MO001/4/2018

Mechanics II

APM1612

Semesters 1 & 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This is a blended **module**; therefore it will also be available on myUnisa.

Please activate your *myUnisa* and *my*Life email in order to access the module site. The Code for this module is APM1612.

This also contains important information about the Learning Units 1 to 5 of your module APM1612.

BAR CODE

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1: WELCOME

1.1 Welcome

We wish to welcome you to the module APM1612 (Mechanics II). We hope that you will enjoy studying this module.

This module is **taught online** and I want to encourage all students of APM1612 to study this module in this way! Apart from Tutorial letter 101 you will also receive other tutorial letters during the semester – for instance, containing feedback on assignments. These tutorial letters (102, 103 and so on; 201, 202 and so on) will not necessarily be available at the time of registration. These tutorial letters will be despatched to you as soon as they are available or needed. They will appear on the myUnisa site.

You will receive a number of tutorial letters during the semester. A tutorial letter is our way of communicating with you about teaching, learning and assessment. Right from the start we would like to point out that **you must read all the tutorial letters** you receive during the semester **immediately and carefully**, as they always contain important, and sometimes urgent information.

There is **no prescribed textbook** and **no recommended readings** for this module. All the material is contained in the learning units.

The myUnisa learning management system is Unisa's online campus that will help you to communicate with your lecturers, with other students and with the administrative departments of Unisa - all through the computer and the internet.
To go to the myUnisa website, start at the main Unisa website, http://www.unisa.ac.za, and then click on the "Login to myUnisa" link on the right-hand side of the screen. This should take you to the myUnisa website. You can also go there directly by typing in http://my.unisa.ac.za. Please consult the publication my Studies @ Unisa which you received with your study material for more information on myUnisa.

1.2 **Purpose of the Module**

This module deals with the dynamics of systems of particles and rigid bodies under the influence of forces. Broadly, the outcomes are as follows.

- Finding the centre of mass of systems of particles, rigid bodies, and more general systems,
- Finding the moments of inertia of various objects, about given axes of rotation,
- Understanding and applying the equations of translation and rotation of bodies, to analyse the motion of objects and to solve problems,
- Understanding and applying the concepts of kinetic and potential energy, and using energy conservation methods to solve problems.

2: INTRODUCTION

2.1 Structure of the module

There are FIVE main Learning Units for this module

LEARNING UNIT 1: PRELIMINARIES
LEARNING UNIT 2: THE CENTRE OF MASS
LEARNING UNIT 3: ROTATION
LEARNING UNIT 4: ROTATION AND TRANSLATION

LEARNING UNIT 5: ENERGY METHODS

2.2 Contact details

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0003

2.3 What do I need to do to pass this module?

This module is a semester module, so please start studying as soon as you receive your study Material.

Very important: Please refer to Tutorial letter 101 section 6 Module specific study plan for detailed instructions regarding your studies.

You need to do and submit all assignments by the due date.

Work through tutorial letter 102 which is a workbook with exercises to practice each section.

Then for exam preparation you need to work through past examination papers.

3: OUTCOMES AND ASSESSMENT CRITERIA

Outcome 1: Calculate centres of mass of systems of particles, rigid bodies of continuous structure, and systems consisting of a combination of these.

Assessment criteria:

1.1. Centre of mass is calculated correctly, in either coordinate or vector format, as appropriate, and either summation or integration is selected correctly according to the situation.
1.2. A reference frame (Cartesian coordinate system) is introduced, if it is not already given.
1.3. The position of the centre of mass can be described in relation to the system or body.

1.4. Standard techniques such as calculating centres of mass of laminas bounded by functions and solids of revolution are applied correctly
1.5. Rules applying to combined objects, objects with parts removed, and simplifying tactics such as planes or axes of symmetry are used correctly where appropriate.

Outcome 2: Calculate moments of inertia of systems of particles, rigid bodies and systems.

Assessment criteria:

3.1. The axis of rotation is understood correctly, and the moments of inertia can be calculated correctly from first principles, using summation or integration as necessary.
3.2. Simplifying tactics such as the parallel and perpendicular axis theorems, symmetry, combined bodies and bodies with parts removed, and utilising previously proved results are used correctly when appropriate.

Outcome 3: Apply the equation of translation of centres of mass of systems

Assessment criteria:

3.1. A sketch of the system is drawn, with all the relevant forces marked in, with correct directions and points of action.
3.2. The equation of motion for describing the motion of the centre of mass is written down correctly, both in vector form and in component form.
3.3. Forces are classified correctly as internal or external to a given system.
3.4. The case of non-acceleration of the centre of mass is recognised when no external forces act on the system.

Outcome 4: Apply the equation of rotation of a rigid body or system

Assessment criteria:

4.1. A sketch of the system is drawn, with all the relevant forces marked in with correct directions and points of action.
4.2. Vector products of vectors expressed in terms of unit vectors are evaluated correctly.
4.3. The angular momentum of a single particle and of systems of particles about a given point is calculated correctly. The special nature of the angular momentum of a rigid body rotating about a fixed axis can be explained.
4.4. The connection between the moments of forces and the angular momentum of a system of particles, and how the equation of rotation of a rigid body rotating about a fixed axis follows from this, is explained.
4.5. The moment of a given force, acting at a given point P, about a given reference point Q, is calculated correctly.
4.6. The equation of motion for describing the rotation of a rigid body rotating about a given axis is written down correctly, both in vector form and in terms of unit vectors.

Outcome 5: Calculate kinetic and potential energies for given systems

Assessment criteria:

5.1. The meaning of the work done by a force is defined, and how the concepts of kinetic and potential energy follow from this is explained.
5.2. The gravitational potential energy of the system is calculated correctly in relation to a given zero energy level.
5.3. The kinetic energy for a system, undergoing translation, rotation or a combination of translation and rotation is calculated correctly.

Outcome 6: Apply the principle of energy conservation in a described situation

Assessment criteria:

6.1. Forces acting in the system are correctly classified as conservative or non-conservative.
6.2. Whether the conservation principle of mechanical energy applies or not in a given situation is judged correctly and justified.
6.3. Given an initial and a final situation where the energy conservation does apply,
  a) a suitable zero potential energy level is selected appropriately,
  b) initial and final potential energies as well as initial and final kinetic energies are calculated correctly,
  c) The energy equation is written down correctly.

Outcome 7: Apply the above skills to solve problems in mechanics.

Assessment criteria:

7.1. The system or situation in the question statement is understood correctly and described in a sketch.
7.2. All relevant forces acting on the system are identified and applied correctly (with correct directions and points of action)
7.3. An appropriate solution method is selected by applying correctly one following laws/principles, or a combination thereof: The equation for the translation of the centre of mass of a system; the equation for the rotation of a rigid body around a fixed axis or about an axis through its centre of mass; the energy conservation principle.
7.4. The motion of a rigid body is correctly classified as pure rotation, pure translation, or a combination of both; and the equations of motion are selected accordingly.
7.5. Where necessary, a suitable coordinate system is selected, and all calculations are done correctly in relation to the selected reference system.
7.6. In applying the energy conservation method, an initial and a later situation are selected appropriately.
7.7. Where applicable, the rolling condition is applied correctly.
7.8. A solution to the problem is found in terms of values given in the problem statement, and correctly interpreted (where applicable).

4: ASSESSMENT PLAN

Refer to section 8 of Tutorial letter 101 for more detailed information on the assessment plan.

4.1 Assignments

Assignments are seen as part of the learning material and assessment for this module. As you do the assignments, discuss with fellow students or tutor, you are actively engaged in learning. It is therefore important that you complete all the assignments. There are six assignments for this module.

4.2 Examinations

The **examination mark** is the percentage mark you get in the examination.
The examination mark contributes 80% to the final mark, and the semester mark contributes 20%.
You pass the module if your final mark is $\geq$ 50, and you pass it with distinction if your final mark is $\geq$75. There is also a subminimum rule, which says that you must get at least 40% in the examination to pass the module.

5: LEARNING UNITS

You will be required to log on to myUnisa and go to the module site for APM1612. Go into each unit on *myUnisa*. These will be included in Additional Tutorial letters.

There are five main learning units for this module.

LEARNING UNIT 1: PRELIMINARIES
LEARNING UNIT 2: THE CENTRE OF MASS
LEARNING UNIT 3: ROTATION
LEARNING UNIT 4: ROTATION AND TRANSLATION
LEARNING UNIT 5: ENERGY METHODS
Each learning unit will have a theory contents sections and activity sections.

We wish you all the best for your studies.

LEARNING UNIT 1

PRELIMINARIES

CONTENTS OF LEARNING UNIT 1

Study unit 1	Introduction to the module and the study guide
Study unit 2	Background physics

Introduction

This first Learning Unit of the study guide contains, firstly (in Unit 1) an introduction to this module and some information on how to best study this module and, secondly (in Unit 2), some background information in physics which you should already be familiar with. Many important examples are provided accordingly.

After reading through Learning Unit 1, you will be ready to start the real work in Learning Unit 2!

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9 July 2015

Unit 1 INTRODUCTION TO THE MODULE AND THE STUDY GUIDE

This unit tells you about the module and about the study guide, and how best to study this module.

Contents of this unit:

3.1 What this module is all about

3.2 How to work through this study guide

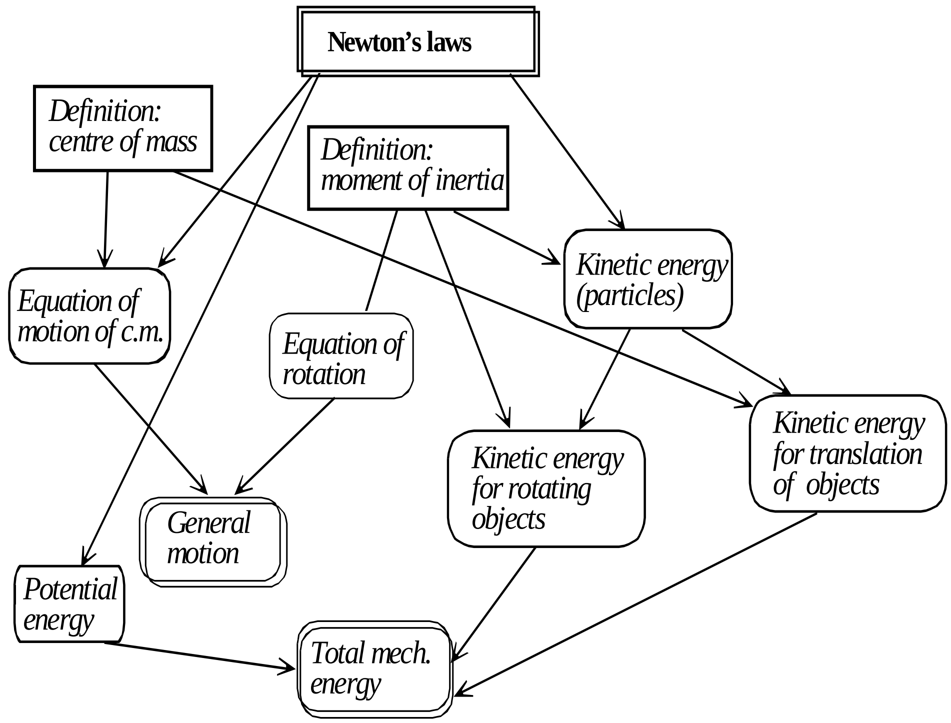
1.1 WHAT THIS MODULE IS ALL ABOUT

Mechanics is a part of physics, and can be defined as the branch of physics which analyses the action of forces on matter. Classical[1] mechanics, which is what we deal with in this module, is one of the main achievements of applied mathematics, and one of the most successful attempts to describe the world around us. And as you might guess, mechanics has proved to be extremely useful! Mechanics does form a very important part of the daily life of all of us: You are applying principles of mechanics every time to open a door, or pick up an object, or play in a see-saw (well, it might have been a while since you last did that) — mechanics comes in everywhere, and a lot of the objects around us, from very simple to quite complex, are built based on an understanding of mechanics.

Mechanics is usually divided into kinematics, statics and dynamics. Kinetics deals with motion as such, with no interest in the forces which cause and modify the motion; statics deals with situations where the actions of forces cancel out so that the object is at rest; and finally, dynamics describes the way that action of forces causes motion. This module deals with rigid body dynamics, more specifically the problem of describing how systems of particles, rigid bodies and systems move under the action of forces. In particular we will spend a lot of time investigating rotational motion. (This is where the concern with rigid bodies, rather than just particles, comes in — particles cannot rotate since they are just point masses!)

All of mechanics is based on Newton's laws of motion, which you probably have come across already. However, what we do in this module is to rewrite those laws in more and more convenient forms, using various concepts such as the centre of mass and moments of inertia, so that we end up with very powerful tools for analysing the motion of objects in various situations. The flow chart below attemts to show how the different concepts we discuss in this module fit together.

[1] as opposed to quantum mechanics, that is!



Hopefully after you have worked through this module, you have been impressed with the power of mathematics and mechanics as tools for understanding the world around us! This module is based on quite simple concepts; if you wish to learn more, you can move on to even more powerful mathematical tools (such as the calculus of variations) as explored in some of our other Applied Mathematics and Physics modules.

1.1.1 Outcomes of this module:

When you have worked through this study guide, you should be able to:

- Calculate centres of mass of systems of particles, rigid bodies of continuous structure, and systems consisting of a combination of these.
- Calculate moments of inertia of systems of particles, rigid bodies and systems.
- Apply the equation of translation of centres of mass of systems.
- Apply the equation of rotation of a rigid body or system.
- Calculate kinetic and potential energies for given systems.
- Apply the principle of energy conservation in a described situation.
- Apply the above skills to solve problems in mechanics.

A more detailed list of the outcomes, with the criteria that will be used in assessing that you have reached them, will be given in Tutorial letter 101 of this module.

To pass this module, you will need to demonstrate that you have achieved these outcomes. Tutorial letter 101 explains how you will demonstrate that you have mastered the outcomes of the module (for instance, by submitting assignments and by writing an examination).

You will master the contents of the module, gaining the necessary knowledge and skills, by working through this study guide. All the necessary information is contained in this study guide, although we have also included a lot of additional material and resources into this module to improve you understanding where necessary. Please see the Tutorial letter 101 on how all these extra resources link to this study guide!

The next section explains how you should work through this study guide, for you to make the most of it!

1.2 HOW TO WORK THROUGH THIS STUDY GUIDE

1.2.1 Some general remarks

Mechanics involves using a fairly small number of definitions and equations to solve problems. The definitions and formulas appear between horizontal lines in the study guide, and if you look through the guide, you will see that there really are not very many of them! But to be able to apply them correctly to solve problems requires an understanding which you can only gain by practice. So, you can only learn this subject by doing lots of exercises yourself. It is important to realise that you can't learn by just reading solutions to problems; you must be able to solve each problem and similar problems on your own!

Studying mechanics does take time. Don't be discouraged if you have trouble understanding a concept or an example at first reading. Similarly, if you are unable to solve a problem the first time, keep on trying! If you feel you really cannot understand something, do contact us, your lecturers, for help.

As with all mathematics and physics, concepts tend to build on other, already introduced concepts. Therefore, you must make sure that you know the previous material well enough before moving on to later topics!

1.2.2 About the study guide

The study guide is divided into five parts, each dealing with one particular topic; and there are 15 study units, plus an appendix about integration and one containing problem-solving strategies. The first Learning Unit, consisting of the two study units that you are going through now, is just and introduction. The content matter you need to master is contained Learning Unit 2 to Learning Unit 5 in units numbered from 3 to 15.

The study guide explains the content matter by deriving and explaining the definitions and results, and illustrates how to use them to solve problems with the help of many worked-out examples. But I should already warn you that you will NOT be able to understand the material if you will just read through it passively! You must study actively, which means that you must really try to understand every step, preferably with a pen and paper on the ready so that you can fill in any missing gaps in calculations or make any other notes for yourself. Indeed you are recommended to dedicate an exercise book to this module, to write your notes and calculations in while working through the study guide.

In particular, when reading through the worked-out examples, you must resist the temptation of just reading through them. When you read through an example, you must make sure that you really understand all the steps in the solution. Ask yourself what rules were applied and why. Whenever a new equation is introduced, you must know exactly where it came from and why it was included! Similarly, you can be sure that each sequence of calculations has a clear purpose – you must understand what that purpose is. Only then will you in future be able to apply the method correctly yourself.

To help you with the necessary active studying, we have included a lot of activities in the study guide. The activities are there to ensure that you have truly mastered the material before you move on to the next section. It is easy to believe that you understand everything when you are just reading through it, so you really should do all the activities to ensure that you do really did understand things correctly, and will be able to apply what you learned in the rest of the study guide! The activities have been carefully selected to guard against common misunderstandings, to ensure that you reach all the outcomes in each unit, and to prepare you for completing the assignment questions. The answer and feedback to each activity is given right after it. Your study material includes a workbook which also contains detailed solutions to many of the activities.

Most of the time the worked examples are followed by an activity which requires you to solve a similar problem. It is recommended that you attempt to solve that problem immediately after going through the worked examples; this is the optimal way to learn from the worked examples, and will ensure that you are ready to tackle similar assignment questions, and ready to move on to the next topic!

It is recommended that you do the activities in writing in your exercise book, so that they are available for future reference. Some activities are simple enough to do in your head, or directly in the study guide, but others, in particular problem-solving ones, will require you to compare you solution to the correct answer, and you might also want to compare your solution to a given model solution, whether it is given in the study guide or the workbook! If you are struggling with the activities in any particular Learning Unit of the study guide, you should make sure to do more exercises from the workbook relating to that section. If you are still having trouble, please do contact your tutor or your lecturer for help! Do this early enough, before you fall behind!

1.2.3 The language of theoretical mechanics

You will soon see that the problems and examples in the study guide are written in a very special kind of language — for instance we will talk about spheres, laminas and cones instead of ordinary everyday objects; and about a rod rotating in a vertical plane about a horizontal axis instead of the more concrete case of a rod fixed to the wall with a nail. Also, most of the time we will discuss a “rod of length a with mass M”, rather than referring to a rod which is 2 metres long and weighs 2.5 kg. We use this abstract language, firstly, because it is more general than if we were to use everyday objects for every example and exercise. Secondly, many of the results that we derive in this study guide only hold for “idealised” objects, such as a rod which has zero thickness. Any real rod that we might imagine has a definite thickness — it might be very thin, having for instance a diameter of only 1 millimeter, but this is still not at all the same thing as zero thickness! By dealing with abstract, theoretical objects we can state very precise results; and we can then always apply these results to everyday objects by assuming that the results hold at least approximately for the real objects as well.

It is one of the objectives of this module to make you familiar with this abstract language of theoretical mechanics. You will have to learn how to recognise and understand keywords describing

- position: flat, horizontal, tangential etc.
- properties: uniform, smooth, massless, thin etc.
- types of motion: smooth, roll without slipping, rotate freely, etc.
- objects: a rod, disc, pulley etc.

If you find the terminology used in the study guide confusing, and have a lot of trouble understanding the problems, you might find it helpful to try the following approaches:

- Try to think of real-life examples of the situations and objects we are talking about — for instance, a coin for a disc, or a ball for a sphere. Using such everyday objects, you can often picture in your mind how a disc or a sphere might move in a certain situation. Or, you could actually do an experiment to see what might happen!
- As you work through the examples of the study guide, draw up a list of these abstract terms and what they mean, with real-life examples. *For instance: horizontal plane — floor, vertical plane — wall, smooth plane — no friction.*
- Read the examples and activity questions carefully! Take care not to assume things that are not specified in the question — for instance, if you are told that a particle is attached to the end of the rod, do not add complications such as assuming that it is attached with a rope.

1.2.4 Problem-solving strategies

To pass this module, you must know and understand certain theoretical principles (the centre of mass and its motion, rotation, kinetic and potential energy and so on), and you must also be able to apply them. You can look through some of the examples and problem solving activities in the study guide to see what kind of applications you should be able deal with. Sometimes this will be very easy— some problems consist simply of a direct application of a definition or a principle that has just been introduced.

But more often you will have to solve **problems** which are more than just straightforward applications of an equation. Typically, a problem will be described in words, rather than in mathematical terms. You will have to introduce the necessary mathematical notation, coordinate systems, and so on by yourself! Also, quite often a problem statement contains no hints of which of the principles you should be applying — and there may be more than one way to solve the problem. Until you get practice in the problem solving, it can be difficult to know where to start.

However, problem solving is one of the skills we wish you to learn in this module. It is a difficult thing to teach, but we would like to suggest to you a strategic way of thinking about problems. In general, there are four separate steps in successful problem solving:

1. **Understanding the problem**
   This should be obvious — before you can start solving a problem, you must correctly understand the situation that is described, what is known and what you are asked to do!

2. **Planning a solution**
   Once you understand the problem, you can decide on how to proceed to solve it. This is of course easier said than done – indeed, this is the tricky bit! You'll need to decide which approach or method applies to the particular problem. Sometimes you'll have to apply several different approaches in the correct order; sometimes more than one method may seem appropriate; and often the correct method may initially seem to be a very unlikely candidate.

3. **Executing the plan**
   This is where you translate the problem into mathematics, write down the equations and solve for the unknown quantities, but actually much more may be involved: you may have to introduce a coordinate system, calculate intermediate results, and so on.

4. **Analysing the solution**
   There are at least two good reasons to take a good look at the solution you have obtained in step 3, rather than just writing it down and rushing to the next problem! Firstly, you

may wish to check your solution to make sure that it is correct. Secondly, if you are solving problems in order to understand a subject, you should see what you can learn from the solution. For instance, does the solution surprise you, or is it as you would have expected?

Of course, if you are solving simple or familiar types of problems, or if you have had lots of practice with problem solving, you may not be consciously aware that you are applying these four steps. It may seem to you that you are jumping straight to step 3 with a statement like *"Let a be the angular acceleration of the disc"*. However, if you think about it, you will realise that you could not have arrived there without first going through steps 1 and 2. Likewise, however much you may try not to, you certainly do learn something from every problem you solve!

As you get better and better at problem solving, you should be able to apply these steps without thinking. However, it is certainly helpful to spell them out, especially if you are still trying to learn how to solve problems in a new field such as theoretical mechanics.

We would like to suggest to you a problem-solving strategy based on these four steps and a problem-solving "toolbox" to help you approach the problems in this study guide in a systematic way.

HOW TO SOLVE IT – A STRATEGY

Any problem can be solved by going through the listed steps 1 to 4. To complete each step, you will use information and skills that you have at your disposal, based on what you have already learned or experienced. Each of these can be thought of as a **tool**, and by writing them down, you can develop a "**toolbox**" for dealing with each of the four steps of the problem-solving strategy.

The tools can be divided into subject-matter tools (i.e. your accumulated knowledge and skills in mechanics results and concepts) and strategic tools (sometimes given in terms of questions which help you to approach the problem).

Your toolbox will expand as you learn new things during the course of the module. Also, different types of problems require different types of tools.

In the following we give you an initial toolbox with general types of tools; later on we shall give you more specialised ones!

GENERAL TOOLBOX

1. **UNDERSTANDING THE PROBLEM**

   Here, you must understand what the object/system/situation is like, and what you are asked to do.

   To make sure that you have understood the problem, answer the following questions:

   - What is given and what is wanted? What conditions apply?
   - Can you describe the situation in your own words?

   You could make use of the following tools:

- Knowledge of the language of mechanics problems, and using keywords for clues about positions, objects and their properties, types of motion, etc.
- Sketches and diagrams
- Real-life examples and experiments
- Listing in standard mathematical notation the known and unknown quantities

2. **PLANNING A SOLUTION**

   Most of the time solving a problem in mechanics involves deciding on the correct principles or results of physics to apply in a given situation. Hence, one important class of tools consists of your knowledge of these:

   - The principles, definitions and results of mechanics – add the tools here
   - Knowledge about when the principles and results apply and when not – add the tools here
   - Sub-toolboxes you may already have designed for other tasks – add here

   To decide which of these you should apply to a particular problem, you may wish to use the following strategic questions as tools:

   - Can you find similar, already solved examples and problems? Can you use their method, or their results? (Similarity could mean dealing with a similar situation, or dealing with the same type of unknown.)
   - Which mechanics principles could be applied in this situation?
   - Which definitions, principles, results deal with the given type of unknown?
   - Do we have all the information necessary to apply the definitions, principles or results we have decided on? If not, can we determine the information from the given? Alternatively, can we introduce the information as another unknown? Which definitions, principles, results deal with the new unknown?

3. **EXECUTING THE PLAN**

   To complete this step, you will probably have to apply the following tools:

   - Sketches and diagrams
   - Mathematical notation, symbols for variables, coordinate systems
   - Equations and formulas
   - Mathematical tools (integration, solving equations, etc.)

4. **ANALYSING THE SOLUTION**

   To check the correctness of the solution, you can

   - see whether the solution makes sense
   - try to think of other ways to solve the same problem
   - compare the end result with other known, similar results
   - compare the result with experiments and guesses based on real-life objects
   - work in a group and compare your results with those of others

   If your solution seems to be wrong, you should

   - find out where you went wrong, by checking the argument and the calculations

- go back to step 1 or step 2

To reflect and learn from the solution, you can

- try to invent similar problems
- compare this problem with other examples and problems that you have come across, and ask yourself what the differences and similarities are

The described strategy does not provide an instant solution to the problem, but it does help you to find the solution by helping you to think systematically about the problem.

Throughout the study guide, you will come across versions of this general toolbox, as well as separate toolboxes for specific tasks. Also, at the end of each unit, we have listed what we think are the most important tools you should have derived from that unit.

The way you use these tools and toolboxes is up to you! To make the most of the tools and your toolbox, you should keep them all close at hand so that they are readily accessible to you. To make this easier, we have repeated all the toolboxes listed in the text at the end of the guide, so that you can cut them out. To these, you must add all the individual tools listed at the end of the units – ideally, you should write all of them on separate pieces of paper! (Of course, also make sure that you know how to use the tools you add to your toolbox...) And also add any additional tools which you think we may have overseen, or which you think will be useful to you. Now, you can actually put all the separate pieces of paper into a box (okay, you can use an envelope) so that every time you have to solve a difficult problem, you can go through the available tools one by one. In time, some of the tools or sub-toolboxes will become so familiar to you that you will be able to put them aside (but keep them around for future reference...).

Of course you may think this approach is just silly, and choose to ignore the toolbox idea completely, but we do believe you will come to realise that this strategy does indeed help in problem solving. We say so from personal experience!

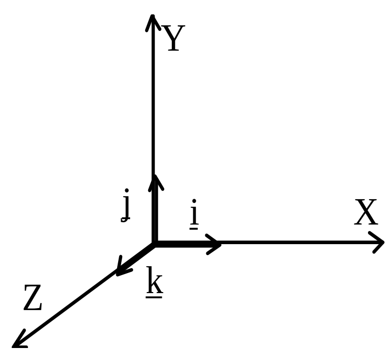
You may, of course, be more comfortable with another type of strategy to solve problems! In that case, why not write it down? There are no right or wrong strategies, only those that work and those that do not work for you. Setting out your favourite strategies will help you to analyse how you like to solve problems, and that will help you to further develop your problem-solving skills.

Unit 2 BACKGROUND PHYSICS

It will be assumed in this module that you are familiar with certain basic concepts and notations of physics and mathematics, some of which are briefly summarised in this first "real" study unit of the study guide. If any of the concepts below give you trouble, you may have to read up on them in any mathematics, mechanics or physics text book!

2.1 Vectors

Most of the time we will operate in the three-dimensional space $\mathbb{R}^3$. To be able to refer to the position of any point in this space, we introduce a coordinate system XYZ which consists of three mutually perpendicular axes called the X-axis, Y-axis and Z-axis, which are oriented according to the so-called right-hand rule. (Later on it will become clear why the orientation is so important!) In this module, we shall usually draw the axes as shown in the "side view" below, that is, the XY-plane is the plane of this page and Z is perpendicular to the plane, towards the reader. (Note that there are, of course, many other ways of depicting the axes. We choose to have the XY-plane on the plane of the page, because most of the time we shall consider objects which move in the XY-plane and rotate about an axis which is parallel to the Z-axis.)



WARNING:

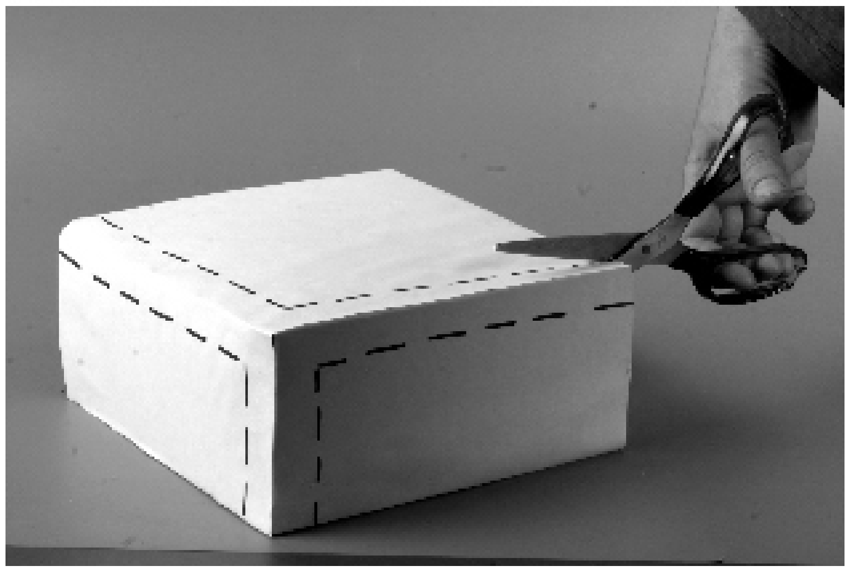
In this module you will occasionally have to be able to "think" in three dimensions, and to understand two-dimensional pictures of three-dimensional things, as in the sketch above! Quite often we shall be dealing with three-dimensional objects and shall have to agree how they are situated in relation to the coordinate axes; and later on, you will have to be able to understand the orientation of certain vectors (such as the orientation of the Y-axis in relation to the X- and Z-axes).

If you were in a lecture room, or sitting in my office, I could explain all these things with a lot of gestures; but many of you will have to figure things out from this study guide.

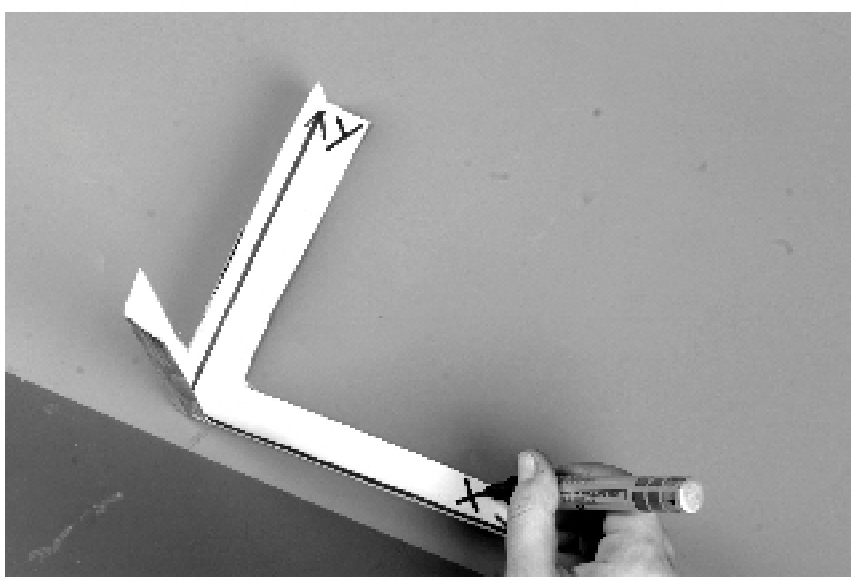
As a compromise, I would like to suggest the following: make a three-dimensional model of the XYZ-coordinate system for yourself, and use that to make sense of the three-dimensional thinking in this module. Instructions follow below!

HOW TO MAKE A THREE-DIMENSIONAL MODEL OF THE XYZ-COORDINATE SYSTEM

- You will need a cardboard box (of any size) of which at least one end has been glued closed. A cereal box, for instance, will work nicely!
- Start from one of the corners which has been firmly glued closed – this will be the origin of the coordinate system.
- We wish to keep just the three edges of the box which extend from this corner – these will be our coordinate axes. So, we cut out everything else as shown below. Leave the edges wide enough to keep the coordinate system rigid!

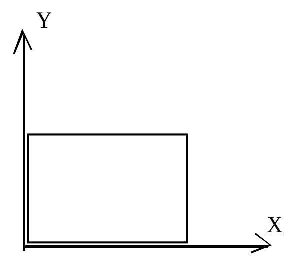


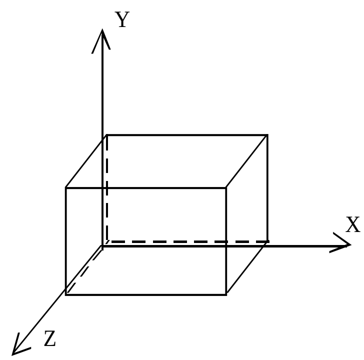
- Finally, we need to label the coordinate axes. Put the model down on the table in front of you so that two of the edges are flat against the table forming an L shape, and the third one goes straight up from the table. Label the ones along the table as X and Y (the one extending away from you is Y and the one going towards the right is X). Label the one sticking up from the table as Z.



Now, you have a concrete example of what the XYZ-coordinate system looks like.

You can now compare it with the two sketches shown below. The one on the left is the "top view" of the coordinate system, in which the X- and Y-coordinates are on the plane of this page, and Z is not shown, but is assumed to be pointing towards the viewer, out of the page. In this way of drawing things, all items are shown "flat", and it is up to the viewer to remember about the third dimension! The one on the right is the "side view", in which we attempt to show things in three dimensions. All the coordinate axes are now shown, and any objects are shown "in depth".

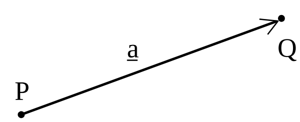




An object (a rectangular box) is included in both sketches for illustration. Compare with your coordinate model – put for instance a matchbox in the corresponding position in your model, and try to duplicate the two views shown above!

The position of a point in $\mathbb{R}^3$ is then fully determined by giving its **coordinates** (x, y, z): the values x, y, and z give the point's position on the X-, Y- and Z-axes, respectively.

A **vector** is a quantity which possesses a direction as well as magnitude (length). In this module, vectors will be denoted by underlined letters. If P and Q are two points, then $\underline{PQ}$ will denote the vector from P to Q (so, in the figure below, we have $\underline{a} = \underline{PQ}$.)



We will denote the magnitude (i.e. the length) of a vector $\underline{a}$ by $|\underline{a}|$. The **zero vector** (with zero length and undefined direction) is denoted by $\underline{0}$. The **resultant** of a collection of vectors is another word for the sum of the vectors, itself also a vector. Once we have fixed the origin O of our coordinate system, the **position vector** $\underline{r}$ of a point P is the vector from the origin O to the point P, that is,

$$\underline{r} = \underline{OP}.$$

(More generally, if Q is any point in $\mathbb{R}^3$, then the position vector of P **from point** Q is $\underline{r} = \underline{QP} = \underline{OP} - \underline{OQ}$.)

In mechanics, it is often useful to refer to a point by its position vector (from the origin), rather than by its coordinates. However, moving from coordinate position to position vector and back is easy, as we will explain next. We denote by $\underline{i}$, $\underline{j}$, $\underline{k}$ the **unit vectors,** that is, vectors of length one in the positive X-, Y- and Z-directions, respectively. Now, if the coordinates of a point P are known: $P = (x, y, z)$ then the position vector of P is simply

$$\underline{r} = \underline{OP} = x\underline{i} + y\underline{j} + z\underline{k}.$$

Going in the reverse direction, from vectors to coordinates, note that every vector A can be expressed in terms of the mutually perpendicular unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ as

$$\underline{A} = a\underline{i} + b\underline{j} + c\underline{k}$$

for some a, b and c, and that the vector is then identical to the position vector from the origin to the point $P = (a, b, c)$.

In fact, it is possible to identify vectors of $\mathbb{R}^3$ with points in $\mathbb{R}^3$, and write for instance $\underline{a} = (x, y, z)$ to mean that $\underline{a} = x\underline{i} + y\underline{j} + z\underline{k}$. (For example, $(4, 2, -3) = 4\underline{i} + 2\underline{j} - 3\underline{k}$. The magnitude of this vector $\underline{a}$ is then given by

$$\left|\underline{a}\right| = \sqrt{x^2 + y^2 + z^2}.$$

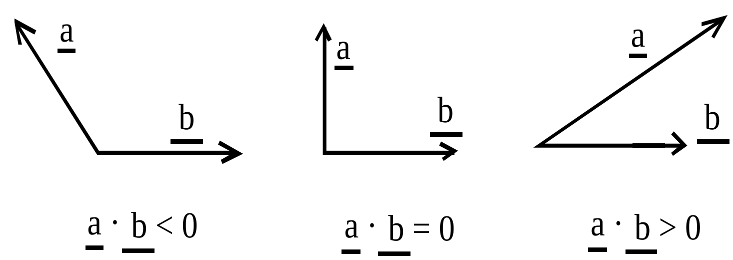
The **scalar product** of two vectors $\underline{a}$ and $\underline{b}$, denoted by $\underline{a} \cdot \underline{b}$ is defined as

$$\underline{a} \cdot \underline{b} = \left|\underline{a}\right| \left|\underline{b}\right| \cos\theta \tag{2.1}$$

where $\left|\underline{a}\right|$, $\left|\underline{b}\right|$ are the magnitudes of the vectors $\underline{a}$, $\underline{b}$ and θ is the angle between them. Note that the value of the scalar product of any two vectors is always a real number, that is, a scalar — hence the name! The order of the vectors in the scalar product does not affect the end result: we have $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$. The scalar product is sometimes also called the "dot product", based on the notation used for it.

From the definition (2.1) we see that the scalar product $\underline{a} \cdot \underline{b}$ of any two non-zero vectors $\underline{a}$ and $\underline{b}$ is positive if and only if the angle θ between them is between $-\frac{\pi}{2}$ and $+\frac{\pi}{2}$ (meaning that the vectors have an acute angle, less that 90°, between them) and negative if and only if the angle between them is between $-\pi$ and $-\frac{\pi}{2}$ or $+\frac{\pi}{2}$ and $+\pi$ (an obtuse angle).

If either one of the vectors has a length of zero then their scalar product is equal to zero. If a and b are non-zero vectors, then the scalar product $\underline{a} \cdot \underline{b}$ is zero if and only if the vectors are at right angles to each other (i.e. orthogonal to each other). This can often be used as a handy test for orthogonality!



When vectors are identified with points in $\mathbb{R}^3$, the following rule provides an easy way to calculate the scalar products: If $\underline{a} = (a_1, a_2, a_3)$ and $\underline{b} = (b_1, b_2, b_3)$ then $\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$. Note that it follows that $\underline{a} \cdot \underline{a} = |a|^2$.

2.2 Kinematics

In this study guide, we will use the "dot" notation for the time differentials of various quantities. That is, if x is any quantity which varies in time, then we use $\dot{x}$ to denote its derivative with respect to time, and $\ddot{x}$ to denote its second derivative with respect to time. That is,

$$\begin{aligned} \dot{x} &= \frac{d}{dt}x = x'(t), \\ \ddot{x} &= \frac{d}{dt}\dot{x} = \frac{d^2}{dt^2}x = x''(t). \end{aligned}$$

If $\underline{r} = \underline{r}\,(t)$ is the position vector of a particle, then its velocity is the vector defined by

$$\dot{\underline{r}} = \frac{d}{dt}\underline{r}$$

and its acceleration is the vector

$$\ddot{\underline{r}} = \frac{d^2}{dt^2}\underline{r}.$$

For one-dimensional motion, where the position of the particle is given by its X-coordinate $x = x\,(t)$, we usually denote velocity by v:

$$v = \dot{x}$$

and acceleration by a:

$$a = \dot{v} = \ddot{x}.$$

In this case the velocity and acceleration are also real values. Remember that the sign of the velocity indicates the direction of the motion along the axis, and its absolute value the speed of the motion. Similarly, the sign of the acceleration indicates the way that the velocity is changing. It is important to understand the difference between the acceleration and the velocity of a moving particle! A particle can have positive velocity but negative acceleration, or negative velocity but positive acceleration (in both these cases it is slowing down) and it can also have negative velocity and negative acceleration, or positive velocity and positive acceleration (in both cases it is speeding up).

You should be familiar with the following results which hold when the acceleration a is constant:

$v = v_0 + at$	(velocity after time t)
$x = x_0 + v_0 t + \frac{1}{2}at^2$	(distance travelled in time t)
$v^2 - v_0^2 = 2a\,(x - x_0)$	(velocity after travelling distance x)

2.3 Newton's laws of motion

Newton's laws of motion can be summarised as follows:

First law: A particle remains stationary or in uniform motion in a straight line unless it is acted upon by external forces.

Second law: The acceleration is proportional to the external force and acts along the same line:

$$\underline{F} = m\ddot{\underline{r}} \tag{2.2}$$

where $\ddot{\underline{r}}$ is the acceleration vector of the particle, m is its mass and $\underline{F}$ is the total external force acting on the particle.

Third law: If one particle exerts a force on another particle (the action force), then there is an equal and opposite force (the reaction force), exerted by the second particle on the first particle. The action and reaction forces act along the line joining the two particles.

Remarks:

- The uniform motion mentioned in the first law means that the particle moves at constant velocity, that is, the acceleration is zero. This also means that the particle does not change its direction of motion!
- The second law states that the acceleration and the force are proportional, with the mass of the particle acting as the constant of proportionality. We say that the mass

measures the inertia of the particle, that is, how well it can resist change in its motion — a heavier particle has a greater mass and thus greater inertia, which means that it takes more force to get it into motion at a certain acceleration than it would a lighter particle with a lower mass.

- In particular it follows that if no forces act on the particle, then the acceleration of the particle is zero, and the particle will remain in uniform motion along a straight line.
- If we introduce the concept of linear momentum *(plural: "momenta")*

$$\underline{p} = m\dot{\underline{r}} \tag{2.3}$$

then Newton's second law can be reformulated as follows:

$$\underline{F} = \frac{d\underline{p}}{dt}. \tag{2.4}$$

This follows directly from the fact that

$$\frac{d\underline{p}}{dt} = m\ddot{\underline{r}}.$$

2.4 Forces

In this module it will be particularly important to remember that forces are **vectors**, that is, they have both magnitude and direction. Since we shall be discussing general objects rather than just particles, it is also important to remember that each force has a specific point of action.

For any given situation, you should be able to identify all the forces acting on a system or an object, giving both their **direction** and their **point of action**. Sometimes this information is given directly in the description of the situation (as in "a downwards force with magnitude F acts on the upper left corner of the box"). Another easy one is the force of gravity which acts downwards, at the centre of mass (centre of gravity) of each object. However, often you have to figure out yourself which forces act on the system!

When drawing forces in our sketches, we shall draw the force as a vector (hence, with a direction and length) which starts at the point of application. See below for examples of such sketches.

The following is a list of the basic categories of forces which you will come across in this module. Pay attention in particular to the so-called contact forces (normal forces and friction), as they are easy to overlook:

- **The force of gravity.** This force acts between any two particles or bodies with non-zero mass. For the sake of this module, we are only interested in the force of gravity that the Earth exerts on everyday objects on its surface (also known as weight).So, in this module, the force of gravity exerted on an object of mass M will have the magnitude Mg, where g is the constant of gravitational acceleration, and is directed downwards. The force of gravity can be taken to act at the centre of mass of the object. (More about centres of mass in Learning Unit 2!)
- **Normal forces**. These forces arise where two solid objects are in contact with each other, and they act to prevent the objects from overlapping. There is always an action-reaction pair of equal but opposite normal forces, consisting of the normal force exerted by object 1 on object 2, and the normal force exerted by object 2 on object 1. (Or, to put it more simply, the force on object 2 from object 1 and the force on object 1 from object 2.) The normal forces act at the joint surface, where the two objects touch each other, and they are always perpendicular to the joint surface of the objects. The magnitude of the normal force can be found from the fact that it must always exactly cancel out all

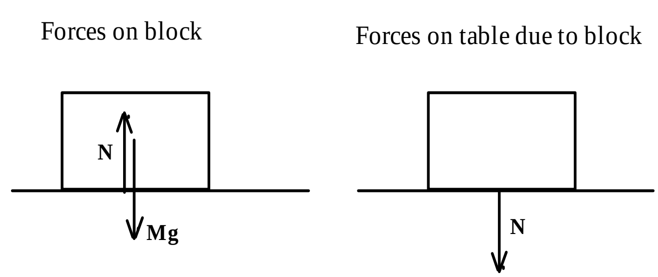
the other forces acting on the object perpendicularly to the surface.

