R is transitive, it follows that (x, z) ∈ R.

This completes the proof that if R is transitive, then $R \circ R \subseteq R$.

Now we have to prove the converse.

Assume $R \circ R \subseteq R$.

Suppose $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R \circ R$ according to the definition of composition.

Because $R \circ R \subseteq R$, it follows that $(x, z) \in R$.

Therefore, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$.
Hence, R is transitive.

We can now conclude that if R is a relation on X, then R is transitive iff $R \circ R \subseteq R$.

9 Learning unit 6 – Special kinds of relations

Study Material

Study guide

You should cover study unit 6 in the study guide. In this learning unit, we will only do activities for sections 6.1 to 6.3. You will revise sections 6.4 and 6.5 again in the next learning unit, where you will be doing the activities for these sections. Where the study unit refers to the CAI tutorial, you should also work through the relevant part in the CAI tutorial. The theory, examples and exercises will help you understand the concepts. If you have not received your CAI tutorial with your study material, please see the *Orientation* section for downloading instructions.

Time allocated

You will need one week to master this learning unit.

Notes

Background

Different kinds of relations have certain properties. We consider the definitions of different kinds of relations.

Applications

We investigate which properties given relations have in order to determine which kinds of relations they are classified as. Equivalence classes and partitions can also be determined for relevant relations.

Activities

Do the activities provided in study unit 6, sections 6.1 to 6.3 to consolidate your knowledge of the work in this learning unit.

Study unit 6, sections 6.1 to 6.3

Activity 6.4

For each of the following relations, determine whether or not the relation is a weak partial order (reflexive, antisymmetric and transitive) on the given set:

(a) Let A = {a, b, {a, b} }. The relation S on A is defined by (c, B) $\in$ S iff c $\in$ B.
We see that a $\in$ {a, b} and b $\in$ {a, b} and, thus, S = { (a, {a, b}), (b, {a, b}) }.

Reflexivity:

Is it the case that for all x $\in$ A, (x, x) $\in$ S?
No, we give a counterexample: (a, a) $\notin$ S. (It is also the case that (b, b) $\notin$ S and ({a, b}, {a, b}) $\notin$ S.)

S is not reflexive; therefore, we can say that it is not a weak partial order.
(You can test whether or not S is antisymmetric and transitive.)

(b) Define R $\subseteq \mathbb{Z} \times \mathbb{Z}$ by x R y iff x + y is even.

If x + y is even then we can say that x + y = 2k for some integer k.

Reflexivity:

Is it the case that for all $x \in \mathbb{Z}$, $(x, x) \in R$?

$x + x = 2x$, ie $x + x$ is an even number for any $x \in \mathbb{Z}$.

Thus, R is reflexive on $\mathbb{Z}$.

Antisymmetry:

If $(x, y) \in R$, is it the case that $(y, x) \notin R$?

Suppose $(x, y) \in R$

then $x + y = 2k$

ie $y + x = 2k$, but this means that $(y, x) \in R$.

Thus, R is not antisymmetric. R is actually symmetric.

Because R is not antisymmetric, it is not a weak partial order.

For interest's sake, let us test whether R is transitive:

Transitivity:

If $(x, y) \in R$ and $(y, z) \in R$, is it the case that $(x, z) \in R$?

Suppose $(x, y) \in R$ and $(y, z) \in R$

then $x + y = 2k$ and $y + z = 2m$ for some $k, m \in \mathbb{Z}$.

ie $x = 2k - y$ and $z = 2m - y$

ie $x + z = 2k - y + 2m - y$

ie $x + z = 2(k + m - y)$

ie $x + z = 2t$ for some integer t

ie $(x, z) \in R$. Thus, R is transitive.

(c) Define R on $\mathbb{Z} \times \mathbb{Z}$ by $(a, b)\ R\ (c, d)$ if either $a < c$ or else ($a = c$ and $b \leq d$).

Reflexivity:

Is it the case that for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, $(a, b)\ R\ (a, b)$?

For any $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ it is not the case that $a < a$, but

($a = a$ and $b \leq b$)

ie $(a, b)\ R\ (a, b)$.

Thus, R is reflexive.

Antisymmetry:

If $(a, b) \neq (c, d)$ and $((a, b), (c, d)) \in R$, is it the case that $((c, d), (a, b)) \notin R$?

Suppose $(a, b) \neq (c, d)$ and $(a, b)\ R\ (c, d)$, then $a < c$ or else ($a = c$ and $b \leq d$).

Firstly, we do not have $c < a$ and, secondly, we do not have that $c = a$ and $d \leq b$, thus, $((c, d), (a, b)) \notin R$.

(In the second case, we cannot have $c = a$ and $d = b$ because we assumed that $(a, b) \neq (c, d)$.

Furthermore, we cannot have $c = a$ and $d < b$, because by our assumption, $b \leq d$.

We can safely say that $((c, d), (a, b)) \notin R$, thus, R is antisymmetric.

Transitivity:

If $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$, is it the case that $((a, b), (e, f)) \in R$?

Suppose $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$

ie $a < c$ or else ($a = c$ and $b \leq d$), and $c < e$ or else ($c = e$ and $d \leq f$).

We can look at the following cases:
$a < c$ and $c < e$,
$a < c$ and $c = e$ and $d \leq f$,
$a = c$ and $b \leq d$ and $c < e$, or
$a = c$ and $b \leq d$ and $c = e$ and $d \leq f$.
In the first three cases we have $a < e$ and in the last case we have $a = e$ and $b \leq f$.
We can deduce that $((a, b), (e, f)) \in R$, thus, R is transitive.

Because R is reflexive, antisymmetric and transitive, we can say that R is a weak partial order.

Activity 6.5

For each of the following relations, determine whether or not the relation is a strict partial order (irreflexive, antisymmetric and transitive) on the given set:

Let $A = \{a, \{a\}, \{b\}\}$ and let S on A be the relation $S = \{ (a, \{a\}), (a, \{b\})\}$.

Irreflexivity:
Is it the case that for all $x \in A$, $(x, x) \notin S$?
Yes. $(a, a) \notin S$, $(\{a\}, \{a\}) \notin S$ and $(\{b\}, \{b\}) \notin S$.

Antisymmetry:
If $(x, y) \in S$, is it the case that $(y, x) \notin S$?
Yes. $(a, \{a\})$ and $(a, \{b\})$ are the elements of S, but $(\{a\}, a), (\{b\}, a) \notin S$.

Transitivity:
If $(x, y) \in S$ and $(y, z) \in S$, is it the case that $(x, z) \in S$?
No, two ordered pairs in S are such that $(x, y) \in S$ and $(y, z) \in S$, so we need not find the pair (x, z) in S. S is thus transitive.
(We cannot prove that S is not transitive. Such a proof actually has a special name: it is vacuously true that S is transitive.)

Because S is irreflexive, antisymmetric and transitive, we can say that S is a strict partial order.

(b) Define $R \subseteq (\mathbb{Z} \times \mathbb{Z}) \times (\mathbb{Z} \times \mathbb{Z})$ by $(a, b)\ R\ (c, d)$ iff $a < c$.

Irreflexivity:
Is it the case that for all $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, $((a, b), (a, b)) \notin R$?
Yes. It is not true that $a < a$, therefore $((a, b), (a, b)) \notin R$.

Antisymmetry:
If $(a, b) \neq (c, d) \in R$ and $((a, b), (c, d)) \in R$, is it the case that $((c, d), (a, b)) \notin R$?
Yes. Suppose $(a, b) \neq (c, d)$ and $(a, b)\ R\ (c, d)$, then $a < c$.
This means that it is not possible that $c < a$, therefore $((c, d), (a, b)) \notin R$.

Transitivity:
If $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$, is it the case that $((a, b), (e, f)) \in R$?
Suppose $((a, b), (c, d)) \in R$ and $((c, d), (e, f)) \in R$
then $a < c$ and $c < e$

ie a < e
We can deduce that ((a, b), (e, f)) $\in$ R, thus R is transitive.

Because R is irreflexive, antisymmetric and transitive, we can say that R is a strict partial order.

Activity 6-7

1. *Let X = {a,b,c}. Write down all strict partial orders on X. Which of them are linear?*

Strict partial orders on X are irreflexive, antisymmetric and transitive. The strict partial orders on X are:
∅, { (a,b) }, { (a,c) }, { (b,a) }, { (b,c) }, { (c,a) }, { (c,b) },
{ (a,b), (a,c) }, { (a,b), (c,b) }, { (a,c), (b,c) }, { (b,a), (b,c) }, { (b,a), (c,a) }, { (c,a), (c,b) },
{ (a,b), (b,c), (a,c) }, { (b,a), (a,c), (b,c) }, { (b,c), (c,a), (b,a) }, { (a,c), (c,b), (a,b) }, { (c,a), (a,b), (c,b) }, and { (c,b), (b,a), (c,a) }.
All the relations containing three elements satisfy trichotomy and are therefore linear.
Note: How do we know we have found all the strict partial orders on X? We were systematic. We wrote down those with one pair then we listed all the ways to add another pair without losing properties like transitivity. Lastly, we listed the relations containing 3 pairs, and each of these can be viewed as a 2-step journey, together with the contraction of that journey required by transitivity. It is not possible to have more than 3 pairs without losing irreflexivity or antisymmetry.

2. *In each of the following cases, determine whether or not R is some sort of order relation on the given set X (weak partial, weak total, strict partial, or strict total). Justify your answer.*

To determine whether R is some sort of order relation on X, we have to examine the relevant properties of R in each case.

(a) X = { ∅, {0}, {2} } and R = { (∅, {0}), (∅, {2}) }:

Reflexivity:
R is not reflexive so we provide a counterexample:
(∅, ∅) $\notin$ R (We also have that ({0}, {0}) $\notin$ R and ({2}, {2}) $\notin$ R.)

Irreflexivity:
For all x $\in$ X we have that (x, x) $\notin$ R:
(∅, ∅) $\notin$ R, ({0}, {0}) $\notin$ R and ({2}, {2}) $\notin$ R.
Hence, R is irreflexive.

Antisymmetry:
R is antisymmetric because the mirror images of (∅, {0}) and (∅, {2}) are not in R.

Transitivity:
R is transitive because it does not contain any members with first co-ordinates
{0} and {2}, so there are no 2-step journeys to worry about.
(Does this proof bother you? If so, remember the definition of transitivity,
"**if** (x, y) $\in$R and (y, z) $\in$R, **then** ...". When the **if** part does not apply, we have no more work to do!)

Trichotomy:

There is no ordered pair comparing {0} and {2}.
Therefore, R does not satisfy trichotomy.
R is a strict partial order on X, because R is irreflexive, antisymmetric and transitive.

(b) *X = {∅, {∅}, {{∅}}} and R = ⊆ (R is the relation of all ordered pairs where each first co-ordinate is a subset of the second co-ordinate, and R ⊆ X × X.)*

For example, ∅ ⊆ {∅}, so (∅, {∅}) ∈ ⊆ or (∅, {∅}) ∈ R.
Because X has only three elements, we can describe R in list notation.
⊆ = { (∅, ∅), (∅, {∅}), (∅, {{∅}}), ({∅}, {∅}), ({{∅}}, {{∅}}) }.

Note: ({∅}, {{∅}}) ∉ ⊆ because {∅} is not a subset of {{∅}}. *{∅} is not a subset of {{∅}} because each element of {∅} is not an element of {{∅}} – the only element of {∅} is ∅ and the only element of {{∅}} is {∅}. We see that ∅ ∈ {∅}, but ∅ ∉ {{∅}} thus {∅} is not a subset of {{∅}}.*

Reflexivity:
It is clear that for every x ∈ X, (x, x) ∈ R: (∅, ∅), ({∅}, {∅}) and ({{∅}}, {{∅}}) are all members of R. Therefore, R (or ⊆, if you prefer) is reflexive on X.

Antisymmetry:
By inspection of R it is clear that for all x, y ∈ X, if x ≠ y and (x, y) ∈ R, then (y, x) ∉ R.
Or to say the same thing in different words, whenever both x ⊆ y and y ⊆ x, then x = y.
Therefore, R (or ⊆, if you prefer) is antisymmetric.

Transitivity:
A typical 2-step journey is (∅, {∅}), followed by ({∅}, {∅}), and its corresponding 1-step journey (∅, {∅}) is also in R, playing a double role.
Clearly, every 2-step journey in R can be performed in a single step.
Can you find them all?
Therefore, R (or ⊆, if you prefer) is transitive.

Irreflexivity:
There are several x in X such that (x, x) ∈ ⊆, for example, x = ∅, where (∅, ∅) ∈ ⊆.
So ⊆ is not irreflexive.

Trichotomy:
⊆ does not satisfy trichotomy, because it is not the case that for all x, y∈ X, if x ≠ y then (x, y) ∈ ⊆ or (y, x) ∈ ⊆, for example, there is no ordered pair containing both {∅} and {{∅}}.

Because ⊆ is reflexive, antisymmetric and transitive, it is a weak partial order.

(c) X = ℤ and R = ≤:

Reflexivity:
For all x ∈ ℤ, (x, x) ∈ ≤, because x ≤ x for every integer x.
Therefore, R is reflexive.

Antisymmetry:
For any x, y $\in \mathbb{Z}$, if x ≠ y and (x, y) $\in \leq$ then x ≤ y.
Therefore it is not the case that y ≤ x. So (y, x) $\notin \leq$.
Therefore, ≤ is antisymmetric.
Another way to say the same thing is that if x ≤ y and, at the same time, y ≤ x, then it must be the case that x = y, because if the value of x appears to the left of the value of y on the number line and the value of y appears to the left of that of x, then they must lie on the same position.

Transitivity:
If (x, y) $\in \leq$ and (y, z) $\in \leq$ for any x, y, z $\in \mathbb{Z}$, then x ≤ y and y ≤ z,
ie x appears to the left of y and y appears to the left of z.
Thus, x appears to the left of z, ie x ≤ z, ie (x, z) $\in$ v. Therefore, ≤ is transitive.

Irreflexivity:
≤ is not irreflexive, because we can find values of x such that (x, x) $\in \leq$,
for example, x = 113, or if you prefer simple values, x = 2.

Trichotomy: For all x, y $\in \mathbb{Z}$, if x ≠ y then either x > y or y > x,
because the one must appear to the left of the other on the number line.
Therefore, either (x, y) $\in \leq$ or (y, x) $\in \leq$. Hence, ≤ satisfies trichotomy.
Because ≤ is reflexive, antisymmetric and transitive, it is a weak partial ordering,
and because ≤ is a weak partial ordering satisfying trichotomy, it is also a weak total (linear) ordering.

(d) X = $\mathbb{Z}$ and R = >:

Reflexivity:
There are values of x such that (x, x) $\notin$ >, since if x = 113, say, then it is not the case that x > x. Therefore, > is not reflexive on $\mathbb{Z}$.

Antisymmetry:
If (x, y) $\in$ >, then x > y, ie x lies to the right of y on the number line.
Thus, it cannot be the case that y > x.
So (y, x) $\notin$ >, and > is therefore antisymmetric.

Transitivity:
Suppose (x, y) $\in$ > and (y, z) $\in$ >, ie x > y and y > z.
This means that x lies to the right of y and y to the right of z on the number line.
Therefore, x lies to the right of z, ie x > z, ie (x, z) $\in$ >.
Thus, > is transitive.

Irreflexivity:
For all x $\in \mathbb{Z}$ it is not the case that x lies to the right of itself, ie it will never be the case that x > x.
Therefore, (x, x) $\notin$ > for all x $\in \mathbb{Z}$, and hence, > is irreflexive.

Trichotomy:

For all x, y $\in \mathbb{Z}$, if x ≠ y then either x lies to the right of y (ie x > y) or x lies to the left of y (ie y > x). So, either (x, y) $\in$ > or (y, x) $\in$ >. Therefore > satisfies trichotomy.
We can conclude that > is a strict partial order relation because it is irreflexive, antisymmetric and transitive. What is more, > is a strict linear ordering because it satisfies trichotomy as well.
Note: Any linear order is also a partial order, but not vice versa.

(e) *X = $\mathbb{Z}^+$ and R is defined by: x R y iff x divides y with zero remainder, ie y = kx for some k $\in \mathbb{Z}^+$.*
(x R y is another way of saying (x, y) $\in$ R.)

This means that x is a factor of y and y is a multiple of x.
Let us synthesize some ordered pairs that belong to R:
How about (2, 6), (3, 6), (5, 35) and (4, 24)?
All of these meet the requirement that y = kx for some k $\in \mathbb{Z}^+$.

Reflexivity:
For each x $\in \mathbb{Z}^+$ we have that x = 1x and 1 $\in \mathbb{Z}^+$, so (x, x) $\in$ R.
R is therefore reflexive on $\mathbb{Z}^+$.

Antisymmetry:
Suppose x ≠ y and (x, y) $\in$ R.
Can (y, x) qualify to belong to R?
If (x, y) $\in$ R, y = kx ① for some k $\in \mathbb{Z}^+$.

Does it ever happen that (y, x) $\in$ R, ie x = my ② for some m $\in \mathbb{Z}^+$?

Substitute ② into ①:

y = kx = k(my) = (km)y, ie y = (km)y, which means km=1.
Hence, k = m = 1, so x = y.
But we specifically assumed that x ≠ y,
so it can never happen that (y, x) $\in$ R, which means that (y, x) $\notin$ R.
Therefore, R is antisymmetric.

Transitivity:
Suppose (x, y) $\in$ R and (y, z) $\in$ R.
Then y = kx for some k $\in \mathbb{Z}^+$.
and z = my for some m $\in \mathbb{Z}^+$.
Hence, z = my = m(kx) = (mk)x,
ie (x, z) $\in$ R.
Thus, R is transitive.

Irreflexivity:
Since we can find values of x such that (x, x) $\in$ R,
for example, x = 113, where (113, 113) $\in$ R,
R cannot be irreflexive.

Trichotomy:

R does not satisfy trichotomy.
Take x = 2 and y = 3, then there does not exist some k, m $\in \mathbb{Z}^+$ such that 3 = k(2) or 2 = m(3), so neither (2, 3) $\in$ R nor (3, 2) $\in$ R.
Thus, R is a weak partial order on $\mathbb{Z}^+$.

Activity 6-10

1. *Let X = {a, b, c}. Write down all equivalence relations on X.*

Equivalence relations on X must be reflexive, symmetric and transitive:
R_1 = { (a, a), (b, b), (c, c) },
R_2 = { (a, a), (b, b), (c, c), (a, b), (b, a) },
R_3 = { (a, a), (b, b), (c, c), (a, c), (c, a) },
R_4 = { (a, a), (b, b), (c, c), (b, c), (c, b) },

R_5 = {(a, a), (b, b), (c, c), (a, b), (b, a), (a, c), (c, a), (b, c), (c, b) },

2. *In each of the following cases, determine whether or not the given relation is an equivalence relation. If it is, describe the equivalence class(es) of R. Justify your reasoning.*

(a) X = {a, b, c} and R = { (c, c), (b, b), (a, a) }:

Reflexivity:
R is reflexive because for every x $\in$ X, we have (x, x) $\in$ R, as we can see by inspecting R. The ordered pairs (a, a), (b, b) and (c, c) are all present in R.

Symmetry:
R is also symmetric, because there is no pair (x, y) $\in$ R, x ≠ y, such that (y, x) $\notin$ R,
in other words, for every (x, y) $\in$ R it is also the case that (y, x) $\in$ R, since the first co-ordinate is equal to the second co-ordinate in all the ordered pairs belonging to R. Each pair in R, namely, (a, a), (b, b) and (c, c), plays a double role; each plays the part of (x, y) as well as that of (y, x).

Transitivity:
Is it the case that for all x, y, z $\in$ X, if (x, y) $\in$ R and (y, z) $\in$ R, then (x, z) $\in$ R?
This is surely the case, because the only 2-step journeys are trivial ones like "From b go to b, and then go from b to b".
If fact, each pair in R plays a triple role; it plays the parts of (x, y), (y, z) and (x, z).