- **Forces of friction.** Friction is a force that resists the motion of one object relative to another, when the surfaces of the two objects are in contact. Frictional forces also come in an action-reaction pair. They act parallel to the surface between the objects, and the direction of the frictional force can be found from the fact that it always acts to resist motion.
- **Tension.** By tension at the end of a rope, cable etc. is meant the force with which the rope, cable etc. pulls on the object which is attached to it. This force is directed along the rope, and acts at the point where the rope is attached. The action/reaction pair then consists of the rope pulling on the object, and the object pulling on the rope. Usually we are not interested in the forces acting on the rope, and therefore interpret the tension as a force pulling an object in the direction of the rope. If the tension at the end of the rope is not zero, then the other end of the rope must also be attached somewhere or pulled. The other end of the rope will therefore also have its own tension. Assuming that the rope has negligible mass, and moves freely, the tension at both ends of the rope can be assumed to be the same. This also holds if the rope passes over a frictionless pulley or the like — such a rope transmits tension without change, meaning that it may change the direction of the tension of the rope but not its magnitude. (The situation is different if the pulley is not frictionless — we shall come across this situation later on!)

Normal and friction forces act over the entire surface of contact, but we can usually assume that they act in the middle of the surface.

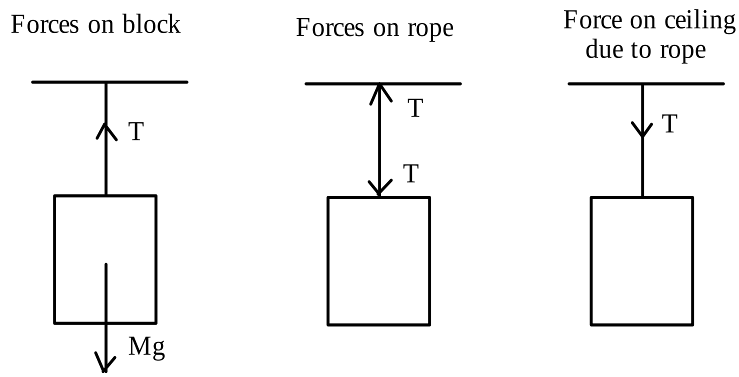
Example 2.1

A block of mass M lies on top of a table. The two forces acting on it are the force of gravity, $M\underline{g}$, acting downwards at the centre of the block and the normal force $\underline{N}$ exerted upwards on the block by the table. The reaction pair of the normal force is another normal force of the same size as $\underline{N}$ pushing down on the table. What about the reaction pair of the force of gravity? That acts on the object that causes the gravity, namely the Earth — the block pulls the Earth towards itself with the force $M\underline{g}$, which, however, due to the huge mass of the Earth, makes hardly any difference in the motion of the Earth!



Example 2.2

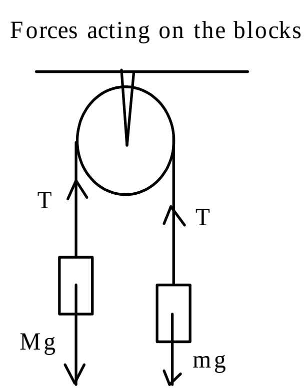
The block in the previous example is attached to one end of a rope, the other end of which is attached to the ceiling. Now the forces acting on the block are the force of gravity, $M\underline{g}$, acting downwards at the centre of the block, and the tension $\underline{T}$ on the rope, acting upwards along the rope at the point where the rope is attached. What is the reaction pair of the tension force? The block is pulling down on the rope with a force of the same magnitude as $\underline{T}$. The rope will transmit this tension to the other end of the rope, so that the rope pulls at the ceiling with the same tension — and there is a corresponding reaction force of the ceiling pulling at the rope with the same force.

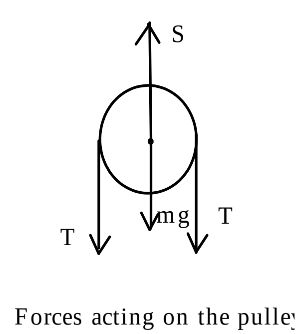


Example 2.3
Two blocks (with masses m and M) are attached to the opposite ends of a light rope. The rope passes frictionlessly over a smooth pulley (mass m), which is attached to the ceiling. What are the forces acting on the blocks and the pulley?

Solution
Let us start with forces of gravity: The forces $m\underline{g}$, $M\underline{g}$ and $m\underline{g}$ act downwards at the centres of the two blocks and the pulley, respectively. The tension of the rope is the same at both ends, since the rope passes frictionlessly over the pulley. Thus we have two forces of tension $\underline{T}$, each pulling one of the blocks upwards (in the direction of the rope), and each acting at the point where the rope is attached to the block. How then does the rope act on the pulley? The rope lies on top of the pulley and pulls it downward, which means that there is in fact a normal force acting on the pulley due to the rope. However, it is convenient to assume that in fact the rope exerts a tension on the pulley on both sides of the pulley, acting on the point where the rope leaves the pulley. The magnitudes of these tensions are again the same as $\underline{T}$.

Is this all? To see that something is still missing, consider the forces listed so far as acting on the pulley: the gravity $m\underline{g}$ downwards and two identical forces of tension, also pulling downwards. So, all forces so far are downwards; if they are indeed all the forces acting on the pulley, then according to Newton's second law, the pulley must accelerate downwards. But we know that is does not; therefore there must be another force or forces acting upwards! This is obviously the force keeping the pulley suspended where it is. We must thus add another force $\underline{S}$ which acts upwards at the point where the pulley is fixed to the ceiling.





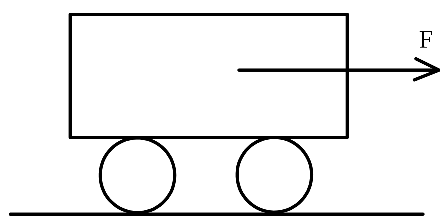
In the examples above we have utilised some of the following hints which you can also use to make sure that you have included all the forces acting on an object or a system:

TOOLBOX: IDENTIFYING ALL THE FORCES ACTING ON A SYSTEM

- Draw a picture of the system as a whole, and, if necessary, separate diagrams for all the components of the system, with the forces acting on them. Draw each force as a vector (an arrow) which starts at the point of action of that force.
- For all forces, consider the corresponding reaction force. Does the reaction force act on one of the objects we are interested in? If so, remember to include it!
- Does the collection of forces make sense? Remember that if the object is motionless, then the forces acting on it must balance out – that is, their sum must be zero. If the object is supposed to be still, but all the forces act in the same direction, then something is wrong – you may have let out some forces.

The following example is quite complicated, but it does give a good illustration of the various kinds of forces!

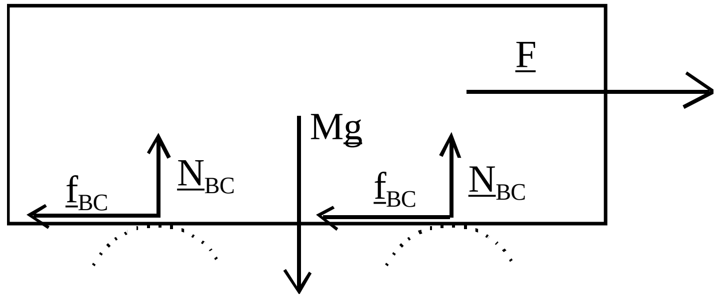
Example 2.4
A rectangular block of mass M lies on top of two identical parallel rough cylinders, each with mass m and radius r, on top of a rough horizontal table. A horizontal force $\underline{F}$ is applied to the block, perpendicular to the axes of the cylinders.



Assume that the mass of the block rests evenly on both the cylinders. Indicate all the forces acting on the block and the two cylinders.

Solution
Let us first take a moment to think what happens in the described system. The force F pulls the block towards the right. It seems obvious that the block will roll on top of the cylinders, and the cylinders in turn will roll on the table. What causes the cylinders to roll? The surface between the block and the cylinders is rough, so that from the point of view of the block there are forces of friction resisting the motion of the block across the cylinders. These forces act on the points where the cylinders touch the block, and are directed towards the left, since the motion of the block is towards the right. As far as the cylinders are concerned, the same friction acts to pull the tops of the cylinders towards the right. That is, forces of friction are acting on the cylinders. These forces are of the same magnitude as the ones acting on the block; they are also acting on the points where the cylinders touch the block, but now towards the right. So, now we have forces acting at the top of the cylinders, towards the right. This causes the cylinders to move towards the right. However, there is also friction between the table and the cylinders, resisting the motion of the cylinders towards the right — this causes the cylinders to roll, rather than to slide.

Block:



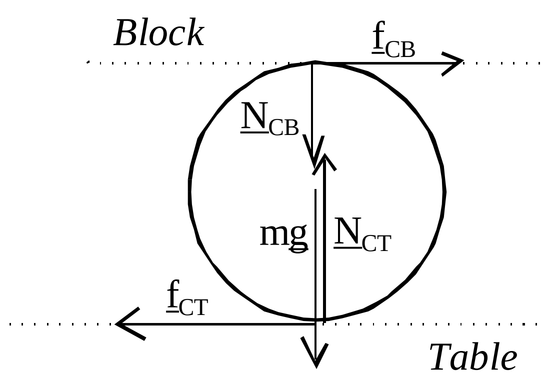
The forces acting on the block are:

- The force $\underline{F}$; we have not been told which point this is applied at.
- Gravity $M\underline{g}$, downwards and acting at the centre of the block.

Acting at the points of contact between the block and each of the two cylinders, we have:

- Friction $\underline{f}_{BC}$. Friction opposes motion, so the direction of $\underline{f}_{BC}$ is opposite to the direction of $\underline{F}$.
- The normal force $\underline{N}_{BC}$ exerted on the block by the cylinder, upwards.

The cylinders:



Identical forces act on both the cylinders. They are:

- Gravity $m\underline{g}$, downwards.

Acting at the point of contact between the block and each cylinder, we have:

- The frictional force between the block and the cylinder, $\underline{f}_{CB}$. Note that $\underline{f}_{CB}$ and $\underline{f}_{BC}$ form an action-reaction pair, so their directions are opposite:

$$\underline{f}_{CB} = -\underline{f}_{BC}.$$

- The normal force that the block exerts on the cylinder, $\underline{N}_{CB}$. Again, $\underline{N}_{BC}$ and $\underline{N}_{CB}$ form an action–reaction pair, so

$$\underline{N}_{BC} = -\underline{N}_{CB}.$$

Acting at the point of contact between the cylinder and the table:

- The normal force that the table exerts on the cylinder, $\underline{N}_{CT}$ (upwards).
- The frictional force $\underline{f}_{CT}$, due to the friction between the cylinder and the table, opposite to the direction of $\underline{F}$.

Note that $\underline{N}_{CT}$ and $\underline{f}_{CT}$ also each form halves of action-reaction pairs — namely, there are corresponding forces $\underline{N}_{TC}$ (the normal force that the cylinder exerts on the table, acting on the table) and $\underline{f}_{TC}$ (the force due to the friction between the table and the cylinder, acting on the table). We are not listing these, since they act on the table, not the system consisting of the cylinders and the block.◀

CONCLUSION

This unit contained a summary of definitions and results involving vectors, coordinates, kinematics, forces and Newton's laws of motion. If you felt unsure about any of this, you might wish to go back to your study guides or textbooks on physics or mathematics for revision, since we do assume that you are fairly familiar with all the topics discussed in this unit!

Remember to add the following tools to your toolbox:

- all the background information given in this unit (make a summary of them yourself!)
- Newton's laws of motion for particles
- the tools for the task of identifying all forces acting on a system

LEARNING UNIT 2

THE CENTRE OF MASS

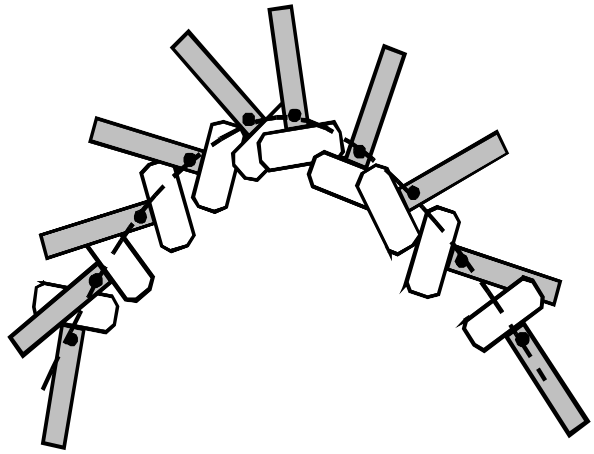
CONTENTS OF LEARNING UNIT 2

Study unit 3	The centre of mass of a system of particles
Study unit 4	The motion of the centre of mass of a system of particles
Study unit 5	The centre of mass of a rigid body
Study unit 6	More on integration
Study unit 7	The motion of the centre of mass — the general case

Introduction

In the simplest applications of mechanics it is customary to treat all bodies, irrespective of their shape and size, as if they were particles, i.e. point masses. But are we really justified to simplify things so much? After all, the motion of rigid bodies has properties which a particle can't share, as the next example shows.

Example 2.1
The figure below shows a sketch of the motion of a hammer tossed in the air. As it travels through the air, it also rotates and it does not seem possible to represent this complicated motion by the path of a particle.



Although this is correct, the remarkable thing is that there is a particular point on the hammer which does move as if it were a particle thrown into the air and acted upon by gravity. The curve superimposed on the sketch shows its path. This point is called the **centre of mass** of the hammer.

In this Learning Unit (Learning Unit 2) of the study guide (Study units 3 to 7), we shall show how it is possible, to some extent, to approximate the motion of a body or a system of particles by the motion of a particle. The particle should have the same mass as the original body/system, and should be situated at the centre of mass of the body/system. We shall first define what is meant by the centre of mass, and how it can be found; and then we shall derive an equation of motion for it. Initially we shall consider systems of particles. This is a natural place to start, since Newton's laws of motion, which we base our results on, are expressed for particles! By adding the properties of rigidity and solidity (a continuous structure), we shall eventually be able to consider ordinary, everyday objects and combinations of them.

The outcomes of Learning Unit 2

When you have worked through this Learning Unit of the study guide, you should be able to

- explain why the centre of mass is important
- calculate the centre of mass of a given system of particles
- find the centre of mass of a rigid body by using integration, as well as the appropriate rules of symmetry and composite bodies
- introduce a suitable coordinate system
- analyse the motion of the centre of mass of a system of particles, of rigid bodies and of systems when the forces acting on parts of the body or the system are known

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Systems of rigid bodies and particles 115

12/7/2015

Unit 3 THE CENTRE OF MASS OF A SYSTEM OF PARTICLES

Key questions:

- *What is meant by the centre of mass of a system of particles?*
- *How can it be found?*

In this unit we consider the centre of mass of a system of particles. But why would we be interested in systems of particles at all, when we really wish to consider objects like hammers? The answer to this is that all objects can be considered to be systems of particles, with the added property that they are rigid (with all the particles staying in fixed positions in relation to each other) and with the property that there are so many particles that they can be considered to form a continuous structure. So, the analysis of the behaviour of objects (usually called "rigid bodies" in mechanics) can be brought back to the analysis of the behaviour of systems of particles, which in turn can be determined by applying Newton's laws of motion to all the individual particles making up the system.

Accordingly, we need to start with systems of particles. In this unit we give a formal definition of the centre of mass of a system of particles, and you will learn how to find centres of mass — which often involves introducing a coordinate system, which you will also find out how to do! In the next unit, we will derive, from Newton's laws for the motion of particles, the law of motion for the centre of mass of the particles.

Contents of this unit:

3.1 Definition of the centre of mass

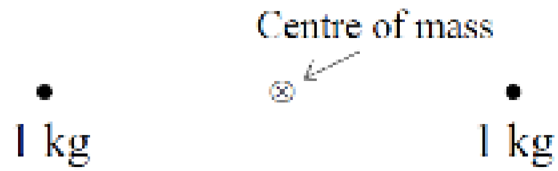
3.2 About coordinate systems

What you are expected know before working through this unit:

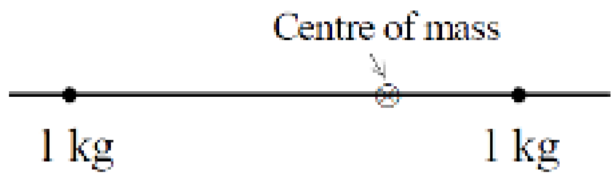
In this module all you will need is straightforward algebra, and an ability to work with points in the one, two and three dimensional coordinate systems!

3.1 Definition of the centre of mass

The position of the centre of mass of a system is merely the "average position" of the mass of the system. For example, if the system consists of two particles, each with a mass of 1 kg, then the centre of mass is halfway between them. (Note that here and in the rest of the study guide, we use the symbol $\otimes$ to denote the centre of mass in the sketches!)



What if one of the particles has a mass of 1 kg and the other has a mass of 2 kg? In that case, we would expect the centre of mass to lie closer to the 2kg mass.



More generally, let us assume that a system consists of n particles 1, 2, 3, ..., n, all of which are situated on the X-axis. Assume that particle number i is in position x_i and has a mass of m_i. What is the "average position" of the mass of the system in this case? It should be the average of all the positions $x_1, x_2, \ldots, x_n$ of the particles, with each position x_i weighed by the mass m_i of that particular particle. That is, the centre of mass should be at the position

$$\frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{m_1 + m_2 + \ldots + m_n}.$$

Of course, we would really wish to consider a more general case, with particles situated anywhere in the three-dimensional space $\mathbb{R}^3$. The position of each of the particles, and of the centre of mass of the system, can then be described by giving their position vectors. The following definition expresses this idea.

Definition 3.1 (The centre of mass of a system of particles)
Let a system consist of n particles with masses $m_1, m_2, \ldots, m_n$ and position vectors $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_n$. We define the **centre of mass** of the system to be the point with the position vector $\underline{R}$ given by

$$\underline{R} = \frac{\sum_{i=1}^{n} m_i \underline{r}_i}{\sum_{i=1}^{n} m_i} = \frac{m_1 \underline{r}_1 + m_2 \underline{r}_2 + \ldots + m_n \underline{r}_n}{m_1 + m_2 + \ldots + m_n}. \tag{3.1}$$

Note that the position vectors $\underline{r}_i$ and $\underline{R}$ must all be from the same origin point O. In actual calculations, the position vectors of the particles are usually expressed in terms of the $\underline{i}$, $\underline{j}$ and $\underline{k}$ unit vectors of some coordinate system, and the end result $\underline{R}$ is then also of this type.

Example 3.1
Find the centre of mass of a system consisting of three particles of masses $m_1 = 4m$, $m_2 = m$ and $m_3 = 3m$ located at the points $\underline{r}_1 = \underline{i} + 3\underline{j}$, $\underline{r}_2 = -2\underline{i} - 2\underline{j}$ and $\underline{r}_3 = 4\underline{i} - 2\underline{j}$, respectively.

Solution: Using (3.1), we get

$$\begin{aligned}
\underline{R} &= \frac{4m\cdot\left(\underline{i}+3\underline{j}\right)+m\cdot\left(-2\underline{i}-2\underline{j}\right)+3m\cdot\left(4\underline{i}-2\underline{j}\right)}{4m+m+3m} \\
&= \frac{14m\underline{i}+4m\underline{j}}{8m} \\
&= \frac{7}{4}\underline{i}+\frac{1}{2}\underline{j}=1\frac{3}{4}\underline{i}+\frac{1}{2}\underline{j}.
\end{aligned}$$

The centre of mass therefore has the position vector $1\frac{3}{4}\underline{i}+\frac{1}{2}\underline{j}$. ◄

Note how the constant m cancelled out in the calculation! Since each position vector in (3.1) is multiplied by the mass of that particle, and the whole thing is divided by the sum of the masses, it follows that the *centre of mass stays the same if all the masses of all the particles are multiplied or divided by the same non-zero amount K*. The centre of mass of a system depends only on the relative distribution of the mass within the system, not on the exact masses of the particles! (But if we multiply just some of the masses by K and not others, then we have changed the mass distribution of the system and the centre of mass may change.)

The position of the centre of mass can of course alternatively be thought of as a point specified by its coordinates. If we select an XYZ-coordinate system, we can find the X-, Y- and Z-coordinates of the centre of mass separately. If particle number i has coordinates (x_i, y_i, z_i) in the chosen coordinate system, then the centre of mass has coordinates $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = \frac{m_1x_1+m_2x_2+\ldots+m_nx_n}{m_1+m_2+\ldots+m_n} \tag{3.2}$$

$$\bar{y} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i} = \frac{m_1y_1+m_2y_2+\ldots+m_ny_n}{m_1+m_2+\ldots+m_n} \tag{3.3}$$

$$\bar{z} = \frac{\sum_{i=1}^{n} m_i z_i}{\sum_{i=1}^{n} m_i} = \frac{m_1z_1+m_2z_2+\ldots+m_nz_n}{m_1+m_2+\ldots+m_n} \tag{3.4}$$

In one- and two-dimensional cases we only need to use one or two of these formulas. If all the particles lie on the XY-plane, for instance, then the centre of mass must also lie on the XY-plane, and we only have to find the values of $\bar{x}$ and $\bar{y}$.

Example 3.2

Particles of masses $m_1 = 2$, $m_2 = 3$, $m_3 = 5$ and $m_4 = 4$ are located at the points $(x_1, y_1, z_1) = (4, 1, 1)$, $(x_2, y_2, z_2) = (0, 2, 1)$, $(x_3, y_3, z_3) = (-1, 1, -2)$ and $(x_4, y_4, z_4) = (1, 0, 1)$ respectively. Find the centre of mass of the system.

Solution: We can use equations (3.2), (3.3) and (3.4):

$$\begin{aligned}
\overline{x} &= \frac{m_1x_1+m_2x_2+m_3x_3+m_4x_4}{m_1+m_2+m_3+m_4} \\
&= \frac{2\cdot 4+3\cdot 0+5\cdot(-1)+4\cdot 1}{2+3+5+4} = \frac{1}{2}, \\
\overline{y} &= \frac{m_1y_1+m_2y_2+m_3y_3+m_4y_4}{m_1+m_2+m_3+m_4} \\
&= \frac{2\cdot 1+3\cdot 2+5\cdot 1+4\cdot 0}{2+3+5+4} = \frac{13}{14}, \\
\overline{z} &= \frac{m_1z_1+m_2z_2+m_3z_3+m_4z_4}{m_1+m_2+m_3+m_4} \\
&= \frac{2\cdot 1+3\cdot 1+5\cdot(-2)+4\cdot(1)}{2+3+5+4} = -\frac{1}{14}.
\end{aligned}$$

That is, the centre of mass is situated at the point $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{1}{2}, \frac{13}{14}, -\frac{1}{14}\right)$. ◀

It is, of course, easy to move between the vector notation and the coordinate notation, using the fact that the point (x, y, z) has the position vector $x\underline{i} + y\underline{j} + z\underline{k}$. The answers using either method (coordinates or position systems) will of course be the same, and you may use either method – whichever you are more comfortable with!

Example 3.3

Find the centre of mass of a system consisting of three particles of masses $4m$, m and $3m$ located at the points $\underline{i} + 3\underline{j}$, $-2\underline{i} - 2\underline{j}$ and $4\underline{i} - 2\underline{j}$, respectively.

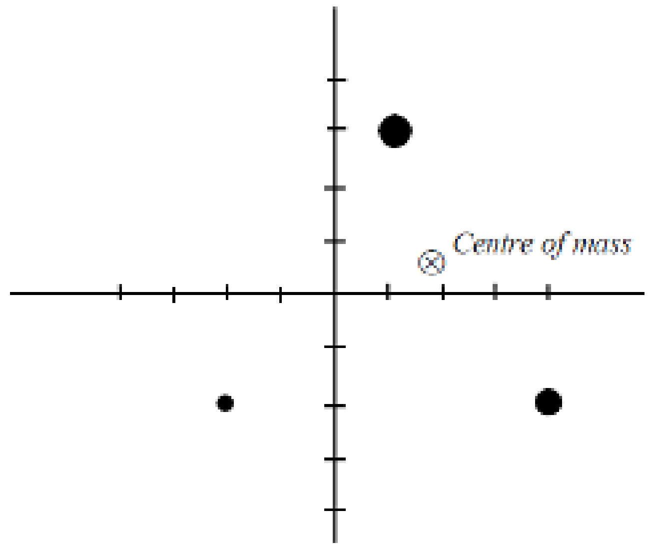
Solution: The system here is identical to that in Example 3.1, but we will repeat the calculations here, this time using (3.2) and (3.3). In coordinate form, the positions of the particles are given by $(1, 3)$, $(-2, -2)$ and $(4, -2)$, and therefore the X- and Y-coordinates of the centre of mass are

$$\begin{aligned}
\overline{x} &= \frac{4m\cdot 1+m\cdot(-2)+3m\cdot 4}{4m+m+3m} = \frac{7}{4}, \\
\overline{y} &= \frac{4m\cdot 3+m\cdot(-2)+3m\cdot(-2)}{4m+m+3m} = \frac{1}{2}.
\end{aligned}$$

So, the centre of mass is at $\left(1\frac{3}{4}, \frac{1}{2}\right)$, which of course corresponds to the previous answer where we obtained the position vector $\underline{R} = \frac{7}{4}\underline{i} + \frac{1}{2}\underline{j}$ for the centre of mass.◀

Note that in this example, we did not explicitly number the particles, as particles number 1, 2 and 3, with x-coordinates x_1, x_2, x_3, y-coordinates y_1, y_2, y_3 and masses m_1, m_2, m_3, but rather applied the formulas in (3.2) and (3.3) directly. A good reason to number the particles is that it ensures that you do take all of them into account, and it also enables you to identify all the particles, their masses and their position vectors in a systematic way. You should only take shortcuts when you are sure you know what you are doing.

The picture below shows the positions of the three particles, as well as the position vector of the centre of mass in the previous example. The heavier particles are denoted by a bigger dot. Note that the calculated centre of mass lies roughly halfway between particles 1 and



3, a bit closer to particle 1. This is what we would expect: particle 2 is much lighter than the other two, so it does not contribute very much to the position of the centre of mass; and the other two particles are of roughly the same size, with particle 1 a little bit heavier.

Activity 3.1

Here is your turn to do the calculations. To practice both methods, find the centre of mass of the following system both using coordinates, and using position vectors expressed in terms of the $\underline{i}$, $\underline{j}$ and $\underline{k}$ unit vectors.

Particles of mass $2m$ and $3m$ have the position vectors $\underline{i} + \underline{k}$ and $3\underline{i} + \underline{j} + 2\underline{k}$, respectively. Find the centre of mass of the system consisting of the two particles.

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Feedback: You should get $(11/5, 3/5, 8/5)$, or identically $\frac{1}{5}\left(11\underline{i} + 3\underline{j} + 8\underline{k}\right)$, for the centre of mass.

Quite often we are faced, not with the job of finding the centre of mass of a given system, but rather with the opposite: the job of positioning objects or particles, or adding new particles, to make the centre of mass coincide with a desired point. There are many real-life situations like this. If you have ever played in a see-saw, you will remember that two children of different sizes will need to adjust their positions on the ends of the see-saw to make the see-saw level — the heavier child will need to move closer to the centre! In this example, the children act as the particles with given masses, and the problem is to select their positions such that the centre of mass of the system coincides with the centre of the see-saw (its pivoting point). Games aside, the centre of mass plays an important role in balancing the wheels of a car. Often in real life the centre of mass is not exactly at the centre of the wheel. In the balancing, a small piece of metal is attached at the rim of the wheel, to ensure that the centre of mass of the wheel coincides with the axis of the wheel. This is important to minimize the moment of inertia, as we will find out later! In this case the problem is to select the correct mass for the added piece of metal.

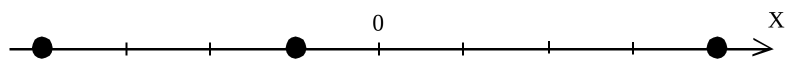
In the following we discuss how to solve these kinds of problems.

Example 3.4
Three particles of mass m are situated on the X-axis, at the positions $x = -4$, $x = -1$ and $x = 4$.

(a) What should the mass of a fourth particle be, to be situated at $x = 2$, if we want the centre of mass of the four-particle system to be at the origin?

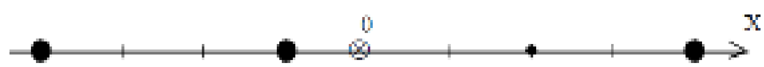
(b) Is it possible to add a fourth particle at $x = -2$ instead, and still have the centre of mass at the origin?

*Solution***:** We will start by looking at a sketch of the situation. Let us first consider the three-particle system. All the particles have the same mass m, and their positions on the X-axis are indicated in the sketch below.

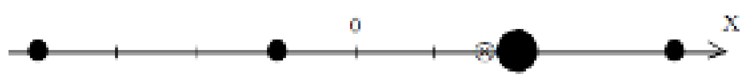


From the sketch, we might guess that the centre of mass of this three-particle system lies somewhere between -1 and 0 on the X-axis, and indeed direct calculation confirms this — the centre of mass of the three-particle system is at $\bar{x} = -1/3$.

(a) Let us now add to this system a fourth particle, situated at $x = 2$, but with an unknown mass. We wish to find the mass of this particle so that the centre of mass of the four-particle system will lie at the origin (at $x = 0$). Adding the new particle will move the centre of mass from its existing position towards the new particle, that is, towards the right. However, if the mass of the fourth particle is very small, then adding it will not change the situation enough to move the centre of mass of the system to the origin,

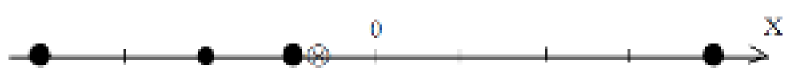


while if the mass of the new particle is too large, then adding it will move the centre of mass too far towards the right, beyond the origin.



There is only one possible mass for the fourth particle which will cause the centre of mass to be at the origin.

(b) What if the new particle is added at point $x = -2$ instead? Again, the addition of the new particle has the effect of moving the centre of mass towards the new particle. But this means that the centre of mass moves towards the left, which is away from the origin.



So, it seems that adding a fourth particle at $x = -2$ cannot cause the centre of mass to move to the origin.

Of course, we cannot find the exact solution to the two questions with these graphical representations. To do that we have to use the definition of the centre of mass, for instance as follows.

(a) Let M denote the unknown mass of the fourth particle. The centre of mass of the four-particle system is then, according to (3.2), at the position $\bar{x}$ on the X-axis, where

$$\bar{x} = \frac{-4 \cdot m - 1 \cdot m + 4 \cdot m + 2 \cdot M}{m + m + m + M}.$$

On the other hand, we wish the centre of mass to be at the origin, that is, we want to have $\bar{x} = 0$. This means that the following equation must hold in case (a):

$$0 = \frac{-4 \cdot m - 1 \cdot m + 4 \cdot m + 2 \cdot M}{m + m + m + M}$$

$$\therefore \quad \frac{-m + 2M}{3m + M} = 0.$$

We will solve the unknown value of M from this. We can multiply both sides by the denominator, $3m + M$ to get

$$-m + 2M = 0 \quad \therefore \quad M = \frac{1}{2}m.$$

That is, a fourth particle of mass $\frac{1}{2}m$ at position $x = 2$ will give us a centre of mass situated at $\bar{x} = 0$.

(b) On the other hand, if a particle of unknown mass M is to be at $x = -2$, then, reasoning as above, for the centre of mass to be at $\bar{x} = 0$ we must have

$$\begin{aligned} 0 &= \frac{-4 \cdot m - 1 \cdot m + 4 \cdot m - 2 \cdot M}{m + m + m + M} \\ \therefore \quad 2M &= -m \quad \therefore \quad M = -\frac{1}{2}m. \end{aligned}$$

This suggests that the fourth particle should have a negative mass, which is of course impossible. Thus, it is not possible to cause the centre of mass to be at the origin by adding a particle at $x = -2$. ◄

Activity 3.2

The following problem is similar to the one above, except that you now need to work in three dimensions! Remember that for the centre of mass to be at the given point, the coefficients of all three unit vectors ($\underline{i}$, $\underline{j}$ and $\underline{k}$) in its expression must be equal to the given values!

Particles of mass $2m$ and $3m$ have the position vectors $\underline{i} + \underline{k}$ and $3\underline{i} + \underline{j} + 2\underline{k}$, respectively.

(a) Show that it is not possible to place a new particle, whatever its mass M, at the position $2\underline{i} + \underline{k}$ and have the new centre of mass at the origin.

(b) At what position $a\underline{i} + b\underline{j} + c\underline{k}$ should a new particle of mass m be positioned to have the new centre of mass at the point $\underline{i} + \underline{j} + \underline{k}$?

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Feedback: In (a), you will get three equations involving M and m which must hold for all at the same time. Two of these equations give a negative value for M; also the values contradict; and one equation only holds if $m = 0$. Any one of these contradictions would be enough to prove that no mass at the given position will give you the required centre of mass! In (b), the third mass should be put at the point $-5\underline{i} + 3\underline{j} - 2\underline{k}$.

Note that the centre of mass can be defined for any system of particles. The particles do not necessarily have to be stationary: they can move around as well. If the position of particle i is given by a position vector $\underline{r}_i$ (or the coordinates (x_i, y_i, z_i)) *at some specified moment* t, then the centre of mass calculated from equations (3.1) or (3.2), (3.3) and (3.4) is the centre of the mass *at that moment* t. If the particles move, then their positions change and the centre of mass might well also change. This idea of viewing the centre of mass of a system of particles as a vector or a point which varies as a function of time will be very important to us in the rest of the study guide — our aim is, after all, to derive laws to describe its motion!

3.2 About coordinate systems

In all the examples above, the coordinate system was already in place — that is, the position vectors were given in terms of the unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$, and the points were referred to by giving their coordinates in some already existing XYZ coordinate system. The fact that the coordinate system is given in a problem is very convenient, but not something you should automatically expect! Remember that a coordinate system is something that we introduce in order to be able to apply mathematical analysis to real-life problems — it is not something that already exists.

Often, in fact most of the time, problem statements are given without any reference to a coordinate system. The following problem, which we will solve later on, is of that type: *Four particles, with masses* m, m, m *and* $2m$, *are positioned in such a way that they form the four corners of a square. Where is the centre of mass of the system?*

To solve this type of a problem, you will first have to introduce a coordinate system. How this can best be done varies from case to case, and will involve a lot of decisions. Below we provide you with a toolbox which should help you get started with this!

TOOLBOX FOR SELECTING A COORDINATE SYSTEM

Before we can even start the task of finding the centre of mass of a system of particles, we need to have a coordinate system in place! If one is already given, fine; if not, then we must decide on a suitable one. The reason why we need a coordinate system is because the formulas (3.1) or (3.2), (3.3) and (3.4) help us to find the position of the centre of mass from the position of the particles. However, we cannot talk about the position of a particle without having a way to refer to it! The formulas (3.1) to (3.4) refer to the position vectors or coordinates of the particles of the system. But position vectors are meaningless unless we have a reference point (position vectors from *where*?) and, similarly, the coordinates of a point do not mean anything unless we have specified our coordinate axes.

There are many possible coordinate systems, any one of which would do; but some are more suitable than others, because they make calculations easier. Here are some guidelines on how you can go about to select a good coordinate system.

- Draw a sketch of the system. The sketch will make sure that you understand the situation, and will make it easier to select a suitable coordinate system! You might want to label the particles in the sketch. You may have to assume values for distances, masses and positions when they are not fully specified!
- Determine the dimension of the system. If all particles are along one straight line, or along one plane, then the system is in fact one- or two-dimensional, respectively. In that case you do not have to introduce a complete three-dimensional coordinate system.
- Now, you can proceed to select the direction of the coordinate axes and the position of the origin. Things to take into account here are symmetry (we will come back to that later, in Unit 5), and the need to be able to find the position vectors or coordinates of all particles as easily as possible!

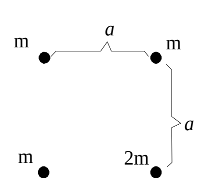
To illustrate these guidelines, we will look at the problem mentioned above.

Example 3.5

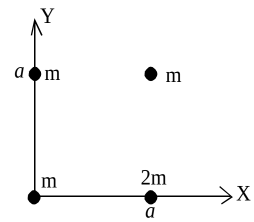
Four particles, with masses m, m, m and $2m$, are positioned in such a way that they form the four corners of a square. Where is the centre of mass of the system?

Solution:

Draw a sketch of the system:



The actual drawing depends on the orientation of the square (we have been unadventurous here and drawn it straight, but it can also be drawn standing on one corner), and also on where we wish to put the heavier particle! Instead of labeling the particles, we have written the mass of each particle next to it. We have assumed that the square has sides of length a. Determine the dimension of the system. All the particles of the system are clearly lying on one plane, and therefore to introduce a coordinate system we just have to decide where on the XY-plane we wish to place the square. Select the direction of the coordinate axes and the position of the origin. It is certainly simplest to assume that the edges of the square are parallel to the X- and Y-axes. Let us introduce the XY coordinate system shown below.



The three lighter particles (mass m) are at the points $(0,0)$, $(0,a)$ and (a,a); and the heaviest one (mass $2m$) is at point $(a,0)$. It follows that in this coordinate system, the centre of mass is at the point $(\overline{x},\overline{y})$ where

$$\overline{x} = \frac{m\cdot 0 + m\cdot 0 + m\cdot a + 2m\cdot a}{m+m+m+2m} = \frac{3}{5}a,$$

$$\overline{y} = \frac{m\cdot 0 + m\cdot a + m\cdot a + 2m\cdot 0}{m+m+m+2m} = \frac{2}{5}a$$

The centre of mass is at $\left(\frac{3}{5}a, \frac{2}{5}a\right)$. ◄

Activity 3.3

Re-do the calculations in the previous example, if we change the coordinate system to the one described below. Remember to draw a sketch of the system and your coordinate system!

Find the centre of mass in the previous example, Example 3.5, if instead we assume that the heaviest particle (with mass $2m$) is situated at the origin, and if we use position vectors expressed in terms of the $\underline{i}$ and $\underline{j}$ unit vectors, rather than the coordinates.

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Feedback: You should get the centre of mass at $R = \frac{2}{5}a\left(\underline{i}+\underline{j}\right)$.

Activity 3.4

Write the full solution to the following problem in your exercise book. Please remember to draw a sketch of the system, select a coordinate system according to the guidelines in the toolbox for selecting a coordinate system, draw a sketch of your coordinate system and the object within it. You can use either position vectors or coordinates in your calculations.

Three particles, all with the same mass m, are situated at three of the corners of a square. Find the centre of mass of the system.

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Feedback: Your answer will depend on how you selected your coordinate system in relation to the three particles. The system is 2-dimensional, so it becomes just a matter of selecting how the particles lie in relation to the X and Y axes. If the three particles lie along the positive XY axes, with the middle one in the origin, then you should get $a\left(\underline{i}+\underline{j}\right)/3$ for the centre of mass where a denotes the length of the sides of the square.

Activity 3.5

The following problem is even more theoretical, in that we do not even give the masses of the two particles! Write the complete solution to your exercise book, and make sure to include a sketch!

Show that in a two-particle system, the ratio of the distances of the two particles from the centre of mass of the system is the inverse of the ratio of their masses.

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Feedback: The system is one-dimensional, and it might be easiest to take origin to coincide with one of the particles! You needed to introduce a coordinate system, find the centre of mass, find the distances from the centre of mass to the two particles, and check that their ratio is indeed the inverse of the ratio of the masses. See the workbook for a fully worked out solution.

CONCLUSION

In this unit you have learned how to

- find the centre of mass of a system of particles, using position vectors and using coordinates
- select a coordinate system to assist you in finding the centre of mass
- solve problems involving centres of mass of systems of particles

Remember to add the following tools to your toolbox:

- the definition of the centre of mass of a system of particles
- the toolbox for the task of selecting coordinate systems

After working through this unit, you should be able to find the centre of mass of systems of particles, introducing a coordinate system if necessary. In the rest of the study guide we will study the centres of mass of more and more complicated objects, but all the subsequent results will build on the simple case of a system of particles we have discussed in this unit!

In the next unit we will look at what the law governing the motion of the centre of mass looks like for our system of particles. After that, we will move on to more complicated objects.

Unit 4 THE MOTION OF THE CENTRE OF MASS OF A SYSTEM OF PARTICLES

Key questions:

- *What then is the equation describing the motion of the centre of mass?*

In this unit, we will show why exactly the centre of mass of a system of particles is so important in describing the behaviour of the system. To this end, we will apply Newton's laws of motion to each particle of the system separately, and end up with a very simple law linking the external forces acting on the system and the acceleration of the centre of mass, in perfect analogy with the forces acting on a single particle and its acceleration.

Contents of this unit:

4.1 The velocity and acceleration of the centre of mass

4.2 The law for the motion of the centre of mass

What you are expected know before working through this unit:

In this unit, you will need the concept of the centre of mass, which you learned in the previous unit, as well as a working knowledge of Newton's laws of motion. You will also need to remember what we mean by taking a derivative with respect to time, and how it links position to velocity and acceleration!

4.1 The velocity and acceleration of the centre of mass

Remember that we defined the centre of mass of a system of n particles as the position vector $\underline{R}$ given by

$$\underline{R} = \frac{\sum_{i=1}^{n} m_i \underline{r}_i}{\sum_{i=1}^{n} m_i} = \frac{m_1\underline{r}_1 + m_2\underline{r}_2 + \ldots + m_n\underline{r}_n}{m_1 + m_2 + \ldots + m_n}. \tag{4.1}$$

If we take into account the fact that the position vectors $\underline{r}_1, \ldots, \underline{r}_n$ of the n particles may change in time, that is, they may be functions of time t, then the position vector $\underline{R}$ of the centre of mass will also change in time, so that it is also a function of t. In particular, as with all position vectors which are functions of time, we can talk about its velocity vector $\underline{\dot{R}}$ and its acceleration vector $\underline{\ddot{R}}$.

Assuming that we are given the masses of the n particles, and we know all the position vectors $\underline{r}_1, \ldots, \underline{r}_n$ as functions of time t, how do we find the vectors $\underline{\dot{R}}$ and $\underline{\ddot{R}}$? We could calculate the expression of $\underline{R}$ as a function of time t and then find its derivative with respect to time t. However, the equation (4.1), which calculates the position vector of the centre of mass from the position vectors of the individual particles, will also readily give us a way of finding the velocity and acceleration vectors of the centre of mass directly from the velocity and acceleration vectors of the individual particles: differentiating both sides of (4.1) once and twice, respectively, with respect to time, we get the two formulas

$$\underline{\dot{R}} = \frac{\sum_{i=1}^{n} m_i \underline{\dot{r}}_i}{\sum_{i=1}^{n} m_i}, \qquad \underline{\ddot{R}} = \frac{\sum_{i=1}^{n} m_i \underline{\ddot{r}}_i}{\sum_{i=1}^{n} m_i}.$$

That is, we get the following results.

Result 4.1

Consider a system of n particles with masses $m_1, m_2, \ldots, m_n$.

(a) If the velocity vectors of the particles are $\underline{\dot{r}}_1, \underline{\dot{r}}_2, \ldots, \underline{\dot{r}}_n$ then the velocity vector $\dot{R}$ of the centre of mass is given by

$$\underline{\dot{R}} = \frac{\sum_{i=1}^{n} m_i \underline{\dot{r}}_i}{\sum_{i=1}^{n} m_i} = \frac{m_1 \underline{\dot{r}}_1 + m_2 \underline{\dot{r}}_2 + \ldots + m_n \underline{\dot{r}}_n}{m_1 + m_2 + \ldots + m_n}. \tag{4.2}$$

(b) If the acceleration vectors of the particles are $\underline{\ddot{r}}_1, \underline{\ddot{r}}_2, \ldots, \underline{\ddot{r}}_n$ then the acceleration vector $\ddot{R}$ of the centre of mass is given by

$$\underline{\ddot{R}} = \frac{\sum_{i=1}^{n} m_i \underline{\ddot{r}}_i}{\sum_{i=1}^{n} m_i} = \frac{m_1 \underline{\ddot{r}}_1 + m_2 \underline{\ddot{r}}_2 + \ldots + m_n \underline{\ddot{r}}_n}{m_1 + m_2 + \ldots + m_n}. \tag{4.3}$$

Note that just as we did in unit 3, we can again write alternative versions of these results in terms of the X-, Y- and Z-components of the vectors in some coordinate system, so that for instance we would find $\ddot{\bar{x}}$ (the X-component of the acceleration vector $\underline{\ddot{R}}$ of the centre of mass) from $\ddot{x}_1, \ddot{x}_2, \ldots, \ddot{x}_n$. However, in this unit it is important to remember that we are dealing with vectors and therefore we will most of the time prefer to work with the vectors $\underline{R}$, $\underline{\dot{R}}$ and $\underline{\ddot{R}}$. Of course, if we have fixed an XYZ coordinate system, then we can always express all the vectors in terms of the unit vectors $\underline{i}$, $\underline{j}$, $\underline{k}$ of the coordinate system!

The following example illustrates the two alternative ways of calculating the velocity and acceleration of the centre of mass, if we know the position vectors as functions of time. This example is just to remind you of the time-dependency of the position, velocity and acceleration vectors; in what follows, we will very rarely come across an example such as this, with the position expressed as a function of time!

Example 4.1

Two particles move around on the YZ-plane. Particle A has mass m and particle B has mass $3m$. The position vector of particle A varies in time such that the position at time t is given by $t\underline{j} - t^2\underline{k}$. Similarly, the position of particle B at time t is given by $\left(1 - t - t^2\right)\underline{j} + 5\underline{k}$. Find the centre of mass, the velocity of the centre of mass and the acceleration of the centre of mass at time t.

Solution: The position vectors of the two particles at time t are

$$\begin{aligned}\underline{r}_A &= t\underline{j} - t^2\underline{k},\\ \underline{r}_B &= \left(1 - t - t^2\right)\underline{j} + 5\underline{k}\end{aligned}$$

and by direct differentiation we see that their velocities are

$$\begin{aligned}\dot{\underline{r}}_A &= \underline{j} - 2t\underline{k},\\ \dot{\underline{r}}_B &= (-1 - 2t)\,\underline{j}\end{aligned}$$

and their accelerations are

$$\begin{aligned}\ddot{\underline{r}}_A &= -2\underline{k},\\ \ddot{\underline{r}}_B &= -2\underline{j}.\end{aligned}$$

Thus, since the masses of the two particles are $m_A = m$ and $m_B = 3m$, the position of the centre of mass of the two-particle system at time t is given by

$$\begin{aligned}\underline{R} &= \frac{m_A\underline{r}_A + m_B\underline{r}_B}{m_A + m_B}\\ &= \frac{m\left(t\underline{j} - t^2\underline{k}\right) + 3m\left(\left(1 - t - t^2\right)\underline{j} + 5\underline{k}\right)}{m + 3m}\\ &= \frac{1}{4}\left(3 - 2t - 3t^2\right)\underline{j} + \frac{1}{4}\left(15 - t^2\right)\underline{k},\end{aligned}$$

its velocity at time t is

$$\begin{aligned}\dot{\underline{R}} &= \frac{m_A\dot{\underline{r}}_A + m_B\dot{\underline{r}}_B}{m_A + m_B} = \frac{m\left(\underline{j} - 2t\underline{k}\right) + 3m\left((-1 - 2t)\,\underline{j}\right)}{m + 3m}\\ &= \frac{1}{4}(-2 - 6t)\,\underline{j} - \frac{1}{2}t\underline{k},\end{aligned}$$

and its acceleration at time t is

$$\begin{aligned}\ddot{\underline{R}} &= \frac{m_A\ddot{\underline{r}}_A + m_B\ddot{\underline{r}}_B}{m_A + m_B} = \frac{m\left(-2\underline{k}\right) + 3m\left(-2\underline{j}\right)}{m + 3m}\\ &= -\frac{6}{4}\underline{j} - \frac{1}{2}\underline{k}.\end{aligned}$$

Note that $\dot{\underline{R}}$ and $\ddot{\underline{R}}$ could, of course, also have been obtained from $\underline{R}$ by direct differentiation, as follows:

$$\underline{R} = \frac{1}{4}\left(3 - 2t - 3t^2\right)\underline{j} + \frac{1}{4}\left(15 - t^2\right)\underline{k}$$

when differentiated once, and twice, gives

$$\begin{aligned}\dot{\underline{R}} &= \frac{1}{4}(-2 - 6t)\,\underline{j} + \frac{1}{4}(-2t)\,\underline{k},\\ \ddot{\underline{R}} &= \frac{1}{4}(-6)\,\underline{j} + \frac{1}{4}(-2)\,\underline{k},\end{aligned}$$

which is the same result as before.◄

The following example is a much more typical example of the kind of calculations you will need to be able to do in this module!

Example 4.2
Two particles, both with mass m, are situated on the XY-plane. At a time t, particle 1 and particle 2 are both at the point with position vector $\underline{j}$. The velocities of both particles are zero. Particle 1 has a acceleration of $\underline{i} - \underline{j}$ and particle 2 has a acceleration of $3\underline{j}$. Calculate the values of the following quantities at time t :

(a) The position vector of the centre of mass of the system consisting of the two particles.

(b) The velocity vector of the centre of mass of the system.

(c) The acceleration vector of the centre of mass of the system.

Solution: At time t, we have the following quantities:

	Particle 1	Particle 2
mass:	$m_1 = m$	$m_2 = m$
position vector:	$\underline{r}_1 = \underline{j}$	$\underline{r}_2 = \underline{j}$
velocity:	$\dot{\underline{r}}_1 = \underline{0}$	$\dot{\underline{r}}_2 = \underline{0}$
acceleration:	$\ddot{\underline{r}}_1 = \underline{i} - \underline{j}$	$\ddot{\underline{r}}_2 = 3\underline{j}$

(a) Position vector of the centre of mass:

$$\underline{R} = \frac{m \cdot \underline{r}_1 + m \cdot \underline{r}_2}{m + m} = \frac{1}{2}\left(\underline{r}_1 + \underline{r}_2\right) = \underline{j}.$$

(b) Velocity of the centre of mass:

$$\dot{\underline{R}} = \frac{m\,\dot{\underline{r}}_1 + m\,\dot{\underline{r}}_2}{m + m} = \underline{0}.$$

(c) Acceleration of the centre of mass:

$$\ddot{R} = \frac{m_1\,\ddot{\underline{r}}_1 + m_2\,\ddot{\underline{r}}_2}{m_1 + m_2} = \frac{m\left(\underline{i} - \underline{j}\right) + m\left(3\underline{j}\right)}{2m} = \frac{1}{2}\underline{i} + 2\,\underline{j}\,.$$

Note that although the centre of mass is not currently moving (its velocity is zero), it has non-zero acceleration which means that the velocity is changing, and will soon be non-zero!◄

Activity 4.1

Answer the following questions.

Two particles are situated on the XY-plane. Particle 1 has mass m and particle 2 has mass $2m$. At a time t, particle 1 has position vector $-2\underline{j}$ and particle 2 has position vector $\underline{j}$. Particle 1 has a velocity of $\underline{i} - \underline{j}$ and particle 2 has a velocity of $\underline{j} - \underline{i}$. Particle 1 has a acceleration of $2\underline{i}$ and particle 2 has a acceleration of $-\underline{i}$. Calculate the position, velocity and acceleration of the centre of mass of the system.

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Feedback: Here, position and acceleration of the centre of mass both vanish, but the velocity is a non-zero vector. $\frac{1}{3}\underline{j} - \frac{1}{3}\underline{i}$. From zero acceleration it follows that the velocity of the system is constant, and the centre of mass will move away from the origin in a straight line in the direction given by the velocity vector, at constant speed.

As you now know, the motion of the centre of mass of a system (its position, velocity, acceleration) is found as a weighted average of the corresponding quantities of the individual particles. In particular, the motion of the centre of mass can be very different from the motion of any of the particles! The next important activity reminds you of this fact.

Activity 4.2

Which of the next statements are true and which are false? You should be able to justify your answers, either by proving the fact, or by giving a counter-example which proves that it is false! Write your answers, with justifications, in your exercise book.

True or false:

1. If the velocities of all the particles are zero, then the velocity of the centre of mass is zero.
2. If the velocity of the centre of mass is zero, then the velocities of all the particles are also zero.
3. If the centre of mass has non-zero velocity, then at least one of the particles must be moving.

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Feedback: Compare your answers with the answers below. Make sure you know and understand the correct reasoning, since misunderstandings about this are very common and can lead to confusion later on! 1 is correct: if all the velocities of the particles are zero then so is their weighed average. 2 is false: velocities of particles can cancel out and give a zero velocity for the centre of mass. 3 is true, and follows from 1.

Again, sometimes the problem is stated without any reference to a coordinate system; we then have to introduce a suitable system. The following example is of this type!

Example 4.3

Two cars are moving along a straight and level road. Car 1 has a mass of 800 kg and it is moving at 100 km/h. It is followed by car 2 with a mass of 1600 kg, moving at 60 km/h. How fast is the centre of mass of the two cars moving?

Solution: Let us take the X-axis to go along the road, with the positive direction in the direction that the cars are travelling. Then car 1 and car 2 have the velocities

$$\begin{aligned} \underline{\dot{r}}_1 &= 100\underline{i}, \\ \underline{\dot{r}}_2 &= 60\underline{i} \end{aligned}$$

(with the units in kilometers per hour). Their masses are given as

$$\begin{aligned} m_1 &= 800, \\ m_2 &= 1600 \end{aligned}$$

(in kilograms). It follows that the velocity of the centre of mass equals

$$\begin{aligned} \underline{\dot{R}} &= \frac{m_1\,\underline{\dot{r}}_1 + m_2\,\underline{\dot{r}}_2}{m_1 + m_2} = \\ &\quad \frac{800 \cdot 100\underline{i} + 1600 \cdot 60\underline{i}}{800 + 1600} \\ &= 73\frac{1}{3}\underline{i} \quad \text{(kilometers per hour).} \end{aligned}$$

Remark: Note that we can work here with the units km/h. However, please do always remember that $\frac{m}{s}$ (metres per second) is the standard unit of velocity!◄

Activity 4.3

Solve the following problem in your exercise book. Draw first a sketch with the positions of the two cars, as well as their velocities and accelerations as vectors - you must get the directions these right! You will need to introduce a coordinate system.

Problem: Car 1 with a mass of 2400 kg is moving along a straight stretch of road. It is followed by car 2 with a mass of 1600 kg. At a certain moment, car 2 is 20 meters behind car 1. At that moment, car 1 travels at 60 km/h, while car 2 is moving at 80 km/h. Where is the centre of mass of the system of the two cars at that moment? How fast is the centre of mass moving? If at that particular moment, car 1 is braking, at acceleration 5 m/s^2, and car 2 is speeding up at 2 m/s^2,what is the acceleration of the centre of mass? Describe the motion of the centre of mass: In what direction is it moving? Is its speed increasing, decreasing or constant?

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Feedback: The centre of mass lies the distance 12 metres in front of car 2, and 8 metres behind car 1. The velocity vector of the centre of mass is $68\underline{i}$ (kilometers per hour) and the acceleration vector is $-2.2\underline{i}$ (m/s^2) if the direction of the unit vector $\underline{i}$ is from car 2 towards car 1 (which is also the direction both the cars are moving). This means that the centre of mass is, at that moment, moving in the direction the cars are moving, with velocity 68 km/h. Since the acceleration is in the negative $\underline{i}$-direction while velocity is in the positive direction, we know that the speed of the centre of mass is decreasing.

4.2 The law for the motion of the centre of mass

For a particle, Newton's laws of motion link the acceleration of the particle to the forces acting on it. Our goal in this section is to see what happens in the case of a system of particles. We will consider a general system of particles with forces acting on it, we will apply Newton's laws of motion to each particle separately, and then see what conclusions we can make about the motion of the centre of mass of the system — the outcome will be a very nice formula indeed!

So, let us consider a system consisting of n particles, with masses $m_1, m_2, \ldots, m_n$ and with position vectors $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_n$, respectively. Unless the system is completely isolated, which is impossible in real life, various forces act on the particles in the system. Let us examine the forces acting on a single particle. We can distinguish between **external** forces, which are the forces acting on the particle from outside the system, and **internal** forces, which are due to the actions of any of the particles which are a part of the system.

Note that what is internal and what is external depends on how the system is defined. For example, if we consider the planets moving around the sun as a system of "particles", then the gravitational force the sun exerts on a planet can be regarded as an external force. On the other hand, the gravitational force on a planet due to another planet would be considered as an internal force in our chosen system. However, if we consider the system consisting of the planets <u>and</u> the sun, then both of the gravitational forces mentioned above are internal forces — an external force in this case would be, for instance, the force of gravity from a nearby star or galaxy!

Let $\underline{F}_i$ be the resultant **external** force on the ith particle in the system, that is, the sum of all the vectors representing external forces acting on the particle. Let $\underline{f}_{ij}$ be the **internal** force on the ith particle exerted by the jth particle, and similarly let $\underline{f}_{ji}$ be the force on the jth particle exerted by the ith particle.

The internal forces must satisfy Newton's third law, the law of action and reaction, which states that the forces that the ith and jth particles exert on each other are equal and opposite. We can express this as

$$\underline{f}_{ij} = -\underline{f}_{ji}$$

so that

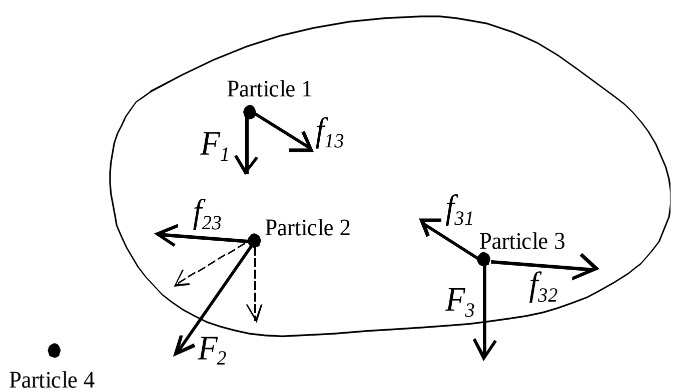
$$\underline{f}_{ij} + \underline{f}_{ji} = \underline{0},$$

that is, all the internal forces cancel each other out in pairs.

Note that according to the notation introduced above, $\underline{f}_{ii}$ would be the force that the particle exerts on itself. This must be zero: $\underline{f}_{ii} = \underline{0}$. This follows from the action-reaction result above, applied in the case $i = j$!

Example 4.4

The sketch below illustrates the external and internal forces that could act on/in a system of particles. The system consists of particles 1, 2 and 3. Particles 1 and 3 attract each other, leading to the internal forces $\underline{f}_{13}$ and $\underline{f}_{31}$. Particles 2 and 3 repulse each other, hence the internal forces $\underline{f}_{23}$ and $\underline{f}_{32}$. There are no internal forces between particles 1 and 2. A downwards directed external force (which could, for instance, be the force of gravity) acts on all three particles. No other external forces act on particles 1 and 3, and therefore the resultant external forces $\underline{F}_1$ and $\underline{F}_3$ are directed downwards. Particle 2, on the other hand, is attracted by another particle, particle 4, which is not a part of the system. Hence the resultant external force $\underline{F}_2$ acting on particle 2 is directed down-and-left, as shown. Note that particle 4 must also be attracted by particle 2, and other forces may also act on it, but since particle 4 is not part of the system we are interested in, we will not analyse the forces acting on it.◀



Let us now apply Newton's second law (2.2) separately to each of the particles in a system. Newton's second law tells us that particle number i satisfies the equation of motion

$$\underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij} = m_i \ddot{\underline{r}}_i. \tag{4.4}$$

Here, the left-hand side is the vector sum of all the forces, the external plus the internal ones, acting on the ith particle; and the right-hand side involves the mass and the acceleration vector of the ith particle.

If we now add up over all the particles, $i = 1, \ldots, n$, on both sides of (4.4) we get

$$\sum_{i=1}^{n} \left(\underline{F}_i + \sum_{j=1}^{n} \underline{f}_{ij} \right) = \sum_{i=1}^{n} m_i \ddot{\underline{r}}_i. \tag{4.5}$$

Let us analyse the left and right sides of this equation. Firstly, by re-arranging the sums which appear on the left, we can re-write the left side of (4.5) as

$$\begin{aligned}
\sum_{i=1}^{n}\left(\underline{F}_i+\sum_{j=1}^{n}\underline{f}_{ij}\right) &= \sum_{i=1}^{n}\underline{F}_i+\sum_{i=1}^{n}\sum_{j=1}^{n}\underline{f}_{ij} \\
&= \sum_{i=1}^{n}\underline{F}_i+\left\{\underline{f}_{11}+\underline{f}_{22}+\cdots+\underline{f}_{nn}\right\}+\left\{\left(\underline{f}_{12}+\underline{f}_{21}\right)+\cdots+\left(\underline{f}_{n,n-1}+\underline{f}_{n-1,n}\right)\right\} \\
&= \sum_{i=1}^{n}\underline{F}_i+\underline{0}+\underline{0} \\
&= \sum_{i=1}^{n}\underline{F}_i
\end{aligned}$$

since all the $\underline{f}_{ii}$ terms are zero, while all the other internal forces cancel each other out in pairs. That is, the sum of all the forces acting on all the particles of the entire system, external or internal, equals the sum of all the external forces acting on all the particles of the system.

We will from now on say that a force acts on a system if the force acts on any particle which is a part of the system.

Let $\underline{F}$ now denote the resultant of all the external forces acting on the entire system, that is, the sum of all external forces acting on all the particles:

$$\underline{F}=\sum_{i=1}^{n}\underline{F}_i.$$

Then, the left-hand side of (4.5) is simply $\underline{F}$.

On the other hand, the right side of (4.5) is equal to $M\ddot{\underline{R}}$ where $\ddot{\underline{R}}$ is the acceleration of the centre of mass of the system, and

$$M=\sum_{i=1}^{n}m_i$$

is the total mass of the system. Therefore, we have arrived at the following result:

Result 4.2 (Equation for the motion of the centre of mass for a system of particles)

Let a system consist of n particles, and let M be the total mass of the system, $\underline{R}$ the position vector of its centre of mass, and $\underline{F}$ the resultant of all the external forces acting on the system, that is,

$$M=\sum_{i=1}^{n}m_i,\quad \underline{F}=\sum_{i=1}^{n}\underline{F}_i,\quad \underline{R}=\frac{\sum_{i=1}^{n}m_i\underline{r}_i}{\sum_{i=1}^{n}m_i}.$$

Then

$$\underline{F}=M\ddot{\underline{R}} \tag{4.6}$$

If you go through the proof of this result, you will see that the result followed directly from applying Newton's second and third laws of motion to the particles in the system.

We have merely introduced the notation of a centre of mass, and ended up with a very simple expression (4.6) to describe the way the centre of mass moves. To find out the acceleration of the centre of mass of a system of particles, all we need to know is the total mass of the system and the sum of all the external forces acting on it.

Notice the striking resemblance between this equation, valid for a system of particles, and Newton's second law of motion for a single particle, (2.2). Indeed, if the system consisted of just one particle, then this would give exactly Newton's second law. The centre of mass of a system is just a hypothetical point in space, describing the average location of the mass of the system, but according to Result 4.2 it moves just like a particle whose mass is equal to the total mass M of the system, when it is acted upon by the force $\underline{F}$, which is the sum of all the external forces acting on the system.

It follows that it is possible to approximate the motion of a system of particles by the motion of a single particle with the mass of the system (at least if we are willing to ignore the way the particles move in relation to each other). However chaotic the motion of the system may be, underlying it is a pattern as uncomplicated as the motion of a single particle.

The following example deals with the straightforward case of finding the acceleration of the centre of mass when all forces acting on the system are known.

Example 4.5
Let us assume that particle 1 has a mass m, particle 2 has a mass $2m$ and particle 3 has a mass $2m$. The force $\underline{F}_1 = \underline{i} + \underline{j}$ acts on particle 1, the force $\underline{F}_2 = -4\underline{k}$ acts on particle 2 and the force $\underline{F}_3 = \underline{i} - \underline{j} + \underline{k}$ acts on particle 3. Find the acceleration vector of the centre of mass.

Solution:
The following information is given:

	Particle 1	Particle 2	Particle 3
Mass:	$m_1 = m$	$m_2 = 2m$	$m_3 = 2m$
Force acting on particle:	$\underline{F}_1 = \underline{i} + \underline{j}$	$\underline{F}_2 = -4\underline{k}$	$\underline{F}_3 = 2\underline{i} - 2\underline{j} + \underline{k}$

We have two alternative ways of finding the acceleration of the centre of mass, namely (a) by first calculating the acceleration of each particle according to Newton's second law and then using (4.3); or (b) by using Result 4.2. We will do both calculations, just to check that they do indeed give the same result!

(a) Applying Newton's second law to each particle separately, we get

$$\begin{aligned}
m_1 \ddot{\underline{r}}_1 &= \underline{F}_1 \quad \therefore\ \ddot{\underline{r}}_1 = \frac{\underline{F}_1}{m_1} = \frac{1}{m}\left(\underline{i} + \underline{j}\right),\\
m_2 \ddot{\underline{r}}_2 &= \underline{F}_2 \quad \therefore\ \ddot{\underline{r}}_2 = \frac{\underline{F}_2}{m_2} = -\frac{2}{m}\underline{k},\\
m_3 \ddot{\underline{r}}_2 &= \underline{F}_3 \quad \therefore\ \ddot{\underline{r}}_2 = \frac{\underline{F}_2}{m_3} = \underline{i} - \underline{j} + \frac{1}{2m}\underline{k},
\end{aligned}$$

and by applying (4.3), we get

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\begin{aligned}
\ddot{R} &= \frac{m_1 \underline{\ddot{r}}_1 + m_2 \ddot{r}_2 + m_3 \underline{\ddot{r}}_2}{m_1 + m_2 + m_3} \\
&= \frac{m\left(\frac{1}{m}\left(\underline{i} + \underline{j}\right)\right) + 2m\left(-\frac{2}{m}\underline{k}\right) + 2m\left(\underline{i} - \underline{j} + \frac{1}{2m}\underline{k}\right)}{m + 2m + 2m} \\
&= \frac{1}{5m}\left(3\underline{i} - \underline{j} - 3\underline{k}\right).
\end{aligned}
$$

(b) We will now calculate the acceleration vector of the centre of mass of the system by utilising Result 4.2. If we consider the system formed by the three particles, then the total mass of the system is $M = 5m$, and the total force acting on the system is

$$\underline{F} = \underline{F}_1 + \underline{F}_2 + \underline{F}_3 = 3\underline{i} - \underline{j} - 3\underline{k}.$$

According to Result 4.2, the acceleration of the centre of mass, $\underline{\ddot{R}}$, obeys the law of motion

$$M\,\underline{\ddot{R}} = \underline{F}$$

from which we get

$$\underline{\ddot{R}} = \frac{1}{M}\,\underline{F} = \frac{1}{5m}\left(3\underline{i} - \underline{j} - 3\underline{k}\right)$$

Methods (a) and (b) did give the same answer for the acceleration of the centre of mass, but I am sure you agree that method (b) was much faster — in (a), we first divided the force acting on each particle by the mass of the particle to find the acceleration of that particle, and then again multiplied by the mass when applying equation (4.3).◀

Activity 4.4

Solve the following problem using <u>*both*</u> *of the methods illustrated earlier.*

Three particles with masses m, $2m$ and $4m$ are at rest at position $2\underline{i} + 3\underline{j}$, $\underline{i} + 2\underline{j}$ and $-\underline{i} - 2\underline{j}$, respectively.

- Find the centre of mass of the system.
- Find the acceleration of the centre of mass when the force $-\underline{i}$ acts on the first particle, the force $\underline{i} + \underline{j}$ on the second particle, and the force $2\underline{j}$ on the third particle.

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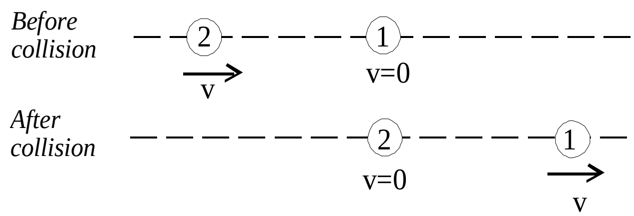
Feedback: The centre of mass is at $-\frac{1}{7}\underline{j}$, and the acceleration is $\frac{3}{7m}\,\underline{j}$.

Note that from the equation for the motion of the centre of mass

$$M\,\underline{\ddot{R}} = \underline{F}$$

in Result 4.2 it follows in particular that if the resultant external force is zero (for instance, if the external forces acting on the system all cancel out), then the acceleration of the centre of mass is zero. This means that the centre of mass has a constant velocity, and it thus moves in a straight line, at constant speed. This must hold, regardless of how many internal forces may be acting in the system! The following example illustrates this.

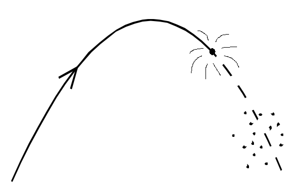
Example 4.6
Two particles of identical masses are involved in an ideal elastic collision, as follows: Initially, particle 1 is at rest and particle 2 moves towards it at constant velocity υ. After the collision, particle 2 is at rest and particle 1 moves away from it at the same constant velocity υ. Investigate the motion of the centre of mass of the system formed by the two particles.



Solution:
Since the two particles have the same mass, the centre of mass is always situated halfway between them. Before the collision particle 2 moves towards particle 1 with velocity υ, and therefore the centre of mass (always being halfway between the two particles) is moving towards the stationary particle 1 with velocity $\frac{\upsilon}{2}$. By similar reasoning, we see that after the collision the centre of mass moves away from the now stationary particle 2 with velocity $\frac{\upsilon}{2}$. That is, regardless of the collision, the centre of mass of the system always moves forward with a constant velocity $\frac{\upsilon}{2}$. This is in agreement with Result 4.2. No external forces act on the system, so the acceleration of the centre of mass is zero, and motion with uniform velocity is predicted. The collision is just an internal rearrangement of matter within the system, and does not change the motion of the centre of mass.◄

In the following example, external forces do act on the system, but again Result 4.2 simplifies an otherwise complicated situation.

Example 4.7
Suppose that, at a fireworks display, a rocket of mass M is launched on a parabolic path. At a certain point along the path, it explodes into fragments. If the explosion had not occurred, the rocket would have continued along the parabolic trajectory shown with a dotted line in the sketch. If the explosion does happen, then the **centre of mass of the fragments** follows the same trajectory that the unexploded rocket would have followed.



To prove this, we note that the forces of the explosion are **internal** to the system formed by the rocket or its fragments. The total **external** force acting on the system is the force of gravity, which is equal to $M\underline{g}$ both before and after the explosion. Thus the rocket and the centre of mass of the fragments follow exactly the same path. This, regardless of the fact that right after the explosion, we can see some of the fragments actually moving upwards, or backwards!◄

The principle that the acceleration of the centre of mass is zero if the total external force acting on the system is zero is often very useful for solving problems. Both of the following activities apply this principle!

Activity 4.5

Solve the following problem, by analysing the motion of the centre of mass. Write the full solution, in complete detail, in your exercise book. Include a sketch of the situation before and after the explosion.

Problem: An isolated particle of mass m is moving in the XY–plane. It moves along the X-axis at a constant speed, when it suddenly explodes into two fragments of masses $\frac{1}{4}m$ and $\frac{3}{4}m$. An instant later, the smaller fragment is the distance ℓ above the X-axis. How far below the X–axis is the larger fragment at this instant?

....

Feedback: The larger fragment will be the distance $\ell/3$ below the X–axis. You should use the fact that no external forces act on the particle, either before or after it explodes, and therefore the centre of mass of the system formed of the two fragments will still be on the X–axis. A full solution should include the following:

- A sketch which should show the X and Y–axes, and the before and after situations (either in the same or separate sketches; you should clearly label each case). The before sketch should show the particle of mass m on X–axis, and the after sketch should show the two particles, the one with mass $m/4$ the distance ℓ above the X–axis and the one with mass $3m/4$ at the same X-coordinate but an unknown distance below the X–axis.
- You must denote the unknown distance asked for by some variable.
- You must explain briefly what leads you to the equation you are going to use: namely, that the centre of mass will still be at the X-axis, since no external forces act on the system.
- Finally, write down the necessary equation, and solve for your unknown variable from it.

If you omitted some of these, please re-write the solution in your workbook! (In an assignment or examination question, you would have lost points had you not included all of these!)

Activity 4.6

Solve the following problem. Write the full solution, in complete detail, in your exercise book.

Two particles A and B are initially at rest, the distance a apart. Assume that A has mass m and B has mass M. Further, assume that A and B attract each other with a constant force F. No external forces act on the system. At what distance from A's original position do the particles collide? Hint: How does the centre of mass move?

....

Feedback: Since only internal forces act on the system, the centre of mass does not move. Therefore, when the particles collide, they must both be at the centre of mass. Make sure you understand this, and are able to explain it clearly in you solution! Your solution must include the following:

- You must decide on a coordinate system, and show a sketch with your coordinate system and the before/after situations of the particles.
- From your initial positions of the particles, you can calculate where the centre of mass is.
- You must explain that the particles will collide at the centre of mass, and explain why this is so.
- The question you need to answer is: at what distance from A's original position do the particles collide?. This is equal to the distance from A's initial position to the centre of mass.

If you omitted some of these, please re-write the solution in your exercise book! The particles collide at a distance $a\frac{M}{m+M}$ from the original position of particle A.

Note that there are other ways to solve this problem: for instance, you can use the forces acting on each particle to analyse how they move and hence where they collide (but this will be more tedious and longer than the solution indicated above, which uses the lack of motion of the centre of mass!)

We have seen how applying Newton's laws of motion to the individual particles of a system, and then introducing the concept of a centre of mass, leads to a very simple equation for the motion of the centre of mass. Before we finish this unit, we wish to look at a similar simplification concerning linear momenta — note however that we will not be using the concept of the linear momentum in this module, so the paragraph below is just for your information!

Remember that Newton's second law describing the motion of a particle can alternatively be expressed in terms of the linear momentum of the particle: If $\underline{p} = m\dot{\underline{r}}$ then

$$\frac{d\underline{p}}{dt} = \underline{F}.$$

Now, a similar simplifying result holds for the linear momentum of a system. A particle with mass m and position vector $\underline{r}$ has a linear momentum $\underline{p} = m\dot{\underline{r}}$.

The total linear momentum of a system of particles, denoted by $\underline{P}$, is the (vector) sum of the linear momenta of the individual particles: if there are n particles in the system and m_i and $\underline{r}_i$ are the mass and position vector of particle number i, then we define

$$\underline{P} = \sum_{i=1}^{n} m_i \dot{\underline{r}}_i. \tag{4.7}$$

It follows directly from the definition of the centre of mass of a system, that we can equivalently define

$$\underline{P} = M\dot{\underline{R}} \tag{4.8}$$

where $\underline{R}$ is the position vector of the centre of mass of the system, and M is the total mass of the system.

If we differentiate (4.8) once with respect to time and compare the result with (4.6), we get an alternative form of the equation for the motion of the centre of mass:

$$\frac{d\underline{P}}{dt} = \underline{F}. \tag{4.9}$$

This result is again very similar to equation (2.4), that is, Newton's second law for the one-particle case, using linear momenta. Again this reflects the fact that the centre of mass of a system moves just like a particle, which is acted on by the sum of all the external forces acting on the system.

But is the result of this unit not an oversimplification? What about the rest of the story, that is, the motion of the particles in relation to each other? We must admit that in some of the examples above, there was certainly a lot of detail lost in the attempt to describe the behaviour of the system using only the relatively simple motion of the centre of mass! We had to leave out completely the way that the particles move in relation to the centre of mass. However, as we will learn later on, the general motion of an object can be described very nicely as a combination of rotation and translation, and the translation will be the translation of the centre of mass, so the motion of the centre of mass will be very important then!

CONCLUSION

In this unit you have learned how to

- find the velocity and acceleration of the centre of mass of a system of particles, from the velocity and acceleration of the particles
- apply the equation for the motion of the centre of mass, which links the acceleration of the centre of mass to the resultant force acting on the system
- solve problems using the results in this chapter

Remember to add the following tools to your toolbox:

- the formula for calculating the velocity and acceleration of the centre of mass from those of the particles
- resultant external force acting on a system is proportional to the acceleration of the centre of mass
- the equation for the motion of the centre of mass

In this and the previous unit, we learned how find the centre of mass of a system of particles, and how the motion of the centre of mass is determined by the action of all the forces acting on the system. In the next unit we will start moving from systems of particles to objects.

Unit 5 THE CENTRE OF MASS OF A RIGID BODY

Key questions:

- *Enough about systems — what about ordinary objects?*
- *How do ordinary objects differ from systems of particles?*
- *How do we find their centres of mass?*

Introduction

So far we have discussed general systems of particles, but of course we are really more interested in objects like hammers or tables. But, everyday things like hammers are just special kinds of systems of particles with the properties of being rigid (rather than freely moving collections of particles), and having a continuous structure (with no spaces between the different particles which they are made of). In this unit we will discuss the special cases of rigid systems of particles, and rigid systems with a continuous structure. When we are done, we will also know how to find the centre of mass of ordinary objects – which is a skill we will need throughout the rest of the module! The unit also includes a collection of helpful rules which can simplify finding centres of mass in many cases.

Contents of this unit:

5.1 Centres of mass of rigid systems of particles

5.2 Using integration to find centres of mass of rigid bodies (continuous structure)

5.3 Helpful rules for finding centres of mass

What you are expected know before working through this unit:

This unit is based on the definition of the centre of mass in unit 3, but we will extend that definition by bringing in integration. Now would be a good time to refresh your knowledge about integrals!

5.1 Centres of mass of rigid systems of particles

A **rigid** system of particles has the property that the particles are kept in fixed positions relative to each other by some strong internal forces. That is, the system has a fixed, unchangeable shape and each one of its particles has a fixed position relative to all the other particles. If a system is rigid, then we can imagine picking it up from any part of it and turning it around while the system stays of exactly the same shape — which is something we can do with a pen or a hammer, but not with a handful of rice or a plastic bag! For a system to be rigid, we should be able to imagine all the particles to be connected to each other with massless rods. (Note that "massless" here implies something that has no effect as far as the centre of mass and actions of forces are concerned, so that it can simply be ignored - its only purpose is to keep the particles in their places!)

Now, one important difference between the centre of mass of a *general* system of particles and that of a *rigid* system of particles is this: **For a rigid system of particles, the centre of mass has a fixed position in relation to the system.** When we "pick up" the system, we can also point to a certain position and say that the centre of mass is exactly there, at all times, whichever way we turn the system! Compare this with the handful of rice: we can never say that the centre of mass lies, say, 2 millimetres to the right of a certain grain of rice, since the rice grains can move around freely and the centre of mass moves along with them. On the other hand, if two particles are fixed so that they are always 1 metre from each other, then we can certainly say that the centre of mass always lies, say, 30 cm from particle 1, towards particle 2.

Thus we can say that the centre of mass of a rigid system is *at a certain point*, often denoted by G, within the system. (Note, of course, that the point G does not have to coincide with any particle of the system - for instance the centre of mass of a ring is in the middle of the ring, where there are no particles!)

Activity 5.1

This activity aims to make sure that you understand correctly the concept of a rigid system of particles.

Which of the following systems are rigid and which are not rigid? Tick the appropriate box!

Object	**rigid**	**not rigid**
A ring		
A rubber band		
A closed book		
An open book		
A bicycle		

....

Feedback: Remember that the decision is based on whether or not all the different parts of the object or system will always stay in the same fixed positions in relation to each other! The following objects are definitely rigid: a ring, a closed book (if it has hard covers, so it can't be bent). The following are usually not rigid: a rubber band, a bicycle or an open book (since various parts can be turned into different positions).

Recall that in Unit 3 you learned how to find the centre of mass of a system of particles, using the following procedure: 1. Introduce a coordinate system, 2. Express the positions of the particles in terms of the chosen coordinate system, 3. Use the formulas given in that unit to find the x, y and z coordinates or the position vector $\underline{R}$ of the centre of mass.

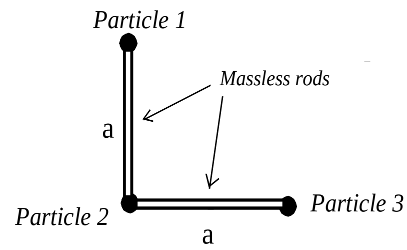
The centre of mass of a rigid system can be found similarly to the centre of mass of a general system. Since the centre of mass is now a particular point of the system, it is often easiest to also select a coordinate system fixed relative to the system. The centre of mass of a rigid system will then be a fixed point within the system, and it will not change in time.

Example 5.1

Three particles, all with mass m, are joined together by two rigid, massless rods of length a to form an L-shaped system. Find the centre of mass of the system.

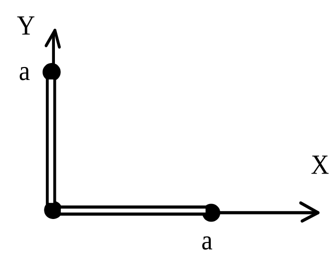
Solution

The system looks like this:



We have drawn the three particles, numbered 1, 2, and 3. (Since all the particles have the same mass, we have not mentioned their masses in the sketch.) We have also drawn in the sketch the massless rods, together with their lengths. However, we must remember that the system consists of the three particles only, and that the rods must be ignored, since they are massless!

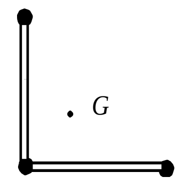
Note that there is no mention of a coordinate system in the statement of the problem. But remember that we must have a coordinate system in place in order to be able to apply the definitions of the centre of mass given in unit 3. Therefore, the first thing we must do is to decide on a coordinate system. Since the system is rigid, and the centre of mass G is at some fixed position in relation to the system, we should also choose our coordinate system in relation to the system. Therefore, let us assume that the X- and Y-axes lie along the arms of the L, with the origin in the corner (at particle 2), as shown below:



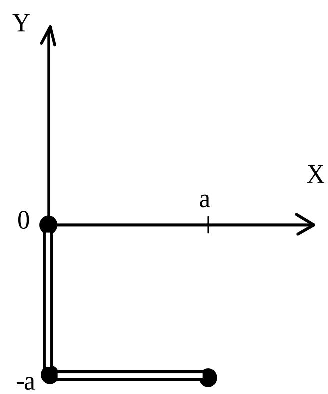
In this coordinate system, the three particles lie at points $(x_1, y_1) = (0, a)$, $(x_2, y_2) = (0, 0)$ and $(x_3, y_3) = (a, 0)$. Now, we can apply equations (3.2) and (3.3). Remember that here we have $m_1 = m_2 = m_3 = m$! We find that the coordinates of the centre of mass are $(\bar{x}, \bar{y})$ where

$$\begin{aligned} \overline{x} &= \frac{m \cdot 0 + m \cdot 0 + m \cdot a}{m + m + m} = \frac{a}{3}, \\ \overline{y} &= \frac{m \cdot a + m \cdot 0 + m \cdot 0}{m + m + m} = \frac{a}{3}. \end{aligned}$$

The centre of mass G lies at the point $(a/3, a/3)$ when the coordinate system is the one indicated above. However, we would like to be able to refer to the point G without having to first explain what the coordinate system is — where does point G lie in relation to the three particles? Since the point $(a/3, a/3)$ lies at the distance $\sqrt{2}a/3$ from the origin of the XY-plane (according to Pythagorean' Theorem), we can refer to point G by saying that the centre of mass of the system lies on the diagonal of the L-shape, the distance $\sqrt{2}a/3$ from the corner.



What happens if we choose another XY-coordinate system? Let us, for instance, assume that the origin is at the upper left corner of the L (at particle 1), but that the X- and Y-axes are still parallel to the arms of the L.



Now, according to the new coordinate system, the three particles lie at points $(x_1, y_1) = (0,0)$, $(x_2, y_2) = (0, -a)$ and $(x_3, y_3) = (a, -a)$. Therefore, the centre of mass is at the point $(\bar{x}, \bar{y})$, where

$$\begin{aligned}\bar{x} &= \frac{m \cdot 0 + m \cdot 0 + m \cdot a}{m + m + m} = \frac{a}{3}, \\ \bar{y} &= \frac{m \cdot 0 + m \cdot (-a) + m \cdot (-a)}{m + m + m} = -\frac{2a}{3}.\end{aligned}$$

The centre of mass G therefore lies at the point $(a/3, -2a/3)$ when the coordinate axes are as indicated in the picture above. However, this still refers to the same point, namely one which lies on the diagonal of the L-shape, the distance $\sqrt{2}a/3$ from the corner.◀

In the example above, we did the calculations with two different coordinate systems. Both the coordinate systems selected gave us the same point G as the centre of mass. But, remember that to find the centre of mass, we need to first find the coordinate positions of all the particles in the selected coordinate system, and often a good choice of the coordinate system can make the calculations easier.

Activity 5.2

Now it is your turn to find the centre of mass of a rigid system. Write a complete solution to the problem below in your exercise book. Pay special attention to the following:

- *Decide on a suitable coordinate system, based on the considerations above.*
- *Draw a sketch of the system, and show your coordinate system clearly in your sketch.*

Four particles, with masses m, $2m$, $2m$ and $2m$ are situated at the corners of a massless square with sides of length a. Find the centre of mass of the system, and explain where on the system the centre of mass lies.

. . . .

Feedback: Did you include the include the following in your answer?

- A sketch of the four particles, at the corners of $a \times a$ square, with the positions of the m and the three $2m$ massed particles clearly indicated.
- Your selected coordinate system marked in the sketch.
- The coordinate positions of all four particles in terms of your selected coordinate system
- The position of the centre of mass of the system, using your coordinate system.
- An explanation of where the in relation to the system the centre of mass lies.

The coordinates you get for the centre of mass depend on your selected coordinate system. But, whatever coordinate system you chose, you must get the following point as the coordinate system: a point along the diagonal from the lightest particle towards the opposite particle, at the distance $\frac{4}{7}a\sqrt{2}$ from the lightest particle. A model solution to this problem can be found in your workbook.

5.2 Using integration to find centres of mass of rigid bodies with continuous structure

A **rigid body** means a rigid system consisting of a very large number of particles which are so closely packed that the body can be assumed to have a **continuous structure**. The object could be three-dimensional (e.g. a hammer or a chair), but it could also be two-dimensional, with negligible thickness, such as a thin sheet of metal, or one-dimensional, such as a thin rod.

Note that such an object can indeed be assumed to be formed of a finite number of particles (atoms), where the particles are kept in fixed positions relative to each other by strong internal forces. This is, of course, only a model, since matter is made up of atoms which behave according to quantum mechanics and not Newtonian mechanics. Nevertheless, the assumption works very well in most practical situations.