To illustrate, let us consider one specific example, say (b, b) in this triple role:
x y y z x z
ie (b,b) and (b,b) and (b,b).
This means that R is transitive.
R is the equality (or identity) relation on X.

What are the equivalence classes of R?
Because X has only three elements, we can consider each element individually:
[a] = {y | (a, y) $\in$ R}
= {a}.

Similarly, [b] = {b} and [c] = {c}.

(b) *X = {a, b, c} and R = X × X:*
It is easy to describe R in list notation because X has only three elements.
R = { (a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c) }.

Reflexivity:
R is reflexive, because (a, a), (b, b) and (c, c) are all contained in R.

Symmetry:
R is symmetric, because for each pair (x, y), its mirror image (y, x) is also in R. This can be checked by inspecting R. We find (a, b) $\in$ R and (b, a) $\in$ R; (a, c) $\in$ R and (c, a) $\in$ R; and (b, c) $\in$ R and (c, b) $\in$ R. Furthermore, the ordered pairs (a, a), (b, b) and (c, c) each play a double role, being itself and its own mirror image.

Transitivity:
Scrutinising R very carefully we see that for all x, y, z $\in$ X, if (x, y) $\in$ R and (y, z) $\in$ R then (x, z) $\in$ R.
For example, among the various 2-step journeys found in R is (b, c) followed by (c, a).
Since (b, a) is in R, the 2-step journey can be contracted to a single step.
We also have (a, a) and (a, b) in R, and (a, b), the single-step journey, is there, playing a double role.
Similarly, we find (b, a), (a, b) and (b, b).
Can you spot **all** the other 2-step journeys?
All must be tested, and the associated single-step journeys must be found to be present, before we can confirm transitivity.
This can be done; thus, R is transitive.

We can now conclude that R is an equivalence relation.

What are the equivalence classes of R?
[a] = {y | (a, y) $\in$ R}
= {a, b, c}.
We do not even bother to work out [b] and [c], because b and c are both in [a],
so we know that [a] = [b] = [c] = X.
In other words, R says "All the elements of X are equivalent to one another",
and there is only one equivalence class.

(c) *X = $\mathcal{P}$ (Y) where Y = {1, 2, 3} and*
R consists of all pairs (C, D) such that C ∩ {2} = D ∩ {2}:

We can use a brute force approach to this problem, because X and R are small sets:
$\mathcal{P}$ (Y) = { ∅, {1}, {2}, {3}, {1, 2}, {1, 3}, {2, 3}, {1, 2, 3} } and
we can work out what R is by noting that the only possible outcome of
C ∩ {2} is ∅, if 2 $\notin$ C,
and C ∩ {2} is {2}, if 2 $\in$ C.

Therefore, all subsets of Y that do not contain the member 2 are related to one another by R, and all subsets of Y that do contain the element 2 are related to one another by R. Thus
R = **{** (∅, ∅), (∅, {1}), (∅, {3}), (∅, {1, 3}), ({1}, ∅), ({1}, {1}), ({1}, {3}), ({1}, {1, 3}),
({3}, ∅), ({3}, {1}), ({3}, {3}), ({3}, {1, 3}), ({1, 3}, ∅), ({1, 3}, {1}), ({1, 3}, {3}), ({1, 3}, {1, 3}),
({2}, {2}), ({2}, {1, 2}), ({2}, {2, 3}), ({2}, Y), ({1, 2}, {2}), ({1, 2}, {1, 2}), ({1, 2}, {2, 3}), ({1, 2}, Y),
({2, 3}, {2}), ({2, 3}, {1, 2}), ({2, 3}, {2, 3}), ({2, 3}, Y), (Y, {2}), (Y, {1,2}), (Y, {2,3}), (Y, Y) **}**.

By inspection, R is reflexive on X, symmetric and transitive. Therefore, R is an equivalence relation.
The equivalence classes of R are
[∅] = **{** ∅, {1}, {3}, {1, 3} **}** and
[{2}] = **{** {2}, {1, 2}, {2, 3}, {1, 2, 3} **}**.

Note: Of course we could call [∅] by the name [{1}] instead, or [{2}] by the name [{2, 3}] and so on.

3. Let R be the relation on $\mathbb{Z}$ such that (x, y) ∈ R iff x − y is a multiple of 4.
We can define R as follows: (x, y) ∈ R iff x − y = 4k for some integer k.

(a)
Reflexivity:
Is it true that (x, x) ∈ R for all x ∈ $\mathbb{Z}$?
Yes, for all x ∈ $\mathbb{Z}$ we have x − x = 0 = 4·0, which is a multiple of 4,
thus, R is reflexive on $\mathbb{Z}$.

Irreflexivity:
Is it the case that for all x ∈ $\mathbb{Z}$, (x, x) ∉ R?
No, there is no integer x such that (x, x) ∉ R.
We give a counterexample:
(1, 1) ∈ R since 1 − 1 = 0 = 4 − 0,
thus, R is not irreflexive.

Symmetry:
If (x, y) ∈ R, is it true that (y, x) ∈ R?
Suppose (x, y) ∈ R, ie x − y is a multiple of 4,
ie x − y = 4k for some k ∈ Z.
ie y − x = − (x − y) = −4k = 4(−k).
So y − x is a multiple of 4, hence, (y, x) ∈ R.
Thus, R is symmetric.

Antisymmetry:
No, if (x, y) ∈ R, it is not necessarily true that (y, x) ∉ R.
We give a counterexample:
(5, 1) ∈ R since 5 – 1 = 4, which is a multiple of 4, but 1 – 5 = –4, which is also a multiple of 4,
so (1, 5) ∈ R.
Thus, R is not antisymmetric.

Transitivity:
If (x, y) ∈ R and (y, z) ∈ R, is it true that (x, z) ∈ R?

Suppose $(x, y) \in R$,

then $x - y = 4k$ for some $k \in \mathbb{Z}$ ①

and suppose $(y, z) \in R$,

then $y - z = 4m$ for some $m \in \mathbb{Z}$. ②

① + ②, then $(x - y) + (y - z) = 4k + 4m$

ie $x - z = 4(k + m)$, which is a multiple of 4, so $(x, z) \in R$.
Thus, R is transitive.

Trichotomy:

Is it true that for all positive integers x, y if $x \neq y$, then either $(x, y) \in R$ or $(y, x) \in R$?

No, it is not true! We show this by using a counterexample:

Choose $1, 2 \in \mathbb{Z}$,

then $2 - 1 = 1$ and $1 - 2 = -1$ and these are not multiples of 4,

so $(2, 1) \notin R$ and $(1, 2) \notin R$.

Thus, R does not satisfy trichotomy.

(b) *What kind of relation is R?*

Since R is reflexive on $\mathbb{Z}$, symmetric and transitive, it follows that R is an equivalence relation.

(c) *R is an equivalence relation, so we can give the equivalence classes of R:*

$[x] = \{y \mid (x, y) \in R\}$

We know $(v, w) \in R$ iff $v - w = 4k$, therefore, $[x] = \{y \mid x - y = 4k \text{ for some } k \in \mathbb{Z}\}$

$[0] = \{y \mid 0 - y = 4k \text{ for some } k \in \mathbb{Z}\}$
$\quad = \{y \mid y = -4k\}$
$\quad = \{..., -8, -4, 0, 4, 8, ...\}$

$[1] = \{y \mid 1 - y = 4k \text{ for some } k \in \mathbb{Z}\}$
$\quad = \{y \mid y = -4k + 1\}$
$\quad = \{... -3, 1, 5, 9, ...\}$

$[2] = \{y \mid 2 - y = 4k \text{ for some } k \in \mathbb{Z}\}$
$\quad = \{y \mid y = -4k + 2\}$
$\quad = \{... -2, 2, 6, 10, ...\}$

$[3] = \{y \mid 3 - y = 4k \text{ for some } k \in \mathbb{Z}\}$
$\quad = \{y \mid y = -4k + 3\}$
$\quad = \{... -1, 3, 7, 11, ...\}$

$[-4]$, $[4]$, etc. are identical to $[0]$,
similarly, $[-3]$, $[5]$, etc. are the same as $[1]$,
similarly, $[-2]$, $[6]$, etc. are the same as $[2]$,
and, similarly, $[-1]$, $[7]$, etc. are the same as $[3]$.
Therefore, R has four different equivalence classes namely, $[0]$, $[1]$, $[2]$ and $[3]$.

4. *Suppose $\mathbb{Q}^+$ is the set of all positive quotients n/m, where $n, m \in \mathbb{Z}^+$ ie $\mathbb{Q}^+$ is the set of positive rational numbers.*

Let R be the relation on $\mathbb{Q}^+$, defined by $(x, y) \in R$ iff $y = (a \cdot x) / b$ for some $a, b \in \mathbb{Z}^+$.
Prove that R is an equivalence relation and describe the equivalence classes of R.
We can get the "feel" of a relation by writing down some of its members. Let us do this with R:
The members of R are ordered pairs of positive rational numbers, such as (1/2, 3/5),
ie x = 1/2 and y = 3/5.

Does this pair meet the entrance requirement for R, namely that $y = (a \cdot x)/b$?

Yes, where a = 6 and b = 5,
y a x b
(Test it to check: $3/5 = (6 \cdot (1/2)) / 5$.)
Another member of R is (4, 5).
(It meets the requirement $y = (a \cdot x) / b$, because $5 = (5 \cdot 4) / 4$, where a = 5 and b = 4.)

Now we want to prove that R is an equivalence relation:
In this kind of proof, we often need to determine whether a certain ordered pair belongs to R. Say, for example, we need to show that $(x, y) \in R$. We must then demonstrate that **x and y meet the requirements of the definition of R**, ie that $y = a \cdot x/b$. (Make sure you use the appropriate sequence for x and y!)
In order to be an equivalence relation, R must be reflexive on $\mathbb{Q}^+$, symmetric and transitive. This gives our "agenda".

Reflexivity:
Goal: to show that for every $x \in \mathbb{Q}^+$, $(x, x) \in R$.
Let us relate the definition of reflexivity to the definition of the specific relation R on $\mathbb{Q}^+$,
ie to show that $(\mathbf{x}, \mathbf{x}) \in R$, we must show that $\mathbf{x} = (a \cdot \mathbf{x}) / b$ for some $a, b \in \mathbb{Z}^+$.)
For all $x \in \mathbb{Q}^+$, we know that x = x,
ie $\mathbf{x} = 1 \cdot \mathbf{x}/1$ and $1 \in \mathbb{Z}^+$.
Thus, $(x, x) \in R$, and so R is reflexive on $\mathbb{Q}^+$.

Symmetry:
Goal: We assume that $(x, y) \in R$, ie $y = (a \cdot x) / b$, and we want to **use** this to demonstrate that $(y, x) \in R$.
Suppose $(x, y) \in R$,
then $y = (a \cdot x) / b$ for some $a, b \in \mathbb{Z}^+$
ie $b \cdot y = a \cdot x$
ie $(b \cdot y) / a = x$
ie $x = (b \cdot y) / a$.
Thus, $(y, x) \in R$ and so R is symmetric.

Transitivity:
Goal: We assume that $(x, y) \in R$, ie $y = (a \cdot x) / b$ and that $(y, z) \in R$,
ie $z = (c \cdot y) / d$, and then set out to **use** these facts to prove that $(x, z) \in R$.
Suppose $(x, y) \in R$, then $y = (a \cdot x) / b$ ① for some $a, b \in \mathbb{Z}^+$, and
suppose $(y, z) \in R$, then $z = (c \cdot y) / d$ ② for some $c, d \in \mathbb{Z}^+$.
Substitute ① into ②, then

$z = c\cdot(a\cdot x/b) / d$

ie $z = (ca\cdot x) / bd$

ie $z = (e\cdot x) / f$ where $e = ca$ and $f = bd$ for some $e, f \in \mathbb{Z}^+$.

Thus, $(x, z) \in R$ and so R is transitive.

Since R is reflexive on $\mathbb{Q}^+$, symmetric and transitive, R is an equivalence relation.

Next we look at the equivalence classes of R:

Note: Remember that equivalence classes are determined by considering sets of the following format:

$[x] = \{y \mid (x, y) \in R\}$ for all $x \in \mathbb{Q}^+$.

In this case, it means that:

$[x] = \{y \mid y = (a\cdot x) / b\}$

Consider $x = 1$ in the above equation.

$[1] = \{y \mid y = (a\cdot 1) / b\}$

$= \{y \mid y = a/b\}$

This is the set of all positive rational numbers, for example, $[1] = [2] = [1/2] = [3/4] = ...$, etc.

In this example, each equivalence class is equal to every other equivalence class, so there is only one equivalence class in R.

5. *Prove that if R is a relation on $\mathbb{Z}^+$, then R is symmetric iff $R = R^{-1}$.*

Let us first try to prove that if R is symmetric, then $R = R^{-1}$.

Assume R is symmetric. We want to show that $R = R^{-1}$.

Suppose $(x, y) \in R$,

then $(y, x) \in R$ because R is symmetric.

So, $(x, y) \in R^{-1}$ according to the definition of R^{-1}.

Hence, $R \subseteq R^{-1}$.

Conversely, suppose $(x, y) \in R^{-1}$,

then $(y, x) \in R$, and because R is symmetric, $(x, y) \in R$.

Hence, $R^{-1} \subseteq R$.

Since $R \subseteq R^{-1}$ and $R^{-1} \subseteq R$, we can conclude that $R = R^{-1}$.

Next we assume that $R = R^{-1}$. Now we need to show that R is symmetric, ie that if $(x, y) \in R$, then $(y, x) \in R$.

Suppose $(x, y) \in R$,

then $(y, x) \in R^{-1}$.

But because $R = R^{-1}$, $(y, x) \in R$.

Therefore, R is symmetric.

This completes the proof that if R is a relation on $\mathbb{Z}^+$, then R is symmetric iff $R = R^{-1}$.

Activity 6-12

Determine whether P is a partition of X in each of the following cases. If so, describe the corresponding equivalence relation.

(a) $X = \{1, 2, 3\}$ and $P = \{ \emptyset, \{1\}, \{2, 3\} \}$:
A partition of X must consist of nonempty subsets of X.
So P is not a partition of X because $\emptyset \in P$.

(b) $X = \{1, 2, 3\}$ and $P = \{ \{1\}, \{2\}, \{1, 3\} \}$:
P is not a partition of X since $\{1\} \cap \{1, 3\} = \{1\} \neq \emptyset$.

(c) $X = \{1, 2, 3\}$ and $P = \{ \{1,3\}, \{2\} \}$:
P satisfies all the requirements to be a partition of X:
P is a collection of nonempty subsets of X,
and for each $x \in X$ there is some $Y \in P$ such that $x \in Y$,
and for all $Y, W \in P$, if $Y \neq W$ then $Y \cap W = \emptyset$.
The equivalence classes of the corresponding equivalence relation (that we call R) are:
$[2] = \{2\}$, so, $(2, 2) \in R$, and
$[1] = [3] = \{1, 3\}$, so (1, 1), (3, 3), (1, 3) and (3, 1) must all be in R.
Therefore, $R = \{ (1, 1), (2, 2), (3, 3), (1, 3), (3, 1) \}$.

(d) $X = \{1, 2, 3\}$ and $P = \{ \{1\}, \{2\} \}$:
P is not a partition of X, because there is no $Y \in P$ such that $3 \in Y$.

(e) $X = \mathbb{Z}$ and $P = \{\{0\}, \mathbb{Z}^+, Neg\}$ where $Neg = \{x \mid x \in \mathbb{Z} \text{ and } x < 0\}$:
P is a partition of $\mathbb{Z}$ with the equivalence classes $\{0\}$, $\mathbb{Z}^+$ and Neg.
The corresponding equivalence relation is:
$\{(x, y) \mid (x = 0 \text{ and } y = 0) \text{ or } (x \in \mathbb{Z}^+ \text{ and } y \in \mathbb{Z}^+) \text{ or } (x \in Neg \text{ and } y \in Neg)\}$.

(f) $X = \mathbb{Z}$ and $P = \{ [0], [1], [2], [3], [4] \}$ where
$[0] = \{x \mid x - 0$ is divisible by 5 with zero remainder$\}$
$[1] = \{x \mid x - 1$ is divisible by 5 with zero remainder$\}$
$[2] = \{x \mid x - 2$ is divisible by 5 with zero remainder$\}$
$[3] = \{x \mid x - 3$ is divisible by 5 with zero remainder$\}$
$[4] = \{x \mid x - 4$ is divisible by 5 with zero remainder$\}$.
P is a partition of $\mathbb{Z}$. The reasons are:

- Every element of P is a nonempty subset of $\mathbb{Z}$. Each of them contains at least the representative given between square brackets.
- For all $Y, W \in P$, if $Y \neq W$, then $Y \cap W = \emptyset$, ie different classes do not have any elements in common. No integer can be in two different sets $Y, W \in P$, because no integer gives two different remainders on integer division by 5. (*Note*: If, say, x - 3 is divisible by 5 with zero remainder, then it means x itself leaves 3 as remainder when divided by 5.)
- For each $x \in \mathbb{Z}$, there is some $Y \in P$ such that $x \in Y$, because, after all, any integer x will, when divided by 5, give a remainder of 0, 1, 2, 3 or 4. Subtracting this remainder from x results in a value, which is divisible by 5 with zero remainder.

 The corresponding equivalence relation is: $\{(x, y) \mid x - y = 5k$, or some integer $k\}$.

10 Assignment 2

Assignment 2 scope

What is covered?
This assignment is a multiple-choice assignment and covers study units 4, 5 and 6 of the study guide.

Assignment submission
This assignment should be submitted either electronically via myUnisa (the preferred route), or by filling in a mark-reading sheet and submitting it via one of the regional centres or the post office.

Time allocated
You will need one week to complete this assignment.

Due date
Check Tutorial Letter 101 for the due date and unique assignment number for this assignment.

PDF version
You can get a PDF version of this assignment question in **Additional Resources**.

Assignment 2 questions

You can download the questions for this assignment from the **Additional Resources** page or get them from Tutorial Letter 101.

Assignment 2 solutions

After the closing date, a discussion of the assignment will be posted to the **Additional Resources** page. You will be informed of this via an announcement on myUnisa.

Sample Assignment 2 questions and solutions

In Tutorial Letter 102, available on the **Additional Resources** page, you will find a sample assignment 2 that you can work through before attempting Assignment 2.

11 Additional self-assessment questions for learning units 4 to 6

Purpose:

It is very important that you do these self-assessment questions. The assignments are multiple-choice assignments. However, in the exam you will be required to draw Venn diagrams, write down proofs and counterexamples, and sets, etc. These exercises will help you practice to write down the correct mathematical notation required. We give two sets of questions, reflected in section 1 and section 2 below.

Self-assessment questions:

Your answers to the following self-assessment questions should not be submitted. The solutions are provided in Tutorial Letter 103.
NB: Special hints are provided so that you can avoid making unnecessary mistakes when solving these problems.

Section 1

Question A
Draw Venn diagrams to show that $(A' + B) = (B \cup C) \cap A$ is not an identity for all subsets A, B and C of U. Draw the diagrams in stages.

Question B
Provide a counterexample, and then use it to show that $(A' + B) \neq (B \cup C) \cap A$.

Question C
Using the sets $X = \{1, 2\}$ and $Y = \{2, 3\}$, show that

$$(X \cup Y) \times (X \cap Y) = (X \times (X \cap Y)) \cup (Y \times (X \cap Y)).$$

Does this example show that this is an identity? Justify your answer.

Question D
Prove that $(X \cup Y) - W' = (X \cap W) \cup (Y \cap W)$ is an identity for all subsets X, Y and W of a universal set U.

HINTS:

- Compile Venn diagrams step by step. Partial credit will be given for those Venn diagrams that are correct, even if some final diagram is incorrect.

- Remember to provide headings for Venn diagrams and to name the sets inside your diagrams. (Otherwise, your diagrams will not have meaning.) Shade only the area relevant to the heading. Draw the universal set for each diagram.

- When two sets are not equal, a counterexample should be provided. Choose small sets that have (a) member(s) in the region(s) where the two final Venn diagrams differ. Sets are indicated by curly brackets { , , }. Note: **First,** determine the LHS, then the RHS, and **then** compare the final sets in your conclusion. Note: Do **not** attempt to provide some formal proof.

- Make sure that you understand how to apply the definitions of union, intersection, difference, complement and symmetric difference in a Venn diagram. It will help if you think of $A + B$ as being the set $(A \cup B) - (A \cap B)$, which has as members those elements that live in A or in B, but not in both.

- If the two final Venn diagrams of a given expression have the same areas shaded, a proof is required to show that the given expression is an identity. Notation in a proof is important. Note that **symbols** (eg "$\cap$") should be used as **connectives** for **sets** (eg $Y \cap W$), and **words** (eg "and") should be used as **connectives** in **sentences** (eg $x \in Y$ **and** $x \in W$).

- How do we prove two sets equal? A formal proof is required. In a proof, we can use the connective "iff" in a bi-directional proof, or "if…then…" when the proof is given in two halves.

- Connectives give form to an argument, so if you leave out the connectives "iff" or "if..., then…", your proof is not convincing. The definitions of union, intersection, difference, complement, symmetric difference and Cartesian product should be applied in proofs.

- Make sure that brackets are included when necessary. Remember, "$x \in Y$ and $x \in W$ or $x \in X$" could have different meanings depending on where brackets are placed.

- Cartesian products have ordered pairs as members. When an expression includes a Cartesian product and a formal proof is required to show that it is an identity, start the proof with
$(u, v) \in$……. (The members of a Cartesian product are ordered pairs.)

Question E
Let R be the relation on $\mathbb{Z}$ (the set of integers) defined by

$(x, y) \in R$ iff $|y| = |x|$.

a) Give (i) an ordered pair in R, showing why it belongs to R, and
(ii) an ordered pair not in R, showing why it does not belong to R.
(Use at least one negative integer in one of the pairs.)

b) By doing the appropriate tests, show that R is an equivalence relation.

HINTS:

- Make sure that you understand the definitions of the properties of relations. You should be able to apply these definitions in proofs. For example, for transitivity, start the proof with:
Suppose $(x, y) \in R$ and $(y, z) \in R$, then *use* this information (refer to the definition of the given relation) to prove that $(x, z) \in R$.
Note: An *example* will *not prove* that a relation has a certain property.

- Use the definition of a given relation (in this case $(x, y) \in R$ iff $|y| = |x|$) when attempting to prove that the relation has certain properties. General statements such as "for each (x, y) there is a (y, x)" will not convince anybody that a relation has some property.

- The definition for reflexivity of a relation does *not* say "If $(x, y) \in R$ then $x = y$" (two variables *x* and *y* are involved here). When it is required to prove that a relation R is reflexive on *A*, you should prove that "for every $\mathbf{x} \in A$, we have $(\mathbf{x}, \mathbf{x}) \in R$". In your proof, only *one* variable should play a role.

- Do not use the contents of the proofs for theorems 6.1 & 6.2 (pp 93 – 96 in the study guide) to try to prove that some relation has properties such as reflexivity, symmetry or transitivity. You should apply the definitions of these concepts to prove that a relation has these properties.

Question F
Let P and R be relations on $A = \{1, 2, 3, \{1\}, \{2\}, \{3\}\}$ given by
$P = \{(1, \{1\}), (\{1\}, 1), (1, 2), (2, 1)\}$ and $R = \{(1, \{1\}), (3, \{3\}), (2, \{2\}), (\{2\}, \{2\}), (\{3\}, 3)\}$.

a) Test whether P has the following properties: irreflexive; reflexive; symmetric; antisymmetric; transitive.
b) Does R satisfy trichotomy?
c) Determine the relations $R \circ R$ and $R \circ P$ (ie P; R).

d) Give the subset T of R where (A, B) $\in$ T iff A $\subseteq$ B.
e) Give a partition B of the set A = { 1, 2, 3, {1}, {2}, {3} }.

Section 2

Question A

Draw Venn diagrams to show that $(A \cup C) + B = (A \cup B \cup C) - (A \cap B)$ is not an identity for all subsets A, B and C of U. Draw the diagrams in stages.

Question B

Provide a counterexample, to show that $(A \cup C) + B \neq (A \cup B \cup C) - (A \cap B)$.

Question C

Using the subsets X = {1, 2, 3}, Y = {1} and W = {1, 2} of the universal set U = {1, 2, 3, 4, 5}, show that

$$(X \cup Y) - W' = (X \cap W) \cup (Y + W).$$

Does this example show that this is an identity? Justify your answer.

Question D

Prove that $(X \cup Y) \times W = (X \times W) \cup (Y \times W)$ is an identity for all subsets X, Y and W of a universal set U.

HINTS:

- Compile Venn diagrams step by step. In the exam, partial credit will be given for those Venn diagrams that are correct, even if some final diagram is incorrect.

- Remember to provide headings for Venn diagrams and to name the sets inside your diagrams, otherwise your diagrams will not have meaning. Shade only the area relevant to the heading. Draw the universal set for each diagram.

- When two sets are not equal, a counterexample should be provided. Choose small sets that have (a) member(s) in the region(s) where the two final Venn diagrams differ. Sets are indicated by curly brackets { , , }. Note: First, determine the LHS, then the RHS, and then compare the final sets in your conclusion. Note: Do **not** attempt to give some formal proof.

- Make sure that you understand how to apply the definitions of union, intersection, difference, complement and symmetric difference in a Venn diagram. It will help if you think of A + B as being the set $(A \cup B) - (A \cap B)$, which has as members those elements that live in A or in B, but not in both.

- If the two final Venn diagrams of a given expression have the same areas shaded, a proof is required to show that the given expression is an identity. Notation in a proof is important. Note that a **symbol** (eg "$\cap$") is used as a **connective** for **sets** (eg $Y \cap W$), and a **word** (eg "and") is used as a **connective** in a **sentence** (eg $x \in Y$ **and** $x \in W$).

- How do we prove two sets equal? A formal proof is required. In a proof, we can use the connective "iff" in a bi-directional proof, or "if…then…" when the proof is given in two halves.

- Connectives give form to an argument, so if you leave out the connectives "iff" or "if..., then…", your proof is not convincing. The definitions of union, intersection, difference, complement, symmetric difference and Cartesian product should be applied in the proofs.

- Make sure that brackets are included when necessary. Remember, "$x \in Y$ and $x \in W$ or $x \in X$" could have different meanings depending on where brackets are placed.

- Cartesian products have ordered pairs as members. When an expression includes a Cartesian product and a formal proof is required to show that it is an identity, start the proof with “(u, v)∈…….” (Venn diagrams cannot be compiled for Cartesian products.)

Question E

Let R be the relation on $\mathbb{Z}$ (the set of integers) defined by
(x, y) ∈ R iff y = (k + 1)x for some integer k ≥ 0.

a) Give (i) an ordered pair in R, showing why it belongs to R, and
(ii) an ordered pair not in R, showing why it does not belong to R.
(Use at least one negative integer in one of the pairs.)

b) By doing the appropriate tests, show that R is a weak partial order.

HINTS:

- Make sure that you understand the definitions of the properties of relations. You should be able to apply these definitions in the proofs. For example, for transitivity, start the proof with:
  Suppose (x, y) ∈ R and (y, z) ∈ R, then *use* this information (refer to the definition of the given relation) to prove that (x, z) ∈ R.
  Note: An *example* will *not prove* that a relation has a certain property.

- Use the definition of a given relation (in this case (x, y) ∈ R iff y = (k + 1)x) when attempting to prove that the relation has certain properties. General statements such as “for each (x, y) there is a (y, x)” will not convince anybody that a relation has some property.

- The definition for reflexivity of a relation does *not* say “If (x, y) ∈ R then x = y” (two variables *x* and *y* are involved here). When it is required to prove that a relation R is reflexive on *A*, you should prove that “for every **x** ∈ A, we have (**x**, **x**) ∈ R”. In your proof, only *one* variable should play a role.

- Do not use the contents of the proofs for theorems 6.1 and 6.2 (pp 93 – 96 in the study guide) to try to prove that some relation has properties such as reflexivity, symmetry or transitivity. You should apply the definitions of these concepts to prove that a relation has these properties.

12 Learning unit 7 – n-ary relations and functions

Study Material

Study guide
You should cover study unit 6, sections 6.4 to 6.5, and study unit 7 in the study guide.

Time allocated
You will need one week to master this learning unit.

Notes

Background
n-ary relations and further properties of relations are provided, and functions as well as different properties of functions are defined.

Applications
Definitions of properties of relations and functions are applied to determine which properties given relations and functions have. Composition relations or functions can be determined.

Activities

Do the activities provided in study units 6.4 to 6.5 and 7 to consolidate your knowledge of the work in this learning unit.

Study unit 6, sections 6.4 to 6.5

Activity 6-14

1. *Give 5 functions from A = {1, 2, 3, 4} to B = {a, b, c}.*

Think of building each function by filling in the template

{ (1,), (2,), (3,), (4,) }

with elements of B.

Some possibilities are:

f_1 = { (1, **a**), (2, **a**), (3, **a**), (4, **a**) }
f_2 = { (1, **b**), (2, **b**), (3, **b**), (4, **b**) }
f_3 = { (1, **c**), (2, **c**), (3, **c**), (4, **c**) }
f_4 = { (1, **a**), (2, **b**), (3, **a**), (4, **b**) }
f_5 = { (1, **c**), (2, **a**), (3, **b**), (4, **c**) }.

2. *Give all the functions from A = {a, b} to B = {5, 6, 7}.*

We can get the functions by filling in the template { (a,), (b,) }
with second co-ordinates chosen from B.

f_1 = { (a, 5), (b, 5) }
f_2 = { (a, 6), (b, 6) }
f_3 = { (a, 7), (b, 7) }
f_4 = { (a, 5), (b, 6) }
f_5 = { (a, 5), (b, 7) }
f_6 = { (a, 6), (b, 5) }
f_7 = { (a, 6), (b, 7) }
f_8 = { (a, 7), (b, 5) }

$f_9 = \{ (a, 7), (b, 6) \}$.

3. *Give 3 functions from $A \times A$ to B if A = {a, b} and B = {5, 6, 7}.*

Each function has as domain the set

$A \times A = \{ (a, a) , (a, b) , (b, a) , (b, b) \}$.

To build such a function, just fill in the template

{ **(**(a, a)**,)**, **(**(a, b)**,)**, **(**(b, a)**,)**, **(**(b, b)**,)** }

with second co-ordinates chosen from B. Three examples are:

f_1 = { **(**(a, a)**, 5)**, **(**(a, b)**, 5)**, **(**(b, a)**, 5)**, **(**(b, b)**, 5)** }

f_2 = { **(**(a, a)**, 5)**, **(**(a, b)**, 6)**, **(**(b, a)**, 7)**, **(**(b, b)**, 5)** }

f_3 = { **(**(a, a)**, 7)**, **(**(a, b)**, 6)**, **(**(b, a)**, 5)**, **(**(b, b)**, 6)** }.

4. *Let R be a relation on A = {1, 2, 3, {1}, {2}} defined by*
R = { (1, {1}), (1, 3), (2, {1}), (2, {2}), ({1}, 3), ({2}, {1}) }.

(a) *Is R a function from A to A?*

First, we have to ask: is R functional? (ie if $(x, y) \in R$ and $(x, z) \in R$ is y = z?) and is dom(R) = A?

No, R is not a function. We give a *counterexample*:

$(1, \{1\}) \in R$ and $(1, 3) \in R$,

so 1 appears twice as first co-ordinate, but with different second co-ordinates, namely {1} and 3 as partners, so R is not functional and thus not a function.

(We also have that dom(R) ≠ A since $3 \notin$ dom(R).)

(b) *Is ran(R) equal to the codomain of R?*

The codomain A = {1, 2, 3, {1}, {2}} is not equal to the range of R

ie ran(R) = { 3, {1}, {2} } $\neq$ A.

5. *Consider the set $\mathcal{P}$ (A) = **{** ∅, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, {a, b, c} **}**.*

Show that the relations f, g and h described below are functional and have as domains $\mathcal{P}$ (A), $\mathcal{P}(A) \times \mathcal{P}(A)$, and $\mathcal{P}(A) \times \mathcal{P}(A)$, respectively.

(a) Let f = {(x, y) | x, y $\in \mathcal{P}$ (A) and y = x′}.

We can follow either of two approaches. The brute force approach involves writing out in list notation the set f, so that we can verify by inspection that f is functional and that each element of $\mathcal{P}$ (A) occurs as a first co-ordinate. This approach is suitable only for smallish sets and the set $\mathcal{P}$ (A) is just barely small enough.

A more sophisticated approach would involve abstract reasoning with the help of variables.

Let us use the brute force approach here and the abstract approach for subsequent questions where the domain is bigger.

F = **{**(∅, {a, b, c}), ({a}, {b, c}), ({b}, {a, c}), ({c}, {a, b}), ({a, b}, {c}), ({a, c}, {b}), ({b, c}, {a}), ({a, b, c}, ∅) **}**.

By inspection it is clear that dom(f) = $\mathcal{P}$ (A) and that every element of $\mathcal{P}$ (A) occurs exactly once as a first co-ordinate, so f: $\mathcal{P}(A) \to \mathcal{P}(A)$.

(b) Let g = { **(**(u, v), y**)** | $(u, v) \in \mathcal{P}(A) \times \mathcal{P}(A)$ and $y = u \cup v$ }.

This time we use abstract reasoning.

Dom(g) $\subseteq \mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A), because g is a relation from $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) to $\mathcal{P}$ (A).

Is it the case that $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) $\subseteq$ dom(g)?

Yes, because if (u, v) $\in \mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A), then u $\cup$ v is a subset of A, ie for each element (u, v) of the set $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) we can find an element y of $\mathcal{P}$ (A) such that y = u $\cup$ v.

Since dom(g) $\subseteq \mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) and $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) $\subseteq$ dom(g), it follows that

dom(g) = $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A).

Is g functional?
Suppose (x, y) $\in$ g and (x, z) $\in$ g.
Then x = (u, v) for some u $\subseteq$ A and some v $\subseteq$ A,
and y = u $\cup$ v = z, so g is indeed functional.
Since dom(g) = $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) and g is functional, it follows that

g: $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) $\rightarrow$ $\mathcal{P}$ (A), ie g is a function from $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) to $\mathcal{P}$ (A).

(c) Let h = { **(**(u, v), y**)** | (u, v) $\in \mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) and y = u $\cap$ v }.

Dom(h) $\subseteq \mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) because h is a relation from $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) to $\mathcal{P}$ (A).

Is it the case that $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) $\subseteq$ dom(h)?

Yes, for if (u, v) $\in \mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) then u $\cap$ v is a subset of A,

so we can find an element y of $\mathcal{P}$ (A) such that ((u, v), y) $\in$ h, namely y = u $\cap$ v.

Since dom(h) $\subseteq \mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) and $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) $\subseteq$ dom(h), it follows that dom(h) = $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A).

Is h functional?
If (x, y) $\in$ h and (x, z) $\in$ h,
then x has the form (u, v) for some subsets u and v of A, and y = u $\cap$ v = z.

Since dom(h) = $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) and h is functional, it follows that

h: $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) $\rightarrow$ $\mathcal{P}$ (A), ie h is a function from $\mathcal{P}$ (A) $\times$ $\mathcal{P}$ (A) to $\mathcal{P}$ (A).

6. *For each of the following relations from X to Y, determine whether or not the relation may be regarded as a function from X to Y, then determine the range of the relation.*

A relation R from X to Y is a function iff R is **functional** and **dom(R) = X**.
In each of the following examples we investigate whether or not R (or S) has these two properties, then we determine ran(R) or ran(S).

(a) X = Y = $\mathbb{Z}$ and R = {(x, y) | y = x}.
Determine dom(R):
Dom(R) = {x | for some y $\in$ Y, (x, y) $\in$ R}

= {x | for some y $\in \mathbb{Z}$, y = x}
= {x | x is an integer}
= $\mathbb{Z}$.

Next, we investigate functionality.
Suppose (x, y) $\in$ R and (x, z) $\in$ R
ie y = x and z = x
ie y = z.
Therefore, every x in $\mathbb{Z}$ that appears as a first co-ordinate does so in only one pair.
Hence, R is functional.

Because R is functional and dom(R) = $\mathbb{Z}$, it follows that R is a function, so we may write R: $\mathbb{Z} \to \mathbb{Z}$.

(R is in fact a very important function, namely the **identity** function on Z. Informally, R is the function that instructs us, no matter in which city we find ourselves, not to go anywhere else but to stay just where we are.)

Determine ran(R):
ran(R) = {y | for some x $\in$ X, (x, y) $\in$ R}
= {y | for some x $\in \mathbb{Z}$, y = x}
= {y | y is an integer}
=$\mathbb{Z}$.

(b) X = Y = $\mathbb{Z}$ and R = {(x, y) | y = x + 1}.

Determine dom(R):
Note: We show that dom(R) $\subseteq$ Z and $\mathbb{Z} \subseteq$ dom(R), ie dom(R) = $\mathbb{Z}$.

By definition we know that R $\subseteq \mathbb{Z} \times \mathbb{Z}$, so dom(R) $\subseteq \mathbb{Z}$.

But, we also have that $\mathbb{Z} \subseteq$ dom(R), since for **every** x in $\mathbb{Z}$ there is an **integer** y of the form x + 1, so we have a pair of the form (x, x + 1) in R and therefore x $\in$ dom(R).

Since dom(R) $\subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq$ dom(R), it follows that dom(R) = $\mathbb{Z}$.

Next, we look at functionality.
Suppose (x, y) $\in$ R and (x, z) $\in$ R
ie y = x + 1 and z = x + 1
ie y = x + 1 = z
ie y = z
Hence, R is functional.

Since R is functional and dom(R) = $\mathbb{Z}$, R is a function from Z to Z.

Note: R is called the successor function because it tells us to go from x to x + 1.

Determine ran(R):
Note: We show that ran(R) $\subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq$ ran(R), ie ran(R) = $\mathbb{Z}$.

Now, we know that ran(R) $\subseteq \mathbb{Z}$, because R $\subseteq \mathbb{Z} \times \mathbb{Z}$.
It is also the case that $\mathbb{Z} \subseteq$ ran(R) because
for **every** integer y we can find an **integer** x such that y = x + 1,
(just take x to be y – 1, then x + 1 = (y – 1) + 1 = y then (y – 1, y) $\in$ R, so y $\in$ ran(R).
Since ran(R) $\subseteq \mathbb{Z}$ and $\mathbb{Z} \subseteq$ ran(R), it follows that ran(R) = $\mathbb{Z}$.

(c) X = Y = $\mathbb{Z}$ and R = {(x, y) | y = 3 – x}.
Determine dom(R):
dom(R) = {x | for some y $\in$ Y, (x, y) $\in$ R}
= {x | for some y $\in$ Z, y = 3 – x}
= {x | 3 – x is an integer}
= $\mathbb{Z}$

Now for functionality.
Suppose (x, y) $\in$ R and (x, z) $\in$ R
ie y = 3 – x and z = 3 – x
ie y = 3 – x = z
ie y = z.
Therefore, R is functional.
Since R is functional and dom(R) = $\mathbb{Z}$, R: $\mathbb{Z} \to \mathbb{Z}$.
Determine ran(R):
ran(R) = {y | for some x $\in$ X, (x, y) $\in$ R}
= {y | for some x $\in \mathbb{Z}$, y = 3 – x}
= {y | for some x $\in \mathbb{Z}$, x = 3 – y}
= {y | 3 – y is an integer}
= $\mathbb{Z}$

(d) X = Y = $\mathbb{Z}$ and R = {(x, y) | y = $\sqrt{x}$ }, where the notation $\sqrt{x}$ refers to the positive square root of x.
Determine dom(R):
dom(R) ={x | for some y $\in$ Y, (x, y) $\in$ R}
= {x | for some y $\in \mathbb{Z}$, y = $\sqrt{x}$ }
Now, we know that dom(R) $\subseteq \mathbb{Z}$ since R $\subseteq \mathbb{Z} \times \mathbb{Z}$.