As it is just a special case of a rigid system of particles, a rigid body also has a centre of mass indicated by some point G fixed in relation to the body. In principle, the centre of mass of a rigid body can be calculated as before, by the equation (3.1) or (3.2) to (3.4), summing up over all the particles forming the rigid body. However, there are so many particles (atoms) that this is not practical. For example, one pencil could contain 10^{23} atoms – that is, 1 followed by 23 zeroes! Instead, we will utilise the fact that the particles in a rigid body are packed so closely together that we can assume that they form a continuous structure with infinitely many particles packed together, and with no spaces left between them. Of course this is not really true, but once we assume it, we can use a very handy mathematical concept called integration, which enables us to deal quite easily with the problem of summing up infinitely many infinitely small terms.

In the following, we will describe how and why we are justified in moving from sums to integrals when calculating the centre of mass of a system with continuous structure. This will lead us to an alternative equation for calculating the centre of mass of a rigid body, based on integrals, rather than sums.

5.2.1 Derivation of the integration formulas for finding centres of mass for rigid bodies

To calculate the centre of mass of a rigid body, we can proceed as follows. Instead of considering all the actual particles which form the body, we divide the body into n small elements, with masses $\triangle m_1, \triangle m_2, \ldots, \triangle m_n$ and position vectors $\underline{r}_1, \underline{r}_2, \ldots, \underline{r}_n$. The small elements of mass could, for instance, be small cubes or squares, and the position vectors could be the mid-points of these objects. If the sizes of the mass elements are small enough, then they can be considered to be approximately particles. And then, in terms of Result 3.1 it follows that the centre of mass is approximately at the position

$$\underline{R} \approx \frac{\sum_{i=1}^{n} \triangle m_i \cdot \underline{r}_i}{\sum_{i=1}^{n} \triangle m_i}.$$

This is only an approximation since each mass element is not really a particle, but rather a small segment of the whole body — remember that a particle must have all its mass concentrated at one point, which is not really the case with our small mass elements! But we will obtain a more accurate result if we reduce the size of each particle, which means that we are making $\triangle m_i$ smaller and smaller, while increasing the total number n. We can do this if we assume that the body has continuous structure! The approximation becomes exact at the limit when the sizes of the mass elements and hence the $\triangle m_i$ approach zero, while n approaches infinity. That is,

$$\underline{R} = \frac{\lim_{n\to\infty} \sum_{i=1}^{n} \underline{r}_i \triangle m_i}{\lim_{n\to\infty} \sum_{i=1}^{n} \triangle m_i}.$$

At limit, the sums become **definite integrals** and we write

$$\begin{aligned} \lim_{n\to\infty} \sum_{i=1}^{n} \underline{r}_i \triangle m_i &= \int \underline{r} dm, \\ \lim_{n\to\infty} \sum_{i=1}^{n} \triangle m_i &= \int dm. \end{aligned}$$

Note that the "dm" is then the mass of a "differential" (infinitely small) mass element and $\underline{r}$ is its position vector. We are "summing", or integrating, over all the infinitely small mass elements which form the object. The following results are then obtained.

Result 5.1 (The centre of mass of a rigid body)
For a rigid body, the position vector of the centre of mass is given by

$$\underline{R} = \frac{\int \underline{r} dm}{\int dm} \tag{5.1}$$

or, in coordinate form, the centre of mass lies at the point $(\overline{x}, \overline{y}, \overline{z})$ where

$$\overline{x} = \frac{\int x dm}{\int dm}, \tag{5.2}$$

$$\overline{y} = \frac{\int y dm}{\int dm}, \tag{5.3}$$

$$\overline{z} = \frac{\int z dm}{\int dm} \tag{5.4}$$

Here $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$ is the position vector of a small particle-like mass element with mass dm, and integration is over all the mass elements of the body.

What exactly happened here? How does this definition, using integration instead of sums, help us? Remember that we started off by complaining that a rigid body has too many particles (atoms) for the original definition of a centre of mass to be usable. But then we went on to make the assumption that the body is made of infinitely many particles. Surely this should make things even worse as the sums are now infinite? Actually, this is not the case — the infinite sums can be expressed as integrals, and as such can be evaluated fairly easily!

About integration

According to Result 5.1 above, to find the centres of mass of rigid (solid) objects, we will have to use definite integrals. This topic is dealt with in detail in the Mathematics modules on Calculus. For the purposes of this module, you should firstly be able to calculate the value of given definite integrals: for instance, calculate the values of the following:

$$\int_0^\ell x\,dx, \qquad \int_a^b R^4\,dR, \qquad \int_0^{12} y^3\,dy.$$

Secondly, you should be able to follow the explanations of how express certain limits of sums as definite integrals.

If you want to refresh your knowledge of them, please take time now to read through Appendix A at the end of this study guide; or revise the material in your calculus textbooks or study guides!

Note that unlike the sums in the earlier formulas for the calculation of a centre of mass, Result 5.1 is not complete and ready to use — to apply it in practice we still have to do a bit of thinking! For instance, for the integral, we have to decide on the integration variable and the range of integration.

5.2.2 How to use integration to find centres of mass

In practice, the calculation of the centre of mass of a rigid body using Result 5.1 would proceed as follows.

1. First, we have to draw a sketch of the body. Drawing a sketch will help you make sure that you know what the object looks like. Remember that you can turn a rigid body around any way you like, and therefore there are many possible ways to draw the object! A well-chosen orientation in you sketch may make it easier for you to decide on a coordinate system later on.

   We also need to select a suitable coordinate system. Remember that position vectors and coordinates are meaningless without the point of reference provided by the chosen coordinate system! What then is a "suitable" coordinate system? The coordinate system just provides a way of referring to the positions of individual mass elements and of the centre of mass — it does not change the end result of where the centre of mass is! Every coordinate system must give the same point as the centre of mass, it may just be referred to in a different way (it may, for instance, have different X-, Y- and Z-coordinates, depending on where the origin of the XYZ system was chosen to be). However, some coordinate systems are better than others. The difference is that some coordinate systems make the calculations needed to find the centre of mass much easier. As we have seen earlier, and as will be discussed again later on in this unit, symmetry linked with a suitable choice of coordinates can reduce the dimension of the problem!

2. Next, we divide the object into small pieces, each approximately a particle with known mass $\triangle m_i$ and known position vector $\underline{r}_i$ (or coordinates x_i, y_i, z_i). Remember that in what follows, we must be able write down exactly the mass and position vector of each small piece! This means that we need a systematic way of cutting the object into smaller and smaller pieces.

   The centre of mass is now approximately given by the sum (3.1), or in coordinate form, by (3.2), (3.3) and (3.4). However, at limit, when we take smaller and smaller pieces, the sums become integrals and we apply instead the integral formulas (5.1), or in coordinate form, (5.2), (5.3), (5.4) where we integrate over all small mass elements dm. In each situation, the integrals in these formulas become specific definite integrals

— that is, depending on the actual situation, we get an integral from some lower limit to some upper limit of a specific variable of integration (this corresponds to summing up over all the particles which form a system).

3. Finally, we find the numerical values of all the definite integrals to evaluate $\underline{R}$ or $\bar{x}$, $\bar{y}$ and $\bar{z}$. These values, when interpreted according to our chosen coordinate system, give the position of the centre of mass.

If the explanation above on how to find the centre of mass by integration still seems very confusing, do not worry — you will learn it by studying examples, and then by doing it yourself! After working through this and the next unit, you will be able to come back to the explanation above, and it will all make sense to you!

About mass and density

Note that when applying the process described above, we will need to find the mass of a small mass element, $\triangle m_i$ or dm. In practice, when we are applying the formulas, we will usually obtain a small mass element by "slicing" the object into smaller objects. This means dividing the volume (or area, or length) of the object into small volume (or area, or length) elements. We would usually be able to determine the volume, or area, or length of such a small element, and the mass of the small elements can then calculated by using the concept of **density,** usually denoted by ρ (this is the Greek letter “rho”). Density links the mass of an object to its size, as follows:

For a one-dimensional object (e.g. a rod), (so density ρ is mass per unit length); $\boxed{mass = \rho \times \ length}$

For a two-dimensional object (e.g. a rectangle), (so density ρ is mass per unit area); $\boxed{mass \ = \rho \times \ area}$

For a three-dimensional object (e.g. a cube), (so density ρ is mass per unit volume). $\boxed{mass \ = \rho \times volume}$

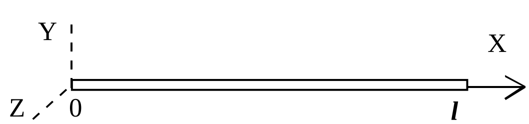
If ρ is constant throughout the object, then we say that the object has uniform density. More generally, ρ could depend on the position within the object.

An example on how to integrate to find centres of mass – showing all the detail

Example 5.2
A thin rod of length ℓ is of uniform density. Find its centre of mass.

Solution (in detail):

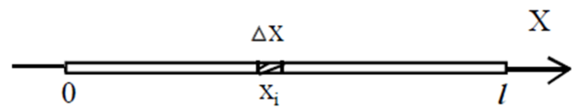
1. Let’s choose the X-axis to go along the rod, as shown.



If the rod is very thin, then we can assume that is lies completely on the X-axis. The centre of mass will then also lie on the X-axis, with the Y- and Z-coordinates equal to zero, and all we have to do is to calculate its X-coordinate.

2. To calculate the X-coordinate of the centre of mass, let us see what happens when we

divide the rod into N short segments of equal length. Let $\triangle x$ be the length of the each of the segments, and let x_i denote the X-coordinate of the centre of the ith segment.



We need to know the mass of segment number i, and for that we will use the density of the rod. The rod has uniform density, which means that the mass per unit length of the rod equals some constant ρ. So, the mass of the ith mass element equals

$$\triangle m_i = \rho \triangle x$$

for all values of i.

What about the values of the x_i? They are N points situated at even intervals on the line from $x = 0$ to $x = l$. The exact values are not too difficult to find for any given value of N, but we are not really interested in the exact values, since we will in any case let N increase without bound! It is enough for us to be able to say what happens to the small segments and to the $\triangle m_i$ and x_i values as N goes to infinity.

We have divided the rod into N small segments, each with a known position and mass; if the segments are small enough, then they are approximately particles, and we can say that the X-coordinate of the centre of mass is approximately

$$\overline{x} \approx \frac{\sum_{i=1}^{N} x_i \triangle m_i}{\sum_{i=1}^{N} \triangle m_i} = \frac{\sum_{i=1}^{N} x_i \rho \triangle x}{\sum_{i=1}^{N} \rho \triangle x}.$$

Now, what happens when we let $N \to \infty$ and $\triangle x \to 0$? The segments of the rod get smaller and smaller, and we have to sum up over more and more of them. At limit, we get integrals instead of sums:

$$x = \frac{\int_0^\ell x \rho dx}{\int_0^\ell \rho dx}. \tag{5.5}$$

The upper and lower limits of integration (from $x = 0$ to $x = \ell$) are determined by the fact that x, the position of each infinitely small mass element of length dx, ranges over all the possible values of x from 0 to ℓ.

3. Evaluating the integrals, we get

$$\overline{x} = \frac{\int_0^\ell x \rho dx}{\int_0^\ell \rho dx} = \frac{\rho \int_0^\ell x dx}{\rho \int_0^\ell dx} = \frac{x^2/2]_0^\ell}{x]_0^\ell} = \frac{\ell^2/2 - 0}{\ell - 0} = \frac{\ell}{2}.$$

In our coordinate system, this is a point in the middle of the rod. So, the centre of mass G of a uniform rod lies at the centre of the rod, as we would expect. ◀

Above, we have derived the formula (5.5) in great detail, in effect re-deriving Result 5.1 (the centre of mass of a rigid body using integrals). In the rest of this study guide, we will not write down all these details, but will instead apply (5.1), (5.2), (5.3) and (5.4) directly. Instead of first dividing the object into N small pieces and then letting N increase without bound, we will work directly with the **differentials** dm, considered to be infinitely small mass elements. To apply these formulas, for instance (5.2):

$$\overline{x} = \frac{\int x dm}{\int dm},$$

we need to decide how we shall divide the object into the small mass elements. For each small mass element dm, we have to be able to find its mass and its position vector $\underline{r}$, or its coordinates x, y and z in the chosen XYZ coordinate system. This means that we must be able to refer to each small mass element by using a variable of integration. The upper and lower limits of integration will then correspond to the range of possible values of the variable of integration. To illustrate this, we shall re-derive the equation (5.5) above, using the integration notation directly.

An example on how to integrate to find centres of mass – the way we will do it

Example 5.3
The previous example revisited: *Solution (a shorter version)*

1. Let the coordinate system be as before, with the rod lying along the X-axis such that the left end of the rod is at the origin.

2. We shall now divide the rod into small mass elements dm, each an infinitely short segment of the rod of length dx. If the rod is of uniform density ρ, then the mass of the element is
$$dm = \rho\, dx.$$
Integration is over x, (that is, x will be the variable of integration), since each small segment corresponds to a different value of x. The positions of the small mass elements range from $x = 0$ to $x = \ell$, and therefore integration is from 0 to ℓ. Then, according to (5.2), the X-coordinate of the centre of mass is given by
$$\overline{x} = \frac{\int_0^\ell x\rho dx}{\int_0^\ell \rho dx}.$$

3. As before, we evaluate the integrals and get $\overline{x} = \frac{\ell}{2}$. ◄

Remarks:

- Note how the unknown density ρ cancelled out in the calculations! This will always happen if the object is of a uniform density. *This is because, as we have stated before, the centre of mass of the system depends on the* <u>*relative*</u> *masses of the different parts of the system, not on the* <u>*actual*</u> *masses.*
- The way the rod is situated on the x-axis determines the limits of integration. For instance, if we decide to place the rod on the negative part of the X-axis, with its right end at the origin, then the integration would be from $x = -\ell$ to $x = 0$ and the integral would be
$$\overline{x} = \frac{\int_{-\ell}^0 x\rho dx}{\int_{-\ell}^0 \rho dx} = \frac{\rho\int_{-\ell}^0 x dx}{\rho\int_{-\ell}^0 dx} = \frac{\frac{1}{2}x^2\Big]_{-\ell}^0}{x\Big]_{-\ell}^0} = \frac{\frac{1}{2}(0) - \frac{1}{2}(-\ell)^2}{0-(-\ell)} = -\frac{\ell}{2}.$$
In our new coordinate system, this again refers to the midpoint of the rod.

Activity 5.3

Here is your chance to practice finding a centre of mass by integration. Write the complete solution down on your exercise book.

A thin rod of length ℓ is of uniform density. Find its centre of mass by integration, when the rod lies on the X-axis between $-\ell/2$ and $+\ell/2$.

Feedback: In this coordinate system, you should get $\overline{x} = 0$ as the centre of mass! If you did not, check that you got the following in your answer: Integration should be from $-\ell/2$ to $+\ell/2$; you should therefore get

$$\overline{x} = \frac{\frac{1}{2}(\ell/2)^2 - \frac{1}{2}(-\ell/2)^2}{(\ell/2) - (-\ell/2)} = 0.$$

You have now seen three different coordinate systems used for finding the centre of mass of the rod. Which do you think had the simplest calculations? If you are like me, and don't like to work with fractions too much, you might have preferred the original solution! But the main thing is that all the choices must give you the same solution! This can be used to check the solution: re-do the calculations with another coordinate choice to make sure the answer you got the first time around is correct!

The next example deals with an object of non-uniform density. You will see that the solution is not much more complicated!

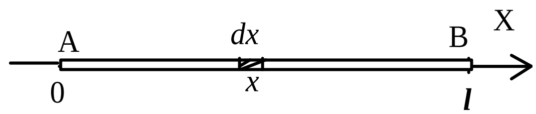
Example 5.4
A rod AB of length ℓ is of non-uniform density. Assume that the density at a point on the rod at the distance x from the end A of the rod is given by

$$\rho(x) = \frac{c}{\ell}x$$

were c is a constant. Find the centre of mass.

Solution:
Note that here the density is smallest at point A and increases towards point B — that is, end B of the rod is heavier than end A! Accordingly we would expect that the centre of mass of the rod would no longer lie in the middle of the rod, but rather somewhere closer to end point B than end point A.

Take the X-axis to go along the rod, as shown.



We divide the rod into small mass elements, each consisting of a short segment of the rod of length dx. If such a segment is situated at position x on the X-axis, then its mass can be calculated by

$$dm = \rho(x)\,dx = \frac{c}{\ell}x\,dx$$

since $\rho(x)$ gives the density at that particular position. The positions of the small mass elements range from $x = 0$ to $x = \ell$. (5.2) now gives:

$$\begin{aligned}
\overline{x} &= \frac{\int_0^\ell x\,dm}{\int_0^\ell dm} = \frac{\int_0^\ell x\frac{c}{\ell}x\,dx}{\int_0^\ell \frac{c}{\ell}x\,dx} \\
&= \frac{\frac{c}{\ell}\int_0^\ell x^2 dx}{\frac{c}{\ell}\int_0^\ell x\,dx} = \frac{\frac{1}{3}x^3\Big]_0^\ell}{\frac{1}{2}x^2\Big]_0^\ell} = \frac{\frac{1}{3}\ell^3}{\frac{1}{2}\ell^2} = \frac{2}{3}\ell.
\end{aligned}$$

That is, the centre of mass G of the rod is situated $2/3$ of the length of the rod from end A. ◄

Activity 5.4

Solve the following problem. Write your complete solution in your exercise book, and pay particular attention to the following: You must select a coordinate system yourself, and you should draw a sketch of the rod and the coordinate system, and make clear which is the end where the x in the density expression is calculated from. Give your answer as a point in your coordinate system, and also explain where it lies on the rod.

The density of a thin rod of length ℓ varies with the distance x from one end as $\rho(x) = \rho_0 x^2/\ell^2$. Find the centre of mass.

....

Feedback: you should get as the centre of mass a point at the distance $\frac{3}{4}\ell$ from the lighter end of the rod.

We can now, in principle at least, calculate the centre of mass of any system of particles (by summing over all the particles), or of a rigid body (by integrating over infinitely small mass elements). We will give many more examples of the calculation of the centre of mass of rigid bodies later on, but first we will introduce some basic rules which often make calculations much easier. The rules in the next section are valid for all rigid systems of particles, but you'll see that they are really helpful for rigid bodies in particular!

5.3 Helpful rules for finding centres of mass

The following general rules can often be used to considerably simplify the calculation of centres of mass. In this section we will mostly use rigid bodies as examples, but since we do not use integration anywhere, the results apply equally well to systems of particles.

5.3.1 Using symmetry and dimension of the object to help find centres of mass

Laminas

If an object is of a negligible thickness (i.e. is very thin), and if it is flat, then we can think of it as a portion of a plane. Such an object is called a **lamina**.

Activity 5.5

The concept of lamina will be used a lot in the rest of the study guide, so it is important that you understand it well.

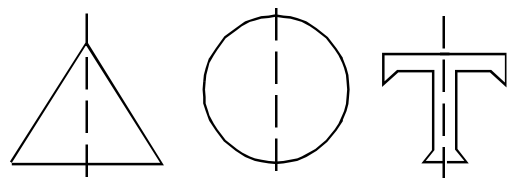
Which of the following everyday objects could we call laminas, and which would have to be treated as three-dimensional objects?

Object	Lamina	3D object
A sheet of paper		
A rubber band		
An empty toilet paper roll		
A balloon		

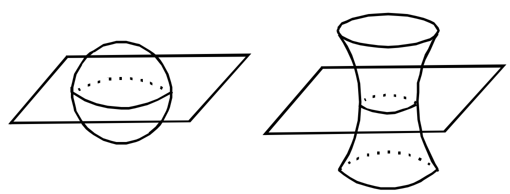
Feedback: Remember that for an object to be a lamina, it has to be flat and very thin! You should be able to imagine it being cut out from a thin sheet of metal. The only object here which can be considered to be a lamina is a sheet of paper, the others are usually not laminas.

About symmetry

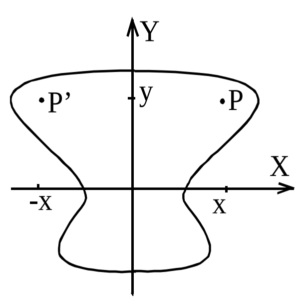
Suppose that a line can be drawn through the lamina so that the part of the body on one side of this line is a mirror image of the part on the other side. Such a line is called an **axis of symmetry**. The dotted lines in the pictures below are all axes of symmetry.



In the case of a body with a definite thickness (that is, it is a three-dimensional object), if a plane can be drawn which cuts the body in such a way that the two parts on either side of this plane are mirror images of each other, then this plane is called a **plane of symmetry**.



Now consider a lamina with an axis of symmetry represented by the Y-axis as in the figure below. Let $P = (x, y)$ be the position of a particle of mass m_i on the right-hand side of the Y-axis. Since Y is an axis of symmetry, there will be a point $P' = (-x, y)$ on the left, which is the position of another particle of mass m_i (the mirror image of the particle at P).



But now, when we calculate the X-coordinate of $\underline{R}$, the contributions of these two particles will be

$$m_i x + m_i (-x) = 0.$$

Each particle on the right has a mirror image on the left, such that the contributions of the particles in calculating the X-coordinate of the centre of mass cancel out. As a result we have that

$$\overline{x} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i} = 0,$$

which means that $\underline{R}$ must lie on the axis of symmetry.

By similar reasoning, we can show that the centre of mass of a body with a plane of symmetry must lie on that plane. This leads to the following result.

Result 5.2

If a body or a system of particles has an axis or a plane of symmetry, then the centre of mass must lie on that axis or plane. If there are more than one axis and/or plane of symmetry, then the centre of mass will lie at their intersection.

Note that for two halves of the object to be mirror images, it is not enough that their shapes are the same – their mass distributions must be mirror images of each other. This is so if the two halves are made from the same, uniform material!

From this result it follows that the centres of mass of a uniform rod, rectangular lamina, circular disc, cube, sphere, etc. lie at the geometric midpoints of these bodies. In these cases the centre of mass can be found directly, using the symmetry argument. At other times symmetry can at least be used to simplify calculations, for instance in order to reduce the dimension of the problem.

About dimensions

Finally, we note the following trivial result that follows directly from the symmetry argument. This deals with the **dimension of the system/body**, and really just states that if the body is one- or two-dimensional, then we don't have to find its centre of mass using a full three-dimensional coordinate system!

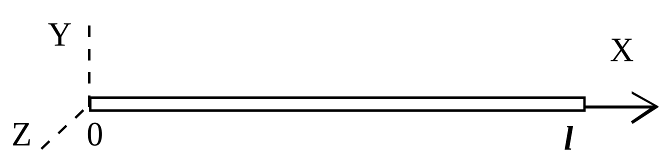
Result 5.3

If the entire mass of a body lies on a line or a plane, then the centre of mass of the body also lies on that line or plane.

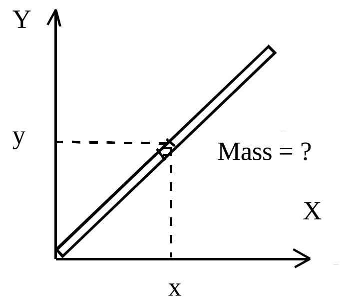
5.3.2 Selecting the best possible coordinates

The choice of the coordinate system can make a lot of difference in the calculation of the centre of mass, as we already learned in Unit 3. The following trick can often be applied to select an optimal coordinate system: If symmetry or other arguments tell us that the centre of mass lies on some line or plane, then we should make sure that line coincides with the X-, Y- or Z-axis; or correspondingly, that the plane where the centre of mass is known to lie coincides with the XY-, YZ- or XZ-plane. Such a choice guarantees that we shall only have to find one (or two) coordinates, rather than all three coordinates. The other coordinates will be zero.

We already used this idea earlier when we found the centre of mass of the rod by integration. Remember that our initial coordinate system was as shown below:



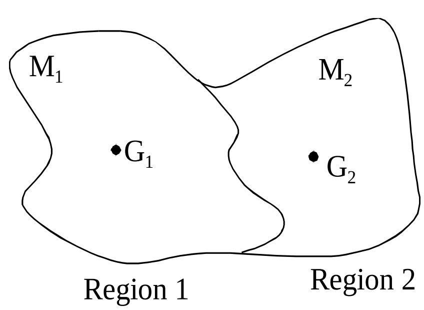
The rod is a one-dimensional object, and we selected it to go along the X-axis. But any other coordinate system would give the same result, namely that the centre of mass lies in the middle of the rod. However, the calculations are quite a bit easier when the rod is chosen to lie along one of the axes. Say, for instance, that we had drawn the rod differently, such it was instead lying along the line $y = x$ on the XY-plane, as shown below.



We can, of course, still divide the rod into small lengths, as shown. However, now it is a bit more difficult to find the x and y coordinates of each length of the rod. Also, what is the mass dm of each piece of the rod, and what should the variable of integration be? (Actually, all such small pieces are uniquely defined by either their x or their y position, so either x or y could be used as the integration variable! We will leave it as an exercise for you to re-do the calculations in this coordinate system — it will be one of the exercises in the workbook.)

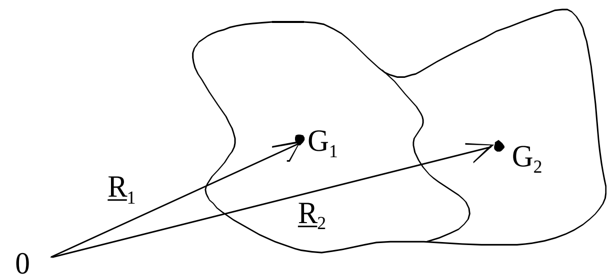
5.3.3 Finding the centre of mass of composite bodies

The following very useful rule makes it very easy to find the centre of mass of bodies which have been put together from objects whose centres of mass we already know. Such a body is called a composite body. Suppose that a body can be divided into two separate parts (components) 1 and 2 with masses M_1, M_2 and centres of mass G_1 and G_2, respectively.



(Another way of looking at this is to consider the body as a system composed of two bodies labelled 1 and 2, respectively.)

Let $\underline{R}_1$ and $\underline{R}_2$ be the position vectors of the centres of mass G_1 and G_2 in our chosen coordinate system.



If $\underline{R}$ is the position vector of the centre of mass of the whole body, then by (3.1) we have

$$\underline{R} \;=\; \frac{\sum_{i=1}^{n} m_i \underline{r}_i}{\sum_{i=1}^{n} m_i} \qquad (5.6)$$

where we sum up over all the particles in both parts. Here, n is the number of particles in the entire system, m_i is the mass of particle i and $\underline{r}_i$ is its position vector.

Let n_1 and n_2 denote the number of particles in part 1 and part 2 respectively, then $n_1 + n_2 = n$, and we can of course number the particles such that particles numbered $i = 1$ to $i = n_1$ are in part 1, and particles $i = n_1 + 1$ to $i = n_1 + n_2$ are in part 2. Then

$$M_1 \;=\; \sum_{i=1}^{n_1} m_i,$$

$$M_2 \;=\; \sum_{i=n_1+1}^{n} m_i.$$

Since we have denoted the position vectors of the parts 1 and 2 by $\underline{R}_1$ and $\underline{R}_2$ respectively, the following must hold:

$$\underline{R}_1 \;=\; \frac{\sum_{i=1}^{n_1} m_i \underline{r}_i}{M_1},$$

$$\underline{R}_2 \;=\; \frac{\sum_{i=n_1+1}^{n} m_i \underline{r}_i}{M_2}.$$

But now, we can re-group the terms of the sums in (5.6) by writing separate sums for the particles which belong to part 1 and part 2:

$$\underline{R} = \frac{\sum_{i=1}^{n_1} m_i \underline{r}_i + \sum_{i=n_1+1}^{n} m_i \underline{r}_i}{\left(\sum_{i=1}^{n_1} m_i + \sum_{i=n_1+1}^{n} m_i\right)}.$$

This means that

$$\underline{R} = \frac{M_1 \underline{R}_1 + M_2 \underline{R}_2}{M_1 + M_2}.$$

This reasoning can be extended to any number of components forming the entire object. We get the following result:

Result 5.4

Suppose that a body is composed of N separate parts, with masses $M_1, M_2, \ldots, M_N$ and centres of mass with position vectors $\underline{R}_1, \underline{R}_2, \ldots, \underline{R}_N$ respectively. Then the position vector for the centre of mass of the whole body is

$$\underline{R} = \frac{\sum_{i=1}^{N} M_i \underline{R}_i}{\sum_{i=1}^{N} M_i} = \frac{M_1 \underline{R}_1 + \cdots + M_N \underline{R}_N}{M_1 + \cdots + M_N}. \tag{5.7}$$

In coordinate form, the centre of mass lies at the point $(\overline{x}, \overline{y}, \overline{z})$ where

$$\overline{x} = \frac{M_1 \overline{x}_1 + \cdots + M_N \overline{x}_N}{M_1 + \cdots + M_N} \tag{5.8}$$

$$\overline{y} = \frac{M_1 \overline{y}_1 + \cdots + M_N \overline{y}_N}{M_1 + \cdots + M_N} \tag{5.9}$$

$$\overline{z} = \frac{M_1 \overline{z}_1 + \cdots + M_N \overline{z}_N}{M_1 + \cdots + M_N} \tag{5.10}$$

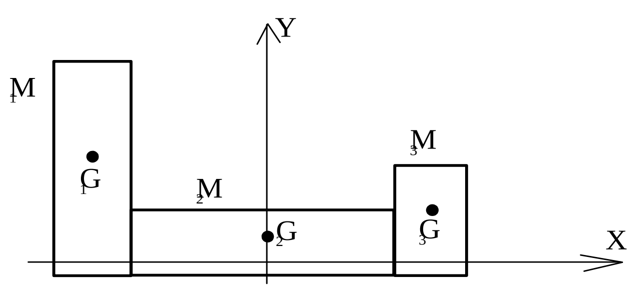
if $(\overline{x}_1, \overline{y}_1, \overline{z}_1), \ldots, (\overline{x}_N, \overline{y}_N, \overline{z}_N)$ give the coordinates of the separate parts.

Remarks

- It follows that if an object can be divided into separate components, we can then find the centre of mass of the whole object by simply treating each of the parts as if it were a particle, located at its centre of mass, and with the mass of the component. (Compare this result with the formulas for calculating the centre of mass of a system of particles – the formulas look very similar!)
- This result can often greatly simplify the job of finding the centre of mass of complicated objects. We can try to divide the object into components whose centres of mass are easy to find (e.g. symmetrical objects, the centre of mass of which is at their geometric midpoint). Once the centres of mass of the components have been found, and expressed in terms of the chosen coordinate system, we just apply equations (5.7) to (5.10) to find the centre of mass of the whole body.

Example 5.5

A uniform lamina can be divided into three rectangles as shown in the figure below.



Assume that the centres of the rectangles are at $G_1 = (-3, 2)$, $G_2 = \left(0, \frac{1}{2}\right)$ and $G_3 = (3, 1)$. The masses of the rectangles are $M_1 = 8$, $M_2 = 4$ and $M_3 = 4$ respectively. Find the centre of mass of the lamina.

Solution:

The centres of mass of the three rectangles are at their midpoints, G_1, G_2 and G_3. If

$\underline{R} = \overline{x}\underline{i} + \overline{y}\underline{j}$ denotes the centre of mass of the whole lamina, then according to (5.8), (5.9)

$$\overline{x} = \frac{8(-3) + 4(0) + 4(3)}{8+4+4} = -\frac{3}{4}$$

$$\overline{y} = \frac{8(2) + 4\left(\frac{1}{2}\right) + 4(1)}{8+4+4} = \frac{11}{8}.$$

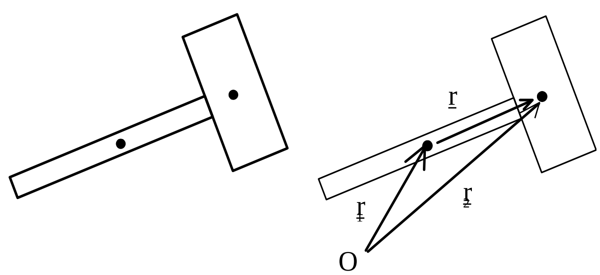
Thus, the centre of mass has the position vector

$$\underline{R} = -\frac{3}{4}\underline{i} + \frac{11}{8}\underline{j}.$$

Example 5.6

A hammer can be considered to approximately consist of a rod of mass m, at one end of which a rectangle of mass M is fixed. Suppose that the centre of mass of the rod has the position vector $\underline{r}_1$ and that the rectangle has the position vector $\underline{r}_2$ from the origin. If $\underline{R}$ is the position vector of the centre of mass, we have, according to (5.7)

$$\underline{R} = \frac{m\underline{r}_1 + M\underline{r}_2}{m+M}.$$



We will re-write this into a more informative form. If we denote by $\underline{r}$ the vector from the centre of the rod to the centre of the rectangle, that is, $\underline{r} = \underline{r}_2 - \underline{r}_1$ (see the figure above), we see that in fact

$$\begin{aligned} \underline{R} &= \frac{m\underline{r}_1 + M\left(\underline{r}_1 + \underline{r}\right)}{m+M} \\ &= \frac{(m+M)\underline{r}_1 + M\underline{r}}{m+M} \\ &= \underline{r}_1 + \frac{M}{m+M}\underline{r}. \end{aligned} \tag{5.11}$$

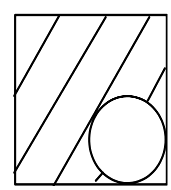
That is, the centre of mass is located by starting at $\underline{r}_1$ (the centre of rod) and moving along the vector $\underline{r}$ towards the centre of the rectangle. This expresses the idea that the centre of mass of the hammer is always along the line joining the centres of mass of the two components, the fraction $M/(m+M)$ of the distance towards the rectangle. Note that if $m << M$ (if m is much smaller than M), then $m+M$ is approximately equal to M, written as $m+M \approx M$. Then

$$\underline{R} \approx \underline{r}_1 + \frac{M}{M}\underline{r} = \underline{r}_1 + \underline{r} = \underline{r}_2.$$

This is as expected; if $m << M$, then most of the mass of the system is concentrated at the centre of mass of the rectangle of mass M, and so the centre of mass of the entire system would be approximately equal to $\underline{r}_2$. More generally, we can see from (5.11) that the smaller M is, the closer $\underline{R}$ is to $\underline{r}_1$; and the larger M is, the closer $\underline{R}$ is to $\underline{r}_2$.◄

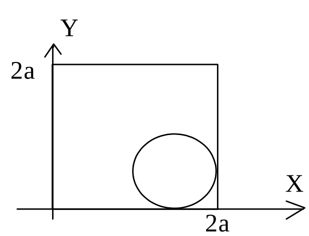
5.3.4 Objects with parts removed

The rule for the centres of mass of composite bodies, Result 5.4, also gives a convenient way to calculate the centres of mass of uniform objects with "holes" in them. For example, consider the object shown below: a square lamina made of uniform material with sides of length $2a$, from which a circle with radius $a/2$ has been cut off, as shown.

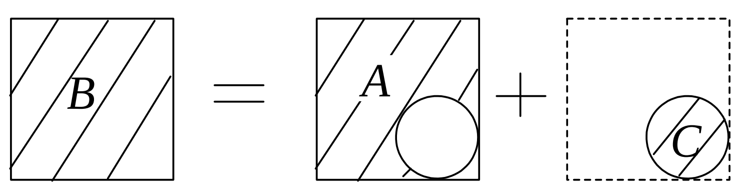


Of course we can still find the centre of mass of the lamina by slicing the lamina into thin strips, but the calculation of the lengths of these strips would be quite complicated! A very simple alternative way to find the centre of mass, based on the idea of composite bodies, uses the fact that we can very easily find the centres of mass of the square and the circle.

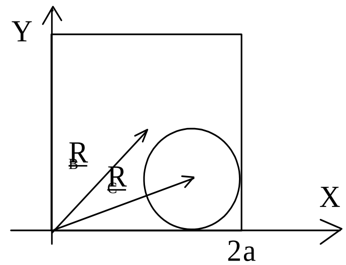
Firstly, let us assume that the object lies on the XY-plane, as shown below.



The lamina, let's call it object A, consists of the entire square (object B) minus the circle (object C). Put in another way, the square B can be composed of our object A and a circle C made of the same material.



Let M_A, M_B and M_C be the masses of the objects A, B and C respectively, and let their centres of mass have the position vectors $\underline{R}_A$, $\underline{R}_B$ and $\underline{R}_C$, respectively.



Then the rule for the centres of mass of composite bodies tells us that

$$\underline{R}_B = \frac{M_A \underline{R}_A + M_C \underline{R}_C}{M_A + M_C}.$$

We wish to find $\underline{R}_A$, so let's solve it from this equation; we get

$$\begin{aligned}\underline{R}_A &= \frac{(M_A + M_C)\,\underline{R}_B - M_C \underline{R}_C}{M_A} \\ &= \frac{M_B \underline{R}_B - M_C \underline{R}_C}{M_B - M_C}\end{aligned}$$

We have thus derived an expression for the centre of mass we wish to find, expressed in terms of the masses of the square and the circle cut off from it, and the centres of mass of the square and the circle which was cut off — both of which are simple to find since the square and the circular hole are nice, straight forward objects!

To find the mass of the square and the circle, we need to find their areas. Let the density of the material be ρ; then (from the well know geometric formulas for the area for a square and a circle) for square B, we have

$$M_B = \rho \cdot 4a^2$$

and for the circle C,

$$M_C = \rho\pi\,(a/2)^2\,.$$

The centres of mass of objects B and C lie at their midpoints, so that we have

$$\begin{aligned}\underline{R}_B &= a\underline{i} + a\,\underline{j}, \\ \underline{R}_C &= \frac{3}{2}a\underline{i} + \frac{1}{2}a\,\underline{j}.\end{aligned}$$

(These are the coordinates of the square and the circle in the coordinate system given above!)

Hence, we have

$$\begin{aligned}\underline{R}_A &= \frac{M_B \underline{R}_B - M_C \underline{R}_C}{M_B - M_C} \\ &= \frac{\rho 4a^2 \left(a\underline{i} + a\,\underline{j}\right) - \rho\pi\,\frac{a^2}{4}\left(\frac{3}{2}a\underline{i} + \frac{1}{2}a\,\underline{j}\right)}{\rho 4a^2 - \rho\pi\,\frac{a^2}{4}} \\ &= \frac{a}{16-\pi}\left[\left(16 - \frac{3}{2}\pi\right)\underline{i} + \left(16 - \frac{\pi}{2}\right)\underline{j}\right].\end{aligned}$$

The steps above can be followed for any object A which consist of an entire object B from which another object C has been removed, in one, two or three dimensions. We therefore get the following general rule for dealing with these kinds of situations.

Result 5.5 (The centre of mass of objects with parts removed)
Let an object A consist of an object B from which a part, another object C, has been removed. Let M_A, M_B and M_C be the masses of the objects A, B and C respectively, and let their centres of mass have the position vectors $\underline{R}_A$, $\underline{R}_B$ and $\underline{R}_C$. Then

$$\underline{R}_A = \frac{M_B\underline{R}_B - M_C\underline{R}_C}{M_B - M_C}$$

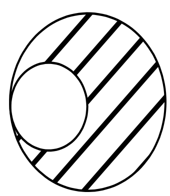
In coordinate form, if the centres of mass of the objects A, B and C have the coordinates (x_A, y_A, z_A), (x_B, y_B, z_B) and (x_C, y_C, z_C) respectively, then

$$\begin{aligned} x_A &= \frac{M_B x_B - M_C x_C}{M_B - M_C}, \\ y_A &= \frac{M_B y_B - M_C y_C}{M_B - M_C}. \\ x_A &= \frac{M_B z_B - M_C z_C}{M_B - M_C}. \end{aligned}$$

Activity 5.6

In this activity, you can use the formulas in Result 5.5 directly. Remember that you have to introduce a coordinate system first!

A disc with radius R has been removed from a circular metal plate with radius $2R$, as shown below. Find the centre of mass.



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Feedback: the centre of mass will be the distance $R/3$ from the centre of the large metal plate.

The result generalises easily to include objects from which several pieces have been removed. We must assume that the object A is made of uniform material, since this enables us to find the mass of the "missing piece" C!

All the methods listed in this section, for simplifying the job of finding centres of mass, should always be considered first, before you rush to integrate or write down sums. In fact, they are so useful that we will write them down as a general toolbox.

TOOLBOX FOR SIMPLIFYING THE TASK OF FINDING CENTRES OF MASS OF RIGID SYSTEMS OR BODIES

1. Draw a sketch of the system (or body). Remember that you can turn a rigid system around any way you like, and therefore there are many possible ways to draw the system! A well-chosen orientation in you sketch may make it easier for you to decide on a coordinate system later on, but you can always re-do your sketch if necessary.

2. Are there axes or planes of symmetry?

3. What is the dimension of the system?

4. Can the system be expressed as a composite body, where the centres of mass of the components are easier to find?

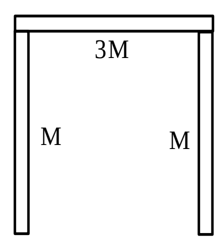
5. Select a final coordinate system, based on all the considerations above. (This may involve re-drawing the system in another orientation.)

The following example illustrates the use of this toolbox.

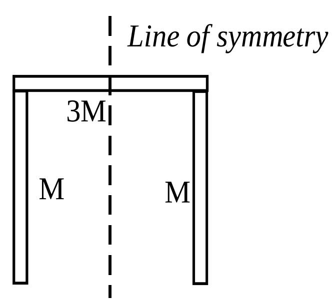
Example 5.7
Three thin rods of equal length are arranged in an inverted ∪. The two rods on the arms of the ∪ each have a mass M; the third rod has a mass $3M$. Where is the centre of mass of the system?

*Solution:*We have assumed that the rods are at right angles with each other. If the rods are very thin, it does not really matter what happens at the corners (whether the rods are neatly joined together as shown here, or whether they just touch each other at the corners). All the rods have the same length, denoted by L.

1. Draw a sketch of the system. Such a sketch could look like this:



2. Are there axes or planes of symmetry? Yes. Looking at the sketch of the system, we see that it is symmetrical about the vertical line through the middle of the $3M$-rod. (The M-rods and the two halves of the $3M$-rod are situated symmetrically on either side of this line, so that one side is a mirror image of the other about this line.)

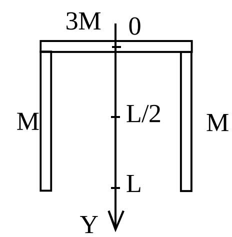


3. What is the dimension of the system? The system here is two-dimensional.

4. Can the system be expressed as a composite body, where the centres of mass of the components are easier to find? Yes. The system consists of three rods, and assuming that the rods are uniform, the centre of mass of each rod is at its midpoint.

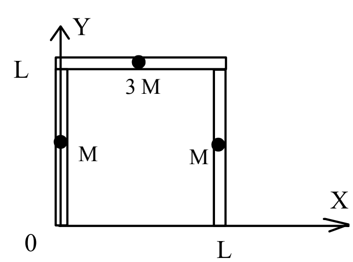
5. Select a final coordinate system, based on all the considerations above. Since the vertical line through the middle of the system is an axis of symmetry, the centre of mass must lie on this axis. So, if we call this axis the Y-axis, then the centre of mass lies on this axis, and we only need to find its Y-coordinate, $\overline{y}$. (Thus by a suitable

choice of the coordinate system, based on symmetry, we have reduced the problem to a "one–dimensional" one!) We still have to decide on the direction of the Y-axis and where the origin of the Y-axis lies. Let us choose the origin to be at the $3M$–rod, and let the Y-axis go downwards, as shown below.



Then the $3M$-rod has 0 as its Y-coordinate, and the two M-rods both have $L/2$ as their Y-coordinates. Hence, according to (5.9), the Y-coordinate of the centre of mass of the entire system is $\overline{y} = \dfrac{3M \cdot 0 + M \cdot \frac{L}{2} + M \cdot \frac{L}{2}}{3M + M + M} = \dfrac{L}{5}$.

Note that although any coordinate system will identify the **same point** as the centre of mass of the assembly, calculations are easier in some coordinate systems than in others. To illustrate this, we will re-do the calculations here with a different coordinate system. Let us for instance see what happens if we simply choose the origin of the XY-plane to lie in the lower left corner of the assembly.