Is it also the case that $\mathbb{Z} \subseteq$ dom(R)? Alas, no. Let us find a counterexample:
Take the integer 2. As we saw in study unit 2, √2 is irrational, so there can be no integer y equal to √2, and hence 2 $\notin$ dom(R).
(Other counterexamples are -1, -2, -3, etc.)
As far as dom(R) is concerned, therefore, we can do no better than to describe dom(R) as
{x | for some y $\in \mathbb{Z}$, y = $\sqrt{x}$ }.

Equivalently, saying the same thing in different words,
dom(R) = {x | x = y^2 for some y $\in \mathbb{Z}$ }.

Note: The advantage of the latter description is that it suggests a way to generate the members of dom(R) in the following way: Start with y = 1, form y^2, then take y = 2 and form y^2, and so on.

Now for functionality.
Suppose (x, y) ∈ R and (x, z) ∈ R
ie $y = \sqrt{x}$ and $z = \sqrt{x}$
ie $y = \sqrt{x} = z$.
Thus, y = z and R is functional.

However, we saw earlier that dom(R) ≠ ℤ, so R is **not** a function from ℤ to ℤ.

Determine ran(R):
ran(R) = {y | for some x ∈ X, (x, y) ∈ R}
= {y | for some x ∈ ℤ, $y = \sqrt{x}$ }.

We know that ran(R) ⊆ ℤ since R ⊆ ℤ × ℤ.

Is it the case that ℤ ⊆ ran(R)?
Well, is it the case that every integer is the square root of some other integer?
Only for integers living in $\mathbb{Z}^{\geq}$, since for every y ∈ $\mathbb{Z}^{\geq}$, y is the square root of the integer y^2,
ie $y = \sqrt{x}$ for some integer x (just take $x = y^2$), ie y ∈ ran(R).
But no negative integer can be written in the form $\sqrt{x}$, because $\sqrt{x}$ refers to the positive square root of x.
As a counterexample, √4 = 2, whereas −2 can only be indicated by −√4, so −2 ∉ ran(R).
Thus, ran(R) ≠ ℤ.

(e) X = Y = ℤ and R = {(x, y) | $y^2 = x$}.
You may have been strongly tempted to say that this relation is the same as the one we dealt with in (d), but in fact, it is not. The present relation contains pairs like (4, 2) **as well as (4, −2)**, since $(-2)^2 = 4$. With every integer x in its domain, the present relation associates both the positive square root $\sqrt{x}$ and the negative square root $-\sqrt{x}$, so we could in fact have described the relation by
R = {(x, y) | x ∈ Z and $y = \sqrt{x}$ or $y = -\sqrt{x}$}.

Determine dom(R):
dom(R) = {x | for some y ∈ Y, (x, y) ∈ R}
= {x | for some y ∈ Z, $y^2 = x$}.
= {x | $+\sqrt{x}$ and $-\sqrt{x}$ are integers}

Is dom(R) = ℤ?
Well, we know that dom(R) ⊆ ℤ, but just as in the previous question, integers like 2 or −5 do not belong to dom(R), so dom(R) ≠ ℤ.

Now for functionality.
Suppose (x, y) ∈ R and (x, z) ∈ R.
Is it necessarily the case that y = z?
No! Take x = 4, as a c*ounterexample*, then we may think of y as 2 and z as −2. It is clear that y ≠ z.

(Informally, R is not functional because it often gives us conflicting instructions: if we are in city number 4, say, R tells us to go directly to city 2 and also to go directly to city −2, and we cannot do both at the same time.)

Determine ran(R):

ran(R) = $\{y \mid \text{for some } x \in X, (x, y) \in R\}$
$= \{y \mid \text{for some } x \in \mathbb{Z}, y^2 = x\}$
$= \{y \mid y^2 \text{ is an integer}\}$
$= \mathbb{Z}$.

(f) $X = Y = \mathbb{R}$ and $S = \{(x, y) \mid x^2 + y^2 = 1\}$.

If you wish, you may visualise S as the circle with its centre at the origin and with a radius of one unit.

Determine dom(S):

dom(S) $= \{x \mid \text{for some } y \in Y, (x, y) \in S\}$
$= \{x \mid \text{for some } y \in R, x^2 + y^2 = 1\}$
$= \{x \mid -1 \leq x \leq 1\}$.

Is S functional?

No. We can find a counterexample. Take x = 0, for example.

Then $(0, 1) \in S$ because $0^2 + 1^2 = 1$, and $(0, -1) \in S$ because $0^2 + (-1)^2 = 1$.

So the element 0 of the domain is used more than once as a first co-ordinate.

We can conclude that S is not a function since it is not functional and dom(S) $\neq \mathbb{R}$.

Determine ran(S):

ran(S) = $\{y \mid \text{for some } x \in X, (x, y) \in S\}$
$= \{y \mid \text{for some } x \in R, x^2 + y^2 = 1\}$
$= \{y \mid -1 \leq y \leq 1\}$.

7. *Is the relation R on $\mathbb{Z}^+$, which consists of all pairs (x, y) such that y = x − 1, a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$?*

Well, R certainly seems functional, but the problem lies with dom(R).

Recall that $R \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$. Is $1 \in$ dom(R), ie can we find an appropriate second co-ordinate inside $\mathbb{Z}^+$ to match with 1?

No, because the only second co-ordinate that is suitable is 0, and $0 \notin \mathbb{Z}^+$. So dom(R) $\neq \mathbb{Z}^+$.

Therefore, R is not a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$.

8. *Let A = {a, b, c}. Consider all the equivalence relations on A. (See Activity 6-**10**(1)). How many relations are also functions from A to A?*

We use brute force and then abstract reasoning in our answer. We recommend the latter.

Brute force:

From the previous example, we have all the equivalence relations on A. Now we can inspect them to see which are functional:

R_1 = { (a, a), (b, b), (c, c) }

R_2 = { (a, a), (b, b), (c, c), (a, b), (b, a) }
R_3 = { (a, a), (b, b), (c, c), (b, c), (c, b) }
R_4 = { (a, a), (b, b), (c, c), (a, c), (c, a) }
R_5 = { (a, a), (b, b), (c, c), (a, b), (b, a), (b, c), (c, b), (a, c), (c, a) }
Clearly, R_1 is the only function from A to A.

Abstract reasoning:
If R is any equivalence relation on A, then R is reflexive on A and therefore dom(R) = A.
But, if R is any equivalence relation other than the identity relation {(a, a), (b, b), (c, c)}, then R will fail to be functional.
To see this, note that every equivalence relation on A must contain the pairs (a, a), (b, b) and (c, c). Any additional pair such as, for example (a, b), will result in a member of A being used more than once as a first co-ordinate.

9. *Let A = {a, b, c}. (See activity 6-**7**(1))*
(In the answers to the questions that follow, we do everything twice, first using brute force and then using abstract reasoning. We recommend the latter.)

(a) *How many weak partial orders on A (reflexive, antisymmetric and transitive relations) are also functions from A to A?*
Brute force:
Here are all the weak partial orders on A. We can inspect them to see which are functional:
S_1 = { (a,a), (b,b), (c,c) }
S_2 = { (a,a), (b,b), (c,c), (a,b) }
S_3 = { (a,a), (b,b), (c,c), (a,c) }
S_4 = { (a,a), (b,b), (c,c), (b,a) }
S_5 = { (a,a), (b,b), (c,c), (b,c) }
S_6 = { (a,a), (b,b), (c,c), (c,a) }
S_7 = { (a,a), (b,b), (c,c), (c,b) }
S_8 = { (a,a), (b,b), (c,c), (a,b), (a,c) }
S_9 = { (a,a), (b,b), (c,c), (a,b), (c,b) }
S_{10} = { (a,a), (b,b), (c,c), (b,a), (b,c) }
S_{11} = { (a,a), (b,b), (c,c), (b,a), (c,a) }
S_{12} = { (a,a), (b,b), (c,c), (c,a), (c,b) }
S_{13} = { (a,a), (b,b), (c,c), (a,c), (b,c) }
S_{14} = { (a,a), (b,b), (c,c), (a,b), (b,c), (a,c) }
S_{15} = { (a,a), (b,b), (c,c), (a,c), (c,b), (a,b) }
S_{16} = { (a,a), (b,b), (c,c), (b,a), (a,c), (b,c) }
S_{17} = { (a,a), (b,b), (c,c), (b,c), (c,a), (b,a) }
S_{18} = { (a,a), (b,b), (c,c), (c,a), (a,b), (c,b) }
S_{19} = { (a,a), (b,b), (c,c), (c,b), (b,a), (c,a) }

Clearly, S_1 is the only function from A to A.
(Incidentally, how do we know that we have found all the weak partial orders on A? Well, because we were systematic: S_1 is the smallest possible chap, with just 3 elements; then we wrote down those with 4 elements; then 5; and finally those with 6.)

Abstract reasoning:
Every weak partial order is reflexive, so every element of A already appears in an ordered pair as first co-ordinate.
The moment a weak partial order has more than just the pairs needed for reflexivity, ie has more pairs than the identity relation on A, some element of A will occur more than once as a first co-ordinate, so the relation will not be functional.
Thus, the identity relation on A is the only weak partial order that is also a function from A to A.

(b) *How many strict partial orders on A (irreflexive, antisymmetric and transitive relations) are also functions from A to A?*

Brute force:
Here are all the strict partial orders on A, so that we can inspect them to see which are functional:

T_1 = { }
T_2 = { (a,b) }
T_3 = { (a,c) }
T_4 = { (b,a) }
T_5 = { (b,c) }
T_6 = { (c,a) }
T_7 = { (c,b) }
T_8 = { (a,b), (a,c) }
T_9 = { (a,b), (c,b) }
T_{10} = { (b,a), (b,c) }
T_{11} = { (b,a), (c,a) }
T_{12} = { (c,a), (c,b) }
T_{13} = { (a,c), (b,c) }
T_{14} = { (a,b), (b,c), (a,c) }
T_{15} = { (a,c), (c,b), (a,b) }
T_{16} = { (b,a), (a,c), (b,c) }
T_{17} = { (b,c), (c,a), (b,a) }
T_{18} = { (c,a) ,(a,b), (c,b) }
T_{19} = { (c,b), (b,a), (c,a) }

Clearly, none of the relations are functions from A to A. Many of the relations are functional, but none have A as domain.
How do we know we have found all the strict partial orders on A? Easy. Each of them is just the corresponding weak partial order with the reflexive pairs thrown away.

Abstract reasoning:
If a strict partial order T is to be a function from A to A, then it must have A as its domain and be functional.
So, every member of A must occur as first co-ordinate of exactly one pair in T.
This means that T must be obtainable by filling in the gaps in the template
{ (a,), (b,), (c,) }
in such a way that the result is irreflexive, antisymmetric and transitive.

Let us try to fill the gaps:
To assign to **a** the value **a** would violate irreflexivity.
So suppose we start by assigning to **a** the value **b**:

{ (a, **b**), (b,), (c,) }.
Then we cannot in the next pair assign to **b** the value **a** (because that would violate antisymmetry), nor can we assign to **b** the value **b** (because that would violate irreflexivity), so we must assign to **b** the value **c**: { (a, **b**), (b, **c**), (c,) }.

To maintain transitivity, we are now forced to add the pair (a, c) even before we can think about filling the gap in the pair (c,). But then, we have two different pairs starting with **a** and this violates functionality.
Our last hope is to start by assigning to **a** the value **c**: { (a, **c**), (b,), (c,) }.
But then, reasoning as before, we find that to **c** we must assign the value **b**, so that transitivity demands the inclusion of the pair (a, b), and the relation loses functionality.
From this, we conclude that there is no way to fill in the template so as to produce a relation that is both a strict partial order and a function.

Study unit 7

Activity 7-4

1. *Write down the possible surjective functions from X to Y:*

(b) X = {a, b} and Y = {c, d}:
To obtain a surjective function from X to Y, we must try to fill in, using all the members of Y, the template { (a,), (b,) }.
This can be done in two ways:
g_1 = { (a, c), (b, d) }
and g_2 = { (a, d), (b, c) }.

(c) X = {a, b} and Y = {c, d, e}:
The template
{ (a,), (b,) }
cannot be completed in a way that uses all the elements of Y, because Y has 3 members and there are only two gaps to be filled, so there are no surjective functions from X to Y in this case.

2. *Let f: $\mathbb{Z} \to \mathbb{Z}$ be defined by f(x) = x + 1.*

(a) *Determine f[$\mathbb{Z}$] (= ran(f)) . (Do not give specific examples.)*

f[$\mathbb{Z}$] = {y | for some x $\in \mathbb{Z}$, (x, y) $\in$ f}
= {y | for some x $\in \mathbb{Z}$, y = x + 1}
= {y | for some x $\in \mathbb{Z}$, x = y – 1}
= {y | y – 1 is an integer}
= $\mathbb{Z}$

(b) *Is f surjective? If f is not surjective, provide a counterexample to show why it is not surjective.*
f is surjective because the range of f is equal to the codomain of f: f[$\mathbb{Z}$] = $\mathbb{Z}$.

3. *Let g: $\mathbb{Z} \to \mathbb{Z}$ be defined by g(x) = 4x + 8.*

(a) *Determine g[$\mathbb{Z}$] (= ran(g)) . (Do not give specific examples.)*

g[$\mathbb{Z}$] = {y | for some x $\in \mathbb{Z}$, (x, y) $\in$ g}
= {y | for some x $\in \mathbb{Z}$, y = 4x + 8}
= {y | for some x $\in \mathbb{Z}$, x = (y – 8)/4}
= {y | (y – 8)/4 is an integer}
$\neq \mathbb{Z}$

(b) *Is g surjective? If g is not surjective, provide a counterexample to show why it is not surjective.*
We give a counterexample: Let y = 9 (9 $\in \mathbb{Z}$), then x = (y – 8)/4 = (9 – 8)/4 = ¼ $\notin \mathbb{Z}$, so if y = 9, no integer x can be found such that (x, 9) $\in$ g. Thus, 9 $\notin$ g[$\mathbb{Z}$].
This means that $\mathbb{Z}$ is not a subset of g[$\mathbb{Z}$] because each element of $\mathbb{Z}$ is not an element of g[$\mathbb{Z}$], so g[$\mathbb{Z}$] $\neq \mathbb{Z}$. Thus, g is not surjective because the range of g is not equal to the codomain.

Activity 7-5

1. *Write down the injective (one-to-one) functions from X to Y.*

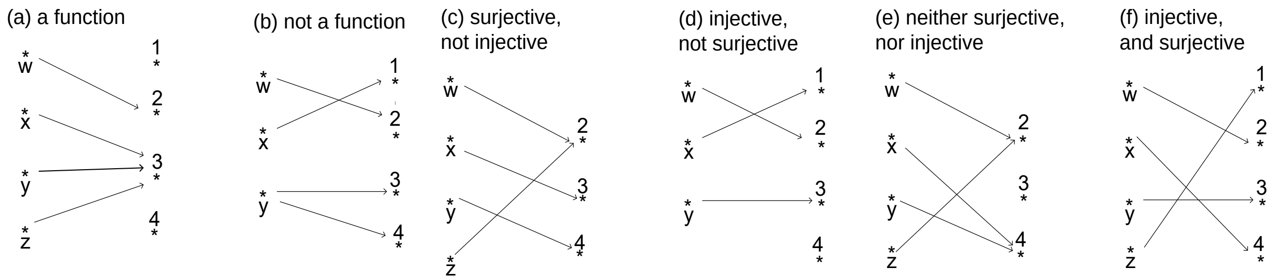
(b) X = {2, 4} and Y = {1, 3}:
We can fill in the gaps in the template
{ (2,), (4,) }
so that different pairs contain different elements of Y in two ways, giving the injective functions:
f_1 = { (2, 1), (4, 3) }
and f_2 = { (2, 3), (4, 1) }.

(c) X = {2, 4} and Y = {1, 3, 5}:
We can fill in the gaps in the template
{ (2,), (4,) }
so that different pairs contain different elements of Y in several ways.
f_1 = { (2, 1) , (4, 3) }
f_2 = { (2, 3) , (4, 1) }
f_3 = { (2, 1) , (4, 5) }
f_4 = { (2, 5) , (4, 1) }
f_5 = { (2, 3) , (4, 5) }
f_6 = { (2, 5) , (4, 3) }.

2. *Consider h: $\mathbb{Z} \to \mathbb{Z}$ be defined by g(x) = 2x – 5. Is h injective?*
Assume g(u) = g(v)
then 2u – 5 = 2v – 5
ie u = v.
Therefore, h is injective.

Activity 7-6

For each of the following diagrams, write down the corresponding relation it represents, then provide the reason(s) why the relation has the given property or properties.



Note: The arrows that should be present in the graphs are not present in the graphs in the study guide.

{(w, 2), (x, 3), (y, 3), (z, 3)} is a function: for each first co-ordinate there is only one corresponding second co-ordinate and the domain is the set {w, x, y, z}.

{(w, 2), (x, 1), (y, 3), (y, 4)} is not a function since it is not functional: y appears twice as first co-ordinate, but y does not have only one corresponding second co-ordinate.
{(w, 2), (x, 3), (y, 4), (z, 2)} is surjective since the range is the set {2, 3, 4} = codomain, but it is not injective since w and z share the same corresponding second co-ordinate, namely 2.

{(w, 2), (x, 1), (y, 3)} is injective since each first co-ordinate has a unique corresponding second co-ordinate, but not surjective, since 4 is an element of the codomain {1, 2, 3, 4}, but not an element of the range {1, 2, 3}.

{(w, 2), (x, 4), (y, 4), (z, 2)} is not injective since x and y share the same corresponding second co-ordinate, namely 4 and w and z share the same corresponding second co-ordinate, namely 2, and the relation is not surjective since 3 is not an element of the range {2, 4} but 3 is an element of the codomain {2, 3, 4}.

{(w, 2), (x, 4), (y, 3), (z, 1)} is injective since each first co-ordinate has a unique corresponding second co-ordinate, and it is surjective since the range is the set {1, 2, 3, 4} = codomain.

Activity 7-10

Determine $f \circ f$, $g \circ g$, $g \circ f$, and $f \circ g$ in the following cases:

(a) $f: Z \to Z$ is defined by the rule $f(x) = x + 1$ and $g: Z \to Z$ is defined by the rule $g(x) = x - 1$:

$f \circ f: Z \to Z$ is defined by $(f \circ f)(x)$:

$(f \circ f)(x)$

$= f(f(x))$
$= f(\mathbf{x} + \mathbf{1})$ (replace $f(x)$ by $\mathbf{x} + \mathbf{1}$)
$= (\mathbf{x} + \mathbf{1}) + 1$ ($f(\mathbf{x}) = \mathbf{x} + 1$, so $f(\mathbf{x} + \mathbf{1}) = (\mathbf{x} + \mathbf{1}) + 1$)
$= x + 2$.

Note: If you want to express what this means in words: if you feed $f \circ f$ an element x, it spits out the same thing you get if you feed x to f and then feed the result to f again.
First, you feed **x** to f to get **x** + 1. Now feeding **x + 1** to f gives you (**x + 1**) + 1 = x + 2, because f takes anything you feed it and adds 1 to it.

$g \circ g: Z \to Z$ is defined by $(g \circ g)(x) = g(g(x))$

$= g(x - 1)$
$= (x - 1) - 1$
$= x - 2$.

$g \circ f: Z \to Z$ is defined by $(g \circ f)(x) = g(f(x))$

$= g(\mathbf{x} + \mathbf{1})$ (replace $f(x)$ by $\mathbf{x} + \mathbf{1}$)
$= (\mathbf{x} + \mathbf{1}) - 1$ ($g(\mathbf{x}) = \mathbf{x} - 1$, so $g(\mathbf{x} + \mathbf{1}) = (\mathbf{x} + \mathbf{1}) - 1$)
$= x$.