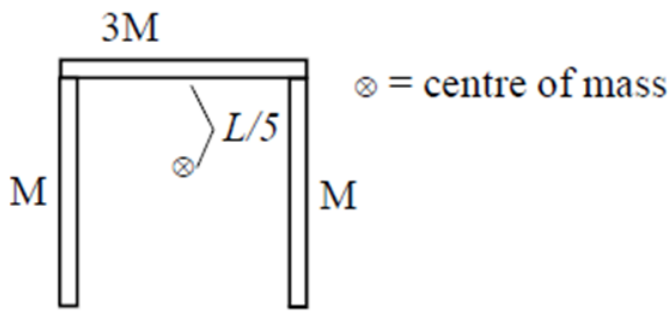
In this coordinate system, the leftmost rod of mass M has its centre of mass at $(x_1, y_1) = \left(0, \dfrac{L}{2}\right)$, the top rod of mass $3M$ has its centre of mass at $(x_2,\ y_2) = \left(\dfrac{L}{2}, L\right)$, and the rightmost rod of mass M has its centre of mass at $(x_3,\ y_3) = \left(L, \dfrac{L}{2}\right)$. Therefore, according to equations (5.8) and (5.9), the whole system has its centre of mass at $(\overline{x}, \overline{y})$ where

$$\begin{aligned} \overline{x} &= \frac{M \cdot 0 + 3M \cdot \frac{L}{2} + M \cdot L}{M + 3M + M} = \frac{1}{2}L, \\ \overline{y} &= \frac{M \cdot \frac{L}{2} + 3M \cdot L + M \cdot \frac{L}{2}}{M + 3M + M} = \frac{4}{5}L. \end{aligned}$$

Alternatively, using unit vectors, the centres of mass of the rods have position vectors

$$\underline{R}_1 = \frac{L}{2}\underline{j}, \quad \underline{R}_2 = \frac{L}{2}\underline{i} + L\underline{j}, \quad \underline{R}_3 = L\underline{i} + \frac{L}{2}\underline{j},$$

and hence, according to (5.7), the centre of mass of the whole system has the position



vector

$$\underline{R} = \frac{M \cdot \frac{L}{2}\underline{j} + 3M\left(\frac{L}{2}\underline{i} + L\underline{j}\right) + M\left(L\underline{i} + \frac{L}{2}\underline{j}\right)}{M + 3M + M}$$

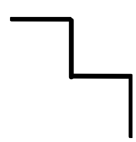
$$= \frac{L}{2}\underline{i} + \frac{4}{5}L\underline{j}.$$

Thus, both coordinate systems identify the **same point** as the centre of mass of the system: the point which lies the distance of $L/5$ below the middle of the $3M$-rod.◀

Activity 5.7

Now you are ready to find the centres of mass of complicated objects, using the rules listed above. Write the full solution to the following problem in your exercise book. Please include the following: Go through and comment on the five steps in the simplifying-toolbox above, as they apply in this case. After that, draw a sketch of your coordinate system and the object within it. Express the centre of mass both as a point in your coordinate system, and explain also where it is in relation to the object.

The object shown below is constructed of four uniform thin rods, each of length a and mass M, joined together at right angles at their ends. Find the centre of mass of the object.



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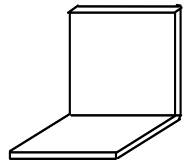
Feedback: The centre of mass is at the point which lies the distance $a/4$ above and to the right of the point where the two middle rods intersect. The coordinate point you get will depend on which coordinate system you selected. What is important is that you made it clear what your selected coordinate system was, and then expressed the centres of mass of all the components in relation to your selected system!

Activity 5.8

Solve the following problem, again using first the simplifying procedures listed in the previous toolbox.

Two uniform squares of sheet metal of dimensions $L \times L$ are joined at a right angle along one edge. One of the squares has twice the mass of the other. Find the centre of mass.



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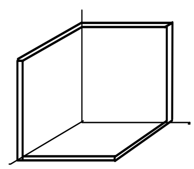
Feedback: the centre of mass will be the distance $L/6$ from the heavier plate and the distance $L/3$ from the lighter one.

Activity 5.9

Solve the following problem, again using first the simplifying procedures listed in the toolbox above.

Three uniform square pieces of sheet metal are joined along their edges so as to form three of the sides of a cube. The dimensions of the squares are $L \times L$. Where is the centre of mass of the object?



. . . .

Feedback: the coordinates will be $(L/3, L/3, L/3)$ if origin is at the corner, with coordinate axes at the joints between the squares.

CONCLUSION

In this unit you have learned:

- what is meant by a rigid system and a rigid body, and how finding the centre of mass of these objects differs from finding it for a general system
- how to use integration to find the centre of mass of a system with continuous structure
- how to use density to link mass to length, area or volume
- how to use symmetry, the dimension of the problem and the selection of coordinates to simplify finding the centre of mass
- how to find the centre of mass of a composite body from the centres of mass of the components
- how to find the centre of mass of objects with parts removed
- how to apply all the tricks you learned in this unit to find centres of mass

For more practice on all these skills, we recommend you do more exercises from the workbook!

Remember to add the following tools to your toolbox:

- the principle of using integration to find the centre of mass of a rigid body
- the principles of symmetry and the dimension of a system
- determining the centre of mass of a rigid body as an integral over particle-like elements
- the formula for the centre of mass of a composite body
- The formula for the centre of mass if an object with parts removed
- the toolbox for simplifying the task of finding the centre of mass of a rigid body/system

After working through this unit, you should have a general idea of how we are going to find the centres of mass of rigid bodies by integration, and you should be able to do the integration in the case of the simplest objects (rods.). You should also be able to use the simplifying rules: a good choice of coordinates, symmetry, and using the rule for composite bodies to reduce finding the centres of mass of complicated objects to their simpler components. You will need all these skills throughout the rest of the module!

In the next unit we will continue to learn how to find centres of mass by integration, for more complicated objects than rods. This we will do by combining the idea of integration you encountered in this unit with the rule of composite bodies – this will lead to a technique of "slicing and integrating" which we can use to find the centres of mass for quite complicated objects!

Unit 6 MORE ON INTEGRATION TO FIND CENTRES OF MASS

Key questions:

- *So far, the only rigid bodies we have encountered have been rods. What about more complicated objects?*

Combining the integration idea with the simplifying rules of the last section of the previous unit, we can tackle the job of finding the centre of mass of almost any object. In this unit we will explain how!

Contents of this unit:

6.1 Finding the centre of mass by slicing and integrating – laminas

6.2 The centre of mass of a lamina bounded by a curve

6.3 Finding the centre of mass by slicing and integrating – three-dimensional objects

6.4 Finding the centre of mass by using polar coordinates

6.5 The centre of mass of a solid of revolution

What you are expected know before working through this unit:

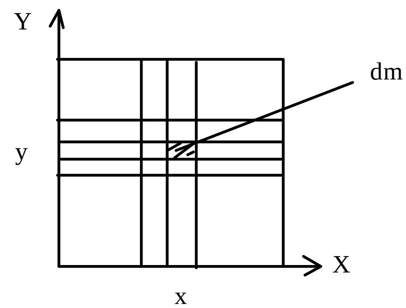
You will need more and more advanced integration skills here, but you will also need to use a bit of trigonometry — that is, using properties of similar triangles, and recalling some facts of polar coordinates and trigonometric functions!

6.1 Finding the centre of mass by slicing and integrating – laminas

We will start by introducing the idea of “slicing and integrating”, and showing how it can be used to find the centre of mass of simple geometrical objects. In this section we will start first with laminas (that is, two-dimensional "flat" objects) and will then move on to more general (three-dimensional) objects in a later section.

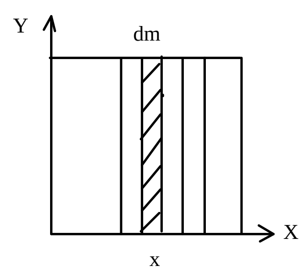
The idea behind slicing and integrating is again to use integration instead of summing up, just as we did previously, but now starting not with a system of particles but rather with the result on finding the centres of mass of compound bodies. That result also gives a summing-up rule for finding the centre of mass of a system to use as a starting point for deriving the integration rules, but now the summing up is for more general objects rather than just particles! Read on to find out how this will work!

Remember that in Result 5.1 (centre of mass by integration) we divided the object into small mass elements dm, each of which is approximately a **particle** with position vector $\underline{r} = \left(x\underline{i} + y\underline{j} + z\underline{k}\right)$, and then integrated over all the elements. While this does in principle always work, the integration can get quite complicated if the object is two- or three-dimensional. For instance, consider the task of finding the centre of mass of a square, using Result 5.1.



We can easily divide the square into small particle-like elements (for example into small squares), but each of them can only be identified by giving both its X- and Y-positions (compare this with the case of the rod, where each small mass element is fully defined by just giving its X-coordinate). Therefore both x and y have to be our integration variables, considering that integrating over all the mass elements now involves integration over all possible x-values and simultaneously also over all possible y-values. This leads to a double integral of the type $\int\int(\cdots)dxdy$, which is beyond the scope of a first-year module!

But all is not lost, as we have one trick still available for us — namely Result 5.4 (how to find the centre of mass of composite bodies)! This result tells us that the mass elements dm do not necessarily have to be particle-like (that is, of very small size), but in fact, they can be *any* parts which the object can be divided into, as long as their *masses* are infinitely small – the move from sums to integrals at limit will then still work! For instance, the small mass elements can be long slices which are infinitely thin. The vector $\underline{r} = \left(x\underline{i} + y\underline{j} + z\underline{k}\right)$ should now be the position vector of the *centre of mass* of dm. As an example, we might consider slicing the square into very thin strips, each of which is then approximately a rectangle. The centre of mass of each slice is easy to find, as it is known to lie at the midpoint of that slice.



Now, each thin slice is fully defined by its X-coordinate x, and therefore we only have to integrate over all the x-values.

Here is the re–formulated result (which, of course, also incorporates Result 5.1, since the centre of mass of a particle is just the position of the particle!)

Result 6.1

For a rigid body, the centre of mass is given by

$$\underline{R} = \frac{\int \underline{r} dm}{\int dm} \tag{6.1}$$

or, in coordinate form, the centre of mass lies at the point $(\bar{x}, \bar{y}, \bar{z})$ where

$$\overline{x} = \frac{\int x dm}{\int dm} \tag{6.2}$$

$$\overline{y} = \frac{\int y dm}{\int dm} \tag{6.3}$$

$$\overline{z} = \frac{\int z dm}{\int dm} \tag{6.4}$$

Here, $\underline{r} = \left(x\underline{i} + y\underline{j} + z\underline{k}\right)$ is the centre of mass of a small mass element dm, and we integrate over all the disjoint mass elements that the object is composed of.

In a typical application of this result, we divide the object into thin slices. The slicing should be done in such a way that the slices are objects for which the following are easy to find:

- The centre of mass of the slice. (If the object is of uniform density, then this is easy to find if the slices are simple, symmetrical geometric objects such as rectangles, discs, etc.)
- The mass of the slice. (Assuming uniform density, this means that we must be able to find the area/volume of the slice easily)

The selection of the coordinates will now often be linked to the way the object will be sliced: For instance, if the slicing is done perpendicularly to one of the coordinate axes, say the X-axis, then each slice can be identified by just giving its X-coordinate x. The integrating variable will then be x, and the integral will be a single integral which we can easily evaluate.

The following toolbox provides a systematic way to approach the task of finding the centres of mass by integration.

TOOLBOX FOR SLICING AND INTEGRATING TO FIND CENTRES OF MASS

1. Draw a sketch of the body.
2. Check whether you can apply any of the simplifying tricks:
   - symmetry; dimension of the problem
   - interpreting the object as a composite body — in which case you should proceed to slice and integrate the components first
   - suitable selection of coordinates; re-drawing the system if necessary.
3. Decide what would be the best way to slice the object.
4. Select the coordinate system, taking into account the considerations above.
5. Identify your integration variable. Find the centre of mass and the position of the

centre of mass of each slice in terms of density and the variable of integration. Identify the upper and lower limits of integration.

6. Evaluate the integrals to find the centre of mass with respect to the chosen coordinate system.

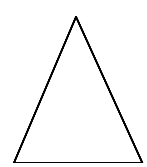
7. Express the position of the centre of mass in relation to the object.

8. Check the solution.

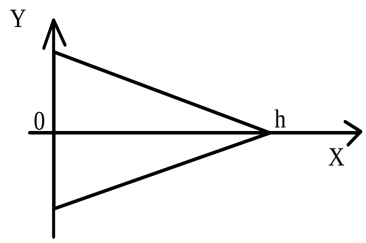
As you will learn, the important thing here is to make sure that the slicing is done such that we can easily find the centre of mass, and the mass, of any given slice. Often the length of a slice varies with the position of the slice – and finding the length might therefore be a bit tricky! The next example illustrates a case where this happens.

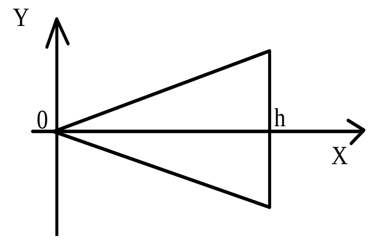
Example 6.1
A uniform triangle with two sides of the same length has height h and a base of length a. Find its centre of mass.

Solution
Draw a sketch of the body. The triangle looks like this (of course, the exact proportions depend on the values of a and h):



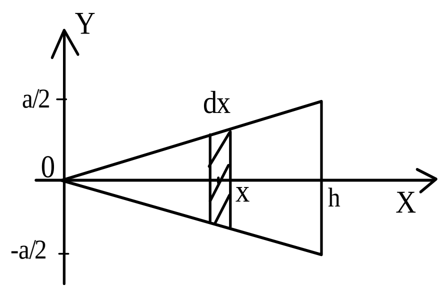
Check whether you can apply any of the simplifying tricks: symmetry; dimension of the problem; interpreting the object as a composite body; suitable selection of coordinates; re-drawing the system if necessary. Interpreting the triangle as a composite body will just lead to more triangles, so we will just have to tackle the triangle as it is! We must now decide on the coordinate system, and also how the triangle should be situated on the coordinate system. The triangle is a two-dimensional object, so we can draw it on the XY-plane. Let us then consider symmetries: clearly a vertical line through the apex of the triangle above is an axis of symmetry. Therefore, we should make sure that one of the coordinate axes, either X or Y, coincides with this line. We have decided here to select the X-axis to concur with the axis of symmetry; which means that we will draw the triangle on its side. (Remember that for a solid body, we can turn the object any way we want!) For convenience we can always draw the triangle so that it starts from the origin and ends at point $x = h$ on the X-axis. Now we have just one more decision to make: should the apex or the base of the triangle be at the origin? That is, which of the following two cases should we choose?





It turns out that the calculations are a little bit easier if we choose the one on the right — see your workbook for an exercise on this!

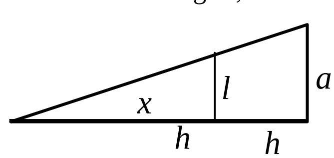
Decide what would be the best way to slice the object. Select the coordinate system. To proceed with the solution, let us therefore assume that the triangle lies on the XY-plane as shown.



The X-axis is an axis of symmetry, so that the centre of mass lies somewhere on the X-axis and we only have to find the X-coordinate of the centre of mass. We will slice the triangle into thin slices perpendicular to the X–axis (that is, parallel to the Y–axis). Is this a good way to slice the triangle? We will need to find the length of each slice which certainly should be possible, using a bit of geometric reasoning.

Identify your integration variable. Find the centre of mass and the position of the centre of mass of each slice in terms of density and the variable of integration. Identify the upper and lower limits of integration. We will divide the triangle into small mass elements dm, each of which consists of a thin strip of the triangle, parallel to the Y-axis. Each slice (strip) is completely defined by its position x along the X–axis, so that x will be our integration variable. The strips are approximately narrow rectangles, and therefore their centres of mass and their areas, and hence their masses, are easy to find: the centre of mass lies in the middle of the rectangle, and the area can be found as length times width. Note that here, unlike in the example of the square we discussed earlier, the length of the strip, and therefore its area and its mass, depend on the position of the strip on the X-axis. The strip shown in the figure, situated at position x on the X-axis has a width dx and length $\ell = ax/h$. This follows from similar triangles, as shown in the sketch below:

$$\frac{h}{a} = \frac{x}{\ell}$$



Thus, this strip has a mass

$$\begin{aligned} dm &= \text{density} \times \text{area} \\ &= \text{density} \times \text{length} \times \text{width} \\ &= \rho \cdot \frac{ax}{h} \cdot dx. \end{aligned}$$

if ρ denotes the density of the object (mass per unit area). The X-coordinate of the centre of mass of this strip is simply x. Finally, if we wish to integrate over all the mass elements (strips), then we must integrate from $x = 0$ to $x = h$.

Evaluate the integrals to find the centre of mass with respect to the chosen coordinate system. Applying (6.2) then gives us as the X-coordinate of the centre of mass of the whole triangle

$$\begin{aligned} \overline{x} &= \frac{\int x\,dm}{\int dm} = \frac{\int_0^h x\rho\frac{ax}{h}dx}{\int_0^h \rho\frac{ax}{h}dx} = \frac{\frac{\rho a}{h}\int_0^h x^2 dx}{\frac{\rho a}{h}\int_0^h x\,dx} \\ &= \frac{\left(\frac{1}{3}x^3\right)\Big]_0^h}{\left(\frac{1}{2}x^2\right)\Big]_0^h} = \frac{\frac{1}{3}h^3}{\frac{1}{2}h^2} = \frac{2}{3}h. \end{aligned}$$

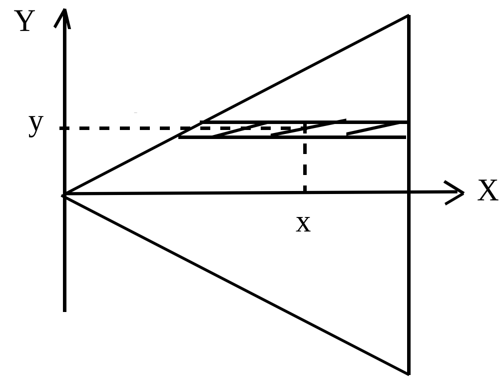
So, the coordinates of the centre of mass in our coordinates system, are $(\overline{x}, \overline{y}) = (\frac{2h}{3}, 0)$.

Express the position of the centre of mass in relation to the object. Interpretation: The centre of mass of the triangle is situated on its axis of symmetry, one-third of the height of the triangle from the base of the triangle. Check the solution. The answer is credible: The centre of mass lies along the axis of symmetry, and we would expect the centre of mass to be closer to the base than to the apex of the triangle!◀

Note that in the calculations in the last example, we sliced the triangle parallel to the basis of the triangle. Usually there are also several ways of slicing an object to find its centre of mass. As with the choice of coordinates, some methods are easier than others — as an illustration, we will re-do the calculations in the previous example, this time slicing the triangle into thin strips parallel to the X-axis, rather than the Y-axis.

Example 6.2

Example 6.1, Solution — Method 2:



We have not changed the coordinate system, so that the X-axis is still an axis of symmetry, and we just have to calculate the X-coordinate of the centre of mass. But this time we have divided the triangle into small mass elements, each of which consists of a thin strip of the triangle, parallel to the X-axis. Note that this time each strip is identified by its position on the Y-axis, so that y will have to be our integration variable! The strip shown in the figure, situated at position $y > 0$ of the Y-axis has a width dy. It reaches from $x = \frac{2h}{a}y$ to $x = h$ on the X-axis, and therefore it has a length $(h - \frac{2h}{a}y)$, and mass

$$dm = \rho(h - \frac{2h}{a}y)dy$$

and the X-coordinate of the centre of mass of this strip is

$$x = \frac{2h}{a}y + \frac{(h - \frac{2h}{a}y)}{2} = \frac{1}{2}h\frac{2y + a}{a}.$$

A similar strip, with the same mass and same X-coordinate for its centre of mass, is situated at $(-y)$. If we wish to integrate over all the mass elements (strips), then we must integrate from $y = -a/2$ to $y = a/2$. Applying (6.2), the X-coordinate of the centre of mass of the whole triangle then is

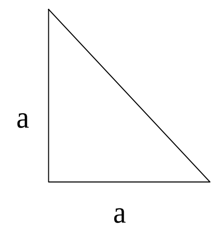
$$\begin{aligned}
\overline{x} &= \frac{\int x dm}{\int dm} \\
&= \frac{\int_{-a/2}^{a/2} \left(\frac{1}{2}h\frac{2|y|+a}{a}\right)\rho(h - \frac{2h}{a}|y|)dy}{\int_{-a/2}^{a/2}\rho(h - \frac{2h}{a}|y|)dy} \\
&= \frac{2}{3}h.
\end{aligned}$$

(The absolute value signs are there because y, marking the position of the strips in our coordinate system, gets both negative and positive values, while x needs to be positive.) This confirms the result that the centre of mass is situated one-third from the base of the triangle.◄

Activity 6.1

This problem also involves a triangle. Hint: do not just try to fit the triangle into the previous example — rather use the toolbox for slicing and integrating to figure out the best way to tackle this particular triangle!

Problem: Find the centre of mass of the triangle shown below by slicing and integrating. Remember to select and indicate in a sketch an appropriate coordinate system.



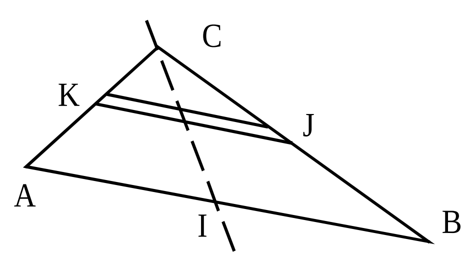
Feedback: the centre of mass will again lie along the line from the apex to the middle of the hypotenuse, the distance $2a/3$ from the apex.

We have now dealt with two different kinds of triangles (isosceles and straight). In each of them, we used the properties of the triangle to select the most appropriate way of orienting and slicing. The next example deals with a general triangle, and uses a clever trick, rather than straightforward calculation, to find the centre of mass!

Example 6.3
Find the centre of mass of an arbitrary uniform triangular lamina.

Solution:
For a general triangle, we cannot assume any symmetries. We could draw the triangle on the XY-plane, e.g. with the longest side along the X-axis to make calculations a bit easier, but to proceed from there we would have to assume that we know the proportions of the triangle. We will, instead, use a clever trick which does not require us to decide on a coordinate system.

Let the corners of the triangle be A, B and C, and let I, J and K then be the centre points of the three sides, as shown below. Imagine now that the triangle is divided into thin strips parallel to side AB, as shown.



If the strips are very narrow, they can be treated as thin rods (or narrow rectangles) with their centres of mass at the centre of each rod. But then the centre of mass of each rod will lie on the dotted line IC, which bisects AB and is therefore a median of the triangle. It follows that the centre of mass of the whole triangle must also lie on this median IC. By repeating similar reasoning, when the triangle is divided into thin strips parallel to BC and AC, we see that the centre of mass must also lie on the two other medians, KB and JA. The conclusion is that the centre of mass of the triangle must lie at the intersection of the three medians of the triangle. And geometry tells us that this intersection lies two–thirds along any median, measured from one of the corners of the triangle. ◀

Note that the result here confirms what we already found as the centres of mass for the other two types of triangles we dealt with earlier!

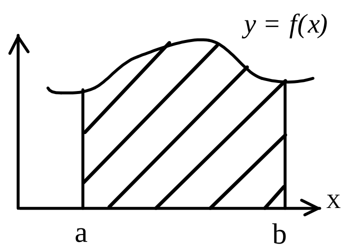
6.2 The centre of mass of a lamina bounded by a curve

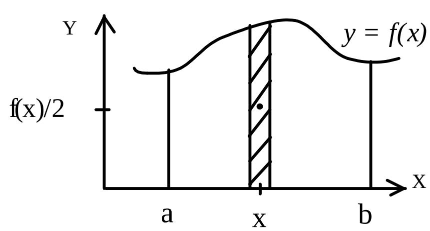
We will next apply the idea of "slicing and integrating" to find a general rule for the centre of mass of a lamina on the XY-plane, which is bounded by the X-axis and the curve of a given function $y = f(x)$ between the lines $x = a$ and $x = b$. This lamina is the shaded region in the sketch below. We will assume the lamina is made of some uniform material. Also, to get a well-defined object, we will assume that $f(x)$ is not negative on the interval $[a, b]$!

To find the centre of mass, we will use the "slicing" principle of Result 6.1. Imagine the lamina divided into thin strips, parallel to the Y-axis. Then each strip is approximately a rod, or a narrow rectangle. The one situated at position x of the X-axis has a height $f(x)$; if its width is dx then the rectangle has an area of $f(x)\,dx$ and a mass of

$$dm = \rho f(x)\,dx$$

where ρ is the surface density of the lamina (the mass per unit area). The centre of mass of this strip is at point $\left(x, \frac{1}{2} f(x)\right)$.





Finally, integrating over all these strips means integrating with respect to x, over the interval $[a, b]$. Result 6.1, equations (6.2) and (6.3), now give as the centre of mass of the lamina the point $(\overline{x}, \overline{y})$ where

$$\overline{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_a^b x\rho f(x)\,dx}{\int_a^b \rho f(x)\,dx} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx}$$

and

$$\overline{y} = \frac{\int y\,dm}{\int dm} = \frac{\int_a^b \frac{1}{2} f(x) \cdot \rho f(x)\,dx}{\int_a^b \rho f(x)\,dx} = \frac{\frac{1}{2}\int_a^b \left[f(x)\right]^2 dx}{\int_a^b f(x)\,dx}.$$

Result 6.2
The centre of mass of a uniform lamina bounded by the X-axis and the curve $y = f(x)$ between the lines $x = a$ and $x = b$ is at the point $(\bar{x}, \bar{y})$ on the XY-plane, where

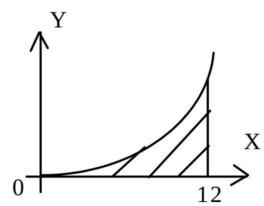
$$\overline{x} = \frac{\int_a^b x f(x)\,dx}{\int_a^b f(x)\,dx} \qquad (6.5)$$

$$\overline{y} = \frac{\frac{1}{2}\int_a^b \left[f(x)\right]^2 dx}{\int_a^b f(x)\,dx} \qquad (6.6)$$

Warning: Do not use these formulas unless you are sure that they apply in a given situation! You should also be able to derive the formulas.

Example 6.4
Find the centre of mass of a uniform lamina bounded by the X-axis, the curve $y = x^2$ and the lines $x = 0$ and $x = 12$.

Solution:
A sketch of the lamina looks like this.



Applying equations (6.5) and (6.6) with $f(x) = x^2$, $a = 0$ and $b = 12$ we get

$$\begin{aligned}
\overline{x} &= \frac{\int_0^{12} x f(x)\,dx}{\int_0^{12} f(x)\,dx} = \frac{\int_0^{12} x \cdot x^2 dx}{\int_0^{12} x^2 dx} \\
&= \frac{\int_0^{12} x^3 dx}{\int_0^{12} x^2 dx} = \frac{\left(\frac{1}{4}x^4\right)\Big]_0^{12}}{\left(\frac{1}{3}x^3\right)\Big]_0^{12}} = \frac{5184}{576} = 9, \\
\overline{y} &= \frac{\frac{1}{2}\int_0^{12} [f(x)]^2\,dx}{\int_0^{12} f(x)\,dx} = \frac{\frac{1}{2}\int_0^{12} (x^2)^2\,dx}{\int_0^{12} x^2 dx} \\
&= \frac{\frac{1}{2}\int_0^{12} x^4 dx}{\int_0^{12} x^2 dx} = \frac{\frac{1}{2}\left(\frac{1}{5}x^5\right)\Big]_0^{12}}{\left(\frac{1}{3}x^3\right)\Big]_0^{12}} = 43.2.
\end{aligned}$$

The centre of mass is therefore at point (9, 43.2). ◄

Activity 6.2

The following exercise is a straightforward application of these rules!

Find the centre of mass of a uniform lamina bounded by the X–axis, the lines $x = 0$, $x = 5$, and the curve $y = 2\sqrt{x}$.

....

Feedback: the centre of mass is at $\left(3, \frac{3}{4}\sqrt{5}\right)$.

Finding the centres of mass for these laminas by integration is something you must learn to do routinely — please do more practice examples from your workbook, until you feel you have mastered the technique!

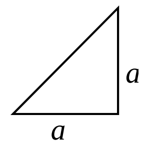
Here is a last chance to apply this technique which also reminds you to always use all the tricks you know to simplify finding centres of mass!

Activity 6.3

Here you can apply the result for laminas bounded by a curve in a novel situation. You must introduce a $XY-$coordinate system, and find a function to describe the triangle as a lamina!

Use the concept of a lamina bounded by a curve, to find the centre of mass of the triangle shown below.

triangle



$_5 4.pdf$

....

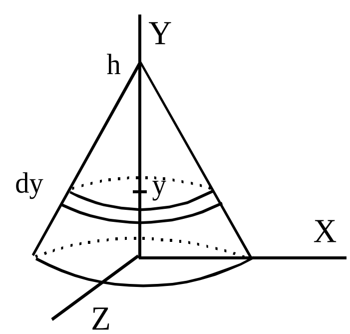
Feedback: compare your answer to where you know the centre of mass of a triangle to be!

6.3 Finding the centre of mass by slicing and integrating – three-dimensional objects

In the previous section we discussed only two-dimensional objects, that is, laminas. However, the procedure of slicing and integrating can equally well be applied to a three–dimensional rigid body. The difference is that the “slices” will then be thin three-dimensional objects (e.g. discs), rather than rectangles etc. (Imagine for instance slicing an apple — in the middle of the apple, slices are roughly thin cylinders!) The process is really very similar to the two-dimensional case. The difference is that since the slices are now three-dimensional, to find their masses we shall have to find their volumes, rather than their areas. And of course you must now think in three dimensions, which might make selecting coordinate system and drawing pictures of the situation a bit more complicated!

Example 6.5
A solid cone has a radius a and height h. Find its centre of mass.

Solution:
Draw a sketch of the body, select the coordinate system and how to slice the object. Let us choose XYZ coordinates as shown in the figure below. Now both the XY-plane and YZ-plane are planes of symmetry (since each of them cuts the cone into two identical parts which are mirror images of each other).Therefore, the centre of mass must be situated at the intersection of the XY- and YZ-planes, that is, on the Y-axis. So, we just have to find the Y-coordinate of the centre of mass; and we can slice the object into thin slices parallel to the XZ-plane, as shown.Identify your integration variable. Find the centre of mass and the position of the centre of mass of each slice in terms of density and the variable of integration. Identify the upper and lower limits of integration. Evaluate the integrals to find the centre of mass with respect to the chosen coordinate system. Each slice is identified by its position on the y–axis, so y will be the integration variable.



For the slice at position y of the Y-axis , the centre of mass is at point y on that axis. Each slice is approximately a disc, with thickness dy. To calculate its mass, we need its radius. By similar triangles,

$$\frac{h-y}{\text{radius}} = \frac{h}{a}$$

so that

$$\text{radius} = \frac{a}{h}(h-y).$$

The mass of the slice/disc is now given by

$$\text{mass} = \text{density} \times \text{volume}$$

where

$$\begin{aligned} \text{volume} &= \text{thickness} \times \text{area} \\ &= \text{thickness} \times \pi \, (\text{radius})^2 . \end{aligned}$$

Thus, the mass is

$$dm = \rho\pi \left(\frac{a}{h}(h-y)\right)^2 dy$$

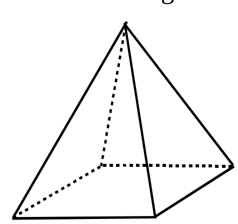
where ρ is the density (mass per unit volume). (We are assuming here that the cone is made of uniform material!) Integrating ("summing") over all of these discs means integrating from $y = 0$ to $y = h$. So, according to (6.3), we get

$$\begin{aligned} \overline{y} &= \frac{\int y\,dm}{\int dm} = \frac{\int_0^h y\rho\pi \left(\frac{a}{h}(h-y)\right)^2 dy}{\int_0^h \rho\pi \left(\frac{a}{h}(h-y)\right)^2 dy} \\ &= \frac{\rho\pi \frac{a^2}{h^2} \int_0^h y\,(h-y)^2\,dy}{\rho\pi \frac{a^2}{h^2} \int_0^h (h-y)^2\,dy} \\ &= \frac{\int_0^h \left(yh^2 - 2y^2h + y^3\right) dy}{\int_0^h \left(h^2 - 2yh + y^2\right) dy} \\ &= \frac{\left(\frac{h^2}{2}y^2 - \frac{2h}{3}y^3 + \frac{1}{4}y^4\right)\Big]_0^h}{\left(h^2y - hy^2 + \frac{1}{3}y^3\right)\Big]_0^h} \\ &= \frac{\left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right) h^4}{\left(1 - 1 + \frac{1}{3}\right) h^3} = \frac{1}{4}h. \end{aligned}$$

The centre of mass of the cone is at position $\left(0, \frac{1}{4}h, 0\right)$, in the coordinate system given above. Express the position of the centre of mass in relation to the object. The centre of mass is one–fourth of the height of the cone, above the base of the cone. Check the solution. The answer does seem likely — we would expect the centre of mass to lie closer to the base that the apex to the cone, since the cone is thicker at the base! Compare this with the similar result for the triangle in two dimensions (Example 6.1), where we found the centre of mass at the distance $\frac{1}{3}h$ above the base. Note also that both for the cone and the triangle, the length/radius of the base makes no difference!◄

Activity 6.4

Find the centre of mass of a uniform solid pyramid with height h and a square $a \times a$ base.



Hint: A cross-section of the cone at the height z above the base (with $0 \leq z \leq h$) forms a square with sides of length $\frac{h-z}{h}a$.

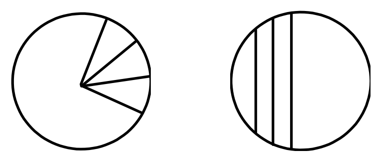
. . . .

Feedback: the centre of mass will be along the axis of symmetry, the distance $h/4$ above the base.

6.4 Finding the centre of mass by using polar coordinates

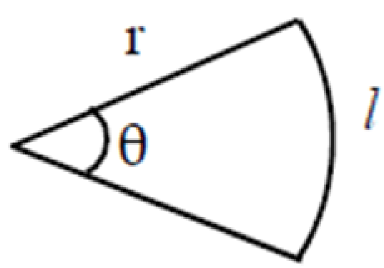
Note: In this section, the integration gets a bit more complicated, since we are dealing with objects where polar coordinates work better than the usual rectangular ones! Indeed the integration techniques in this section do not form a part of the outcomes of this module, so you will not be asked to do integration using polar coordinates in the assignments nor in the examination. This section is only for your information; and later on you will need to refer to some of the results in this section.

In some cases, the "slicing and integrating" is easiest to do if we use a **polar coordinate** approach. This is especially the case with objects such as circles or sectors of circles, spheres, etc. The idea is then to divide the object into small mass elements, each corresponding to a **small angle** $d\theta$. So, we are subdividing the object into mass elements as in the figure on the left, rather than as in the figure on the right.



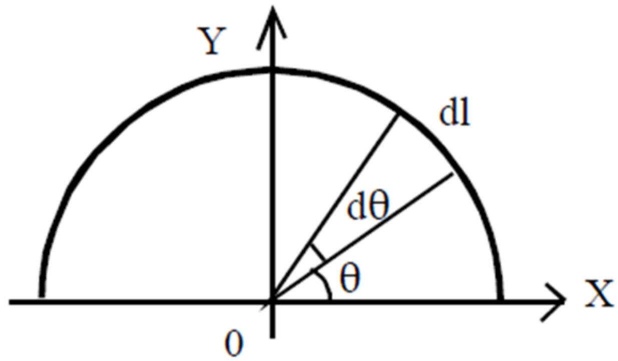
Each small mass element now corresponds to a unique angle θ, and integration over all the elements will correspond to integration over all the relevant θ-values — that is, our integration variable is now θ. The following examples show various modifications of this idea. Note that various geometric results will be needed in this polar coordinate approach; some of these are listed below.

If a circle has a radius r, then an angle θ determines a sector of the circle. The length of the corresponding arc is $\ell = \theta r$, and the area of the sector is $A = \frac{1}{2}r^2\theta$.



Example 6.6
A thin uniform rod is bent in the shape of a semicircle of radius R. Where is the centre of mass of the rod?

Solution:
Let us assume that the semicircle lies on the XY-plane, as shown.



The Y-axis is then the axis of symmetry, and the centre of mass therefore lies somewhere on the Y-axis. To find $\overline{y}$, the Y-coordinate of the centre of mass, we will divide the rod into small mass elements, each consisting of a short segment of the semicircle. If a segment corresponds to a small angle $d\theta$, then its length is

$$d\ell = Rd\theta$$

and its mass is

$$dm = \rho d\ell = \rho Rd\theta$$

where ρ denotes the density of the rod (the mass per unit length). The Y-coordinate of the segment, the position of which is given by the angle θ, (see the figure above) is

$$y = R\sin\theta.$$

Finally, if we wish to integrate over all the small segments, then we have to integrate over $\theta \in [0, \pi]$. Now, (6.3) gives

$$\begin{aligned}\overline{y} &= \frac{\int ydm}{\int dm} = \frac{\int_0^\pi R\sin\theta \cdot \rho Rd\theta}{\int_0^\pi \rho Rd\theta} \\ &= \frac{\rho R^2 \int_0^\pi \sin\theta d\theta}{\rho R\int_0^\pi d\theta} = \frac{R\left(-\cos\theta\right)]_0^\pi}{\theta]_0^\pi} \\ &= \frac{2}{\pi}R.\end{aligned}$$

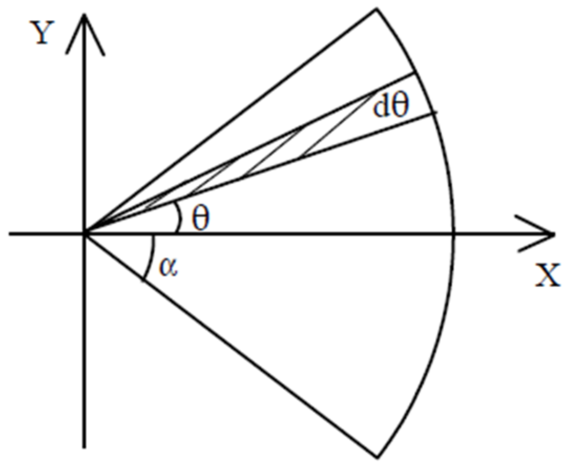
Thus, in our coordinate system the centre of mass lies at the point $(0, 2R/\pi)$. Interpreted with respect to the object itself, we see that the centre of mass of a thin rod bent into a semicircle with radius R lies on its axis of symmetry, the distance $2R/\pi$ from the diameter of the semicircle.◀

Note that you can also solve the problem in the previous example by using the usual tactic of slicing the rod parallel to the Y-axis, into thin slices of width dx. But in that case, you must consider very carefully the question of what is the length, and hence the mass, of each such slice!

Example 6.7
Find the centre of mass of a uniform lamina with the shape of a sector which forms an angle 2α of a circle of radius r.

Solution:
Let us assume that the sector of the circle corresponds to an angle 2α where $\alpha \leq \pi/2$. If

the XY-plane is chosen as in the figure, then the X-axis is the axis of symmetry and the centre of mass lies on the X-axis. To find where on the X-axis the centre of mass is, let us divide the sector into small strips, each corresponding to a small angle $d\theta$, as shown in the figure.



If $d\theta$ is very small, then each of these strips is approximately a triangle, with a centre of mass situated at a distance $\frac{2}{3}r$ from the origin (according to Example 6.1). For the strip which forms the angle θ with the positive X-axis (see picture), the X-coordinate of the centre of mass is $\frac{2}{3}r\cos\theta$. The mass of this sector is obtained by

$$\text{mass} = \text{density} \times \text{area}$$

so that the mass is

$$dm = \rho\frac{1}{2}r^2 d\theta$$

where ρ denotes the density.

Finally, we have to integrate from $\theta = -\alpha$ to $\theta = +\alpha$. From equation (6.2), we can determine the X-coordinate of the centre of mass as

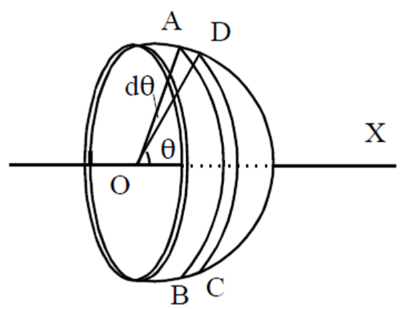
$$\begin{aligned}
\bar{x} &= \frac{\int x dm}{\int dm} = \frac{\int_{-\alpha}^{\alpha}\frac{2}{3}r\cos\theta\cdot\rho\frac{1}{2}r^2 d\theta}{\int_{-\alpha}^{\alpha}\rho\frac{1}{2}r^2 d\theta} \\
&= \frac{\frac{1}{3}r^3\rho\int_{-\alpha}^{\alpha}\cos\theta d\theta}{\frac{1}{2}r^2\rho\int_{-\alpha}^{\alpha}d\theta} \\
&= \frac{2}{3}r\frac{(\sin\theta)]_{-\alpha}^{\alpha}}{(\theta)]_{-\alpha}^{\alpha}} = \frac{2}{3}\frac{\sin\alpha}{\alpha}r.
\end{aligned}$$

The centre of mass lies on the axis of symmetry of the sector, at a distance $\frac{2}{3}\frac{\sin\alpha}{\alpha}r$ from the origin (the centre of the corresponding circle).◀

Example 6.8
Of special interest is the case where $\alpha = \pi/2$, giving us a lamina shaped like a semicircle. We then get the following result: the centre of mass of a uniform lamina, in the shape of a semicircle with radius R, lies along its axis of symmetry, at a distance $\frac{4R}{3\pi}$ from the centre of the circle. (Note that this is approximately $0.4R$.)◀

Example 6.9
Find the centre of mass of a thin, uniform hollow hemisphere with radius r.

Solution:
Make sure that you can imagine in your mind what the object here looks like! Think of a tennis ball or soccer ball, hollow inside, cut in half. Let us assume that the hemisphere lies on the XYZ system, as shown. In terms of symmetry, the centre of mass lies on the X-axis, so $\overline{y} = 0$ and $\overline{z} = 0$, and we only have to find the X-coordinate $\overline{x}$ of the centre of mass. To calculate $\overline{x}$, we divide the hemisphere into thin strips, each corresponding to a small angle $d\theta$ on the XY-plane.



The position of the strip shown in the figure is fully described by the angle θ on the XY-plane. In the figure above, the angle between the X-axis and OD is θ, and the angle between OA and OD and between OB and OC is the small angle $d\theta$. Then this strip is approximately a thin band, with width $AD = rd\theta$, with radius $DC/2 = r\sin\theta$, and a centre of mass situated on the X-axis at the position $x = r\cos\theta$. If the density is ρ, then the mass of this band is given by

$$\begin{aligned} \text{mass} &= \text{density} \times \text{area}, \\ \text{area} &= \text{circumference} \times \text{width} \end{aligned}$$

so that the mass is

$$\begin{aligned} dm &= \rho \cdot 2\pi r \sin\theta r d\theta \\ &= \rho 2\pi r^2 \sin\theta d\theta. \end{aligned}$$

Note that here, $0 \leq \theta \leq \pi/2$. From (6.2)

$$\begin{aligned} \overline{x} &= \frac{\int x dm}{\int dm} = \frac{\int_0^{\pi/2} (r\cos\theta) \cdot \rho 2\pi r^2 \sin\theta d\theta}{\int_0^{\pi/2} \rho 2\pi r^2 \sin\theta d\theta} \\ &= \frac{\rho 2\pi r^3 \int_0^{\pi/2} \cos\theta \sin\theta d\theta}{\rho 2\pi r^2 \int_0^{\pi/2} \sin\theta d\theta}. \end{aligned}$$

But $\frac{d}{d\theta}\sin^2\theta = 2\sin\theta\cos\theta$; so

$$\begin{aligned} \overline{x} &= \frac{r\left(\frac{1}{2}\sin^2\theta\right)\Big]_0^{\pi/2}}{(-\cos\theta)\big]_0^{\pi/2}} = \frac{r\left(\sin^2\left(\frac{\pi}{2}\right) - \sin^2(0)\right)}{2\left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right)} \\ &= \frac{r}{2}. \end{aligned}$$

The centre of mass lies at the position $\overline{x} = \frac{r}{2}$ on the X-axis, that is, the distance $r/2$ from the base of the hollow hemisphere. ◄

This concludes our section on finding centres of mass by slicing and integrating, using polar coordinates. Again, please note that this section you can just read through since you will not be asked to use integration using polar coordinates to find centres of mass in this module! We showed the calculations here to derive the centres of mass of certain special objects, and you will need to use the results derived in this section later on in the study guide.