$f \circ g: Z \to Z$ is defined by $(f \circ g)(x) = f(g(x))$

$= f(x - 1)$
$= (x - 1) + 1$
$= x$.

(b) $f: R \to R$ is defined by $f(x) = 3x - 2$, and
$g: R \to R$ is defined by $g(x) = x^2 + x$:

$f \circ f: R \to R$ is defined by $(f \circ f)(x) = f(f(x))$

= f(**3x** – **2**) (replace f(x) by **3x** – **2**)
= 3(**3x** – **2**) – 2 (f(**x**) = 3**x** – 2, so f(**3x** – **2**) = 3(**3x** – **2**) – 2)
= 9x – 8.

g ∘ g:R→R is defined by (g ∘ g)(x) = g(g(x))
= $g(x^2 + x)$
= $(x^2 + x)^2 + (x^2 + x)$
= $(x^2 + x)(x^2 + x) + (x^2 + x)$
= $x^4 + 2x^3 + x^2 + x^2 + x$
= $x^4 + 2x^3 + 2x^2 + x$.

g ∘ f:R → R is defined by (g ∘ f)(x) = g(f(x))
= g(3x – 2)
= $(3x - 2)^2 + (3x - 2)$
= $9x^2 - 12x + 4 + 3x - 2$
= $9x^2 - 9x + 2$.

f ∘ g: R→R is defined by

(f ∘ g)(x) = f(g(x))
= $f(x^2 + x)$
= $3(x^2 + x) - 2$
= $3x^2 + 3x - 2$.

(c) f: $\mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by f(x) = 113, and
g: $\mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by g(x) = x + 1:

f ∘ f: $\mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by (f ∘ f)(x) = f(f(x))
= f(113) = 113.

g ∘ g: $\mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by (g ∘ g)(x) = g(g(x))
= g(x + 1)
= (x + 1) + 1
= x + 2.

g ∘ f: $\mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by (g ∘ f)(x) = g(f(x))
= g(113)
= 113 + 1
= 114

f ∘ g: $\mathbb{Z}^{\geq} \to \mathbb{Z}^{\geq}$ is defined by (f ∘ g)(x) = f(g(x))
= f(x + 1)
=113. (because f does not care what you feed it is constantly going to spit out 113 nothing else).

Activity 7-12

1. *Write down the bijective (one-to-one correspondence) functions from X to Y in each case:*

(a) X = { ∅, {113} } and Y = { {1} }:
To obtain a bijective function from X to Y we must fill in the template
{ (∅,), ({113},) }
in such a way that different pairs get different elements of Y (for injectivity) and all elements of Y are used up (for surjectivity).
This is not possible, since Y has only one element and there are two pairs needing *different* second co-ordinates.
So there is no bijective function from X to Y in this case.

(b) X = { ∅, {113} } and Y = { {1}, {2} }:
We need to fill in the template
{ (∅,), ({113},) }
so that we use, once only, each element of Y. This is possible in 2 ways:
h_1 = { (∅, {1}) , ({113}, {2}) }
and h_2 = { (∅, {2}) , ({113}, {1}) }.

(c) X = { ∅, {113} } and Y = { {1}, {2}, {7} }:
There is no way to fill in the template { (∅,), ({113},) } so that we use each member of Y exactly once, because there are 3 elements in Y and only 2 gaps to be filled in.
Therefore, there are no bijective functions from X to Y in this case.

2. *Check the following functions for injectivity (one-to-one), surjectivity (onto) and bijectivity (a one-to-one correspondence) (ie functions that are both injective and surjective), and give the inverse function of each:*

(a) f: $\mathbb{Z} \to \mathbb{Z}$ is defined by the rule f(x) = x + 1:
f is **injective**, because
if f(u) = f(v)
then u + 1 = v + 1
ie u = v.

f is **surjective**, because
ran(f) = {y | for some x $\in \mathbb{Z}$, (x, y) $\in$ f}
= {y | for some x $\in \mathbb{Z}$, y = x + 1}
= {y | for some x $\in \mathbb{Z}$, x = y − 1}
= {y | y − 1 is an integer} = $\mathbb{Z}$

Since f: $\mathbb{Z} \to \mathbb{Z}$ is **bijective**, f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

We can determine the inverse function f^{-1}:
(y, x) $\in f^{-1}$ iff (x, y) $\in$ f
iff y = x + 1

iff $x = y - 1$

so, $f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(y) = y - 1$.

Note: We can also write this as
$f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(x) = x - 1$ or
$f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(z) = z - 1$, etc.
It is important to note that it does not matter which variable we use.

(b) $f: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = x^2$:

f is **not injective**, because we can find a counterexample:
If we choose $u = 2$ and $v = -2$, then $u \neq v$ but
$f(u) = f(2) = 4$ and $f(v) = f(-2) = 4$
ie $f(u) = f(v)$.

f is **not surjective**, because we can find a counterexample:
If we choose $y = -9$,
then there is no $x \in \mathbb{Z}$ such that $x^2 = y$ (since $x^2 \geq 0$ for every integer x),
so $-9 \notin \text{ran}(f)$.
Hence, $\text{ran}(f) \neq \mathbb{Z}$. Since $f: \mathbb{Z} \to \mathbb{Z}$ is **not bijective**, f^{-1} is not a function from $\mathbb{Z}$ to $\mathbb{Z}$.

(c) $f: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = 3 - x$:

f is **injective** because
if $f(u) = f(v)$
then $3 - u = 3 - v$
ie $u = v$.

f is **surjective** because

$$\begin{aligned}
\text{ran}(f) &= \{y \mid \text{for some } x \in \mathbb{Z}, (x, y) \in f\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, y = 3 - x\} \\
&= \{y \mid \text{for some } x \in \mathbb{Z}, x = 3 - y\} \\
&= \{y \mid 3 - y \text{ is an integer}\} \\
&= \mathbb{Z}
\end{aligned}$$

Since $f: \mathbb{Z} \to \mathbb{Z}$ is **bijective**, f^{-1} is a function from $\mathbb{Z}$ to $\mathbb{Z}$.

$(y, x) \in f^{-1}$ iff $(x, y) \in f$
iff $y = 3 - x$
iff $x = 3 - y$.
So $f^{-1}: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f^{-1}(y) = 3 - y$.

(d) $f: \mathbb{Z} \to \mathbb{Z}$ is defined by the rule $f(x) = 4x + 5$:

f is **injective** because

if $f(u) = f(v)$
then $4u + 5 = 4v + 5$
ie $4u = 4v$
ie $u = v$.

f is **not surjective,** because, if we choose y to be even, say $y = 8$,
then y cannot be written in the form $4x + 5$ (Can you remember why not?)
ie $y \notin \text{ran}(f)$.
Hence, $\text{ran}(f) \neq \mathbb{Z}$ (although $\text{ran}(f) \subseteq \mathbb{Z}$).
So $f: \mathbb{Z} \to \mathbb{Z}$ is **not bijective**.

Note: If f were defined to be a function from $\mathbb{R}$ to $\mathbb{R}$, then f would be bijective (can you show this?) with an inverse calculated as follows:

$(y, x) \in f^{-1}$ iff $(x, y) \in f$
iff $y = 4x + 5$
iff $y - 5 = 4x$
iff $(y - 5)/4 = x$.
Hence, $f^{-1}: \mathbb{R} \to \mathbb{R}$ is defined by the rule $f^{-1}(y) = (y - 5)/4$.

3. Consider an identity function $i_C: C \to C$.

(a) Prove that $i_C: C \to C$ is bijective.
A function is **bijective** if it is **injective** and **surjective**.

$i_C: C \to C$ is defined by the rule $i_C(x) = x$ (i_C is the identity function on C):

i_C is **injective**, because
if $i_C(u) = i_C(v)$
then $u = v$.

i_C is **surjective**, because
$\text{ran}(i_C) = \{y \mid \text{for some } x \in C, (x, y) \in i_C\}$
$= \{y \mid \text{for some } x \in C, y = x\}$
$= \{y \mid y \in C\} \quad = C$
i_C is injective and surjective, thus it is a bijective function.

(c) *Prove that i_C is an equivalence relation on C.*

In order to prove that i_C is an equivalence relation, we have to prove that i_C **is reflexive, symmetric and transitive.**

Reflexivity:
Is it the case that for all $x \in C$, $(x, x) \in i_C$?
Yes, for any $x \in C$, $x = x$,
ie $(x, x) \in i_C$.

Thus, i_C is reflexive.

Symmetry:

If (x, y) $\in i_C$, is it the case that (y, x) $\in i_C$?

Suppose (x, y) $\in i_C$,
then   y = x
ie   x = y,
therefore, (y, x) $\in i_C$.
Thus, i_C is symmetric.

Transitivity:

If (x, y) $\in i_C$ and (y, z) $\in i_C$, is it the case that (x, z) $\in i_C$?
*(Assume (x, y) $\in i_C$ and (y, z) $\in i_C$ then **use** this information to prove that (x, z) $\in i_C$.)*

Suppose (x, y) $\in i_C$,
then   y = x   ①
and
suppose (y, z) $\in i_C$,
then   z = y.   ②

From ① and ② it follows that:
z = y = x
ie z = x,
therefore, (x, z) $\in i_C$.
Thus, i_C is transitive.

i_C is reflexive, symmetric and transitive, thus it is an equivalence relation.

13 Learning unit 8 – Operations

Study Material

Study guide
You should cover study unit 8 in the study guide.

Time allocated
You will need one week to master this learning unit.

Notes

Background
Ordered pairs determined by the Cartesian product of two sets can act as first co-ordinates (inputs) of a function and a single element/object can play the role of the output. Such functions, which "eat" ordered pairs and "spit out" single objects are called binary operations, which have certain properties. These operations are written in the form f: $(A \times A) \rightarrow A$. Matrices are also defined.

Applications
Examples of binary operations can be compiled and properties of binary operations can be determined. Certain computations are possible with matrices.

Activities

Do the activities provided in study unit 8 to consolidate your knowledge of the work in this learning unit.

Study unit 8

Activity 8-3

1. Let X be the set {2, 7}.

(a) *Give 3 binary operations on X, both in list notation and in tabular form.*
Let us compile the table first and then the set of ordered pairs in each case.
There are many possible examples, of which we give just three.

+	**2**	**7**
2	2	2
7	2	2

This table represents the very simple operation that does not care what you feed it, because it has made up its mind to constantly spit out the value 2.

Note: Although we have chosen to call the operation "+", it has *no* connection with ordinary addition whatsoever.
In list notation, + = **{ (**(2, 2), 2**)**, **(**(2, 7), 2**)**, **(**(7, 2), 2**)**, **(**(7, 7), 2**)** }.

Our next example is:

$*$	**2**	**7**
2	7	2

7	2	7

In list notation, ∗ = { **(**(2, 2), 7**)**, **(**(2, 7), 2**)**, **(**(7, 2), 2**)**, **(**(7, 7), 7**)** }.

Our final example is:

□	**2**	**7**
2	7	7
7	2	7

In list notation: □ = {**(**(2, 2), 7**)**, **(**(2, 7), 7**)**, **(**(7, 2), 2**)**, **(**(7, 7), 7**)** }.

(b) *Check these operations for commutativity and associativity.*

Commutativity:
+ and * are commutative, whereas □ is not.
A quick way to see this is to look at the tables and see which are symmetric about the diagonal from the top left to the bottom right. In the case of □, the triangle above the diagonal is not a mirror image of the triangle below, because, for example, 2 □ 7 = 7 whereas 7 □ 2 = 2.

Associativity:
+ must be associative, because it always spits out 2,
so, for all x, y and z in X, x + (y + z) = 2 = (x + y) + z.
To see that * is associative we have to check all the various cases (8 of them), and if you do, you will find that everything works out.

□ fails to be associative. A counterexample is provided by the values x=2, y=2, and z=2:
x □ (y □ z) = 2 □ (2 □ 2) = 2 □ 7 = 7, but (x □ y) □ z = (2 □ 2) □ 2 = 7 □ 2 = 2.

2. *Give 2 binary operations on X = {a, b, c} and check them for commutativity and associativity.*
Bear in mind that there are many examples of such operations, and we will choose two more or less at random.
First, an example using list notation. Just fill in the template
{ ((a, a),), ((a, b),), ((a, c),), ((b, a),), ((b, b),), ((b, c),), ((c, a),), ((c, b),), ((c, c),) }
to get, for instance,
{ ((a, a), **b**) , ((a, b), **b**) , ((a, c), **b**) , ((b, a), **b**) , ((b, b), **b**) , ((b, c), **b**), ((c, a), **b**) , ((c, b), **b**) , ((c, c), **b**) }.

This operation is commutative and associative – do you agree? Think about it.

Finally, the following is an example of an operation that fails to be either commutative or associative:

*	**a**	**b**	**c**
a	b	c	b
b	a	b	b
c	b	b	b

Commutativity fails, because a * b = c whereas b * a = a.
Associativity fails, because a * (a * a) = a * b = c, whereas (a * a) * a = b * a = a.

3. *Consider the dot operation, "•", defined in section 8.1. Let us compare the dot operation on {a, b, c, d} with ordinary multiplication.*

•	**a**	**b**	**c**	**d**
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

(a) *We know that ordinary multiplication on R is commutative. Examine x • y and y • x for each x, y ∈ A. What do you conclude?*

a • b = b = b • a
a • c = c = c • a
a • d = d = d • a
b • a = b = a • b
b • c = d = c • b
b • d = c = d • b
c • d = b = d • c
The *dot* operation is commutative.

(b) *We know that R has an identity for multiplication, namely 1.*
This means that 1 • x = x = x • 1 for all x ∈ R. Does A have an element that behaves similarly?

a • a = a
a • b = b
a • c = c
a • d = d
b • a = b
c • a = c
d • a = d It appears that the element a is an identity element for the *dot* operation.

Activity 8-6

Consider the following vectors: u = (3, 1), v = (−4, −4), and w = (0, −1). Determine the following:

(a) 2u + v = 2(3, 1) + (−4, −4)
= (6, 2) + (−4, −4)
= (2, −2)

(b) u − 3v = (3, 1) − 3(−4, −4)
= (3, 1) + (12, 12)
= (15, 13)

(c) −3(v + w) = −3[(−4, −4) + (0, −1)]
= −3(−4, −5)
= (12, 15)

Activity 8-7

Consider the following vectors: u = (1, 2, 5) and v = (2, 3, 5). Determine the following:

(a) u · v = (1, 2, 5) · (2, 3, 5) = 1·2 + 2·3 + 5·5 = 2 + 6 + 25 = 33.

(b) v(2u) = (1, 2, 5) · (2×(2, 3, 5)) = (1, 2, 5) · (4, 6, 10) = 1·4 + 2·6 + 5·10 = 4 + 12 + 50 = 66.

Activity 8-8

(a) A + B = $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 5 & 5 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ 4 & 0 \end{bmatrix}$

(b) This addition is impossible because the sizes of the matrices (2 × 2 and 2 × 3) do not correspond.

(c) A + B = $\begin{bmatrix} 2 & 0 & 3 \\ 0 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -2 \\ 2 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 7 & 7 \end{bmatrix}$

(d) This addition is impossible because the sizes of the matrices (2 × 2 and 2 × 3) do not correspond.

Activity 8-9

$$2\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 26 \end{bmatrix}$$

Activity 8-10

Two matrices A and B can only be multiplied if the sizes of A and of B match up in the following way:

Schematically: $A \cdot B = C$

$m \times n \quad n \times k \quad m \times k$

(equal)

Determine the following:

1. $\begin{bmatrix} 31 & -3 & 2 \\ 2 & 5 & 1 \\ 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 10 \\ 0 \end{bmatrix}$

2. $\begin{bmatrix} 9 & 3 \\ 1 & 5 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 5 & 1 \end{bmatrix} = ?$

This multiplication cannot be performed because the sizes of the matrices do not match up as explained above. Both matrices are 3 $\times$ 2.

3. $\begin{bmatrix} 1 & -3 & 2 \\ 0 & 6 & 4 \\ 3 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 & 3 \\ 1 & 1/3 & 1 \\ 1/2 & 5 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 8 & 0 \\ 8 & 22 & 6 \\ 3/2 & 12 & 9 \end{bmatrix}$

4. *Provide examples of matrices X and Y such that XY is a 3 × 3 matrix, but YX is a 2 × 2 matrix.*

Let X = $\begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$

and Y = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ for example, then

XY = $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$ and YX = $\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}$.

5. *Provide examples of matrices X and Y such that both X and Y contain at least some nonzero entries, but XY is the 2 × 2 zero matrix,*

ie *XY* = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Take X = $\begin{bmatrix} 3 & 0 \\ 5 & 0 \end{bmatrix}$ and Y = $\begin{bmatrix} 0 & 0 \\ 6 & 2 \end{bmatrix}$ for example.

6. *Prove that addition is a commutative operation on the set of 2 × 2 matrices and that there is a 2 × 2 matrix that acts as an identity element in respect of addition.*

Let X = $\begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and Y = $\begin{bmatrix} s & t \\ u & v \end{bmatrix}$.

Then X + Y = $\begin{bmatrix} x+s & y+t \\ z+u & w+v \end{bmatrix}$

= $\begin{bmatrix} s+x & t+y \\ u+z & v+w \end{bmatrix}$

= Y + X

Notice we use the commutativity of ordinary addition inside the matrix.

Let O be the identity element: O = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

7. *Prove that multiplication is **not** a commutative operation on the set of 2 × 2 matrices, and that there is a 2 × 2 matrix that acts as an identity element in respect of multiplication.*

Let X = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and Y = $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

Then XY = $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ but YX = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Let I be the identity element: I = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

14 Learning units 9 and 10 – Truth tables, quantifiers, predicates and proof strategies

Study Material

Study guide
You should cover study units 9 and 10 in the study guide.

Time allocated
You will need one week to master these learning units.

Notes

Background
Truth values can be assigned to certain statements which can be connected with defined connectives. Different proof strategies involving quantifiers and predicates are described.

Applications
Truth values can be determined for compound statements. It can be determined whether some given statement is a tautology, a negation or neither. Different proof strategies can be used to determine whether or not a given statement is true or not.

Activities

Do the activities provided in study units 9 and 10 to consolidate your knowledge of the work in this learning unit.

Study Unit 9

Activity 9-5

1. *Suppose that p represents the statement "It is sunny" and q the statement "It is humid". Write each of the following in abbreviated form:*

(a) *It is sunny and it is not humid*: $p \wedge \neg q$

(b) *It is humid or it is sunny*: $q \vee p$
(c) *It is false that it is humid*: $\neg q$

(d) *It is false that it is sunny and humid*: $\neg (p \wedge q)$

(e) *It is neither sunny nor humid*: $\neg p \wedge \neg q$

(f) *It is not the case that if it is sunny then it is humid*: $\neg (p \to q)$

(g) *It is humid if it is sunny*: $p \to q$

(h) *It is humid only if it is sunny*: $q \to p$

(i) *It is sunny if and only if it is humid*: $p \leftrightarrow q$

(j) *If it is false that it is either sunny or humid, then it is not sunny*: $\neg [(p \vee q) \wedge \neg (p \wedge q)] \rightarrow \neg p$

2. *Construct the truth tables for the following compound statements*:

(a) $[(\neg q) \rightarrow (\neg p)] \rightarrow (p \rightarrow q)]$

p	**q**	**¬p**	**¬q**	**(¬q) → (¬p)**	**p → q**	**[(¬q) → (¬p)] → (p → q)**
T	T	F	F	T	T	T
T	F	F	T	F	F	T
F	T	T	F	T	T	T
F	F	T	T	T	T	T

(b) $[\neg p \rightarrow (q \wedge (\neg q))] \rightarrow p$

p	**q**	**¬ p**	**¬q**	**q ∧ (¬q)**	**¬p→(q ∧ (¬q))**	**[¬p → (q ∧ (¬q))] → p**
T	T	F	F	F	T	T
T	F	F	T	F	T	T
F	T	T	F	F	F	T
F	F	T	T	F	F	T

(c) $p \vee (\neg p)$

p	**¬p**	**p ∨ (¬p)**
T	F	T
F	T	T

(d) $[p \wedge (p \rightarrow q)] \rightarrow q$

p	**q**	**p → q**	**p ∧ (p → q)**	**[p ∧ (p → q)] → q**
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

(e) $(p \vee q) \wedge (\neg p \vee \neg q)$

p	**q**	**p ∨ q**	**¬p**	**¬q**	**(¬p ∨ ¬q)**	**(p ∨ q) ∧ (¬p ∨ ¬q)**
T	T	T	F	F	F	F
T	F	T	F	T	T	T
F	T	T	T	F	T	T

F	F	F	T	T	T	F

(f) $(\neg p \rightarrow [\, q \wedge r\,]\,) \vee r$

p	**q**	**r**	**$\neg$ p**	**$q \wedge r$**	**$\neg p \rightarrow [\, q \wedge r\,]$**	**$(\neg p \rightarrow [\, q \wedge r\,]\,) \vee r$**
T	T	T	F	T	T	T
T	T	F	F	F	T	T
T	F	T	F	F	T	T
T	F	F	F	F	T	T
F	T	T	T	T	T	T
F	T	F	T	F	F	F
F	F	T	T	F	F	T
F	F	F	T	F	F	F

(g) $(p \rightarrow [\, q \wedge r\,]\,) \leftrightarrow (\,[\, p \rightarrow q\,] \vee [\, p \rightarrow r\,]\,)$

p	**q**	**r**	**$q \wedge r$**	**$p \rightarrow [\, q \wedge r\,]$**	**$p \rightarrow q$**	**$p \rightarrow r$**	**$[\, p \rightarrow q\,] \vee [\, p \rightarrow r\,]$**	**$(p \rightarrow [\, q \wedge r\,]) \leftrightarrow ([\, p \rightarrow q\,] \vee [\, p \rightarrow r\,])$**
T	T	T	T	T	T	T	T	T
T	T	F	F	F	T	F	T	F
T	F	T	F	F	F	T	T	F
T	F	F	F	F	F	F	F	T
F	T	T	T	T	T	T	T	T
F	T	F	F	T	T	T	T	T
F	F	T	F	T	T	T	T	T
F	F	F	F	T	T	T	T	T

Activity 9-6

1. *Express the following sentence symbolically and then determine whether or not it is a tautology*:
If demand has remained constant and prices have been increased, then turnover must have decreased.

*Use **p** to represent the sentence "demand has remained constant", let **q** represent "prices have been increased" and let **r** represent "turnover must have decreased",* then this sentence as a whole is represented by $(p \wedge q) \rightarrow r$.

To determine whether this is a tautology, one can compile a truth table. Sometimes, as in this case, there is also a faster way: one works backward from a truth value of F for the whole sentence and determines whether truth values for the sentences can be found to support it.

From our knowledge of the truth table of "$\rightarrow$", we know that
if $(p \wedge q) \rightarrow r$ is F then $p \wedge q$ is T and r is F, ie p is T, q is T and r is F.

Therefore, the sentence $(p \wedge q) \rightarrow r$ is not a tautology, because allocating the values mentioned above to p, q, and r would make the sentence as a whole false.

2. Refer to the truth tables in Activity 9-5. Determine whether each of the statements is a tautology, a contradiction or neither of the two.

(a) $[(\neg q) \rightarrow (\neg p)] \rightarrow (p \rightarrow q)]$ is a tautology. For all possible combinations of the truth values p and q, $[(\neg q) \rightarrow (\neg p)] \rightarrow (p \rightarrow q)$ is true as can be seen from the final column of the table.

(b) $[\neg p \rightarrow (q \wedge (\neg q))] \rightarrow p$ is a tautology. An interesting observation can be made from the fourth column where all the truth values are F: The statement $q \wedge (\neg q)$ is a contradiction.

(c) $p \vee (\neg p)$ is a tautology. Feel free to practice your truth table technique on this one. However, here is a second method: For $p \vee (\neg p)$ to be F, both p and $\neg p$ have to be F, which is impossible, so $p \vee (\neg p)$ is always T.

(d) $[p \wedge (p \rightarrow q)] \rightarrow q$ is a tautology.
In order for the statement as a whole to assume the value F, q must be F while
$p \wedge (p \rightarrow q)$ is T.

For $p \wedge (p \rightarrow q)$ to be T, p must be T and $p \rightarrow q$ must be T.
But, since q is F, $p \rightarrow q$ can only be T if p is F, whereas we know p is T.
Therefore, it is impossible for $[p \wedge (p \rightarrow q)] \rightarrow q$ to be F.

(e) From the truth table, it is clear that $(p \vee q) \wedge (\neg p \vee \neg q)$ is neither a tautology nor a contradiction.
Let p be T and q be F, then $(p \vee q) \wedge (\neg p \vee \neg q)$ is T, whereas if p is T and q is also T, then
$(p \vee q) \wedge (\neg p \vee \neg q)$ is F.

(f) This is neither a tautology nor a contradiction. There are F and T truth values in the final column.

(g) The final column tells us that this is not a tautology. We also know that two statements p and q are **logically equivalent** iff the statement $p \leftrightarrow q$ is a tautology.

The two given statements are not logically equivalent because the eighth and ninth columns are not identical, ie $(p \rightarrow [q \wedge r])$ is not logically equivalent to $([p \rightarrow q] \vee [p \rightarrow r])$.

The left-hand side (of the biconditional, $\leftrightarrow$) does not have exactly the same truth value as the right-hand side.

Activity 9-9

1. *Rewrite $p \leftrightarrow q$ as a statement built up using only $\neg$, $\wedge$ and $\vee$:*

$p \leftrightarrow q \quad \equiv (p \to q) \wedge (q \to p)$
$\equiv (\neg p \vee q) \wedge (\neg q \vee p)$

2. *Refer to study unit 6 for the definition of equivalence relations. This kind of relation is reflexive, symmetric and transitive. Show that $\equiv$ is an equivalence relation*.

Let's think about the meaning of the symbol "$\equiv$". In the context of this question, it states that $p \equiv q$ means that p is equivalent to q, where p and q are two statements.
It must be the case that $p \equiv p$ (p is equivalent to itself), so the relation is reflexive.
If $p \equiv q$, it is also the case that $q \equiv p$ (p and q are equivalent statements), so the relation is symmetric.
If $p \equiv q$ and $q \equiv r$, it must be the case that $p \equiv r$, so the relation is transitive.
This means that $\equiv$ is an equivalence relation on statements.

3. *Truth table for the exclusive OR (XOR)*:

p	**q**	**$p \oplus q$**
T	T	F
T	F	T
F	T	T
F	F	F

4.

p	**q**	**$\neg p$**	**$\neg q$**	**$p \vee \neg q$**	**$\neg (p \vee \neg q)$**	**$\neg p \wedge q$**
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	F	T	T	T	F	F

5. *Use the property of double negation and De Morgan's laws to rewrite the following statements so that the not symbol ($\neg$) does not appear outside parentheses:*

(a) $\neg [(p \vee q \vee \neg q) \wedge (q \wedge \neg p)]$
$\equiv \neg (p \vee q \vee \neg q) \vee \neg (q \wedge \neg p)$ *De Morgan's law*
$\equiv (\neg p \wedge \neg q \wedge \neg \neg q) \vee (\neg q \vee \neg \neg p)$ *De Morga''s law*
$\equiv (\neg p \wedge \neg q \wedge q) \vee (\neg q \vee p)$ *Double negation*

(a) $\neg [(p \vee (p \to q)) \vee (p \wedge q)]$
$\equiv \neg [(p \vee (\neg p \vee q)) \vee (p \wedge q)]$ *Refer to the study guide, comment p 147, and second table p 148*.
$\equiv \neg (p \vee (\neg p \vee q)) \wedge \neg (p \wedge q)$

$\equiv (\neg p \wedge \neg(\neg p \vee q)) \wedge (\neg p \vee \neg q)$	De Morgan's law
$\equiv (\neg p \wedge (\neg\neg p \wedge \neg q)) \wedge (\neg p \vee \neg q$	De Morgan's law
$\equiv (\neg p \wedge (p \wedge \neg q)) \wedge (\neg p \vee \neg q)$	Double negation

6. *Determine whether or not the following statements are equivalent:* $\neg p \wedge (\neg p \wedge \neg q)$ *and* $\neg(p \vee (p \to q))$.

$\neg(p \vee (p \to q))$	
$\equiv \neg(p \vee (\neg p \vee q))$	*Refer to the study guide, comment p 147, and second table p 148*.
$\equiv \neg p \wedge \neg(\neg p \vee q)$	*De Morgan's law*
$\equiv \neg p \wedge (\neg\neg p \wedge \neg q)$	*De Morgan's law*
$\equiv \neg p \wedge (p \wedge \neg q)$	*Double negation*

Clearly, the two given expressions are equivalent.

Study Unit 10

Activity 10-3

1. *Write down the English equivalent of each of the following statements. Give an opinion on whether or not the statement is true.*

(a) $\exists\, y \in \mathbb{Q},\ y = \sqrt{2}$

There exists some rational number y, which is equal to $\sqrt{2}$.
This is not true, since $\sqrt{2}$ is not a rational number.

(b) $\forall\, x \in \mathbb{R},\ 2x < x^2$

For all real numbers it holds that $2x < x^2$.
We give a counterexample to show that this statement does not hold.
Choose $x = 0$. $2{\cdot}0 = 0$ and $0^2 = 0$. In this case it does not hold that $2x < x^2$.

(c) $\forall\, x \in \mathbb{Z},\ x > 0$

For all integers x, it holds that $x > 0$. The set $\mathbb{Z}$ includes all integers, so if we choose $x = 0$ or x equal any negative integer, the statement does not hold.

(d) $\exists\, x \in \mathbb{Z}^+,\ x = 0$

There exists a positive integer, which is equal to 0.
The set $\mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. The value 0 does not belong to $\mathbb{Z}^+$, so the statement is not true.

Activity 10-4

Prove by means of truth tables that $\neg (p \wedge q) \equiv (\neg p) \vee (\neg q)$.

p	q	¬ p	¬ q	p ∧ q	¬ (p ∧ q)	(¬ p) ∨ (¬ q)
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Activity 10-5

Prove by means of truth tables that $\neg (p \vee q) \equiv (\neg p) \wedge (\neg q)$.

p	q	¬ p	¬ q	p ∨ q	¬ (p ∨ q)	(¬ p) ∧ (¬ q)
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Activity 10-6

Write down the negations of the following in a useful form.

(a) The negation of $\forall x \in \mathbb{Z}^+, x > 3$:

$\neg (\forall x \in \mathbb{Z}^+, x > 3)$

$\equiv \exists x \in \mathbb{Z}^+, \neg (x > 3)$

$\equiv \exists x \in \mathbb{Z}^+, x \leq 3$

(b) The negation of $\exists x \in \mathbb{R}, 2x = x^2$:

$\exists x \in \mathbb{R}, 2x = x^2)$

$\equiv \forall x \in \mathbb{R}, \neg (2x = x^2)$

$\equiv \forall x \in \mathbb{R}, 2x \neq x^2$

(c) The negation of $\forall x \in \mathbb{Z}, (x > 0) \vee (x^2 > 0)$:

$\neg [\forall x \in \mathbb{Z}, (x > 0) \vee (x^2 > 0)$

$\equiv \exists x \in \mathbb{Z}, \neg [(x > 0) \vee (x^2 > 0)]$

$\equiv \exists x \in \mathbb{Z}, \neg (x > 0) \wedge \neg (x^2 > 0)$

$\equiv \exists x \in \mathbb{Z}, (x \leq 0) \wedge (x^2 \leq 0)$

(d) The negation of $\exists y \in \mathbb{Z}^+, (y \leq 10) \wedge (y \neq 0)$:

$\neg [\exists y \in \mathbb{Z}^+, (y \leq 10) \wedge (y \neq 0)]$

$\equiv \forall y \in \mathbb{Z}^+, \neg [(y \leq 10) \wedge (y \neq 0)]$

$\equiv \forall y \in \mathbb{Z}^+, \neg (y \leq 10) \vee \neg (y \neq 0)$

$\equiv \forall y \in \mathbb{Z}^+, (y > 10) \vee (y = 0)$

(e) The negation of $\exists x \in A, P(x) \wedge Q(x)$:

$\neg (\exists x \in A, P(x) \wedge Q(x))$

$\equiv \forall x \in A, \neg (P(x) \wedge Q(x))$

$\equiv \forall x \in A, \neg P(x) \vee \neg Q(x)$

(f) $\forall x \in \mathbb{Z}^+, (x \leq 3) \rightarrow (x^3 \geq 1)$

$\neg (\forall x \in \mathbb{Z}^+, (x \leq 3) \rightarrow (x^3 \geq 1))$

$\equiv \neg (\forall x \in \mathbb{Z}^+, \neg (x \leq 3) \vee (x^3 \geq 1))$

$\equiv \exists x \in \mathbb{Z}^+, \neg (\neg (x \leq 3) \vee (x^3 \geq 1))$

$\equiv \exists x \in \mathbb{Z}^+, \neg \neg (x \leq 3) \wedge \neg (x^3 \geq 1)$

$\equiv \exists x \in \mathbb{Z}^+, (x \leq 3) \wedge (x^3 < 1)$

Activity 10-7

For each of (a) to (d) of activity 10-6, try to decide whether the original statement is true, whether its negation is true or whether neither of the two is true.

(a) The original statement is false. 1, 2 and 3 are elements of $\mathbb{Z}^+$ and they are not greater than 3. The negation of the statement is true for x = 1, 2 and 3.

(b) The original statement is true. Choose x = 2. Then 2x = 4 and $x^2 = 4$.

(c) The negation of the statement is true. Choose x = 0. Then $x \leq 0$ and $x^2 \leq 0$.

(d) The original statement is true. It holds for the elements 1, 2, …, 10. All elements of $\mathbb{Z}^+$ are > 0.

Activity 10-10

1. *Prove each of the following by direct proof, contrapositive and contradiction (reductio ad absurdum), respectively. Which strategy worked best?*

(a) If $x^2 - 3x + 2 < 0$ then $x > 0$.

Direct proof:
Suppose $x^2 - 3x + 2 < 0$,
then $(x - 1)(x - 2) < 0$
The result is < 0, so one factor must be < 0 and the other > 0.
ie either $x - 1 < 0$ and simultaneously $x - 2 > 0$
OR else $x - 1 > 0$ and simultaneously $x - 2 < 0$
ie either $x < 1$ and simultaneously $x > 2$ (which is impossible)
OR else $x > 1$ and simultaneously $x < 2$ ie $1 < x < 2$
We can conclude that $1 < x < 2$
Thus, $x > 0$.

Contrapositive:
Suppose $x \leq 0$,
then $x^2 \geq 0$ and $-3x \geq 0$ (A minus times a minus, remember?)
thus $x^2 - 3x + 2 \geq 0$ (The sum of non-negative numbers is also ≥ 0).

Contradiction (*Reductio ad absurdum*):
Suppose $x^2 - 3x + 2 < 0$,
then $(x - 1)(x - 2) < 0$
and this means that one of the factors is < 0 and the other is > 0.

Now there are just 2 possibilities for x: either $x > 0$ or $x \leq 0$.
The former is the good possibility, so let us try to eliminate $x \leq 0$.

Suppose, just for the moment, that $x \leq 0$ (bad possibility).
Then $x - 1 < 0$ and $x - 2 < 0$, but this contradicts the original fact that one of these factors must be > 0, so we can discard the bad possibility.

Thus, we conclude that $x > 0$.

Which worked best? Well, contrapositive was the shortest, but the one that worked best is the one you feel most comfortable with. You decide. (Most beginners feel safest with direct proof; after a year or two of practice, many people grow to love *reductio ad absurdum* (contradiction), perhaps because so many of the opinions you meet in daily life lead to some absurd conclusion!)

(b) If $x^2 - x - 6 > 0$ then $x \neq 1$:

Direct proof:
Suppose $x^2 - x - 6 > 0$,
then $(x - 3)(x + 2) > 0$
ie either $x - 3 < 0$ and $x + 2 < 0$ (both factors are negative)
OR
$x - 3 > 0$ and $x + 2 > 0$ (both factors are positive)
ie either $x < 3$ and $x < -2$, ie $x < -2$
OR
$x > 3$ and $x > -2$, ie $x > 3$
ie $x < -2$ or $x > 3$
Thus, $x \neq 1$.

Contrapositive:
Suppose $x = 1$,
then $x^2 - x - 6 = 1 - 1 - 6 = -6$
ie $x^2 - x - 6 \leq 0$

Contradiction:
Suppose $x^2 - x - 6 > 0$.

Now there are just 2 possibilities for x: either $x = 1$ or $x \neq 1$.
The latter is the good possibility, so let us eliminate $x = 1$.

Suppose, just for the moment, that $x = 1$.
Then $x^2 - x - 6 = 1 - 1 - 6 = -6$
ie $x^2 - x - 6 < 0$. But this contradicts our starting assumption.

Hence, we conclude that $x \neq 1$.

The easiest? Direct proof, for most people.

(c) *If a + b is odd, exactly one of a and b is odd.*

Direct proof:
Suppose $a + b$ is odd (assume $a, b \in \mathbb{Z}$),
then $a + b = 2n + 1$ for some integer n.
Now there are exactly 2 cases: a is either even or odd.
Case 1: Suppose a is even,

then $a = 2k$ for some integer k

so, $b = (a + b) - a$

$= 2n + 1 - 2k$

$= 2(n - k)+1$

Thus, b is odd.

Case 2: Suppose a is odd,

then $a = 2k + 1$ for some integer k.

Now $b = (a + b) - a$

$= 2n + 1 - (2k + 1)$

$= 2(n - k)$

Thus, b is even.

(Note that we have considered the two cases; not in order to eliminate one of them, as we would in a proof by contradiction, but in order to show that **in both cases** exactly one of a and b is odd.)

Contrapositive:

Suppose a and b are both even or both odd,

then either $a = 2m$ and $b = 2k$ for some integers m and k,

or else $a = 2m + 1$ and $b = 2k + 1$ for some $m, k \in Z$.

So either $a + b = 2m + 2k = 2(m + k)$

which means that $a + b$ is even,

or else $a + b = (2m + 1) + (2k + 1) = 2m + 2k + 2 = 2(m + k + 1)$

which also means that $a + b$ is even.

Contradiction:

Suppose $a + b$ is odd.

There are exactly two possibilities:

either it is the case that exactly one of a and b is odd, or this is not the case.

The former is the good possibility, so we eliminate the latter.

Suppose, just for a moment, that it is **not** the case that exactly one of a and b is odd,

then it may be that a and b are both odd, or else it may be that a and b are both even.

Suppose that a and b are both odd,

then we may write $a = 2m+1$ and $b = 2k + 1$ for some integers m and k.

Now $a + b = 2m + 2k + 2 = 2(m + k + 1)$, ie an even number, contradicting the fact that $a + b$ is odd.

On the other hand, suppose that a and b are both even, ie $a = 2m$ and $b = 2k$, for some integers m and k.

Then $a + b = 2(m + k)$, ie an even number, also contradicting the fact that $a + b$ is odd.

So, exactly one of a and b must be odd.

(Part of this proof is rather similar to the proof by contrapositive.)

The easiest method? Maybe proof by contrapositive.

(d) *If x is even then $x^2 + 4x + 2$ is even (assume $x \in \mathbb{Z}$).*

Direct proof:
Suppose x is even. Then x = 2k for some integer k.
So $x^2 + 4x + 2 = (2k)^2 + 4(2k) + 2$
$= 4k^2 + 8k + 2$
$= 2(2k^2 + 4k + 1)$
Thus, $x^2 + 4x + 2$ is even.