We have dealt with arcs and sectors of circles, and a hollow hemisphere. What about the centre of mass of a corresponding solid hemisphere? In that case, we can actually slice the object parallel to the X-axis, using x as the integration variable; polar coordinates are not needed! The reason is that we are going to get thin discs, the volume of which is fully determined by the width of the disc! We will deal with the solid hemisphere in the next section, and will then further generalise the technique used for it for more general objects.

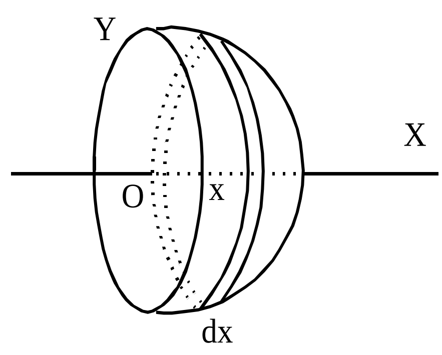
6.5 The centre of mass of a solid of revolution

In the last subsection, we used slicing and integration, with polar coordinates, to find the centre of mass of a hollow hemisphere. You might have wondered what we would then do to deal with a solid hemispheres. We now proceed to deal with that object; it turns out that we can just apply straight forward slicing and integrating. Further, it will turn out that this is a special case of a more general type of object, called a solid of revolution. Following the procedure for the solid hemisphere, we will derive a result for finding the centre of mass of any such solid of revolution!

But let us first start, as promised, with finding the centre of mass of a solid hemisphere. To imagine what this object is like, consider for instance an orange cut in half!

Example 6.10
Find the centre of mass of a uniform, solid hemisphere with radius r.

Solution
The object is shown below, together with a good choice of the coordinate system.



Choosing the X-axis as in the figure, it follows from symmetry considerations that the centre of mass lies on the X-axis. We will take the origin to be on the "cut" surface of the hemisphere (at what would be the centre of the corresponding complete sphere). To find $\overline{x}$, we proceed as follows.

Consider the object being sliced into thin discs, parallel to the YZ-plane. The figure shows one of these thin discs, situated at the position x on the X-axis and with thickness dx. The radius of this disc is $\left(r^2 - x^2\right)^{1/2}$ (this follows from the equation of a circle, $y^2 + x^2 = r^2$).

Now we can calculate the mass of the disc:

$$\begin{aligned} \text{mass} &= \text{density} \times \text{volume}, \\ \text{volume} &= \text{area} \times \text{thickness} \\ &= \pi\ (\text{radius})^2 \times \text{thickness}, \end{aligned} \tag{6.7}$$

so that

$$\begin{aligned} dm &= \rho\pi \left(\left(r^2 - x^2 \right)^{\frac{1}{2}} \right)^2 dx \\ &= \rho\pi \left(r^2 - x^2 \right) dx. \end{aligned}$$

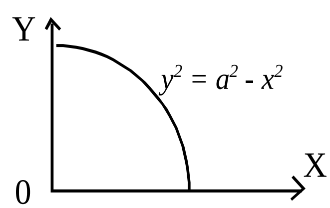
To integrate over all the slices, we need to integrate over x from 0 to r. Then, according to (6.2), the X-coordinate of the centre of mass is

$$\begin{aligned} \overline{x} &= \frac{\int x dm}{\int dm} = \frac{\int_0^r x\rho\pi \left(r^2 - x^2\right) dx}{\int_0^r \rho\pi \left(r^2 - x^2\right) dx} \\ &= \frac{\rho\pi \int_0^r \left(r^2 x - x^3\right) dx}{\rho\pi \int_0^r \left(r^2 - x^2\right) dx} \\ &= \frac{\left(\frac{r^2}{2}x^2 - \frac{1}{4}x^4\right)\Big]_0^r}{\left(r^2 x - \frac{1}{3}x^3\right)\Big]_0^r} = \frac{r^4\left(\frac{1}{2} - \frac{1}{4}\right)}{r^3\left(1 - \frac{1}{3}\right)} \\ &= \frac{3}{8}r. \end{aligned}$$

The centre of mass of the solid hemisphere lies a distance $3r/8$ from the base of the hemisphere.◄

Now, in the calculations above we sliced the object into small mass elements in the shape of discs, and then used the expression (6.7) to determine the volume of these discs. This method is not restricted to the hemisphere, but can also be applied more generally to a special type of symmetrical objects called **solids of revolution.**

To see what this means, consider the curve of a function shown below.



This represents a part of a circle in the first quadrant of the XY-plane. Imagine now this curve and the region below it being rotated through a full circle about the X-axis. (That is, imagine picking the figure above up by the Y-axis, and spinning it around the X-axis). The resulting three-dimensional object will be a hemisphere, as in the previous example.

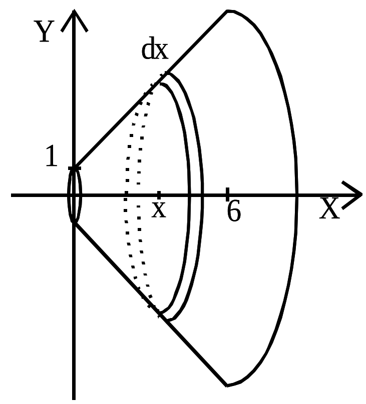
Any object which can be formed by rotating a region bounded by a curve of the type $y = f(x)$ is called a **solid of revolution**. What all these objects have in common is that they have an axis of rotation about which the object is symmetric, and that if they are cut in slices perpendicular to the axis of rotation, the slices will be thin discs. The method used in Example 6.10 can then be adapted to find the centre of mass of any such solid. If we assume uniform density then to find the masses of the thin slices we just need their volumes. Since the slices are discs, we need to know the radius of each disc, which is given directly by the expression of the curve, $y = f(x)$.

The following example illustrates the process.

Example 6.11

Find the centre of mass of the solid of revolution formed when the curve $y = x + 1$ is rotated about the X-axis in the region $0 \leq x \leq 6$.

Solution:



Since rotation is about the X-axis, the X-axis is the axis of symmetry and therefore $\overline{y} = 0$. To calculate $\overline{x}$, we divide the solid of revolution into thin discs. The one at position x on the X-axis (see picture) has a thickness dx and radius $f(x) = x + 1$. We want to calculate its mass, dm. Because we assumed uniform density,

$$\text{mass} = volume \times density$$

and for a disc,

$$\begin{aligned} \text{volume} &= \text{thickness} \times \text{area} \\ &= \text{thickness} \times \pi\ (\text{radius})^2 . \end{aligned}$$

So, if ρ denotes density,

$$dm = \rho \cdot dx \cdot \pi\ (x+1)^2 .$$

The centre of mass of this small element has the X-coordinate x. To calculate the X-coordinate of the whole solid, we use (6.2):

$$\begin{aligned} \overline{x} &= \frac{\int_0^6 x\,dm}{\int_0^6 dm} = \frac{\int_0^6 \rho\pi x\,(x+1)^2\,dx}{\int_0^6 \rho\pi\ (x+1)^2\,dx} = \frac{\int_0^6 \left(x^3+2x^2+x\right)dx}{\int_0^6 \left(x^2+2x+1\right)dx} \\ &= \frac{\left(\frac{1}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2\right)\Big]_0^6}{\left(\frac{1}{3}x^3+x^2+x\right)\Big]_0^6} = \frac{81}{19}. \end{aligned}$$

So, the centre of mass of the solid of revolution is at $\left(\frac{81}{19}, 0, 0\right)$. ◄

Activity 6.5

Here is your chance to find the centre of mass of a solid of revolution.

Problem: Find the centre of mass of the solid of revolution formed by rotating the curve $y = 2a\sqrt{x}$ about the X-axis for $0 \leq x \leq b$.

....

Feedback: the centre of mass will be at $\left(\frac{2}{3}b, 0, 0\right)$.

The approach given in this section gives another way to approach certain types of objects. Namely, if an object has the property of having an axis of symmetry such that any cross section of it cut perpendicular to the axis is a circle, then it is a solid of revolution and the approach given in this section can be used to find its centre of mass. All that needs to be done is to identify its "profile" in terms of the function $f(x)$ which determines its radius at any given point x along its axis! The following activity is of this kind.

Activity 6.6

Use the technique of finding the centre of mass of a solid of revolution, by slicing it into discs, to locate again the centre of mass of a solid cone with a base with radius a and height h. *Hint: you must place the cone in the XYZ coordinate system such that its axis of symmetry lies along the x axis! A well-chosen decision on where the apex should be will make calculations easier.*

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Feedback: Compare your result with the findings in Example 6.5!

Warning: you have now come across two specific integration formulas dealing with functions: those for finding the centre of mass for a lamina bounded by a function, and those for finding the centre of mass of a solid of revolution determined by a given function. Note that these two objects are very different! Do not try to memorise the formulas, since you may then use the wrong one — rather, you should be able to derive each set of formulas from the definitions of the objects!

CONCLUSION

In this unit you have learned

- how to find the centres of mass of rigid bodies by slicing and integrating
- how to find the centres of mass of laminas bounded by curves, and solids of revolution

Remember to add the following tools to your toolbox:

- determining the centre of mass of a rigid body as an integral over general mass elements
- the principle of using slicing and integrating to find the centre of mass of a rigid body
- the toolbox for the task of slicing and integrating
- the concept of objects formed as solids of revolution
- determining the centre of mass of laminas bounded by curves
- determining the centre of mass of solids of revolution

In this unit, we continued finding centres of mass by slicing and integrating, using more general and more effective techniques. You should now have an idea on how to find the centre of mass of quite complicated rigid bodies. But, it is time to remember why we are so interested in finding the centres of mass: we wish to use it to help us analyse the motion of objects. In the next unit we will return to this topic.

Unit 7 THE MOTION OF THE CENTRE OF MASS — THE GENERAL CASE

We are now able to tie up Learning Unit 2 of the study guide. Remember that our main goal in this Learning Unit of the study guide was to derive, and show how to apply, the law of motion which describes how the centre of mass of any object or system of objects moves.

So far in this section we have done the following: In Unit 3, we defined the centre of mass for a system of particles, and showed how to find it, and in Unit 4 we explained why the centre of mass is important: because the motion of the centre of mass of a collection of particles can be described by a very simple equation when all the external forces acting on the system are known. We then turned our attention to rigid bodies (that is, everyday objects) and reasoned that since they are really just collections of very many particles, the concept of a centre of mass is valid for them as well, but due to the large number of particles, we should rather use the mathematical shortcut of slicing-and-integrating to find their centres of mass. Units 5 and 6 were then dedicated to learning how to slice and integrate in various situations to find the centres of mass. And along the way we discussed various shortcuts which make finding the centres of mass easier, such as using symmetry or selecting the coordinate system suitably. Also, we learned to use the ideas of viewing the object as a composite body, or an object with parts removed, to simplify the calculations.

We can now find the centre of mass of any rigid body or system of rigid bodies, and can analyse the motion of the centre of mass. We shall first state the results for a rigid body, and later we will do one final generalisation into a system of rigid bodies. Applications of all these skills will then follow!

Contents of this unit:

7.1 The motion of the centre of mass of a rigid body

7.2 Systems of rigid bodies and particles

What you are expected know before working through this unit:

Into this unit, you need to bring the idea you came across in Unit 4 about combining Newton's second law describing the motion of individual particles, to one law describing the motion of the centre of mass directly. Here we will extend this result to rigid bodies and systems! Also, you will need all the techniques you have learned so far for finding centres of mass in various situations, since this unit combines everything you have done so far in this Learning Unit 2 of the study guide!

7.1 The motion of the centre of mass of a rigid body

A rigid body is also a system of particles, so Result 4.2 can be applied to describe the motion of its centre of mass. For a rigid body, the only internal forces that can act on the body are the forces holding all the particles of the body together, so that the body does not fall apart. All other forces, such as the force of gravity or hitting it with a hammer, are external forces. The following result follows directly from what happens in the case of a system of particles.

Result 7.1
Let a rigid body have a mass M, and let $\underline{R}$ be the position vector of its centre of mass. Then

$$\underline{F} = M\ddot{\underline{R}} \tag{7.1}$$

where $\underline{F}$ is the sum of all the external forces acting on the rigid body.

It follows that to find the acceleration of the centre of mass of the body, we only have to find the total mass of the body and the sum of all the forces acting on the system. Note that, according to this result, the motion of the centre of mass only depends on the resultant force $\underline{F}$, that is, the sum of all the forces acting on the body. The motion of the centre of mass does **not** depend on the points of application of the forces! The following examples illustrate this.

Example 7.1
A uniform rod AB with mass m moves on a frictionless plane. Initially the rod is at rest, parallel to the Y-axis, with the centre of the rod at the origin. Then two identical constant forces

$$\underline{F}_1 = \underline{i}, \quad \underline{F}_2 = \underline{i}$$

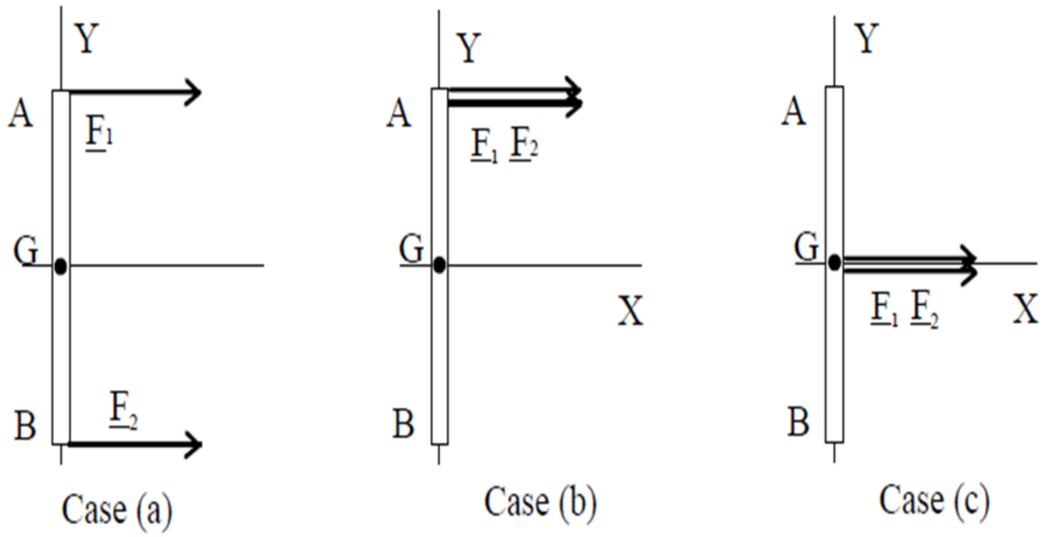
parallel to the X-axis are applied to it. Describe the motion of G, the centre of mass of the rod, in the following cases:

(a) If $\underline{F}_1$ acts at point A, and $\underline{F}_2$ at point B,

(b) if $\underline{F}_1$ and $\underline{F}_2$ act at point A,

(c) if $\underline{F}_1$ and $\underline{F}_2$ act at point G.

Solution:
The centre of mass of the rod, G, is initially at the origin of the XY-plane.



The resultant force acting on the rod in all four cases is

$$\underline{F} = \underline{F}_1 + \underline{F}_2 = 2\underline{i}.$$

According to Result 7.1, in all four cases the centre of mass then moves with acceleration $\underline{a}$ where

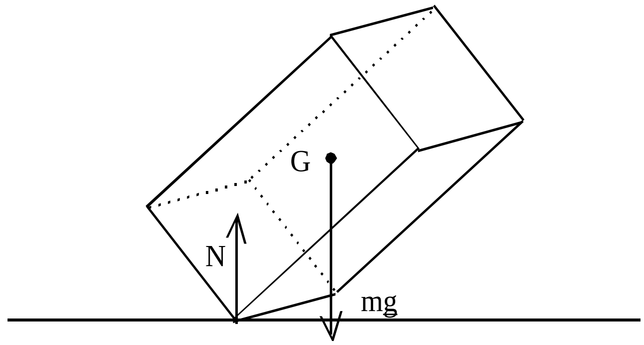
$$m\underline{a} = 2\underline{i},$$

that is, G moves along the X-axis with the same acceleration $2/m$ in all four cases. ◀

It follows from the result above that if no external forces act on a rigid body, then the centre of mass will not move; and if only vertical external forces act on a rigid body, then the centre of mass can only move vertically; it can not move horizontally (sideways)! This is quite important to keep in mind, since sometimes you need to use this fact to solve problems. The following example, and the activities that follow, all use this principle!

Example 7.2
A rectangular box with mass m is held with one corner resting on a smooth (frictionless) table and is gently released. Find the trajectory of the centre of mass of the box.

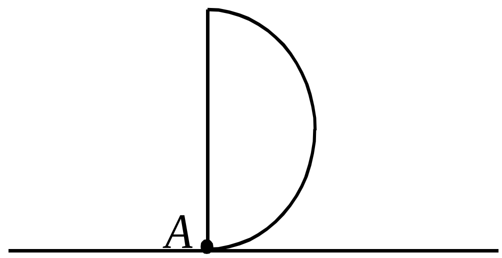
Solution:
The external forces acting on the box are the force of gravity $m\underline{g}$ downwards, and the normal force of the table $\underline{N}$, upwards. (Since the table is frictionless, no frictional forces apply!) Neither of these forces has a horizontal component, so that the centre of mass can only accelerate vertically. For a uniform box, the centre of mass is at the geometrical midpoint. If the box is released from rest, then it can only fall in such a way that the centre of the box falls **straight down**!◀


G
N
mg

Activity 7.1

Solve the following problem, by using the way that the centre of mass of the rigid body behaves. You must introduce a coordinate system! Use the result of Example 6.8 to find the centre of mass of the object.

Problem: The sketch below shows a lamina in the shape of a semicircle with radius R.



Assume that initially the rigid body is kept at rest as shown, on top of a smooth table top, with only corner A of the semicircle touching the table. The object is then released. Assuming that it stays upright, find the vertical and horizontal displacement of point A of the lamina when the lamina comes to a standstill.

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Feedback: If the origin of the coordinate system is at the original position of A, then after the object comes to a standstill, A will be at the point $(\frac{4R}{3\pi} - R, R)$.

7.2 Systems of rigid bodies and particles

In this section we will combine all the results we have up to now, to deal with a final, completely general case of systems consisting of one or more rigid bodies and/or particles put together. We will learn how to find the centre of mass in such a case, and also how the centre of mass behaves in such as case. Note that this is just the final, trivial generalisation of all we have done up to now, but the situations will be more complex in the problem solving, and to deal with them we will introduce several tools for you to use!

7.2.1 Finding the centre of mass of a general system

Up to now we have learned how to find the centre of mass of a rigid body (with a continuous structure) on the one hand, and a system of particles, on the other hand. A combination of both, or of several separate rigid bodies must be handled by the rule for the centres of mass of composite bodies.

When we adapt our standard toolbox in Unit 1 to the particular problem of finding the centre of mass of a general system, it can be re-written as follows:

TOOLBOX FOR FINDING CENTRES OF MASS

1. UNDERSTANDING THE PROBLEM

- What is the system like? What are the shapes, sizes, masses, compositions, positions of the parts? Where are the parts in relation to each other?
- Can you describe the system in your own words?

You could make use of the following tools:

- Knowledge of the language of mechanics problems, and using keywords for clues about the positions and properties of objects
- Sketches and diagrams
- Real-life examples
- Using symbols for referring to parts of the system, positions, distances etc.

2. PLANNING A SOLUTION

We have the following principles, definitions, results and sub-toolboxes available to us for finding centres of mass:

- The original definition, using a sum, for a system of particles
- Slicing and integrating, applicable to bodies with continuous structure
- The equation for the centre of mass of a composite system, put together from any kinds of components
- We have sub-toolboxes for
  - selecting coordinate systems
  - simplifying the task of finding centres of mass
  - slicing and integrating
- Also, we have a special trick for dealing with objects with parts removed.
- If we need to integrate, we may use polar coordinates.
- Finally, we have ways of finding the positions of the centres of mass of various objects, including those of solids of revolution and laminas bounded by functions.

To decide on which of these you should apply to a particular system, ask yourself:

- Can you find similar, already solved examples and problems?
- Are all the components particles? If not, we shall have to find the centres of mass of the components with continuous structure, and then the following list of questions applies:
  - Do we already know where the centre of mass of the object is, based on an already solved example?
  - Can the toolbox for simplifying be applied? Is this an object with parts removed?
  - If all else fails, we can use slicing and integrating. If this is necessary, is the case similar to one we have already done?
- Can the toolbox for simplifying be applied to the entire system?

3. EXECUTING THE PLAN

To complete the calculation of the centre of mass, you will have to

- introduce mathematical notation
- find the centres of mass and the masses of the components , if necessary
- introduce a suitable coordinate system, draw a sketch of the entire system with the coordinates, and express the centres of mass of the components in terms of this coordinate system
- apply the relevant formula to find the centre of mass of the entire system in terms of the coordinates

- express the centre of mass in relation to the system itself.

4. **ANALYSING THE SOLUTION**

   To check the correctness of the solution you can

   - see whether the solution makes sense. Compare the end result to the centre of mass of other similar objects
   - try to think of alternative ways to find the centre of mass
   - re-do the calculations with a different coordinate system
   - compare your solution with experiments and guesses based on real-life objects
   - work in a group and compare your results with those of others

   If your solution seems to be wrong, you should

   - find out where you went wrong, by checking the argument and the calculations
   - go back to step 1 or step 2

   To reflect and learn from the solution, you can

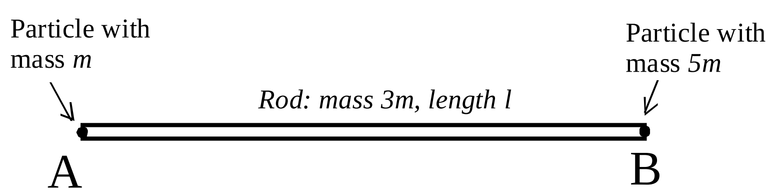
   - think of other systems where a similar approach would work; compare this problem with other systems that you have come across: what are the differences and similarities?

The following examples and activities illustrate the use of this toolbox.

Example 7.3
A uniform rod of length 2ℓ and mass $3m$ has a particle of mass m attached at one end of it, and a particle of mass $5m$ attached at the other end. Find the centre of mass of the object.

Solution:

1. UNDERSTANDING THE SYSTEM: The system consists of the rod and the two particles which are attached at its ends. The rod is made of a uniform material, and the masses of the rod and the two particles are given. We are just told that the particles are attached at the end points, which means that we can assume that they are, for instance, glued on. We should be able to draw the system quite easily, as shown below.

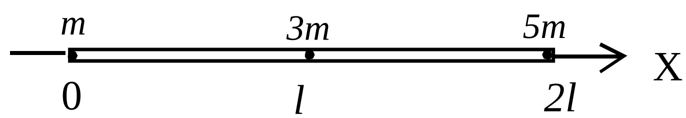
   

   A and B denote the ends of the rod. So, now we can say that the system consists of the uniform rod AB with length l and mass $3m$, a particle of mass m at point A, and a particle of mass $5m$ at point B.

2. PLANNING A SOLUTION: Can we find similar, already solved examples? We do know how to find the centre of mass of a uniform rod, which we can perhaps utilise here. However, we certainly cannot use that result directly — we cannot just pretend that this system is a uniform rod with mass $3m+m+5m$, since that would be a different

case altogether! Are all the components particles? No, there is also the rod which has a continuous structure. We will have to find the centre of mass of the rod. But we do already know where the centre of mass of a uniform rod is: it is in the middle of the rod. Can the toolbox for simplifying be applied to the entire system? Yes, it certainly can: the system is clearly one-dimensional. However, beyond that there are no axes of symmetry.

3. EXECUTING THE PLAN: We already know where the centres of mass of the three components are. Since the system is one-dimensional, we need only one coordinate axis, which should go along the rod. One possibility is to take the origin of this axis to coincide with the end point A of the rod. This gives us the sketch below:

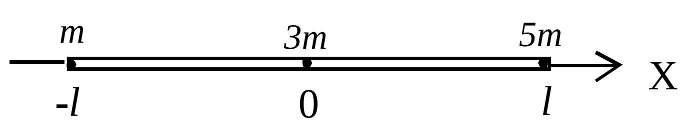


In terms of this coordinate system, the object is a compound body consisting of a particle of mass m at position $x = 0$, a particle of mass $5m$ at position $x = 2\ell$, and the rod which has a mass of $3m$ and its centre of mass at $x = \ell$. Applying the rule for finding the centre of mass of a compound body, we find that the centre of mass of the whole object is at

$$\overline{x} = \frac{m \cdot 0 + 5m \cdot 2\ell + 3m \cdot \ell}{m + 5m + 3m} = \frac{13}{9}\ell = 1\frac{4}{9}\ell.$$

This means that the centre of mass is at a distance $4l/9$ or $2/9$ of the length of the rod (remember that the rod had length $2l$), from the centre of the rod towards the heavier particle.

4. ANALYSING THE SOLUTION – checking the correctness: Does the solution make sense? It certainly does – adding the two particles at the ends of the rod, the centre of mass of the system should no longer lie in the middle of the rod, but rather a distance further, towards the heavier particle. Can we think of alternative ways to find the centre of mass? Not really – the only rule to use here is that of composite bodies. Re-do the calculations with a different coordinate system. This we can certainly do. Assume, for instance, that we take the origin of the X-axis to lie at the midpoint of the rod, so that the system and the coordinate axis look as follows:



In terms of this coordinate system, the system consists of a particle of mass m at position $x = -l$, a particle of mass $5m$ at position $x = \ell$, and the rod with mass $3m$ and centre of mass at $x = 0$. Applying the rule for the centre of mass of a compound body, we find that the centre of mass of the whole object is now at

$$\overline{x} = \frac{m \cdot (-l) + 5m \cdot \ell + 3m \cdot 0}{m + 5m + 3m} = \frac{4}{9}\ell.$$

Again, this means that the centre of mass is at a distance $4l/9$.or $2/9$ of the length of the rod (remember that the rod had a length of $2l$), from the centre of the rod, towards the heavier particle. So, we do get the same result. Think of other systems where a similar approach would work. The same approach will work in any case where particles of mass are attached to objects of continuous structure. Compare this problem with other systems that you have come across: what are the differences and similarities? This is the first time we have come across a rod with particles attached to it, but we can compare it with other rods with a continuous structure. For instance, we have already established that the situation here is very different from the case of a uniform rod of length $2l$ and mass $3m + m + 5m = 9m$, since the centre of mass of such a rod would be at the centre of the rod.◀

Activity 7.2

Solve the following problem, working through the steps of the toolbox given earlier.

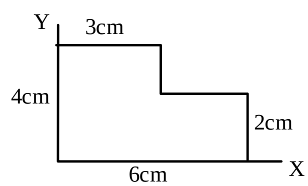
Problem: A uniform circular disc with radius r and mass M has two particles attached to it. One of the particles has mass $2M$ and is attached at the centre of the disc, and the other particle has mass M and is attached at a point at the rim of the disc. Where is the centre of mass of the object?

....

Feedback: You should get a point which lies on the line from the centre of the disc towards the particle on the rim, at the distance $\frac{1}{4}r$ from the centre.

Example 7.4

Find the centre of mass of the lamina of uniform density in the figure below.



Solution:

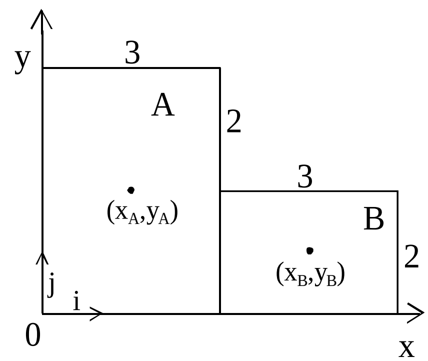
1. UNDERSTANDING THE SYSTEM: There is just one object here, a uniform lamina (thin plate) of the shape shown in the figure. We already have a picture, and the coordinate system is also already in place.

2. PLANNING A SOLUTION: We have to find the centre of mass of the lamina with continuous structure. Do we already know where the centre of mass of the object is? No. Can the toolbox for simplifying be applied? There are no symmetries involved here. The object is two-dimensional, but then it is already set in a two-dimensional coordinate system. But we certainly could use the trick of considering this to be a composite body, put together of rectangles: 4×3 plus a 2×3 rectangle; or a 2×6 plus a 2×3 rectangle; or three 2×3 rectangles! Is this an object with parts removed? This could also apply here: the lamina could be considered to be a 4×6 rectangle with a 2×3 rectangle cut off from the corner.

   We have identified two possible approaches: we can either consider the lamina as put together from two rectangles (the centres of mass of which are very easy to find), or a rectangle with another rectangle removed from it (again, these objects are easy to deal with). Either way, we do not expect to have to do any integration!

3. EXECUTING THE PLAN. We will show both approaches here.
   Method 1 Assume that the lamina is a composite body. Introduce mathematical notation; find the centres of mass of the components and their mass; introduce a suitable coordinate system; draw a sketch of the entire system with the coordinates; and express the centres of mass of the components in terms of this coordinate system. We shall assume that the lamina is put together from a rectangle

A and a rectangle B as shown below, and we shall let (x_A, y_A) and (x_B, y_B) denote the centres of mass of these components, and let M_A and M_B be the masses of A and B. Because the lamina is of a uniform density, the centres of masses of the rectangles are simply the geometric midpoints of the rectangles. We already have a coordinate system in place.



From the sketch above, we can easily see that the centres of mass of A and B are at

$$\begin{aligned}(x_A, y_A) &= (1.5, 2)\\ (x_B, y_B) &= (4.5, 1)\,.\end{aligned}$$

To apply the equation for the centre of mass of composite bodies, we also need the masses of A and B. But because of the uniform density, the mass of a rectangle is density times area, so when we let ρ denote the (unknown) density, we get

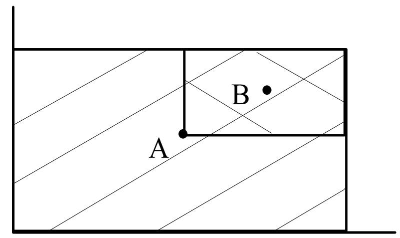
$$\begin{aligned}M_A &= \rho\cdot(3\cdot 4) = 12\rho\\ M_B &= \rho\cdot(3\cdot 2) = 6\rho.\end{aligned}$$

The rule for finding the centre of mass of composite bodies now gives as the centre of mass for the whole lamina the point $(\overline{x}, \overline{y})$, with

$$\begin{aligned}\overline{x} &= \frac{M_A\overline{x}_A + M_B\overline{x}_B}{M_A + M_B}\\ &= \frac{12\rho(1.5) + 6\rho(4.5)}{18\rho} = \frac{5}{2},\\ \overline{y} &= \frac{M_A\overline{y}_A + M_B\overline{y}_B}{M_A + M_B}\\ &= \frac{12\rho(2) + 6\rho(1)}{18\rho} = \frac{5}{3}.\end{aligned}$$

Method 2: Alternatively, we can view the lamina as an object with a part of it removed.

Let A denote the big rectangle with sides 4 cm and 6 cm, with its midpoint at $(3, 2)$ and let B denote the cut-off piece, a rectangle with sides 2 cm and 3 cm with its midpoint at $\left(4\frac{1}{2}, 3\right)$.



Our lamina L consists of A minus B, or, put in another way, A can be composed of L and B. Let M_A, M_B and M_L be the masses of the three objects, and let $\underline{R}_A$, $\underline{R}_B$ and $\underline{R}_L$ denote the position vectors of their centres of mass, respectively. Then, according to the rule for calculating the centres of mass of compound bodies, we have

$$\underline{R}_A = \frac{M_L\,\underline{R}_L + M_B\,\underline{R}_B}{M_L + M_B} \tag{7.2}$$

We know $\underline{R}_A = 3\underline{i} + 2\underline{j}$, $\underline{R}_B = 4\frac{1}{2}\,\underline{i} + 3\,\underline{j}$. Since the lamina is uniform, the mass of each object is (density) $\times$ (area). Denoting density by ρ, we get $M_A = 24\rho$, $M_B = 6\rho$. Since $L = A$ minus B, $M_L = M_A - M_B = 18\rho$. Thus (7.2) becomes

$$\begin{aligned}
3\underline{i} + 2\underline{j} &= \frac{18\rho\,\underline{R}_L + 6\rho\left(\frac{9}{2}\underline{i} + 3\underline{j}\right)}{24\rho} \\
\therefore\ \underline{R}_L &= \frac{1}{18\rho}\left[(24\rho(3\underline{i} + 2\underline{j}) - 6\rho\left(\frac{9}{2}\underline{i} + 3\underline{j}\right)\right] \\
&= \frac{5}{2}\,\underline{i} + \frac{5}{3}\,\underline{j}
\end{aligned}$$

Therefore, both methods have given as the centre mass

$$(\overline{x}, \overline{y}) = \left(\frac{5}{2}, \frac{5}{3}\right).$$

4. ANALYSING THE SOLUTION: Is it correct? Does the solution make sense? Compare the end result to the centres of mass of other similar objects. Yes, the result does make sense. For a whole 4×6 rectangle without the corner cut out, the centre of mass would be at the point $(3, 2)$. Now, the centre of mass still lies along the diagonal of the large 4×6 rectangle, but some distance from the centre, away from the cut-off corner, as we would expect.

   Try to think of alternative ways to find the centre of mass. We have already used two different methods. Just for the sake of illustration, let us try another method — direct slicing and integrating.

   Is the case similar to one we have already done? Yes, in fact, we can view the object as the lamina bounded by the X-axis and the step function

$$y = f(x) = \begin{cases} 4, & 0 \le x \le 3 \\ 2, & 3 \le x \le 6 \end{cases}$$

   between $x = 0$ and $x = 6$. Then, in terms of equations (6.5) and (6.6),

$$\begin{aligned}
\overline{x} &= \frac{\displaystyle\int_0^6 x \cdot f(x)\,dx}{\displaystyle\int_0^6 f(x)\,dx} = \frac{\displaystyle\int_0^3 x \cdot 4dx + \int_3^6 x \cdot 2dx}{\displaystyle\int_0^3 4dx + \int_3^6 2dx} = \frac{5}{2} \\
\overline{y} &= \frac{\frac{1}{2}\int_0^6 [f(x)]^2 dx}{\int_0^6 f(x)dx} = \frac{\frac{1}{2}\int_0^3 (4)^2 dx + \frac{1}{2}\int_3^6 (2)^2 dx}{\int_0^3 4dx + \int_3^6 2dx} = \frac{5}{3}
\end{aligned}$$

Again, we got the same result.

Re-do the calculations with a different coordinate system – this should also be easy to do. If we, for instance, take the origin to be in the middle of the big 4×6 rectangle, while keeping the directions of the X- and Y-axis the same, then the centre of mass will have the coordinates $(\overline{x}, \overline{y}) = \left(-\frac{1}{2}, -\frac{1}{3}\right)$, which clearly refers to exactly the same point on the lamina.

Compare with experiments and guesses based on real-life objects. We could roughly check our solution by using a cardboard cut-out for the lamina. Cut out an object with the same proportions as our lamina here from thick, uniform cardboard, and check that you can balance it at the end of a pencil situated exactly at the alleged centre of mass.

Think of other systems where a similar approach would work. Try to generalise the result. Our approach (all three of them) will of course apply in many other situations. One interesting question might deal with generalisations of the result we have derived here. Firstly, what if we have any rectangular lamina, and out of one of its corners we cut a smaller rectangle with the same proportions as the big one. Does the centre of mass of the lamina obtained always lie on the diagonal of the big rectangle, as it did here? Secondly, where did the values $5/2$ and $5/3$ come from? To clarify this, we could see what happens if we replace the "number problem" we have here with a more general "letter problem". So, we could instead solve the following problem: a $\frac{1}{2}a \times \frac{1}{2}b$ rectangle has been cut off from the corner of an $a \times b$ rectangle. Find the centre of mass of the lamina thus obtained.

Another possible generalisation could go as follows: cut off a $ca \times cb$ rectangle from an $a \times b$ rectangle, where c can be any number between 0 and 1. We can find the centre of mass of such a lamina for all values of c and b, and check that taking $c = 1/2$ gives the same value as here; and that the case $c = 0$ gives the midpoint of the $a \times b$ rectangle. *(What about $c = 1$?)*◄

Activity 7.3

A 2×2 square has been cut off from one corner of a 8×10 rectangle made from thin uniform metal. Find the centre of mass of the remaining object.

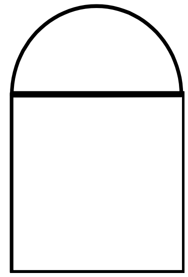
Remember to specify your coordinate system! In addition to giving the coordinate position of the centre of mass, describe also where on the rectangle it lies.

. . . .

Feedback: The centre of mass is at the distance $4/19$ parallel to the longer side, and the distance $3/19$ parallel to the shorter side, away from the cut–off corner.

Example 7.5

The uniform lamina shown below consists of a semicircle of diameter $2a$, on top of a square with sides $2a$. Find the distance of the centre of mass from the base of the square.



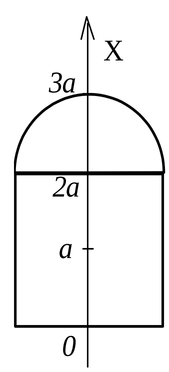
Solution:

1. UNDERSTANDING THE PROBLEM: The sketch shows clearly the shapes, size and relative positions of the two parts forming the lamina. Since the object is described as a lamina, we know that the semicircle here is half of a circular lamina, not just a thin rod in the shape of a semicircle. We are not given the masses of the two parts. But, we are told that the lamina is uniform, so that we can assume a value for the density ρ, and then should be able to find the masses from the areas of the objects.

2. PLANNING A SOLUTION: We shall have to use the formula for finding the centre of mass of composite bodies, and for that we need to find the centres of mass of the two laminas: the square and the semicircle. But we already know where their centres of mass are: the centre of mass of the square is at its centre (in terms of symmetry) and for the semicircle, we can use the result in Example 6.8, which states that the centre of mass of a uniform lamina, in the shape of a semicircle with radius a, lies at a distance $4a/3\pi$ from the diameter.

   Can the toolbox for simplifying be applied to the entire system? Yes, it is clear that a vertical line through the centre of the lamina is a line of symmetry, and if we select a coordinate axis to go along this line, then we only have to find one coordinate of the centre of mass.

3. EXECUTING THE PLAN: Introduce a suitable coordinate system, and draw a sketch of the entire system with the coordinates. Take the X-axis to go along the middle of the lamina, as shown. In terms of symmetry, the centre of mass must be on the X-axis. Since we are told to find the distance of the centre of mass from the base, we might just as well take the origin to be at the base of the square.

   Introduce mathematical notation. Let x_1 and x_2 denote the X-coordinates of the square and the semicircle, and let M_1 and M_2 denote their masses, respectively. Express the centres of mass of the components in terms of this coordinate system. The centre of mass of the square is in the middle of the square, so that it has the X-coordinate $x_1 = a$. The centre of mass of the semicircle lies a distance of $\frac{4a}{3\pi}$ above its diagonal, so in terms of our coordinate system it has the X-coordinate $x_2 = 2a + \frac{4a}{3\pi}$.

(Remember that the X-axis starts at the bottom of the square!) Find the masses of the components. The area of the square is $4a^2$, so that its mass is $M_1 = 4\rho a^2$ where ρ is density (mass per unit area). The semicircle has an area of $\frac{1}{2}\pi a^2$ and therefore its mass is $M_2 = \frac{1}{2}\rho\pi a^2$. Hence, for the whole lamina the centre of mass is

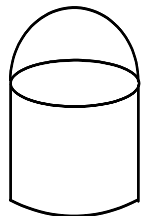
$$\begin{aligned}
\overline{x} &= \frac{M_1 x_1 + M_2 x_2}{M_1 + M_2} \\
&= \frac{\left(4\rho a^2\right) a + \left(\frac{1}{2}\rho\pi a^2\right)\left(2a + \frac{4a}{3\pi}\right)}{4\rho a^2 + \frac{1}{2}\rho\pi a^2} \\
&= \frac{2}{3}\frac{14 + 3\pi}{8 + \pi} a
\end{aligned}$$

The centre of mass of the lamina lies a distance of $\frac{2}{3}\frac{14+3\pi}{8+\pi}a$, or approximately 1.4016 a from the bottom of the square.

4. ANALYSING THE SOLUTION: The solution is credible: we would expect the centre of mass to lie somewhere within the upper half of the square. To check the solution, we could try alternative approaches (e.g. view this as a lamina bounded by a curve), or change the coordinate system, but all the other methods are much more complicated than the one presented here!◀

Activity 7.4

The uniform solid shown below consists of a hemisphere of radius a on top of a cylinder of height h with a base radius a. Find the distance of the centre of mass from the base of the cylinder.



....

Feedback: you should get $(3a^2 + 8ah + 6h^2)/(8a + 12h)$.

7.2.2 The motion of the centre of mass of a general system

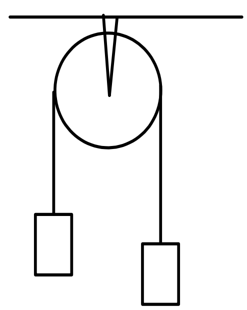
The motion of the centre of mass is again described by the same equation,

$$\underline{F} = M\ddot{\underline{R}}$$

where $\underline{F}$ is the sum of all the external forces acting on the system (that is, of all the external forces acting on any bodies or particles which form a part of the system), M is the total mass of the system and $\ddot{\underline{R}}$ is the acceleration of the centre of mass.

However, in this case we need to be careful about what forms part of the system and what does not; and which forces are internal and which external to the system. To illustrate the possible problems, consider the following situation.

Example 7.6
Atwood's machine consists of two masses of masses m and M hanging from a string which runs smoothly over a pulley.



For the system consisting of the two masses, there are no internal forces; all the forces are external. (At least, if we ignore the very very small gravitational pull that the two masses exert on each other!)

If, on the other hand, we decide to look at the system consisting of the two masses plus the string, then the tension of the rope is an internal force, while the external forces are the forces of gravity and a normal force from the pulley on the rope. Note that if the string is very light, the centre of mass of this system is more or less the same as that of the system consisting of the two masses alone!

For the system consisting of the pulley, the string and the two masses, the external forces are the forces of gravity on all the components, plus a normal force at the centre of the pulley (the force which holds the pulley in the place).◀

Here is a checklist of what you have to do in order to apply the equation of motion above.

TOOLBOX FOR APPLYING THE EQUATION FOR THE MOTION OF THE CENTRE OF MASS OF A SYSTEM

The equation for the motion of the centre of mass of a system, given by $\underline{F} = M\underline{\ddot{R}}$ links

- the acceleration of the centre of mass,
- the external forces acting on the system, and
- the total mass of the system.

Thus, given two of these we can find the third. Usually, we wish to find the acceleration of the centre of mass from the mass and forces acting on the system, so we shall design our toolbox around that problem and leave it for you to modify the toolbox for the other two cases!

To find the acceleration of the centre of mass of a system, we shall have to identify the mass and the resultant external force acting on the system. The following checklist will help you do that.

1. **UNDERSTANDING THE SYSTEM**

   Here, you must understand what the system is like:

   - What are the components? (size, shape, consistency: uniform, massless,..)
   - How are the components related to each other? (relative positions, do they touch each other, linked by a string, is there friction between them...)

   You must also make sure that the system is what you think it is. Change your definition of the system if necessary.

   You might make use of the following tools:

   - knowledge of the language of mechanics problems, and using keywords for clues about positions, objects and their properties, types of motion etc.
   - sketches and diagrams
   - real-life examples and experiments
   - mathematical notation for known and unknown quantities

2. **PLANNING A SOLUTION**

   To be able to apply the equation of motion, we need to find the mass of the system and the external forces acting on it. Do we have the information necessary for doing that?

   MASSES:

   - Are we given the masses of all the components? If not, can we calculate them? Or do we know the relative sizes of the masses?

FORCES:

- For each component which forms a part of the system, identify all the forces acting on it. (List them, and also draw them in your sketch.) Categorise the forces acting on the component into internal ones (due to another component which forms part of the system) and external ones.
- Check your categorisation: All internal forces should appear in action-reaction pairs.
- Now, ignore all the internal forces, but list all the external forces acting on the various components. These all form the external forces acting on the system.
- Are some of the external forces unknown? If so, then we may need further information linking the motions of the components of the system.

If you cannot identify the masses and/or the forces, you may have to check that you have chosen your system correctly.

3. **EXECUTING THE PLAN**

To apply the equation of motion, you will have to

- introduce mathematical notation and symbols for the masses and forces.
- calculate the masses of the components, if necessary
- introduce a suitable coordinate system; draw a sketch of the entire system and the external forces with the coordinate system; express the vectors of the external forces in terms of this coordinate system; introduce notation for the acceleration of the centre of mass based on this system
- write down the equation for the motion of the centre of mass of the entire system in terms of the coordinate system
- add equations describing extra information about the motion of the system if some of the forces are not known
- solve the equation(s) for the acceleration of the centre of mass
- express this acceleration in relation to the system itself, if required

4. **ANALYSING THE SOLUTION**

- Does the solution make sense? Compare it with experiments and guesses based on real-life objects.
- Try to think of alternative ways to solve the problem.

If your solution seems to be wrong, you should

- find out where you went wrong, by checking the argument and the calculations
- make sure that your system is as you intended

To reflect and learn from the solution, you can

- think of other systems where a similar approach would work; try to generalise the result; compare this problem with other systems that you have come across: what are the differences and similarities?

If some of the forces are unknown, and if you need information relating to the relative motion of the components, then you may also have to write down the equations of motion for the individual components!

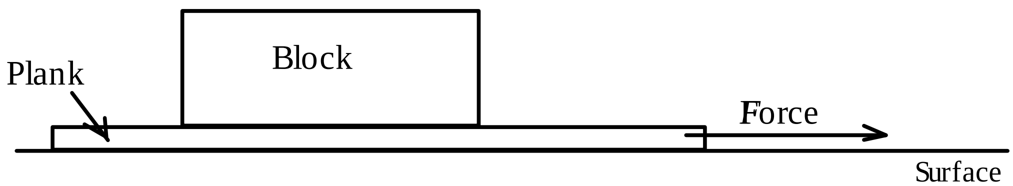
Example 7.7
A system consists of a block with mass M placed on top of a plank of mass $2M$. The coefficient of friction between the plank and the block is μ. The plank in turn rests on a smooth, horizontal surface. Initially, the system is at rest. The plank is pulled with a constant horizontal force $\underline{F}$.

(a) Find the horizontal acceleration of the centre of mass of the system.

(b) How does the plank move if there is no friction between the plank and the block?

Solution:

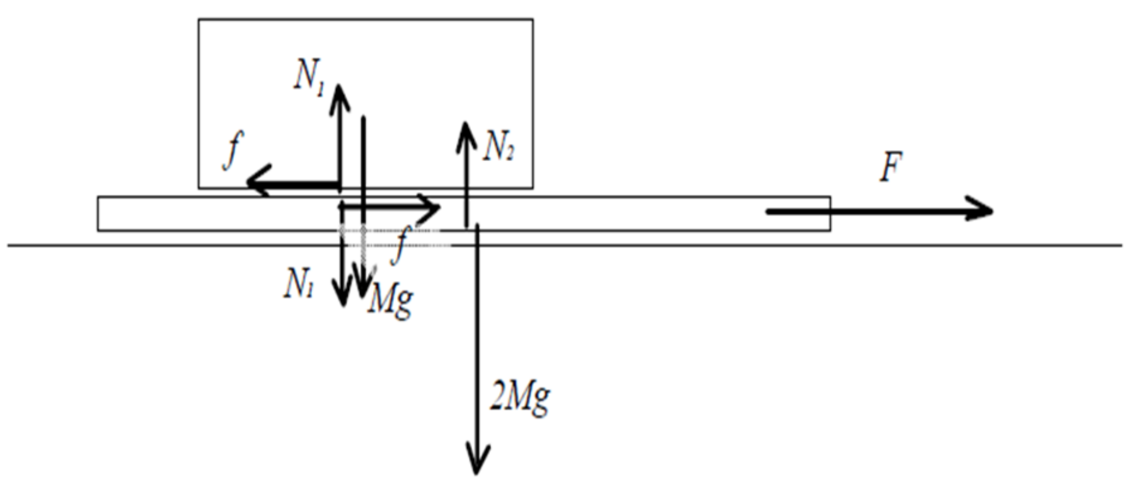
UNDERSTANDING THE SYSTEM: The system consists of the plank (mass $2M$) and the block (mass M). The block is on top of the plank, and the plank is on top of a smooth surface. Thus, the following sketch describes the situation:



The coefficient of friction between the block and the plank is given, and there is no friction between the plank and the surface. The plank is now pulled with a constant horizontal force $\underline{F}$. The direction of $\underline{F}$ is not given, but we can assume that it is towards the right.

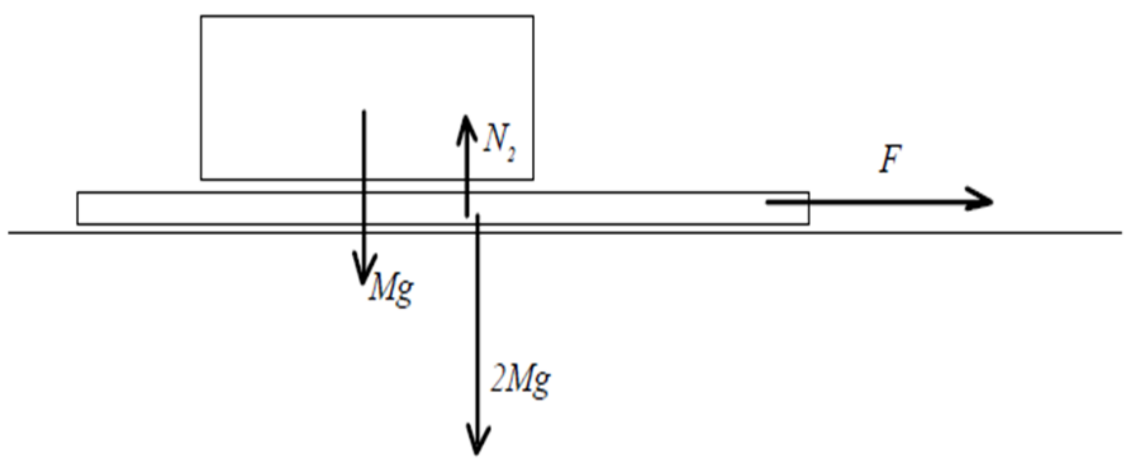
(a) Find the horizontal acceleration of the centre of mass of the system. To answer this question, we shall apply our toolbox.

PLANNING A SOLUTION: We have to find the mass of the system and the external forces acting on it. Masses: The masses of the plank and the block are given. Forces: All the forces acting on the block and the plank are shown below. Note that we have left a gap between the block, the plank and the surface, in order to make it easier to see which object each force acts on.



The forces on the plank are the horizontal force $\underline{F}$ towards the right, gravity $2M\underline{g}$ downwards, the normal force $\underline{N}_1$ from the block downwards, the normal force $\underline{N}_2$ from the surface upwards, and friction $\underline{f}$ from the block, in the direction opposite to $\underline{F}$, that is, towards the left. The forces on the block are gravity $M\underline{g}$ downwards, the normal force $\underline{N}_1$ from the plank upwards, and friction $\underline{f}$ from the plank, towards the right. Categorise the forces into internal and external one: The external forces are the force $\underline{F}$, the forces of gravity $2M\underline{g}$ and $M\underline{g}$, and the normal force N_2 on the plank from the surface. All the other forces are internal. Check: The forces $\underline{f}$ and $\underline{N}_1$ appear in action-reaction pairs, as internal forces should. Are some of the external forces unknown? The normal force $\underline{N}_2$ is in principle unknown, but can be solved very quickly from the fact that we know that the plank-block system will travel along the surface, and therefore the upwards force $\underline{N}_2$ must be just sufficient to counter the downwards forces $M\underline{g}$ and $2M\underline{g}$; that is, $\underline{N}_2 = 3M\underline{g}$. (Similar reasoning gives $\underline{N}_1 = M\underline{g}$.) Anyway, we are only told to find the horizontal acceleration of the centre of mass, so we could just ignore the vertical forces.

We have already introduced notation for the forces. The total mass of the system is $(M + 2M) = 3M$. Introduce a suitable coordinate system, draw a sketch of the entire system and the external forces with the coordinate system, and express the vectors of the external forces in terms of this coordinate system. Introduce notation for the acceleration of the centre of mass based on this system. Let us introduce a coordinate system where the X-axis lies along the surface, and the Y-axis is perpendicular to it, as usual. A sketch with the external forces shown in it looks as follows:



The forces can be written as $\underline{F} = F\underline{i}$, $2M\underline{g} = -2Mg\underline{j}$, $M\underline{g} = -Mg\underline{j}$ and $\underline{N}_2 = N_2\underline{j}$. Let $\ddot{x}$ and $\ddot{y}$ denote the horizontal and vertical acceleration of the centre of mass, so that $\underline{\ddot{R}} = \ddot{x}\underline{i} + \ddot{y}\underline{j}$. Write down the equation of motion:

$$F\underline{i} - 2Mg\underline{j} - Mg\underline{j} + N_2\underline{j} = 3M\left(\ddot{x}\underline{i} + \ddot{y}\underline{j}\right)$$

or, in component form (combining and matching the coefficients of $\underline{i}$ and $\underline{j}$),

$$\begin{aligned} F &= 3M\ddot{x}, \\ -2Mg - Mg + N_2 &= \ddot{y}. \end{aligned}$$

After applying the result $N_2 = 3Mg$, this gives us

$$\begin{aligned} \ddot{x} &= \frac{F}{3M}, \\ \ddot{y} &= 0. \end{aligned}$$

That is, the centre of mass of the system starts to move towards the right, with acceleration

$\frac{F}{3M}$.

(b) How does the plank move if there is no friction between the plank and the block? So far, we have only found a description of how the centre of mass of the entire system moves. To investigate motion of the individual components, we would have to apply the equations of motion to the components separately. In this case of zero friction between the components, this turns out to be particularly easy. Namely, looking at the sketch above, showing all the internal and external forces acting on the block and the plank, we see that if we remove the forces f towards the left and the right (no friction acting between the block and the plank), then the only horizontal force acting on the plank is the force $\underline{F}$ pulling it towards the right. But then the horizontal acceleration $\ddot{x}_{plank}$ of the plank alone must be given by the equation of motion

$$F = (2M)\,\ddot{x}_{plank}$$

and therefore, if there is no friction between the plank and the block, the plank will have the acceleration

$$\ddot{x}_{plank} = \frac{F}{2M}$$

towards the right. To check this result, we can compare it with the acceleration of the centre of mass we have calculated earlier. If there is no friction, then no horizontal forces act on the block, and therefore the horizontal acceleration of the block is zero (it stays in its initial position). It follows that the acceleration of the centre of mass should be equal to

$$\ddot{x} = \frac{(2M)\,\ddot{x}_{plank} + M\ddot{x}_{block}}{2M + M} = \frac{2M \cdot \frac{F}{2M} + M \cdot 0}{3M} = \frac{F}{3M}$$

and this is exactly what we got earlier.◀

Example 7.8

Two identical containers of sugar are connected by a massless cord that passes over a massless, frictionless pulley with a diameter of 50 mm. The two containers are at the same level. Each originally has a mass of 500 g.

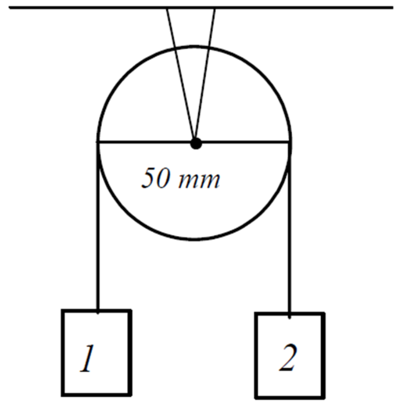
(a) Locate the horizontal position of the centre of mass of the two containers.

(b) 20 g of sugar is transferred from one container to the other, but the containers are prevented from moving. Locate the new horizontal position of their centre of mass.

(c) The two containers are then released. In what direction does the centre of mass of the two containers move? What is its acceleration?

Solution:

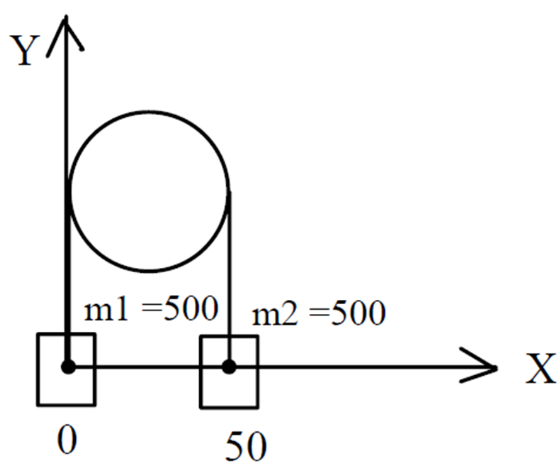
A sketch of the system might look like this:



The shapes of the containers do not matter, but it is important to note that their centres of mass are a distance of 50 mm apart. Since the masses of the containers differ in (a) and (b), we have just labelled the containers as 1 and 2. The containers are shown in their initial positions, which is valid for (a) and (b) — in (c), the containers will move.

Example 7.9

We will have to introduce a coordinate system to be able to answer the questions. Let us choose a coordinate system such that the origin is in the middle of the left–hand container, container 1, in the initial position, as shown below. The units on the X- and Y-axes are in millimetres.



(a) The positions of the centres of mass of the two containers are $(x_1, y_1) = (0, 0)$ and $(x_2, y_2) = (50, 0)$. Since initially the containers have the same mass, their centre of mass is halfway between them, at position $(\overline{x}, \overline{y}) = (25, 0)$.

(b) Let us now move $20g$ from container 1 to container 2, but keep the positions of the containers the same as before. The two containers will then have the masses $m_1 = 480$, $m_2 = 520$. We still have $\overline{y} = 0$, but now

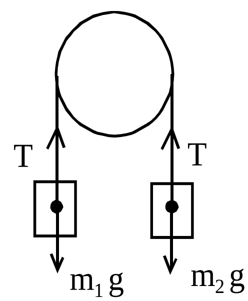
$$\overline{x} = \frac{0 \cdot 480 + 50 \cdot 520}{480 + 520} = 26.$$

The new centre of mass is at $(26, 0)$. That is, the centre of mass is now 1 millimetre from the midpoint, towards the heavier container (container 2).

(c) We will use our toolbox to answer this question.

UNDERSTANDING THE SYSTEM: What are the components? How are the components related to each other? We wish to analyse the system consisting of the two containers, so the components are the two containers. The masses of the two components are $m_1 = 480$ and $m_2 = 520$ grams, respectively. The size and shape are irrelevant here. The components are linked by the following condition: they are suspended at the two ends of a massless cord which passes over a massless, frictionless pulley, with a diameter of 50 mm. This means, firstly, that there is always a link between the positions of the two containers: they always move vertically, not horizontally, so that the horizontal distance between their centres of mass is always 50 mm; and they always move in such a manner that if one container goes up by the distance d, then the other container must go down by the same distance d. Secondly, the fact that the cord passes smoothly over the pulley means that the tensions on the rope on each side of the pulley are the same. This result we shall need later on!

PLANNING THE SOLUTION: Masses: We do know the masses of the components: $m_1 = 480$ and $m_2 = 520$ grams, respectively. Forces: The forces acting on the containers are shown in the sketch below.



On container 1, they are the tension of the cord, T, upwards and gravity, $m_1 g$ downwards; and on container 2, the tension of the cord, T, upwards and gravity, $m_2 g$ downwards. The tensions on the cord are the same for both containers, since the cord passes over the pulley without friction. All the forces listed here are external to the system, there is no internal force on one container directly due to the other container! The containers affect each other only indirectly, through the tension on the cord. Note that the tensions T acting on the two containers do not form an action-reaction pair – they act in the same direction, not in opposite directions! Thus, the resultant external force on the entire system is the sum of $2T$ upwards, plus $(m_1 + m_2)g$ downwards.

EXECUTING THE PLAN: We already have a coordinate system in place and, in terms of this coordinate system, the total external force acting on the system is given by

$$\underline{F} = 2T\,\underline{j} - (m_1 + m_2)g\,\underline{j}.$$

The total mass of the entire system is $M = m_1 + m_2$. Finally, let $\underline{R}$ denote the centre of mass of the system and let $\underline{\ddot{R}}$ denote its acceleration. The equation for the motion of the centre of mass then tells us that

$$M\underline{\ddot{R}} = \underline{F}$$

which in our case gives us the equation

$$(m_1 + m_2)\underline{\ddot{R}} = 2T\,\underline{j} - (m_1 + m_2)g\,\underline{j}. \tag{7.3}$$

If $\underline{\ddot{R}} = \left(\ddot{x}\underline{i} + \ddot{y}\underline{j}\right)$, so that $\ddot{x}$ and $\ddot{y}$ are the vertical and horizontal components of the acceleration of the centre of mass, respectively, then by separating (7.3) into its $\underline{i}$ and $\underline{j}$ components, we see that

$$\begin{aligned} (m_1 + m_2)\ddot{y} &= 2T - (m_1 + m_2)g, \\ \ddot{x} &= 0. \end{aligned}$$

Now, the force T is still unknown, which means that we have to add at least one other condition about the way that the containers move. (Remember that the number of equations should be the same as the number of unknowns!) We need another condition where T appears. This can be included by also considering the equation for the motion of one of the containers. Let us look at the lighter one, container 2. The equation of motion for that one is

$$T - m_1 g = m_1 a$$

if a denotes the acceleration of the container (positive acceleration being upwards). Adding this equation added another unknown quantity, a, so we need yet another condition. But, we have not yet used the fact that one container must always go up at exactly the same rate that the other container goes down. This fact states that if the vertical acceleration of container 1 is a, the vertical acceleration of container 2 must be $-a$. We would like to link this condition to our already existing variable $\ddot{y}$, the vertical acceleration of the centre of mass of the two containers. This is easy to do, simply by expressing the acceleration $\ddot{y}$ in terms of the acceleration of the containers:

$$\ddot{y} = \frac{m_1 a + m_2 (-a)}{m_1 + m_2} = \frac{m_1 - m_2}{m_1 + m_2} a.$$

Now we have three unknowns ($\ddot{y}$, a, T) and three equations,

$$\begin{cases} (m_1 + m_2)\ddot{y} = 2T - (m_1 + m_2)g \\ \\ \ddot{y} = \frac{m_1 - m_2}{m_1 + m_2} a \\ \\ T - m_1 g = m_1 a \end{cases}$$

Solving these equations for $\ddot{y}$, we find that the acceleration of the centre of mass is

$$\underline{\ddot{R}} = -\frac{(m_2 - m_1)^2}{(m_1 + m_2)^2}\, g\, \underline{j}. \tag{7.4}$$

In particular, using the values $m_1 = 480$, and $m_2 = 520$ and $g \approx 9.81\ m/s^2$, we get

$$\underline{\ddot{R}} \approx -0.016 \underline{j}\, \frac{m}{s^2}.$$

We see that the acceleration of the centre of mass is in the negative y direction, which means that the centre of mass moves straight down.

ANALYSING THE SOLUTION: Does the solution make sense? We would expect the centre of mass to move downwards, since no horizontal forces act on the system. Also, if we look at the solution in (7.4) in terms of m_1 and m_2, we see that

- if $m_1 = m_2$ (i.e. if both containers have the same mass) then the acceleration of the centre of mass is zero, which is as we would expect (neither of the containers will move when the system is released)
- regardless of the sizes of m_1 and m_2, the centre of mass will always accelerate downwards when $m_1 \neq m_2$
- the bigger the difference between the masses m_1 and m_2, the bigger the acceleration of the centre of mass

All these observations give credibility to the result (7.4), and again show that it is often preferable to solve a “letter problem” rather than a “number problem”!

Try to think of alternative ways to solve the problem. The acceleration of the centre of mass can alternatively be found as

$$\underline{\ddot{R}} = \frac{m_1 \underline{\ddot{r}}_1 + m_2 \underline{\ddot{r}}_2}{m_1 + m_2}$$

where $\underline{\ddot{r}}_1$ and $\underline{\ddot{r}}_2$ are the accelerations of the centres of mass containers 1 and 2. So, one alternative approach here is to apply the equations of motion separately to the two containers, and then apply this equation to find the acceleration of the centre of mass of the system. In the following we will give the solution using this method.

Since the containers are joint together by a rope, clearly the accelerations are in the Y-direction, and $\ddot{\underline{r}}_1 = -\ddot{\underline{r}}_2$ (if one container moves the distance x up, the other one must move down by the same distance). Denoting $\ddot{\underline{r}}_1 = a\,\underline{j}$, we have

$$\ddot{\underline{R}} = \frac{m_1 - m_2}{m_1 + m_2}\,a\,\underline{j}.$$

To find the value of a, we apply Newton's second law to both containers. The forces acting on each container are gravity downwards, and the tension of the rope upwards. The tension of the rope is the same $(T\,\underline{j})$ for both containers, because the rope passes over a massless, frictionless pulley.

The equations of motion are therefore

$$\begin{cases} T\,\underline{j} - m_1\,g\,\underline{j} = m_1\,a\,\underline{j} \\ T\,\underline{j} - m_2\,g\,\underline{j} = -m_2 a\,\underline{j} \end{cases} \Rightarrow \begin{cases} T - m_1 g = m_1 a \\ T - m_2 g = -m_2 a \end{cases}$$

from which we can solve a to get

$$a = \frac{m_2 - m_1}{m_1 + m_2}\,g.$$

This will again give

$$\ddot{\underline{R}} = \frac{(m_1 - m_2)^2}{(m_1 + m_2)^2}\,\underline{j}.$$

Note that in this case, it was perhaps easier to use the second method, and just look at all the components separately! This is because we had to sort out the unknown force T, which is an external force on our system.

Think of other systems where a similar approach would work. Try to generalise the result. Compare this problem with other systems that you have come across: what are the differences and similarities? We have already generalised the problem, when we solved it for general masses m_1 and m_2, rather than the given specific numerical values.◀

The following activities give you a chance to practice all these ideas.

Activity 7.5

Solve the following problem, by using the way that the centre of mass of the whole system behaves.

Problem: Two people, one with a mass of 60 kg and the other one with a mass of 80 kg, stand on a smooth, frictionless surface holding a pole with a length of 10 m and negligible mass. Starting from the ends of the pole, they pull themselves along the pole until they meet. How far will the 60 kg person move?

....

Feedback: The answer is: 5.7 metres.

Activity 7.6

(Note: In this question, consider carefully what you coordinate system will be!)

A 75 kg man climbs the stairs from the ground to the 20th floor of a building, a height of 60 m. How far does the Earth recoil in the opposite direction? (The mass of the Earth is $M = 5.98 \times 10^{24}$ kg and its radius, $R = 6.4 \times 10^6 m$.)

....

Feedback: 7.53×10^{-22} metres.

CONCLUSION

In this unit you have learned

- how to find the centres of mass of a rigid body and of a system consisting of rigid bodies and/or particles
- how the centre of mass of a rigid body moves, and how the centre of mass of a general system moves

Remember to add the following tools to your toolbox:

- The principle that the resultant external force acting on a body, or a system of bodies/particles, is proportional to the acceleration of the centre of mass
- the fact that zero external force implies zero acceleration for the centre of mass
- the toolbox for finding the centre of mass of a general system
- the toolbox for analysing the motion of the centre of mass of a system

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LEARNING UNIT 3

ROTATION

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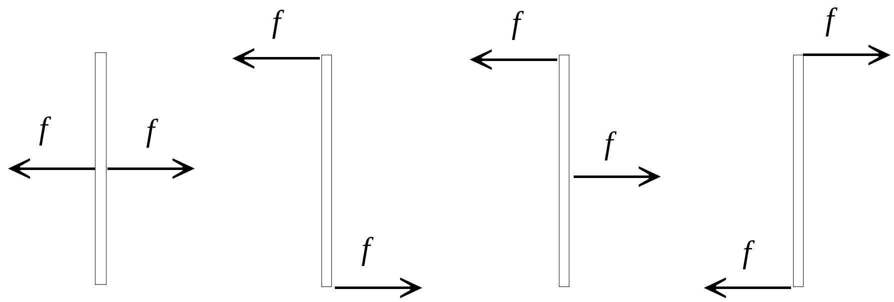
Introduction

In Learning Unit 2 we learned how the motion of a system of particles or a rigid body can partially be described by the simpler motion of its centre of mass. But of course the behaviour of the centre of mass does not tell the whole story, as it cannot describe the motion of the particles in the system relative to each other.

For a general system of particles, the motion of the particles with respect to each other and with respect to the centre of mass can be very complicated (consider, for instance, a handful of rice tossed into the air!) We will not even try to describe the situation for a general system. However, we certainly wish to be able to understand and describe the motion of simpler objects such as pencils or bottles, which should be easier to deal with! Even for these kinds of objects, it is clear that knowing the motion of the centre of mass is not enough for a full description of the motion of the object. The reason for this is that the motion of the centre of mass does not take into account **where** on the object the forces are applied – but this does clearly make a big difference! Consider, for instance, the action of kicking an empty cardboard box: depending on where you kick it, you can make it rotate clockwise or counterclockwise, or you can make it move straight forward. Yet, if the force in your kick does not vary, the centre of mass of the cardboard will always move in an identical way.

Another dramatic illustration of this is seen by applying two equal, but opposing forces to an object. As an example, consider the rod shown below, acted on by two opposing forces of magnitude f. In all the cases shown below, the total force acting on the rod is zero, so that according to the results in Learning Unit 2, the centre of mass has zero acceleration. However, the actual motion of the rod is very different in each case.



In this section, we will concentrate on the case of **rigid systems or bodies**, where all the particles which form the system are held in fixed positions with respect to each other and with respect to the centre of mass. In this case, the only way the particles or parts of the system can move with respect to each other is by **rotation**.

For a rigid system or body, the situation is simple. It can be proved that the motion of a rigid body can be fully described by a combination of two independent motions, **translation** and **rotation**. Each of these can be analysed separately. The translational motion can be described in terms of the motion of the centre of mass, as described in Learning Unit 2. The rotation is described in terms of the rotation of the rigid body about its centre of mass. Here, in Learning Unit 3, we will develop techniques to deal with the rotational motion of rigid bodies.

To describe and analyse translational motion, we used the concepts of force, linear moment, mass and acceleration, and Newton's laws of motion were used to describe how acceleration is caused by force. For rotational motion we shall introduce the analogous concepts of the moment of a force, angular momentum, moment of inertia and angular acceleration, and we shall derive an **equation of rotation** to describe how angular acceleration is caused by the moment of a force.

To describe how force causes rotation, we shall have to introduce various concepts: vector products, the angular momentum, the moment of a force, and moments of inertia. Some of these concepts may well be new to you, and may appear difficult, but the final equation of rotation is very simple to use — so please be patient!

The outcomes of Learning Unit 3

When you have worked through this Learning Unit of the study guide, you should be able to

- calculate the vector product of given vectors by using the direct definition and by using the unit vectors
- calculate the moment of a force acting at a point about any given point of reference
- find the angular momenta of particles and systems of particles, explain the connection between the moments of forces and the angular momentum of a system of particles, and explain how the equation of rotation of a rigid body rotating about a fixed axis follows from this
- find the moment of inertia of a system of particles or a rigid body by using the definition, together with the techniques of integration, parallel and perpendicular axis theorems, and the rule for composite bodies
- apply the equation of rotation for a rigid body rotating about a fixed axis, to solve problems

Unit 8 THE MOMENT OF A FORCE

Key questions:

- *How is the moment of a force found?*
- *What does it mean?*

To change the way the centre of mass of a rigid body moves, a non-zero resultant *force* has to act on the body. Similarly, to change the way that a rigid body rotates about a point, a non-zero resultant *moment of force* must act on it. The moment of a given force about a given point measures the turning effect of the force relative to that point, and will depend not only on the magnitude and direction of the force, but also on where on the body it acts. It turns out that the perfect way to measure the turning effect of a force is by means of a mathematical concept called the vector product. In this unit, we will introduce the vector product and define the moment of a force.

Contents of this unit:

8.1 The vector product

8.2 The moment of a force

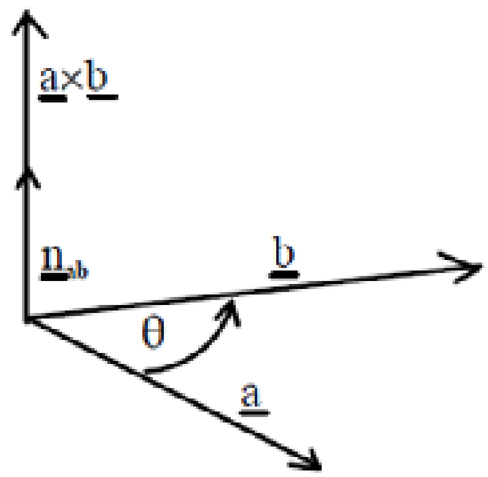
What you are expected know before working through this unit:

In this unit, we introduce to you the vector product; to be able to make sense of it, you should know how to work with vectors in three dimensions. Also, it will be important that you can express and understand forces as vectors.

8.1 The vector product

Various quantities relating to rotation are defined as **vector products**, so we will start by introducing this concept. The vector product of two vectors is also a vector, with a direction and a magnitude. (Compare this with the scalar product of two vectors, which just gives a real number as a result).

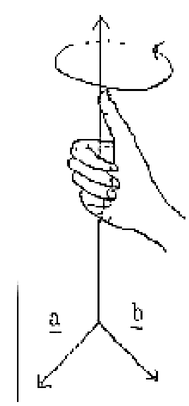
Consider two vectors $\underline{a}$ and $\underline{b}$, as shown in the figure below.



To define the vector product of the two vectors $\underline{a}$ and $\underline{b}$, denoted by $\underline{a} \times \underline{b}$, we need the following:

- the magnitudes of the two vectors, that is, the values $|a|$ and $|b|$
- the smaller angle from $\underline{a}$ to $\underline{b}$, which we will denote by θ
- a unit vector $\underline{n}_{ab}$, also shown in the diagram

The vector $\underline{n}_{ab}$ is a unit vector at right angles to the plane containing $\underline{a}$ and $\underline{b}$, such that the vectors $\underline{a}$, $\underline{b}$ and $\underline{n}_{ab}$ form a **right-handed system**. What this means is illustrated below.



Three vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ (in that order) are said to form a right-handed system if the following holds: Put the fingers of your right hand along vector $\underline{a}$, and curl them toward $\underline{b}$ in the direction of the smaller angle from $\underline{a}$ to $\underline{b}$; then your thumb points in the direction of $\underline{c}$. Or, to put it another way, if vector $\underline{c}$ is grasped by the right hand with the thumb lying along $\underline{c}$, then the fingers wrap around $\underline{c}$ in the direction from $\underline{a}$ to $\underline{b}$ through the **smaller** angle. Note that, depending on the orientation of the vectors, the hand may have to be upside down!

Alternatively, if we imagine looking "down" from the end of vector $\underline{c}$ to the plane where $\underline{a}$ and $\underline{b}$ are, then the motion from $\underline{a}$ to $\underline{b}$ through the smaller angle between them is always **counterclockwise**.

Note that the X-, Y- and Z-coordinate axes, in the way we draw them in this study guide, form a right-handed system, and so do the combinations Y, Z, X and Z, X, Y.

Use your three-dimensional model of the coordinate system to make sure that you understand this! See unit 2.

However, the vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ do not have to be at right angles to each other to form a right-handed system. What is important is their "relative directions". Also, always remember that the order of the vectors is significant here! For instance, Y, X, Z is not a right-handed system!

The **vector product** $\underline{a} \times \underline{b}$ is now defined as follows:

Definition 8.1 (vector product)
Let $\underline{a}$ and $\underline{b}$ be vectors. Then the vector product of $\underline{a}$ and $\underline{b}$ is defined as the vector

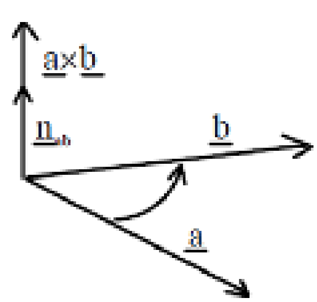
$$\underline{a} \times \underline{b} = |\underline{a}|\,|\underline{b}| \sin\theta \; \underline{n}_{ab}, \qquad (8.1)$$

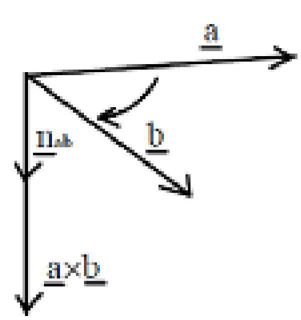
where $|\underline{a}|$, $|\underline{b}|$ are the magnitudes of the vectors $\underline{a}$ and $\underline{b}$, θ is the smaller angle from $\underline{a}$ to $\underline{b}$, and the vector $\underline{n}_{ab}$ is a unit vector such that $\underline{a}$, $\underline{b}$ and $\underline{n}_{ab}$ form a right–handed system.

Thus, the vector product $\underline{a} \times \underline{b}$ is a **vector**, with a **magnitude** and a **direction**. Hence the name "vector product". (Another name sometimes used for the vector product is the cross product, because of the notation.)

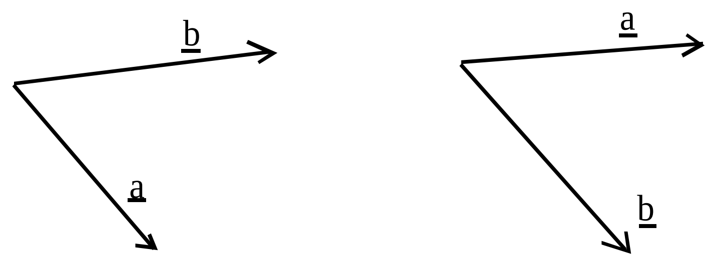
The magnitude of the vector $\underline{a} \times \underline{b}$ is $|\underline{a}|\,|\underline{b}| \sin\theta$, so that it depends on the lengths of the vectors $\underline{a}$ and $\underline{b}$ and on the angle between them.

The direction of the vector $\underline{a} \times \underline{b}$ is determined by the direction of the unit vector $\underline{n}_{ab}$. The unit vector $\underline{n}_{ab}$ must be perpendicular to the plane formed by the vectors $\underline{a}$ and $\underline{b}$. There are two possible unit vectors which are perpendicular to that plane. Which one is $\underline{n}_{ab}$ depends on the relative positions of $\underline{a}$ and $\underline{b}$, as we must choose the unit vector which satisfies the condition that $\underline{a}$, $\underline{b}$ and $\underline{n}_{ab}$ form a right-handed system. Compare the following (these are side-views of a three-dimensional system):





To make this even more clear, we will re-draw these pictures such that $\underline{a}$ and $\underline{b}$ are on the plane of this page.

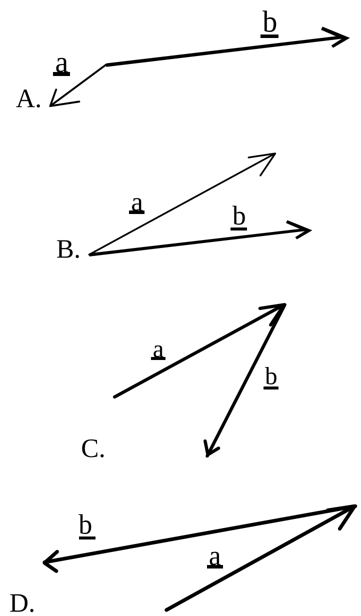


Now, since $\underline{a}$ and $\underline{b}$ are parallel to the page, the unit vector $\underline{n}_{ab}$ must be perpendicular to the page and must thus be either directed out of the page or into the page. The right-hand rule (that is, requesting that we get a right-handed system) states that in the case shown on the left, the unit vector is directed out of the page (towards you) and in the case shown on the right, the unit vector is directed towards/into the page (away from you).

Activity 8.1

Use the right-hand rule to answer the following question.

The two vectors $\underline{a}$ and $\underline{b}$ lie on the plane of this page, as shown. In each case, is the vector product $\underline{a} \times \underline{b}$ oriented towards the viewer or away from the viewer? (You may wish to re-draw some of the vectors to start from the some point!)



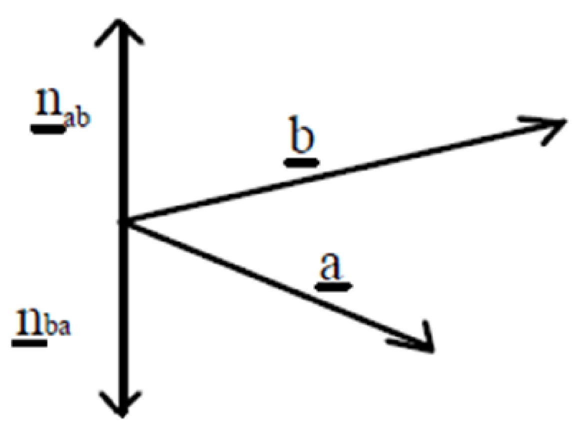
....

Feedback: Towards the viewer in A and D, away in B and C.

Properties of the vector product

1. From the definition of the vector product, it follows that the vector $\underline{a} \times \underline{b}$ is always perpendicular to both vector $\underline{a}$ and vector $\underline{b}$.
2. From the way the unit vector $\underline{n}_{ab}$ was defined, it is at once clear that

$$\underline{n}_{ab} = -\underline{n}_{ba}.$$



Also, the smaller angle θ from $\underline{a}$ to $\underline{b}$ is trivially the same as the smaller angle from $\underline{b}$

to $\underline{a}$. Therefore it follows that

$$\underline{b} \times \underline{a} = \left|\underline{b}\right| \left|\underline{a}\right| \sin\theta \left(-\underline{n}_{ab}\right).$$

Thus, we have

$$\underline{a} \times \underline{b} = -\left(\underline{b} \times \underline{a}\right). \tag{8.2}$$

Note that this means that the vector product is not "commutative": $\underline{a} \times \underline{b}$ is not the same vector as $\underline{b} \times \underline{a}$, but is instead its opposite (that is, a vector with the same magnitude as $\underline{a} \times \underline{b}$, but with opposite direction). It is very important to remember that the order of the two vectors is significant (unlike, for instance, in the ordinary product of real numbers, where the order does not matter: $3 \cdot 2 = 2 \cdot 3 = 6$, or the scalar product or two vectors, where $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ also always holds).

3. If either $\underline{a}$ or $\underline{b}$ or both of them are zero vectors, that is, have length zero, then of course

$$\underline{a} \times \underline{b} = \underline{0}.$$

4. If $\underline{a}$ and $\underline{b}$ are **parallel** (travelling in the same or opposite directions) to each other, that is, their directions are either the same or opposite, then the smaller angle between them is either $\theta = 0^\circ$ or $\theta = 180^\circ$. Either way, $\sin\theta = 0$, and thus in this case we also have

$$\underline{a} \times \underline{b} = \underline{0}.$$

That is, the vector product of any two parallel vectors is zero. In particular, for any vector $\underline{a}$, the vector product of the vector with itself vanishes:

$$\underline{a} \times \underline{a} = \underline{0}. \tag{8.3}$$

5. Inversely, if $\underline{a} \times \underline{b} = 0$ then there are two possibilities: Either at least one of the vectors $\underline{a}$ or $\underline{b}$ is zero, or else $\sin\theta = 0$, which means that the vectors $\underline{a}$ and $\underline{b}$ are parallel. This gives us a useful criterion for determining whether two non-null vectors $\underline{a}$ and $\underline{b}$ are parallel, namely, we must have $\underline{a} \times \underline{b} = 0$. (Compare this with the scalar product of two vectors: there, $\underline{a} \cdot \underline{b} = 0$ for two non-zero vectors if, and only if the vectors are perpendicular to each other.)

6. Let $\underline{a}$ and $\underline{b}$ be two vectors of magnitudes $\left|\underline{a}\right|$ and $\left|\underline{b}\right|$. The magnitude of $\underline{a} \times \underline{b}$, given by $\left|\underline{b}\right| \left|\underline{a}\right| \sin\theta$, will then still depend on the smaller angle θ between the two vectors. We see that the magnitude of $\underline{a} \times \underline{b}$ obtains its maximum value when $\underline{a}$ and $\underline{b}$ are perpendicular to each other, since $\sin\theta$ obtains its maximum absolute value ($\sin\theta = 1$) when $\theta = 90^\circ$.

7. The following properties hold for the vector product and are stated here without proof:

   (a) For any vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ the following equalities hold:

$$\begin{aligned} \underline{c} \times \left(\underline{a} + \underline{b}\right) &= \underline{c} \times \underline{a} + \underline{c} \times \underline{b} \\ \left(\underline{a} + \underline{b}\right) \times \underline{c} &= \underline{a} \times \underline{c} + \underline{b} \times \underline{c} \end{aligned} \tag{8.4}$$

   This means that, just like with ordinary products, we can take the cross products of the terms of a sum separately — just as we do in ordinary multiplication, where $2 \cdot (3+4) = 2 \cdot 3 + 2 \cdot 4$ and $(2+3) \cdot 4 = 2 \cdot 4 + 3 \cdot 4$.

   (b) If A is any number, then

$$\begin{aligned} \left(A\underline{a}\right) \times \underline{b} &= A\left(\underline{a} \times \underline{b}\right) \\ \underline{a} \times \left(A\underline{b}\right) &= A\left(\underline{a} \times \underline{b}\right). \end{aligned} \tag{8.5}$$

From property number 7 above, we see that the vector product does behave very much like any other product — but with the very important difference that the order of the vectors in the product does make a difference to the end result. Also, please keep in mind that in the notation of the vector product we always use the "cross" for the product — you can't just drop it or replace it by a dot (even if this can be done in ordinary products). The "dot" or scalar product $\underline{a} \cdot \underline{b}$ of two vectors is a completely different thing from the vector product!

8.1.1 How to calculate a vector product directly

Assume that we are given two vectors $\underline{a}$ and $\underline{b}$. To calculate their vector product $\underline{a} \times \underline{b}$ directly from the definition, we need the magnitudes of the two vectors and the size θ of the smaller angle between them. From these we can determine the magnitude of the vector product $\underline{a} \times \underline{b}$. To find the unit vector $\underline{n}_{ab}$, which gives the direction of the vector product, we can, for instance, first find the two unit vectors which are perpendicular to both $\underline{a}$ and $\underline{b}$. (This can, for example, be done by setting up the equations $\underline{x} \cdot \underline{a} = 0$ and $\underline{x} \cdot \underline{b} = 0$ which must be satisfied by a vector x which is perpendicular to both vectors; or by determining first the plane generated by $\underline{a}$ and $\underline{b}$ and then finding the corresponding "normal unit vectors", that is, the unit vectors perpendicular to the plane.) We can then decide which one of the two unit vectors satisfies the right-hand rule.