Contrapositive:
Suppose $x^2 + 4x + 2$ is odd.

So, $x^2 + 4x + 2 = 2m + 1$ for some integer m.
ie $x^2 + 4x + 4 = 2m + 2 + 1$ (completing the square)
ie $(x + 2)(x + 2) = 2(m + 1) + 1$
This means that x + 2 must be odd. (Product of two integers is odd, means both integers must be odd.)

Write this as $x + 2 = 2k + 1$ for some integer k.
ie $x = 2(k – 1) + 1$, which means that x is odd.
We can conclude that, if x is even, then $x^2 + 4x + 2$ is even.

Contradiction:
Suppose x is even.
Then x = 2k for some integer k.

There are two possibilities: either $x^2 + 4x + 2$ is even (the good possibility) or it is odd.
Let us eliminate the bad possibility. Assume $x^2 + 4x + 2$ is odd.
Then $x^2 + 4x + 2 = 2m + 1$ for some integer m.
ie $x^2 + 4x + 4 = 2m + 2 + 1$ (completing the square)
ie $(x + 2)(x + 2) = 2(m + 1) + 1$
This means that x + 2 must be odd. (The product of two integers is odd, means both integers must be odd.)
Write this as $x + 2 = 2k + 1$ for some integer k.
ie $x = 2(k – 1) + 1$, which means that x is odd.
This contradicts our initial assumption.
We can conclude that $x^2 + 4x + 2$ is even.

(e) *If n is a multiple of 3, then $n^3 + n^2$ is a multiple of 3.*

Direct proof:
Suppose n is a multiple of 3.
Then n = 3k for some integer k.
So, $n^3 + n^2 = (3k)^3 + (3k)^2$
$= 3(9k^3) + 3(3k^2)$
$= 3(9k^3 + 3k^2)$

It follows that $n^3 + n^2$ is a multiple of 3.

Contrapositive:
Suppose $n^3 + n^2$ is a not a multiple of 3

Then $n(n^2 + n)$ can be written as $3k + 1$, or $3k + 2$ for some integer k.
Let us look at the first alternative:
$n(n^2 + n) = 3k + 1$
This means that both n and $(n^2 + n)$ are not multiples of 3.
We can conclude that n is not a multiple of 3.

Contradiction:
Suppose n is a multiple of 3.
Then $n = 3k$ for some integer k.

Now there are two possibilities: Either $n^3 + n^2$ is a multiple of 3 (the good possibility) or $n^3 + n^2$ is not a multiple of 3.

Let us eliminate the bad possibility, so we assume that $n^3 + n^2$ is not a multiple of 3
then $n(n^2 + n)$ can be written as $3k + 1$, or $3k + 2$ for some integer k.
Let us look at the first alternative:
$n(n^2 + n) = 3k + 1$
This means that both n and $(n^2 + n)$ are not multiples of 3.

We can conclude that n is not a multiple of 3.
But, this contradicts our initial assumption.
We can conclude that $n^3 + n^2$ is a multiple of 3.

2. *Provide a counterexample to show that the following is not true for all integers $x > 0$:*
If $x > 0$, then $x^2 - 3x + 1 < 0$.
Choose $x = 3$.
Then $(3)^2 – 3(3) + 1 = 9 – 9 + 1$, which is greater than 0.

15 Assignment 3

Assignment 3 scope

What is covered?
This assignment is a multiple-choice assignment, and covers study units 6.4 to 10 of the study guide.

Assignment submission
This assignment should be submitted either electronically via myUnisa (the preferred route), or by filling in a mark-reading sheet and submitting it via one of the regional centres or the post office.

Time allocated
You will need one week to complete this assignment.

Due date
Check Tutorial Letter 101 for the due date and unique assignment number for this assignment.

PDF version
You can get a PDF version of this assignment question in **Additional Resources**. You will also find the questions in Tutorial Letter 101.

Assignment 3 questions

You can download the questions for this assignment from the **Additional Resources** page.

Assignment 3 solutions

After the closing date, a discussion of the assignment will be posted to the **Additional Resources** page. You will be informed of this via an announcement on myUnisa.

Sample Assignment 3 questions and solutions

In Tutorial Letter 102, available on the **Additional Resources** page, you will find a sample assignment 3 that you can work through before attempting Assignment 3.

16 Additional self-assessment questions for learning units 7 to 10

Purpose:

It is very important that you do these self-assessment questions. The assignments are multiple-choice assignments. However, in the exam, you will be required to write down proofs and counterexamples, different relations, and you will be required to draw truth tables, etc. These exercises will help you practice to write down the correct mathematical notation required. We give three sets of questions in sections A, B and C below.

Self-assessment questions:

Your answers to the following self-assessment questions should not be submitted. The solutions are provided in Tutorial Letter 102, available on the Additional Resources page. You should also have received this tutorial letter with your study material.

NB: Special hints are provided so that you can avoid making unnecessary mistakes when solving these problems.

SECTION A

Question A.1

Let P and R be relations on A = $\{1, 2, 3, \{1\}, \{2\}, \{3\}\}$ given by

P = $\{(1, \{1\}), (\{1\}, 1), (1, 2), (2, 1)\}$ and R = $\{(1, \{1\}), (3, \{3\}), (2, \{2\}), (\{1\}, \{2\}), (\{2\}, \{2\}), (\{3\}, 3)\}$.

a) Is P functional? Justify your answer.
b) Is P a function from A to A? Justify your answer.
c) Provide the range of R.
d) Is R surjective? Justify your answer.
e) Is R injective? Justify your answer.

Question A.2

Let *f* be a function on $\mathbb{Z}$ (the set of integers) defined by

$$(x, y) \in f \text{ iff } y = 30 - x$$

and let g be a function on $\mathbb{Z}$ defined by

$$(x, y) \in g \text{ iff } y = 31x^2 + 2.$$

a) Determine dom(g).
b) Is f injective (one-to-one)? Justify your answer.
c) Is g a surjective function (onto) from $\mathbb{Z}$ to $\mathbb{Z}$? Justify your answer.
d) Determine the inverse function f^{-1}.

HINTS:

- When it is required to prove that f is a function from A to B, one has to show that f is functional and that dom(f) = A.

Use the definition of f when proving that f is functional and that dom(f) = A. Note that **examples do not constitute a proof**. A statement such as “for every x there is only one y” does not prove anything; it does not even refer to the given relation, nor is it a proper definition.

- Make sure that you understand the definitions of injectivity and surjectivity. You should provide formal proofs when it is required to show that a function has these properties. Provide counterexamples when you want to show that a function does not have these properties.

- Show all the steps when determining an inverse function, starting with the definition of the inverse function. Should an inverse function f^{-1} be determined, the full description of the function should be provided in the end, eg $f^{-1}: A \rightarrow B$ is the function defined by $f^{-1}(y) = \ldots$.

Question A.3
Use a truth table to determine whether the following proposition is a tautology, a contradiction or neither:

$$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \vee r)$$

Question A.4
Provide a counterexample to indicate that the following statement is not true for all $x \in \mathbb{Z}^+$:

If x is even or divisible by 3, then $x^2 + 2x - 3$ is odd.

Question A.5
Prove by contradiction (Reductio ad absurdum) that for all $n \in \mathbb{Z}$ if $n^2 + 6n$ is odd, then n is odd.

Hint: Refer to the study guide, section 10.2.2.

Start your proof as follows:

Assume $n^2 + 6n$ is odd. *(Initial assumption)*

Now there are just two possibilities for n: either n is odd or n is even.
The former is the good possibility, so let us eliminate the latter.

Suppose n is even. *(This is our questionable assumption)*
i. e. n =……… ① *(Give the general form of an even number)*
Now substitute ① into $n^2 + 6n$.
.
.
.
You can continue with the proof to arrive at some contradiction and then come to a conclusion.

Question A.6
Prove by contrapositive that for all $n \in \mathbb{Z}$ if $n^2 + 6n$ is odd, then n is odd.

Hint: Refer to the study guide, section 10.2.3.

Start your proof as follows:
Suppose n is even,
ie n =……… ① (Give the general form of an even number.)
.
.
.
Use ① to show that $n^2 + 6n$ is even.

Question A.7
Consider the following statement:

If $x^3 = x$ then $x^2 = 1$, for all $x \in \mathbb{Z}$.

If the statement appears to be true, give a direct proof; if not, give a counterexample.

Question A.8
Provide a direct proof to show that the following statement holds for all $n \in \mathbb{Z}$.

If n is even, then the product of n and its successor is even.
Hint: The successor of an integer n is the integer n +1.

Question A.9
Consider the following statement:
If 3n is odd, then n is odd.

(i) Write the contrapositive statement.
(ii) Prove the original statement by proving that its contrapositive is true.

Question A.10
Prove by contradiction that, for all $n \in \mathbb{Z}$, if n^2 is odd then n is odd.

SECTION B

Question B.1

Given vector u = (1, 2, 3, 4, 5) and vector v = (5, 4, 3, 2, 1), which one of the following represents the sum u + v?

1. (5, 8, 9, 8, 5)
2. 35
3. (6, 6, 6, 6, 6)
4. (4, 2, 0, 2, 4)

Question B.2

Given vector u = (1, –3, 5) and vector v = (0, –1, 1), which one of the following represents the dot product uv?

1. 8
2. (0, 3, 5)
3. (1, –4, 6)
4. 2

Question B.3

Given matrix A = $\begin{bmatrix} 1 & -1 \\ 2 & 2 \\ 3 & -3 \end{bmatrix}$ and matrix B = $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, which one of the following represents AB?

1. $\begin{bmatrix} 7 \\ -7 \end{bmatrix}$

2. $\begin{bmatrix} 7 & -7 \end{bmatrix}$

3. $\begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 6 & -6 \end{bmatrix}$

4. It is not possible to determine AB.

Question B.4

Let p, q and r be simple declarative statements. Which alternative provides the truth values for the last column of the following table?

p	q	r	[(¬ p) → (q ∨ r)] → p
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

1.

1.

→
T
T
T
F
T
T
T
F

2.

→
F
F
F
T
T
T
T
T

3.

→
T
T
T
T
T
F
T
F

4.

→
T
T
T
T
F
F
F
F

Question B.5

Compile a truth table for the following expression:

$(p \vee q) \vee \neg [p \vee (q \wedge r)]$

From the truth table, it is clear that the expression is

1. only a tautology
2. only a contradiction
3. neither a tautology nor a contradiction
4. both a tautology and a contradiction

Question B.6

Use the property of double negation and De Morgan's properties to rewrite the following expression as an equivalent statement that does **not** have the not symbol (¬) outside parentheses.

$\neg [(\neg p \vee q) \wedge \neg (q \wedge \neg r)]$

1. $(p \wedge \neg q) \vee (\neg q \vee r)$
2. $(p \wedge \neg q) \vee (q \wedge \neg r)$
3. $(\neg p \vee q) \vee (q \wedge \neg r)$
4. $(p \wedge \neg q) \wedge (q \wedge \neg r)$

Question B.7

Let *p* denote the expression "$x^2 - 9 < 0$" and q denote the expression "$y^2 - 4 < 0$". Use p and q along with connectives $\wedge$, $\vee$ or $\neg$ to write down the following statement in propositional logic:

"$x^2 - 9$ is negative if and only if $y^2 - 4$ is negative."

1. $p \wedge (\neg p \vee \neg q)$
2. $(\neg p \vee q) \wedge (p \vee \neg q)$
3. $(\neg p \wedge q) \vee (p \wedge \neg q)$
4. $p \vee (\neg p \wedge \neg q)$

Question B.7

Which one of the following quantified statements is NOT TRUE?

1. There exists an integer x such that $x > 1$ and $x < 4$ if $x^2 - 5x + 6 = 0$.
2. There exists an integer x such that $x < 0$ and $x^2 + 4 = 0$.
3. There exists an integer $x \in \{10, 100, 1000\}$ such that $x^2 \in \{10, 100, 1000\}$.
4. For all integers $x \in \{10, 100, 1000\}$, x^2 is an even number.

Question B.9

Consider the following expression:

$[(p \wedge q) \vee \neg r] \to \neg [(q \vee \neg r) \to (\neg p \wedge r)]$

Which one of the following expressions is the converse of the expression above?

1. $[(q \vee \neg r) \to (\neg p \wedge r)] \to \neg [(p \wedge q) \vee \neg r]$
2. $\neg [(\neg p \wedge r) \to (q \vee \neg r)] \to [(p \wedge q) \vee \neg r]$
3. $\neg [(q \vee \neg r) \to (\neg p \wedge r)] \to [(p \wedge q) \vee \neg r]$
4. $[(q \vee \neg r) \to (\neg p \wedge r)] \to [(p \wedge q) \vee \neg r]$

Question B.10

Consider the following expression with $x \in \mathbb{Z}$:

If $x^2 + 2x - 3 = 0$, then $x < 2$.

Which one of the following expressions is the contrapositive of the expression above?

1. If $x \geq 2$, then $x^2 + 2x - 3 \neq 0$
2. If $x^2 + 2x - 3 \neq 0$, then $x \geq 2$
3. If $x < 2$, then $x^2 + 2x - 3 = 0$
4. If $x^2 + 2x - 3 = 0$, then $x \geq 2$

SECTION C

Question C.1

Given vector u = (3, 0, –2, 1, 4) and vector v = (1, –3, 2, 5). Which one of the following represents the sum u + v?

1. (4, –5, 3, 9)
2. (4, –3, 0, 6, 4)
3. (4, 0, –5, 3, 9)
4. Not one of alternatives 1, 2 and 3 is correct.

Question C.2

Given vector u = (–3, 4, 2) and vector v = (2, –1, 5). Which one of the following represents the product u · v?

1. (–6, –4, 10)
2. –21
3. 0
4. (–1, 3, 7)

Question C.3

Given matrix A = $\begin{bmatrix} 1 & 0 & -3 \\ 2 & -5 & 1 \end{bmatrix}$ and matrix B = $\begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$, which one of the following represents AB?

1. $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$
2. $\begin{bmatrix} 4 & 3 \end{bmatrix}$
3. $\begin{bmatrix} 4 & 8 \\ 0 & -5 \\ 0 & 0 \end{bmatrix}$
4. It is not possible to determine AB.

Question C.4

Which alternative provides the truth values for '∨' in the following table?

p	q	r	p ∨ [(¬ p) ∧ (q→ r)]
T	T	T	
T	T	F	
T	F	T	
T	F	F	
F	T	T	
F	T	F	
F	F	T	
F	F	F	

1.

$\vee$
T
T
T
T
T
T
T
T

2.

$\vee$
T
T
T
T
T
F
T
T

3.

$\vee$
T
T
T
T
T
F
T
F

4.

$\vee$
T
T
T
T
T
F
F
F

Question C.5

Compile a truth table for the following expression:

$$[(p \rightarrow q) \wedge r] \leftrightarrow [(q \rightarrow p) \wedge r]$$

From the truth table, it is clear that the expression is:

1. only a tautology
2. only a contradiction
3. neither a tautology nor a contradiction
4. both a tautology and a contradiction

Question C.6

Use the property of double negation and De Morgan's properties to rewrite the following expression as an equivalent statement that does **not** have the not symbol (¬) outside parentheses.

$$\neg [(p \wedge \neg q \wedge \neg r) \vee (\neg p \vee \neg r)]$$

1. $(p \vee \neg q \vee \neg r) \wedge (\neg p \wedge \neg r)$
2. $(\neg p \wedge q \wedge r) \vee (p \vee r)$
3. $(p \wedge \neg q \wedge \neg r) \vee (\neg p \vee \neg r)$
4. $(\neg p \vee q \vee r) \wedge (p \wedge r)$

Question C.7

Which of the following pairs of propositions are equivalent?

A. $\neg [(p \vee q) \wedge \neg r]$ and $(\neg p \wedge \neg q) \vee r$
B. $(p \vee q) \rightarrow r$ and $\neg (\neg p \wedge \neg q) \vee r$
C. $\neg [(p \vee q) \wedge \neg r]$ and $(\neg p \vee r) \wedge (\neg q \vee r)$

Alternatives:
1. Only A
2. Only A and C
3. Only B and C
4. A, B and C

Question C.8

Let P(x) denote '$x^2 - 3x + 2 < 0$'. Write the following statement using some quantifiers and predicates, as well as a negation:

It is not the case that $x^2 - 3x + 2 < 0$ for all integers x.

1. $\forall x \in \mathbb{Z}, \neg P(x)$
2. $\neg\forall x \in \mathbb{Z}, P(\neg x)$
3. $\exists x \in \mathbb{Z}, \neg P(x)$
4. $\exists x \in \mathbb{Z}, P(\neg x)$

Question C.9

Consider the following expression:

$$\neg [(p \wedge q) \rightarrow (q \vee \neg r)] \rightarrow [(\neg p \wedge r) \vee \neg q]$$

Which one of the following expressions is the contrapositive of the expression above?

1. $\neg [(\neg p \wedge r) \vee \neg q] \rightarrow [(p \wedge q) \rightarrow (q \vee \neg r)]$
2. $\neg [(\neg p \wedge r) \vee \neg q] \rightarrow \neg [(p \wedge q) \rightarrow (q \vee \neg r)]$
3. $\neg [(\neg p \wedge r) \vee \neg q] \rightarrow [\neg (q \vee \neg r) \rightarrow \neg (p \wedge q)]$
4. $\neg [(\neg p \wedge r) \vee \neg q] \rightarrow \neg [\neg (q \vee \neg r) \rightarrow \neg (p \wedge q)]$

Question C.10

Consider the following expression where $x \in \mathbb{Z}$:

If $x > 0$, then $x^2 - 4x - 5 \leq 0$.

Which one of the following expressions is the converse of the expression above?

1. If $x^2 - 4x - 5 > 0$, then $x \leq 0$
2. If $x^2 - 4x - 5 \leq 0$, then $x > 0$
3. If $x \leq 0$, then $x^2 - 4x - 5 \leq 0$
4. If $x \leq 0$, then $x^2 - 4x - 5 > 0$

17 2009 Exam paper and solutions

You will find many old exam papers that you can work through on myUnisa. In this section, we give the 2009 exam paper and the solutions. When you have prepared for the exam, do the paper to the best of your ability without looking at the solutions. Then mark your paper using the solutions. Work through the solutions for all the answers that you got wrong. Go back to the study material to study the concepts that you still are not familiar with. Now rewrite the paper and mark yourself again. Continue this process until you feel confident writing the paper.

Contact your e-tutor for a discussion of any other exam paper on myUnisa.

2009 EXAM PAPER

SECTION 1 : TRUTH TABLES AND SYMBOLIC LOGIC
Write your answer to each question out in full in the answer book. [16 marks]

Question 1.1 **(7)**

a) Give the truth table for the following symbolic compound statement:

$$((p \vee q) \wedge \neg (p \rightarrow r) \wedge (q \rightarrow r)) \leftrightarrow r$$

b) Is the expression a tautology, a contradiction or neither?

Question 1.2 **(5)**

Write down the negation of the following expression. Simplify the expression so that the *not* symbol (¬) does not occur to the left of any quantifier. The *not* symbol may also not occur outside of parentheses.

$$\forall x \in \mathbb{Z}, (x > 0) \vee (x \leq 12)$$

N.B.: Show all steps. Which is true, the original statement, the negation of it or neither?

Question 1.3 **(4)**

Given the implication $P(x) \rightarrow Q(x)$, write down

a) the converse, and
b) the contrapositive

of the implication. After you have written down the converse and the contrapositive, simplify the expressions where possible, e.g. where negations are involved.

(**Hint:** Use the identity $A(x) \rightarrow B(x) = \neg A(x) \vee B(x)$ in the simplification process.)

SECTION 2 : MATHEMATICAL PROOFS
Write your answer to each question out in full in the answer book. [12 marks]

Question 2.1 **(3)**

Provide a counterexample to prove that the following statement is not true for all integers

$x \geq 0$.

If $x \geq 0$ then $x^2 - 2x + 3$ is a multiple of 3.

Question 2.2 **(4)**

Provide a direct proof to show that, for all $n \in \mathbb{Z}$,

if n is even, then $n^3 + 2n$ is divisible by 4.

Question 2.3 **(5)**

Provide a *proof by contrapositive* to prove that, for all $x \in \mathbb{Z}$, if $x^2 + x + 1$ is even, then x is even.

SECTION 3 : SET THEORY
Write your answer to each question out in full in the answer book. [28 marks]

Question 3.1 **(10)**

Use Venn diagrams to determine whether, for all sets X, Y and W.

$(X \cap Y) \cup (W - X) = (X \cup W) \cap (Y + X)$.

If it seems to be an identity, give a proof. Otherwise, give a counterexample.

Question 3.2 **(8)**

Determine whether or not the following holds for all sets X, Y and W:

$(W \times X') \cap (W \times Y') = W \times (X \cup Y)'$

(**Hint:** Start your proof with $(a,b) \in W \times (X \cup Y)'$)

Question **(10)**

A car manufacturer uses 32 robots. Of these robots

24 are used for welding,
10 are used for painting, and
10 are used for sanding.

Furthermore, some of these robots can be programmed as follows:

5 to weld and to paint,
6 to weld and to sand, and
4 to sand and to paint.

Answer the following two questions:

a) How many robots can perform all three functions?
b) How many can only sand?

SECTION 4 : SETS AND RELATIONS (Multiple-choice questions)
Each question comprises 2 marks.
Choose only one alternative per question and then write the question number and the alternative that you regard as the correct answer in the answer book. [14 marks]

Suppose U = {1,2,3,4,5,6}, A = {1,3,5} and B ={1,2,4}. Answer questions 4.1 to 4.4 using these three sets.

Question 4.1
Which one of the following sets represents A + B?

1. {1,2,3,4,5}
2. {3,5}
3. {2,5,9}
4. {2,3,4,5}

Question 4.2
Which one of the following sets represents A - B?

1. {0,1,1}
2. {3,5}
3. {1}
4. {1,2,3,4,5}

Question 4.3
Which one of the following sets represents U - A'?

1. {2,4}
2. {1,2,3,4,5}
3 {1,3,5}.
4. ∅

Question 4.4
Which one of the following sets represents A' ∩ B'?

1. {6}
2. {1,2,3,4,5}
3. {1}
4. {2,3,4,5}

Question 4.5
Let A = {∅, {1},{3}}. Which one of the following sets represents the power set of A, ie $\mathcal{P}$(A)?

1. { ∅, {∅}, {{1}}, {{3}} }
2. { ∅, {1}, {3} }
3. { ∅, {∅}, {{1}}, {{3}}, {∅,{1}}, {∅,{3}}, {{1},{3}}, {∅,{1},{3}} }
4. { {∅}, {{1}}, {{3}}, {{1},{3}} }

Question 4.6
Consider the following relation on T = {1, 2, 3}: R = {(1, 1), (2, 1), (3, 2), (3, 1)}.
Which one of the following statements regarding the relation R is TRUE?

1. R is reflexive on T and antisymmetric.
2. R is symmetric and transitive.
3. R is irreflexive and antisymmetric

4. R is antisymmetric and transitive.

Question 4.7

Consider the following relation on T = {1, 2, 3}: S = {(1, 2), (2, 3), (1, 3)}.
Which one of the following statements regarding the relation S is TRUE?

1. S is a partial order on T.
2. S is a strict total order on T.
3. S is an equivalence relation on T.
4. S is a strict equivalence relation on T.

SECTION 5 :FUNCTIONS AND RELATIONS
Write your answer to each question out in full in the answer book. [30 marks]

Question 5.1 **(7)**

Let A = {1, 2, 3} and B = {1, 2, {a}, {b}}.

a) Provide an example of a relation on B that satisfies trichotomy.

b) Provide an example of a partition of A.

c) Consider the following relation S on the set B: S = { (1, 2), ({b}, {a}), (2, {b}) }

(i) Determine the inverse relation S^{-1}.

(ii) Determine the composition $S;S^{-1}$ (i.e. $S^{-1} \circ S$).

Question 5.2:

This question in the paper is included as question E in self-assessment 2, therefore, we provide another question: Let X be the set of letters of the English alphabet and let R be a relation on X such that (x, y) $\in$ R iff x has as its value a letter coming before the value of y in the usual alphabet rhyme. By doing the appropriate tests, prove that R is a total strict order relation. (Note: Do not use examples in your proof.)

Question 5.3 **(7)**

Suppose f: $\mathbb{Z} \rightarrow \mathbb{Z}$ is defined by the rule $f(x) = x - 2$, and
g: $\mathbb{Z} \rightarrow \mathbb{Z}$ is defined by the rule $g(x) = x^2 - 1$.

a) Is f injective (one-to-one)? Justify your answer fully.

b) Determine ran(f) (i.e. the range of f).

c) Determine the rule $f \circ g(x)$ for the composition $f \circ g$.

Question 5 **(6)**

Consider the binary operation $\cup$ (union) in the following table:

$\cup$	$\emptyset$	$\{\emptyset\}$
$\emptyset$	$\emptyset$	

		{∅ }
{∅}	{∅}	{∅}

a) Does the operation have an identity element? Justify your answer fully.
b) Write the operation in list notation.

SOLUTIONS OF 2009 EXAM PAPER

SECTION 1 [16 MARKS]

Question 1.1 **(7)**

(a)

P	**q**	**r**	**(p∨q)**	**¬(p→r)**	**q→r**	**(p∨q) ∧ (¬(p→r)) ∧ (q→r)**	**↔**	**r**
T	T	T	T	F (T)	T	F	F	T
T	T	F	T	T (F)	F	F	T	F
T	F	T	T	F (T)	T	F	F	T
T	F	F	T	T (F)	T	T	F	F
F	T	T	T	F (T)	T	F	F	T
F	T	F	T	F (T)	F	F	T	F
F	F	T	F	F (T)	T	F	F	T
F	F	F	F	F (T)	T	F	T	F

↑

(b) The final column in the truth table shows that the expression is neither a tautology nor a contradiction.

Question 1.2 **(5)**

$\neg(\forall x \in \mathbb{Z}, (x > 0) \vee (x \leq 12))$

$\equiv \exists x \in \mathbb{Z}, \neg((x > 0) \vee (x \leq 12))$

$\equiv \exists x \in \mathbb{Z}, \neg(x > 0) \wedge \neg(x \leq 12)$

$\equiv \exists x \in \mathbb{Z}, (x \leq 0) \wedge (x > 12)$

Question 1.3 **(4)**

Given: $P(x) \rightarrow Q(x)$

(a) Converse: $Q(x) \rightarrow P(x)$

(b) Contrapositive: $\neg Q(x) \rightarrow \neg P(x)$

i.e.$\neg\neg Q(x) \vee \neg P(x)$

i.e. $Q(x) \vee \neg P(x)$

SECTION 2 [12 MARKS]

Question 2.1 **(3)**

Let $x = 4$, then

$x^2 - 2x + 3 = (4)^2 - 2(4) + 3$
$= 16 - 8 + 3$
$= 11$, which is not a multiple of 3.

Question 2.2 **(4)**

Let $n = 2k$ for some $k \in \mathbb{Z}$,
then $n^3 + 2n = (2k)^3 + 2(2k)$
$= 8k^3 + 4k$
$= 4(2k^3 + k)$, which is divisible by 4.

Question 2.3 **(5)**

Suppose x is an odd integer, i.e. $x = 2k + 1$ for some $k \in \mathbb{Z}$.

Now $x^2 + x + 1 = (2k + 1)^2 + (2k + 1) + 1$
$= 4k^2 + 4k + 1 + 2k + 1 + 1$
$= 4k^2 + 6k + 2 + 1$
$= 2(2k^2 + 3k + 1) + 1$

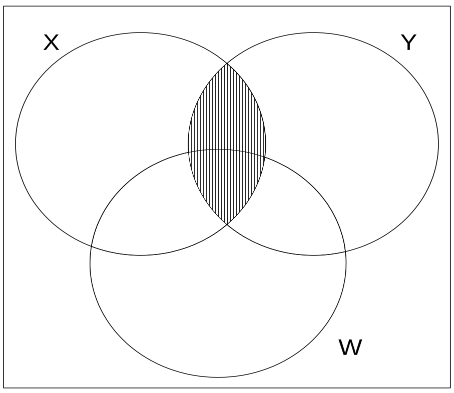
ie $x^2 + x + 1$ is an odd number.

SECTION 3 [28 MARKS]

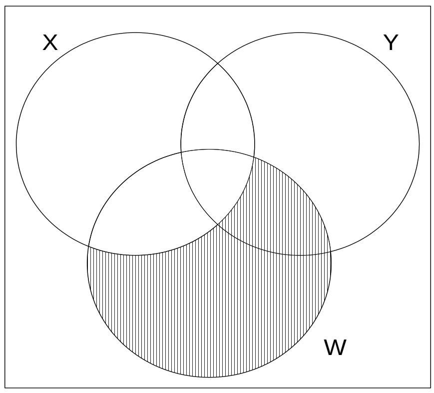
Question 3.1 **(10)**

Left-hand side:

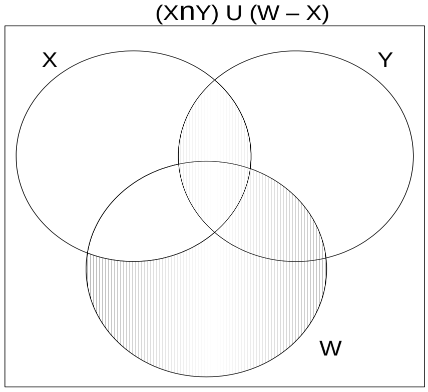
$X \cap Y$



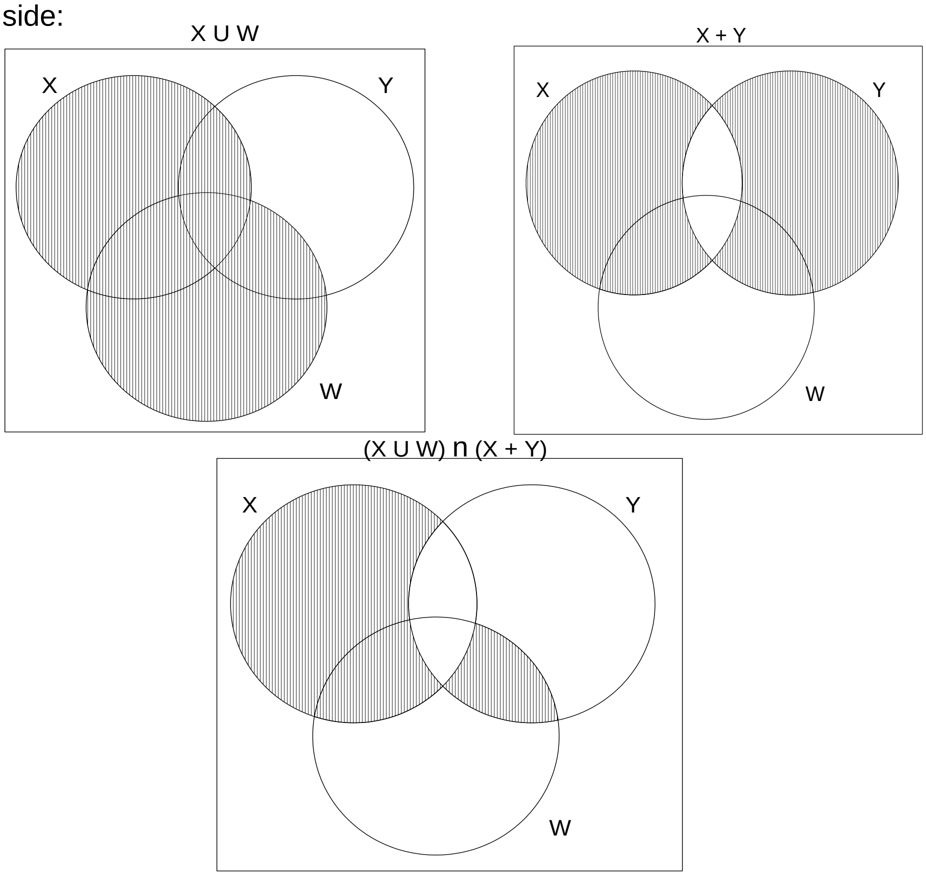
$W - X$



$(X \cap Y) \cup (W - X)$



Right-hand side:



From the Venn diagrams, it is clear that these are not equivalent sets because the colours in areas in the final diagrams differ. A counterexample is required. Note that we use *small sets* in the counterexample.

Let U = {1, 2, 3, 4, 5}, X = {1, 2, 3}, Y = {3} and W = {3, 4, 5}

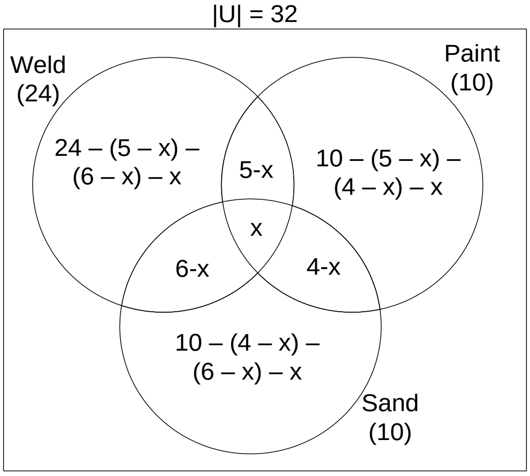
$(X \cap Y) \cup (W - X) = \{3\} \cup \{4, 5\}$
$= \{3, 4, 5\}$

$(X \cup W) \cap (Y + X) = \{1, 2, 3, 4, 5\} \cap \{1, 2\}$
$= \{1, 2\}$

Question 3.2 **(8)**

(a, b) $\in W \times (X \cup Y)'$
iff $a \in W$ and $b \in (X \cup Y)'$
iff $a \in W$ and $b \notin (X \cup Y)$
iff $a \in W$ and ($b \notin X$ and $b \notin Y$)
iff $a \in W$ and ($b \in X'$ and $b \in Y'$)
iff ($a \in W$ and $b \in X'$) and ($a \in W$ and $b \in Y'$)
iff $(a, b) \in (W \times X')$ and $(a, b) \in (W \times Y')$
iff $(a, b) \in (W \times X') \cap (W \times Y')$

Question 3.3 **(10)**



(a) $32 = (24 - (5 - x) - (6 - x) - x) + (10 - (5 - x) - (4 - x) - x) + (10 - (4 - x) - (6 - x) - x) + (5 - x) + (6 - x) + (4 - x) + x$

$= 24 + (10 - 5) + (10 - 10) + x$

$= 24 + 5 + 0 + x$

$= 29 + x$

Thus, $x = 3$ performs all three functions.

(b) $(10 - (4 - x) - (6 - x) - x)$

$= 10 - 1 - 3 - 3$

$= 3$

Thus, 3 can only sand.

SECTION 4 [14 MARKS]

U = {1, 2, 3, 4, 5, 6}, A = {1, 3, 5} and B = {1, 2, 4}.

Question 4.1 **Alternative 4**

A + B = {1, 3, 5} + {1, 2, 4}
= {2, 3, 4, 5}

Question 4.2 **Alternative 2**

A – B = {1, 3, 5} – {1, 2, 4}
= {3, 5}

Question 4.3 **Alternative 3**

U – A' = {1, 2, 3, 4, 5, 6} – {2, 4, 6}
= {1, 3, 5}

Question 4.4 **Alternative 1**

$A' \cap B'$ = {2, 4, 6} $\cap$ {3, 5, 6}
= {6}

Question 4.5 **Alternative 3**

A = {∅, {1}, {3}}. Determine all the subsets of A – these are all the members of $\mathcal{P}$(A).

$\mathcal{P}$(A) = { ∅, {∅}, {{1}}, {{3}}, {∅,{1}}, {∅,{3}}, {{1},{3}}, {∅,{1},{3}} }

Question 4.6 **Alternative 4**

R = {(1,1), (2,1), (3,2), (3,1)} is antisymmetric and transitive.

Question 4.7 **Alternative 2**

S = {(1,2), (2,3), (1,3)} is a strict total order on T because it is irreflexive, antisymmetric and transitive, and also satisfies trichotomy.

SECTION 5 [30 MARKS]

Question 5.1 **(7)**

a) B = {(1, 2), (1, {a}), (1, {b}), (2, {a}), (2, {b}), ({a}, {b})}.

b) {{1, 2, 3}}

c) (i) S^{-1} = { (2, 1), ({a}, {b}), ({b}, 2) } (ii) $S;S^{-1}$ = { (1, 1), ({b}, {b}), (2, 2) }

Question 5.2 **(10)**

To be a strict total order relation, R must be **irreflexive**, **antisymmetric**, **transitive** and satisfy **trichotomy**.

<u>Irreflexivity</u>: *Is it true that (x,x) ∉ R for all x ∈ X?*

R is irreflexive: Let x be any letter of the alphabet. No letter x of the alphabet comes before itself in the usual ordering of the alphabet.

Thus, (x, x) ∉ R for all x ∈ X.

Thus, R is irreflexive.

<u>Antisymmetry</u>: *Suppose (x, y) ∈ R and x ≠ y, does it follow that (y, x) ∉ R?*

Suppose (x, y) ∈ R. Then x comes before y in the usual ordering of the alphabet.

But then, y does not come before x. Therefore, (y, x) ∉ X.

Thus, R is antisymmetric.

<u>Transitivity</u>: *Suppose (x, y) ∈ R and (y, z) ∈ R. Does it follow that (x, z) ∈ R?*

Suppose (x, y) ∈ R and (y, z) ∈ R then x comes before y in the alphabet and y comes before z in the alphabet.

Therefore, x must also come before z and, hence, (x, z) $\in$ R.

Thus, R is transitive.

Trichotomy:
For all x, y $\in$ X, either x comes before y in the alphabet or y comes before x.
So, either (x, y) $\in$ R or (y, x) $\in$ R. R, therefore, satisfies trichotomy.

It follows that R is a strict total order on the alphabet.

Question 5.3 **(7)**

a) Suppose $f(u) = f(v)$ for some $u, v \in \mathbb{Z}$.
Then $u - 2 = v - 2$
ie $u = v$
Thus, f is injective.

b) $\text{ran}(f) = \{y \mid \text{for some } x \in \mathbb{Z}, y = x - 2\}$
$= \{y \mid \text{for some } x \in \mathbb{Z}, x = y + 2\}$
$= \{y \mid y + 2 \text{ is an integer}\}$
$= \mathbb{Z}$

c) $f \circ g(x) = f(g(x))$
$= f(x^2 - 1)$
$= (x^2 - 1) - 2$
$= x^2 - 3$

Question 5.4 **(6)**

a) The identity element is $\emptyset$, because:
$\emptyset \cup \emptyset = \emptyset = \emptyset \cup \emptyset$
$\emptyset \cup \{\emptyset\} = \{\emptyset\} = \{\emptyset\} \cup \emptyset$

b) The operation in list notation:
$\{((\emptyset, \emptyset), \emptyset), ((\emptyset, \{\emptyset\}), \{\emptyset\}), ((\{\emptyset\}, \emptyset), \{\emptyset\}), ((\{\emptyset\}, \{\emptyset\}), \{\emptyset\})\}$

18 Final Word

When preparing for the exam, please work through all of the following:

- Study guide, including all the activities
- Learning units on myUnisa or in this letter
- Sample assignments 1 – 3
- Assignments 1 – 3 and solutions
- Additional self-assessment assignments in Tutorial Letter 102
- CAI tutorial
- Old examination papers on myUnisa
- PowerPoint presentation in **Additional Resources** on myUnisa, summarising the course content

We trust that you have enjoyed this semester studying theoretical computer science in the context of discrete mathematics, and that you will find that the concepts in this module are weaved into other modules in this school.