Example 8.1

Find the following vector product directly from the definition:

$$\left(2\underline{i}\right) \times \left(-\underline{j}\right)$$

where $\underline{i}$, $\underline{j}$ and $\underline{k}$ are the unit vectors of the XYZ coordinate system.

Solution: The vector $\left(2\underline{i}\right)$ has a magnitude of 2, and the vector $-\underline{j}$ has a magnitude of 1. The smaller angle between the two vectors is 90°. Both the vectors are on the XY-plane, and therefore any vectors which are perpendicular to both of them (and perpendicular to the XY-plane) must be parallel to the Z-axis. This gives us two candidates for the unit vector giving the direction of the vector product, namely $\underline{k}$ and $-\underline{k}$. But $\left(2\underline{i}\right)$, $\left(-\underline{j}\right)$ and $\left(-\underline{k}\right)$ form a right-handed system, while $\left(2\underline{i}\right)$, $\left(-\underline{j}\right)$ and $\underline{k}$ do not. (Check it out yourself! For instance, looking down along from the end of the unit vector $\underline{k}$, the vectors $\left(2\underline{i}\right)$ and $\left(-\underline{j}\right)$ are situated like this:

We see that motion from $\left(2\underline{i}\right)$ to $\left(-\underline{j}\right)$ over the shorter angle is clockwise, as indicated in the sketch.) We conclude that

$$\left(2\underline{i}\right) \times \left(-\underline{j}\right) = 2 \cdot 1 \cdot \sin\left(90^\circ\right)\left(-\underline{k}\right) = -2\underline{k}.$$

The vector product is $-2\underline{k}$. ◄

Activity 8.2

Find the vector product $\left(\underline{k}\right) \times \left(3\underline{j}\right)$ directly from the definition (that is, by finding the magnitudes of the vectors, the angle between them, the perpendicular unit vectors, and by using the right hand rule).

....

Feedback: you should get $-3\underline{i}$.

8.1.2 How to calculate a vector product using unit vectors

In practice, we rarely use (8.1) directly to calculate vector products, but instead work with the XYZ-components of the vectors. That is, we first express both the vectors $\underline{a}$ and $\underline{b}$ in terms of the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$. The vector products of the unit vectors are easily calculated and remembered:

$$\underline{i} \times \underline{j} = \underline{k}, \quad \underline{j} \times \underline{k} = \underline{i}, \quad \underline{k} \times \underline{i} = \underline{j}.$$

This follows from the fact that X, Y, Z, as well as Y, Z, X and Z, X, Y all form right-handed systems.

Note that these are the only ones we need to remember; (8.2) can then be applied to get

$$\begin{aligned} \underline{j} \times \underline{i} &= -\left(\underline{i} \times \underline{j}\right) = -\underline{k}, \\ \underline{k} \times \underline{j} &= -\left(\underline{j} \times \underline{k}\right) = -\underline{i}, \\ \underline{i} \times \underline{k} &= -\left(\underline{k} \times \underline{i}\right) = -\underline{j}, \end{aligned} \tag{8.6}$$

and rule (8.3) gives

$$\underline{i} \times \underline{i} = 0, \quad \underline{j} \times \underline{j} = 0, \quad \underline{k} \times \underline{k} = 0.$$

Now, using (8.4) and (8.5), it is very easy to reduce the vector product of any two vectors to the vector products of these unit vectors. The following examples should make this clear.

Example 8.2
If $\underline{a} = \left(2\underline{i} - 3\underline{j} - \underline{k}\right)$ and $\underline{b} = \left(\underline{i} + 4\underline{j} - 2\underline{k}\right)$, find

(a) $\underline{a} \times \underline{b}$

(b) $\left(\underline{a} + \underline{b}\right) \times \left(\underline{a} - \underline{b}\right)$

Solution:

(a)

$$\begin{aligned} \underline{a} \times \underline{b} &= \left(2\underline{i} - 3\underline{j} - \underline{k}\right) \times \left(\underline{i} + 4\underline{j} - 2\underline{k}\right) \\ &= 2\left(\underline{i} \times \underline{i}\right) + 8\left(\underline{i} \times \underline{j}\right) - 4\left(\underline{i} \times \underline{k}\right) - 3\left(\underline{j} \times \underline{i}\right) \\ &\quad -12\left(\underline{j} \times \underline{j}\right) + 6\left(\underline{j} \times \underline{k}\right) - \left(\underline{k} \times \underline{i}\right) - 4\left(\underline{k} \times \underline{j}\right) + 2\left(\underline{k} \times \underline{k}\right) \\ &\quad \text{(rules (8.4) and (8.5))} \\ &= 2 \cdot \underline{0} + 8\underline{k} - 4\left(-\underline{j}\right) - 3\left(-\underline{k}\right) - 12 \cdot \underline{0} + 6\underline{i} - \underline{j} - 4\left(-\underline{i}\right) + 2 \cdot \underline{0} \\ &\quad \text{(rules (8.2), (8.3) and (8.6))} \\ &= 10\underline{i} + 3\underline{j} + 11\underline{k} \end{aligned}$$

(b)

$$\begin{aligned}
\left(\underline{a}+\underline{b}\right)\times\left(\underline{a}-\underline{b}\right) &= \left(\underline{a}\times\underline{a}\right)-\left(\underline{a}\times\underline{b}\right)+\left(\underline{b}\times\underline{a}\right)-\left(\underline{b}\times\underline{b}\right)\\
&\quad \text{((8.4) and (8.5))}\\
&= \underline{0}-\left(\underline{a}\times\underline{b}\right)-\left(\underline{a}\times\underline{b}\right)-\underline{0}\\
&\quad \text{((8.2) and (8.3))}\\
&= -2\left(\underline{a}\times\underline{b}\right)=-2\left(10\underline{i}+3\underline{j}+11\underline{k}\right)\\
&= -20\underline{i}-6\underline{j}-22\underline{k}
\end{aligned}$$

Note in the calculations above how we used parentheses to clarify the order of the calculations. Remember that it is very important to pay attention to the **order** of the vectors in the vector products! Also, remember to always denote the vector product with the "×" symbol — you must not forget to write it down!

Example 8.3
If

$$\begin{aligned}
\underline{a} &= 3\underline{i}-\underline{j}+2\underline{k}\\
\underline{b} &= 2\underline{i}+\underline{j}-\underline{k}\\
\underline{c} &= \underline{i}-2\underline{j}+2\underline{k},
\end{aligned}$$

calculate

(a) $\left(\underline{a}\times\underline{b}\right)\times\underline{c}$,

(b) $\underline{a}\times\left(\underline{b}\times\underline{c}\right)$

where the product in the brackets should be evaluated first.

Solution:

(a)

$$\begin{aligned}
&\left(\underline{a}\times\underline{b}\right)\times\underline{c}\\
&= \left(\left(3\underline{i}-\underline{j}+2\underline{k}\right)\times\left(2\underline{i}+\underline{j}-\underline{k}\right)\right)\times\left(\underline{i}-2\underline{j}+2\underline{k}\right)\\
&= \left(-\underline{i}+7\underline{j}+5\underline{k}\right)\times\left(\underline{i}-2\underline{j}+2\underline{k}\right)\\
&= 24\underline{i}+7\underline{i}-5\underline{k}
\end{aligned}$$

(b)

$$\begin{aligned}
&\underline{a}\times\left(\underline{b}\times\underline{c}\right)\\
&= \left(3\underline{i}-\underline{j}+2\underline{k}\right)\times\left(\left(2\underline{i}+\underline{j}-\underline{k}\right)\times\left(\underline{i}-2\underline{j}+2\underline{k}\right)\right)\\
&= \left(3\underline{i}-\underline{j}+2\underline{k}\right)\times\left(-5\underline{j}-5\underline{k}\right)\\
&= 15\underline{i}+15\underline{i}-15\underline{k}
\end{aligned}$$

Note that these results prove that the vector product is not associative — compare with the ordinary multiplication of real numbers, where $(1\cdot 2)\cdot 3 = 1\cdot(2\cdot 3)$ always holds. In ordinary multiplication, we can just write $1\cdot 2\cdot 3$ to mean either one of the products, since we know that the order does not matter. This cannot be done with the vector product!

Again we are reminded of the fact that when calculating vector products, we must pay close attention to the order of the vectors!

8.1.3 How to calculate vector products using determinants

If you are familiar with matrices and their determinants, you can use the following method to evaluate the vector products of three-dimensional vectors: if $\underline{x} = a\underline{i} + b\underline{j} + c\underline{k}$ and $\underline{y} = d\underline{i} + e\underline{j} + f\underline{k}$, then the vector product $\underline{x} \times \underline{y}$ is given by the determinant:

$$\underline{x} \times \underline{y} = \det\begin{pmatrix} \underline{i} & \underline{j} & \underline{k} \\ a & b & c \\ d & e & f \end{pmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a & b & c \\ d & e & f \end{vmatrix}.$$

That is, to find the vector product of two vectors we create a matrix with the unit vectors as the top row, the coordinates of the first vector as the next row, and the coordinates of the second vector as the last row. The determinant of this matrix will then give the vector product of the two vectors. The following example illustrates this.

Example 8.4

$$\begin{aligned}
& \left(3\underline{i} - \underline{j} + 2\underline{k}\right) \times \left(2\underline{i} + \underline{j} - \underline{k}\right) \\
= \; & \left(3\underline{i} + (-1)\,\underline{j} + 2\underline{k}\right) \times \left(2\underline{i} + \underline{j} + (-1)\underline{k}\right) \\
= \; & \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & -1 & 2 \\ 2 & 1 & -1 \end{vmatrix} \\
= \; & \underline{i}\begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} + \underline{k}\begin{vmatrix} 3 & -1 \\ 2 & 1 \end{vmatrix} \\
= \; & (1-2)\underline{i} - (-3-4)\,\underline{j} + (3+2)\,\underline{k} \\
= \; & -\underline{i} + 7\underline{j} + +5\underline{k}.
\end{aligned}$$

The following activity gives you a chance to practice finding vector products. This is something you must learn to do accurately, without much thought, so if you struggle, please do more practice using the exercises provided in your workbook!

Activity 8.3

Calculate the following vector products:

(a) $\left(3\underline{i} + \underline{j} - \underline{k}\right) \times (2\underline{k})$

(b) $\left(\left(\underline{i} \times \underline{j}\right) \times \underline{i}\right) \times \underline{i}$

(c) $(2\underline{k}) \times \left((2\underline{i} + \underline{k}) + \left(\underline{j} - \underline{k}\right)\right)$

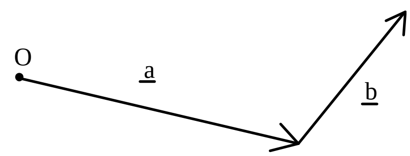
(d) $\left(2\underline{i} + 2\underline{j}\right) \times \left(-\underline{i} - \underline{j} + 5\underline{k}\right)$

(e) $((-\underline{i} + 5\underline{k}) \times (6\underline{i} + \underline{k})) \times \left(\underline{i} + \underline{j}\right)$

....

Feedback: (a) $2\underline{i} - 6\underline{j}$,(b) $-\underline{k}$; (c) $-2\underline{i} + 4\underline{j}$; (d) $10\underline{i} - 10\underline{j}$; (e) $-30\underline{k}$.

By now, you should be able to calculate the vector products of any two vectors, but you may have trouble visualizing what exactly the vector product means. (We certainly do!) Perhaps one good way to describe the concrete meaning of a vector product is in terms of the "turning effect", as follows: Assume that we arrange the vectors $\underline{a}$ and $\underline{b}$, both lying on the plane of this page, one after the other, such that vector $\underline{b}$ starts the end point of vector $\underline{a}$.



If we now imagine that $\underline{a}$ represents a thin rod, nailed down to this page at its initial point O, and that vector $\underline{b}$ is a force pulling at the end point of the rod $\underline{a}$, then the end result is that rod $\underline{a}$ will rotate about point O. The direction of the rotation (clockwise or counterclockwise) now corresponds to the direction of the vector product $\underline{a} \times \underline{b}$ as follows: the rotation will be counterclockwise if and only if $\underline{a} \times \underline{b}$ is directed out of the page, towards the reader, and clockwise if and only if $\underline{a} \times \underline{b}$ is directed into the page, away from the reader. If $\underline{a}$ and $\underline{b}$ are parallel to each other, there is no rotation, and $\underline{a} \times \underline{b} = 0$. The magnitude of the vector product $\underline{a} \times \underline{b}$ measures how effectively $\underline{b}$ will be able to turn $\underline{a}$ — this depend on the lengths of the two vectors, but also the angle between them.

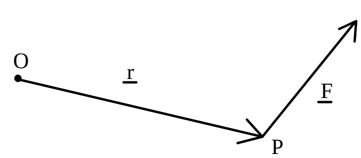
8.2 The moment of a force

The moment of force is now defined as a vector product, as follows. Let a reference point O be given, and assume that a force $\underline{F}$ acts at a point P. Then the moment of the force about point O is defined as follows:

Definition 8.2 (The moment of a force)
The moment of $\underline{F}$ about a point O is defined to be

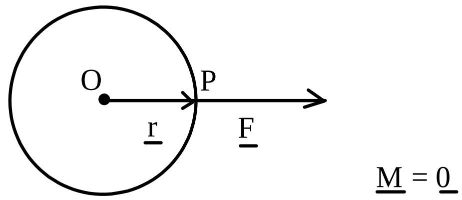
$$\underline{M} = \underline{r} \times \underline{F} \tag{8.7}$$

where $\underline{r} = \underline{OP}$ is the position vector from O to the point P where the force $\underline{F}$ acts. In particular, the moment of a force is a vector, perpendicular to the plane containing both $\underline{r}$ and $\underline{F}$.



Note that the moment of a given force, acting at a point P, **depends on the choice of the reference point** O**:** we must talk about the moment of $\underline{F}$ **about** a point O. If O is changed, then the moment of the force will also change, even if it still acts at the same point P. Likewise, for any force $\underline{F}$, we can make the moment of the force about O disappear simply by choosing $O = P$, i.e. by taking the point of action P itself as the reference point, since then the position vector is a zero vector: we have $\underline{r} = \underline{0}$!

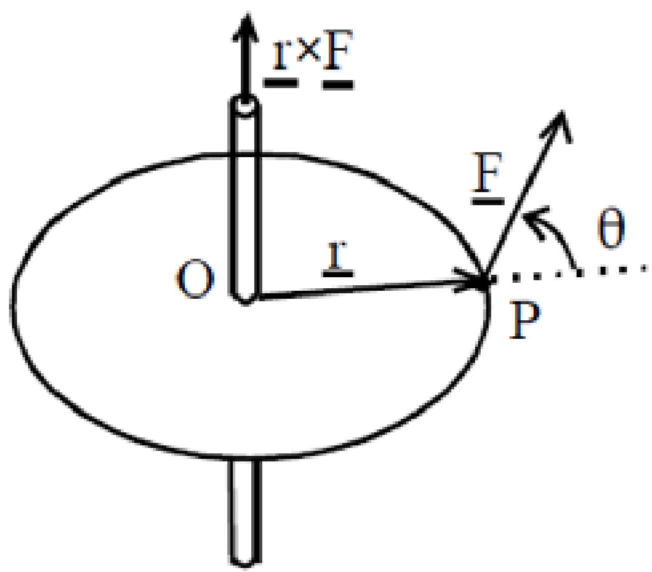
An important consequence of Definition 8.2 is that if the vectors $\underline{F}$ and $\underline{r}$ are **parallel**, then $\underline{r} \times \underline{F} = \underline{0}$ and thus $\underline{M} = \underline{0}$.



That is, if the line of action of the force passes through point O, then the moment of the force about O is zero. (The line of action of a force being a line you get if you extend the vector indefinitely in both direction.)

Since the moment of a force is defined as a vector product, it is a **vector**, perpendicular to the plane containing both $\underline{r}$ and $\underline{F}$. Since it is a vector, it has both a magnitude and a direction.

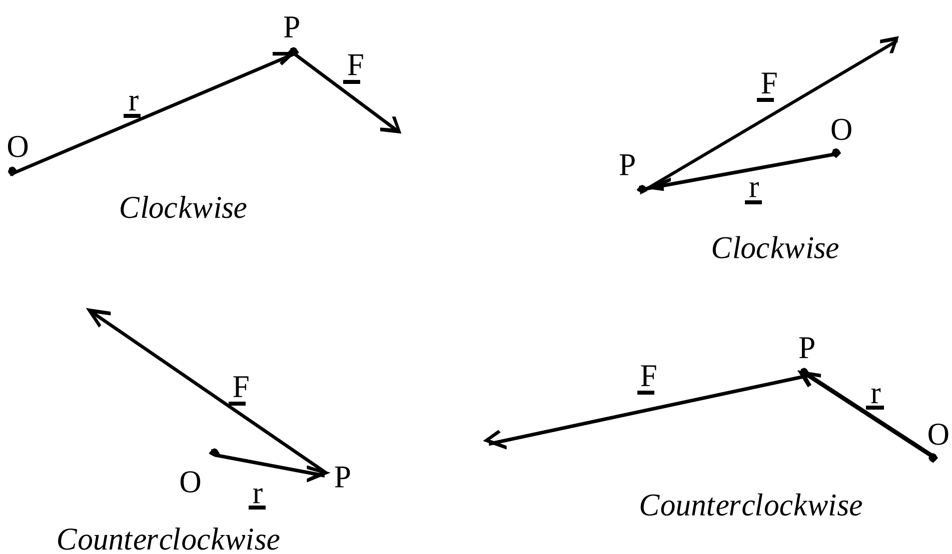
For an illustration of the physical meaning of the moment of a force as a vector, consider the situation where O and P are two separate points on a rigid body. If the body is fixed at point O but free to turn in any direction around that point, then the effect of $\underline{F}$ acting on the body at point P is to cause the body to rotate about O. The **magnitude** of $\underline{M}$, i.e. $|\underline{r}|\,|\underline{F}|\sin\theta$, gives a measure of the "turning effect" of the force on the body**.** The magnitude is maximal when $\underline{r}$ is perpendicular to $\underline{F}$ (that is where $\sin\theta$ reaches its largest value, $\sin\theta = \pm 1$) and the magnitude is minimal (equal to zero) when $\underline{r}$ and $\underline{F}$ are parallel (that is where $\sin\theta = 0$).



The magnitude of $\underline{M}$ in this case agrees with the **torque** of the force $\underline{F}$ about point O, which you may have come across before ("the length of the lever arm times the size of the perpendicular force" — note that $|\underline{F}|\sin\theta$ gives the length of the component of $\underline{F}$ which is perpendicular to vector $\underline{r}$). The significance of the **direction** of the vector $\underline{M}$ on the other hand is that it indicates the **axis of rotation** as well as the **direction of rotation**. By the axis of rotation we mean a line through the point O about which the object will rotate at the given moment (the axis of rotation may change in time). The direction of rotation tells us whether the rotation is clockwise or counterclockwise relative to some frame of reference.

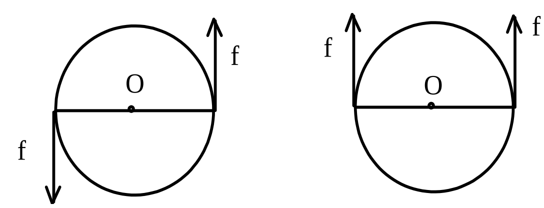
Assume, for instance, that both $\underline{r}$ and $\underline{F}$ are parallel to the XY-plane, in which case $\underline{M}$ is parallel to the Z-axis. If O is the origin, then in fact the axis of rotation is the Z-axis itself. If O is some other point on the XY-plane, then the axis is some other line parallel to the Z-axis. Now, if $\underline{M}$ is in the direction of the positive Z-axis, then the rotation is counterclockwise on the XY-plane (as seen from the direction of positive Z-values). If $\underline{M}$ is in the direction of the negative Z-axis, then the rotation is clockwise. This direction of rotation (that a moment of force which is a positive multiple of the $\underline{k}$ unit vector corresponds to counterclockwise rotation on the XY-plane) follows logically if we assume that our XYZ coordinate system obeys the right-hand rule.

Here is another way to look at the link between the direction of the moment of a force and the corresponding direction of rotation: Assume that the plane of this page is the XY-plane, with the Y-axis going up and the X-axis extending towards the right. Then the Z-axis is situated in such a way that it goes up from the page, towards the reader. Assume now that O and P are on the plane of the page, so that the position vector $\underline{r}$ from O to P is also along the plane of the page; and assume also that the force $\underline{F}$ is along the plane of this page. Then the moment of the force is always either up from the page, towards the reader, or into the page, away from the reader. The former holds if the rotation is counterclockwise, and the latter if the rotation is clockwise. To determine whether the rotation is clockwise or counterclockwise, imagine that the line from O to P is one of the hands of a clock, with the fixed end at O. Now, if it is pulled at point P in the direction of the force $\underline{F}$, does the hand of the clock move clockwise or counterclockwise? Look at the cases shown below, and make sure that you agree with the directions of rotation!



Note that the moment of a force as such does not necessarily have to be related to rotation; a constant force acting on a freely moving particle will also have a non–zero moment about a point O (e.g. the origin), as long as the force is not parallel to the position vector of the particle. But in this case, the moment of the force will not cause the particle to rotate about the origin; rather, it will move along a straight line with an acceleration determined by Newton's second law. Actual rotation can only arise when the force acts on a rigid system. (However, even in the case of a particle the moment of a force does involve a certain sense of rotation with respect to O, in the same way that we may say that an object went past "clockwise" if it went past from left to right, and "counterclockwise" if it went past from right to left.)

It is important to remember that the forces and moments of forces are entirely different quantities, despite the fact that moments of forces are caused by forces. For one thing, a force is an absolute quantity, whereas the moment of a force is always defined with respect to some specified point O. In particular, it is possible to have a system where the sum of the forces (as vectors) is zero, but the sum of the moments of the forces is not zero, and it is also possible to have a system where the sum of the moments of forces is zero but the sum of the forces themselves is not zero. These two possibilities are illustrated below.



How can we calculate the moment of a force, then, if we have to do so? We might proceed like this:

HOW TO FIND THE MOMENT OF A FORCE

- Firstly, we must make sure that the question is well defined — remember the moment of a force is only defined with respect to a point of reference O. We'll have to identify the point O (about which we plan to take the moment), the point P (at which the force acts), and the force $\underline{F}$ (its magnitude and direction).
- If a coordinate system is already given, fine; otherwise we will have to introduce one.
- Next, we shall express the vectors $\underline{r} = \underline{OP}$ and $\underline{F}$ in terms of the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$.
- Finally, we shall calculate the value of the cross product $\underline{M} = \underline{r} \times \underline{F}$.

Must we really always introduce a coordinate system to find the moment of a force? After all, the vector product itself was originally defined without any reference to a system of coordinates! It is indeed possible to figure out the magnitude of the moment of a force directly if we know just the lengths of the vectors $\underline{r}$ and $\underline{F}$ and the angle between them. However, introducing a coordinate system and using the unit vectors to calculate the vector product gives us a systematic way of finding the moment, decreasing the likelihood of errors. Also, a coordinate system enables us to describe the moment vector in a more exact way, without having to use phrases such as "the vector perpendicular to those two ones, towards the left".

Example 8.5
A force $\underline{F} = \underline{i} + 2\underline{k}$ acts at a point P with position vector $\underline{i} + 2\underline{j} - \underline{k}$. Find the moment of $\underline{F}$ about (i) the origin, (ii) the point with position vector $\underline{i} - \underline{j}$.

Solution:
Identify the point O (about which we plan to take the moment), the point P (at which the force acts), and the force $\underline{F}$ (its magnitude and direction). Here, P and $\underline{F}$ are given, but O has different values in (i) and (ii). If a coordinate system is already given, fine. The information in the question is already given in terms of a coordinate system: we have $\underline{F} = \underline{i} + 2\underline{k}$, and the point P has the position vector $\underline{i} + 2\underline{j} - \underline{k}$.

(i) If we wish to find the moment about $O =$ origin, then $\underline{r} = \underline{OP} = \underline{i} + 2\underline{j} - \underline{k}$, and therefore the moment of $\underline{F}$ acting at P about $O =$ origin is

$$\begin{aligned}
\underline{M} &= \underline{r} \times \underline{F} \\
&= \left(\underline{i} + 2\underline{j} - \underline{k}\right) \times \left(\underline{i} + 2\underline{k}\right) \\
&= \left(\underline{i} \times \underline{i}\right) + 2\left(\underline{i} \times \underline{k}\right) + 2\left(\underline{j} \times \underline{i}\right) + 4\left(\underline{j} \times \underline{k}\right) \\
&\quad - \left(\underline{k} \times \underline{i}\right) - 2\left(\underline{k} \times \underline{k}\right) \\
&= \underline{0} + 2\left(-\underline{j}\right) + 2\left(-\underline{k}\right) + 4\underline{i} - \underline{j} - 2\underline{0} \\
&= 4\underline{i} - 3\underline{j} - 2\underline{k}
\end{aligned}$$

(ii) If O is the point with position vector $\underline{i} - \underline{j}$, then

$$\underline{r} = \underline{OP} = \left(\underline{i} + 2\underline{j} - \underline{k}\right) - \left(\underline{i} - \underline{j}\right) = 3\underline{j} - \underline{k},$$

and this time the moment is

$$\begin{aligned}
\underline{M} &= \underline{r} \times \underline{F} = \left(3\underline{j} - \underline{k}\right) \times \left(\underline{i} + 2\underline{k}\right) \\
&= 3\left(\underline{j} \times \underline{i}\right) + 6\left(\underline{j} \times \underline{k}\right) - \left(\underline{k} \times \underline{i}\right) - 2\left(\underline{k} \times \underline{k}\right) \\
&= -3\underline{k} + 6\underline{i} - \underline{j} - 2 \cdot \underline{0} \\
&= 6\underline{i} - \underline{j} - 3\underline{k}.
\end{aligned}$$

Activity 8.4

Two forces $\underline{F} = \underline{i} - \underline{j} - \underline{k}$ and $\underline{G} = 2\underline{i} + \underline{k}$ act at P which has the position vector $\underline{i} + 2\underline{j}$. Find the moments of (i) $\underline{F}$, (ii) $\underline{G}$ and (iii) $\underline{F} + \underline{G}$ about the origin.

....

Feedback: (i) $-2\underline{i} + \underline{j} - 3\underline{k}$; (ii) $2\underline{i} - \underline{j} - 4\underline{k}$; (iii) $-7\underline{k}$.

Activity 8.5

A force $\underline{F} = \underline{i} - \underline{j} + 2\underline{k}$ acts through a point P with position vector $2\underline{i} + \underline{k}$. Find the *magnitude* of the moment of $\underline{F}$ about the point Q with position vector $\underline{j} - 2\underline{k}$. *(Remember that the magnitude of a vector is its length!)*

....

Feedback: the magnitude is $\sqrt{3}$.

Activity 8.6

In the following calculation, remember that the force is in two dimensions, so you have to find both components of the force! So, you will need to write $\underline{F} = x\underline{i} + y\underline{j}$*, for instance, and set up two equations from the given moments to solve* x *and* y *from!*

A force $\underline{F}$ acts parallel to the XY-plane and has moments

$$2aP\underline{k} \quad \text{and} \quad -4aP\underline{k}$$

about the origin when it acts at the points

$$a\underline{i} - a\underline{j} \quad \text{and} \quad 2a\underline{i} + 3a\underline{j},$$

respectively. Calculate the magnitude of the force $\underline{F}$.

....

Feedback: the magnitude will be $2\sqrt{17}P/5$.

Example 8.6
Consider a horizontal rod AB of length ℓ.

(a) Calculate the moment about point A of a downwards force $\underline{F}$ of magnitude 1, which is applied at the midpoint of the rod.

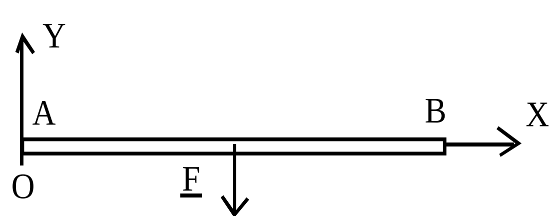
(b) We wish to achieve the same moment about A by applying a downwards force $\underline{G}$ at point B. What should the magnitude of $\underline{G}$ be to achieve this?

(c) Assume that we wish, instead, to achieve the same moment about A by applying a force $\underline{G}$ with unit magnitude at point B. In what direction should $\underline{G}$ be applied to achieve this?

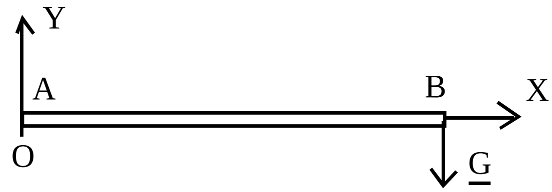
Solution:
Assume that the rod lies along the X-axis, with the endpoint A at the origin.

(a) The force $\underline{F} = -\underline{j}$ is applied at the point with position vector $\frac{1}{2}\ell\underline{i}$ from the origin; hence the moment of $\underline{F}$ about the origin is

$$\begin{aligned} \underline{M} &= \underline{r} \times \underline{F} = \left(\frac{1}{2}\ell\underline{i}\right) \times \left(-\underline{j}\right) = -\frac{1}{2}\ell\left(\underline{i} \times \underline{j}\right) \\ &= -\frac{1}{2}\ell\underline{k}. \end{aligned}$$



(b) If $\underline{G}$ is applied at point B, then its moment about the origin is $\left(\ell\underline{i}\right) \times \underline{G}$.



For this to equal the moment of the force in (a), we must have

$$\left(\ell\ \underline{i}\right) \times \underline{G} \quad = \quad -\frac{1}{2}\ell\ \ \underline{k}$$

$$\therefore \qquad \underline{i} \times \underline{G} = -\frac{1}{2}\underline{k}. \tag{8.8}$$

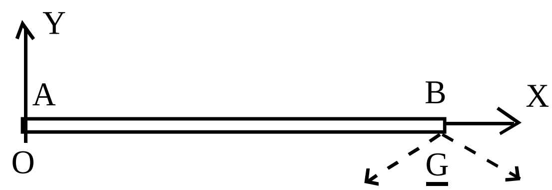
If $\underline{G}$ is parallel to force $\underline{F}$, then we must in fact have $\underline{G} = a\underline{j}$ for some a. But then (8.8) tells us that we must have

$$\underline{i} \times a\underline{j} \quad = \quad -\frac{1}{2}\underline{k}$$

$$\Leftrightarrow \quad a = -\frac{1}{2}.$$

The force $\underline{G}$ should be equal to $\underline{G} = -\frac{1}{2}\underline{j}$.

(c) Assume that the force $\underline{G}$, of unit length, forms the unknown angle θ with the rod.



Then, by applying (8.1), we see that for (8.8) to hold, we must have

$$\begin{cases} \left|\underline{i}\right| \left|\underline{G}\right| \sin\theta = \frac{1}{2} \\ \\ \underline{n}_{iG} = -\underline{k} \end{cases}$$

where θ is the smaller angle between $\underline{i}$ and $\underline{G}$, and $\underline{n}_{iG}$ is the unit vector such that $\underline{i}$, $\underline{G}$, $\underline{n}_{iG}$ form a right-handed system. The first condition means that

$$\sin\theta = \frac{1}{2} \quad \Leftrightarrow \quad \theta = \frac{1}{6}\pi$$

(since $\underline{i}$ and $\underline{G}$ both have unit length). The second condition tells us that the Y-component of $\underline{G}$ must be negative, that is, $\underline{G}$ must point "downwards" in the figure. Together, these facts tell us that $\underline{G}$ can be either one of the two vectors marked with a dotted line in the figure, which form the angle $\pi/6$ with the X-axis.

Alternatively, let us write $\underline{G} = a\underline{i} + b\underline{j}$ for some unknown values a and b. Then (8.8) tells us that $b = -\frac{1}{2}$ must hold, and the value of a can be solved from the condition that $a^2 + b^2 = 1$. ◄

This example illustrated the following two facts, which can easily be proved from the definition of the moment of a force:

- The moment of any force about a point can be duplicated by using a force with half the magnitude of the original force, but applied at a point twice as far away; or by applying a force with twice the magnitude of the original force at a point at half the distance.
- Assume that various forces, all with the same magnitude, are applied at the point P. The ones which give the largest magnitude to the moment about a point O are the ones where the force acts perpendicularly to the line $\underline{OP}$.

CONCLUSION

In this unit you have learned

- how to calculate the vector product of two vectors
- what is meant by the moment of a force about a point, and how to calculate it

Remember to add the following tools to your toolbox:

- the definition of a vector product
- the definition of the moment of a force
- the toolbox for how to find the moment of a force

Unit 9 ANGULAR MOMENTUM

We will see in the rest of this Learning Unit of the study guide that the *moment of a force* plays a role in the rotational motion of rigid bodies analogous to that played by the *force* in translational motion — it produces angular acceleration, just as force produces linear acceleration. The equation of motion describing this can easiest be derived in terms of the angular momentum of the system, so we will introduce this concept next — first for particles, then for systems of particles, and finally, in the next unit, for rigid bodies. Also, we will first introduce the more general concept of angular momentum about a point, before limiting ourselves to the more straightforward and practical case of rotation about an axis.

Angular momentum is a property that describes to rotational motion of a particle or a system of particles; the next step is to explain how it changes when forces act on the system. We will derive the relevant result (the law of change of angular momentum) here; again it is really just Newton's laws of motion re-written by utilizing the definition of angular momentum.

The concept of angular momentum is quite a complex one, and for many students this unit is easily the most difficult to follow in the entire study guide — especially the last section of this unit, where we consider the angular momentum with respect to the fixed axis and the law of how it changes. However, in the next unit, everything becomes concrete again, when we shall see what all this means for actual rigid bodies — we end up deriving the equation of rotation of rigid bodies.

What you must understand thoroughly from this unit is the definitions of the angular momentum for particles and for systems of particles (sections 9.1 and 9.2). The laws of how the angular momentum changes, as described in these two sections, and all of section 9.3 you should read through as background information on how we get to the very practical formulas in the next unit.

Contents of this unit:

9.1 The angular momentum of a particle

9.2 The angular momentum of a system of particles

9.3 The angular momentum with respect to a fixed axis

What you are expected know before working through this unit:

In this unit, we also make some references to the concept of linear momentum, which should be familiar to you from previous physics modules; however reference is all it is, the concepts in this module do not build on that at all. Instead, they build on the concepts of position and velocity vectors. You will also need Newton's second law here. We will again be using the moments of forces, and vector products, as defined in Unit 8.

9.1 The angular momentum of a particle

Let a particle P have a mass m and position vector $\underline{r} = \underline{OP}$ with respect to some fixed origin O. Remember that the **linear momentum** of the particle is then

$$\underline{p} = m\underline{\dot{r}},$$

and that Newton's second law of motion can then be expressed as

$$\underline{F} = \frac{d\underline{p}}{dt}. \tag{9.1}$$

Now, we define the **angular momentum** of the particle as follows:

Definition 9.1 (The angular momentum of a particle)
The angular momentum of the particle P (above) about point O is

$$\underline{\ell} = \underline{r} \times \underline{p} = \underline{r} \times m\underline{\dot{r}} = m\left(\underline{r} \times \underline{\dot{r}}\right).$$

The angular momentum depends on the choice of the point O. As with the moment of a force, the angular momentum is only defined with respect to some specified reference point O. Changing O changes the position vector $\underline{r}$ and thus the angular momentum $\underline{\ell} = m\underline{r} \times \underline{\dot{r}}$. The linear momentum, on the other hand, is an **absolute** quantity which does not depend on the chosen coordinates (since the vector $\underline{\dot{r}}$ does not change, even if the coordinate system does).

By its definition, $\underline{\ell}$ is a vector, with a magnitude and a direction. The vector $\underline{\ell}$ is always perpendicular to the plane where $\underline{r}$ and $\underline{\dot{r}}$ are. If $\underline{r}$ and $\underline{\dot{r}}$ lie on the XY-plane, then $\underline{\ell}$ is parallel to the Z-axis.

(Does this remind you of the moment of a force? It should, since we arrived at the angular momentum from the linear momentum through the same process that we arrived at the moment of the force from the force! This process is one of "taking moments" — we take moments of a vector $\underline{X}$ "acting" at a point P about a point O, by finding the vector (cross) product of the position vector from O to P and the vector $\underline{X}$. In fact, another name for the angular momentum is the moment of momentum!)

Activity 9.1

A particle with mass m moves on the XY-plane with velocity $30\underline{i} + 60\underline{j}$ as it passes through the point $3\underline{i} - 4\underline{j}$.

(a) What is its angular momentum relative to the origin at this moment?

(b) What is its angular momentum relative to the point with position vector $-2\underline{i} - 2\underline{j}$ at this same moment?

....

Feedback: (a) $300m\underline{k}$, (b) $360m\underline{k}$.

9.2 The angular momentum of a system of particles

The angular momentum of a system of particles is defined to be the sum of the angular momenta of the individual particles — of course all must be about the same point of reference O.

Definition 9.2 (The angular momentum of a system)
Let the point O be fixed. If a system consists of n particles, where m_i and $\underline{r}_i$ are the mass and the position vector from O of the ith particle, then the angular momentum of the system about point O is

$$\underline{L} = \sum_{i=1}^{n} \underline{\ell}_i = \sum_{i=1}^{n} \underline{r}_i \times m_i \dot{\underline{r}}_i. \tag{9.2}$$

Example 9.1
Particle 1 at point $(0, 1)$ on the XY-axis and particle 2 at point $(0, -1)$ both move in the direction of the positive X-axis with a velocity of 1. Both particles have a mass m. Find the following:

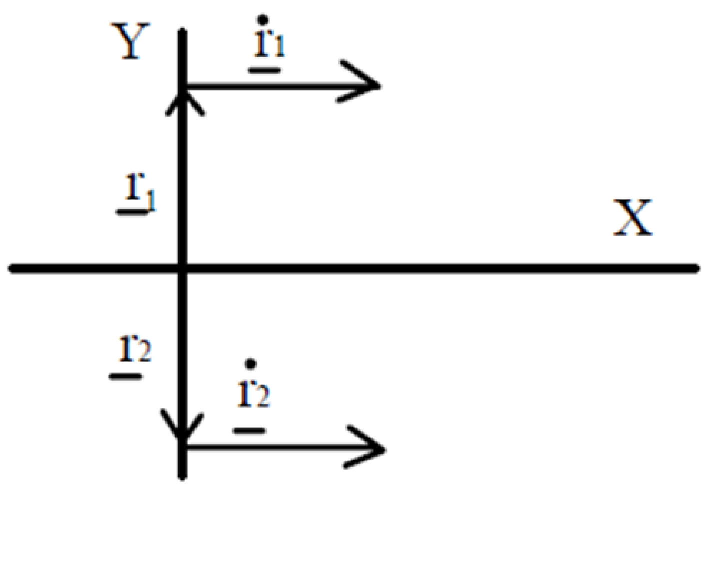
(a) The angular momentum of particle 1 about origin.

(b) The angular momentum of particle 2 about a point O with position vector $2\underline{i}$.

(c) The angular momentum of the system consisting of the two particles,

(i) about the origin,

(ii) about the point O with position vector $-\underline{j}$.



Solution:
We have $\dot{\underline{r}}_1 = \dot{\underline{r}}_2 = +\underline{i}$, but the position vectors $\underline{r}_1$ and $\underline{r}_2$ depend on which point is the reference point O.

(a) If O is the origin, then the position vector from O to particle 1 is $\underline{r}_1 = \underline{j}$. Therefore,

$$\underline{\ell}_1 = m\underline{r}_1 \times \dot{\underline{r}}_1 = m\underline{j} \times \underline{i} = -m\underline{k}.$$

(b) If O has the position vector $2\underline{i}$, then the position vector from O to particle 2 is $\left(-\underline{j}\right) - \left(2\underline{i}\right) = -2\underline{i} - \underline{j}$. In this case,

$$\underline{\ell}_2 = m\underline{r}_2 \times \underline{\dot{r}}_2 = m\left(-2\underline{i} - \underline{j}\right) \times \underline{i} = m\underline{k}.$$

(c)

(i) When O is the origin, the position vector from O to particle 1 is $\underline{r}_1 = \underline{j}$ and the position vector from O to particle 2 is $\underline{r}_2 = -\underline{j}$. Therefore,

$$\begin{aligned} \underline{L} &= m\underline{r}_1 \times \underline{\dot{r}}_1 + m\underline{r}_2 \times \underline{\dot{r}}_2 \\ &= m\underline{j} \times \underline{i} + m\left(-\underline{j}\right) \times \underline{i} = -m\underline{k} + m\underline{k} = \underline{0} \end{aligned}$$

(ii) The position vector from O to particle 1 is now $\underline{r}_1 = \left(\underline{j}\right) - \left(-\underline{j}\right) = 2\underline{j}$ and the position vector from O to particle 2 is $\underline{r}_2 = \left(-\underline{j}\right) - \left(-\underline{j}\right) = \underline{0}$. Hence

$$\underline{L} = m\left(2\underline{j}\right) \times \underline{i} + m\left(\underline{0}\right) \times \underline{i} = -2m\underline{k}.$$

Activity 9.2

Find the velocity of the centre of mass $\underline{\dot{R}}$ and the angular momentum $\underline{L}$ of the following systems of two particles:

(a)

$$\begin{aligned} \text{Particle 1, mass} &= m : \underline{r}_1 = +\underline{i},\ \underline{\dot{r}}_1 = +\underline{j} \\ \text{Particle 2, mass} &= m : \underline{r}_2 = -\underline{i},\ \underline{\dot{r}}_2 = -\underline{j}. \end{aligned}$$

(b)

$$\begin{aligned} \text{Particle 1, mass} &= m : \underline{r}_1 = +\underline{i},\ \underline{\dot{r}}_1 = +\underline{j} \\ \text{Particle 2, mass} &= m : \underline{r}_2 = -\underline{i},\ \underline{\dot{r}}_2 = +\underline{j}. \end{aligned}$$

....

Feedback: In (a), $\underline{\dot{R}} = 0$ and $\underline{L} = 2m\underline{k}$, in (b) $\underline{\dot{R}} = \underline{j}$ but $\underline{L} = \underline{0}$. Note that you have to calculate and add together $\underline{\ell}_1$ and $\underline{\ell}_2$, there is no shortcut here!

The above activity reminds you that it is NOT true that $\underline{L} = \underline{R} \times M\underline{\dot{R}}$! This is one place where you cannot generalise from particles to the system — which should in fact be obvious, since the motion of the centre of mass summarises the average motion of the particles in the system, but loses all detail in how the particles might move in relation to the centre of mass – and this includes any rotation around the centre of mass! In particular, you have now seen that it is possible for a system to have zero velocity for the centre of mass, but non-zero angular momentum (so that although the centre of mass does not move, there is "rotation" in the system), and alternatively, it is possible to have non-zero velocity for the centre of mass but zero angular momentum (so that the system as a whole moves but does not rotate).

The concept of angular momentum is not a particularly "natural" or easily understandable one. It can be interpreted to measure a sense of rotation of a moving particle or an entire collection of moving particles about the point O.

Imagine you are situated at point O. Particles may pass by you in any direction. But, for each particle at any particular moment you can define a momentary imaginary "axis *of rotation" as a line which is perpendicular to both the position vector from* O *to the particle, and the particle's velocity vector. For a concrete interpretation, imagine a rod attached at* O *with the other end at the position of the particle. For the loose end of the rod to be able to follow the particle along its path, how should the hinge at O be situated?*

Now, the vector $\underline{l}$ *for the particle about O is parallel to that "axis of rotation" and its magnitude and sign depend on the momentary "angular velocity" of the motion. For instance, a particle passing directly over your head from left to right would have an angular momentum vector which extends behind you, and a particle which overtakes you from the right would have an angular momentum vector which goes up. But of course the "axis of rotation" of the particle may change from moment to moment, except in the case where the particle always moves on the same plane!*

However, the very nice thing about $\underline{L}$ is that it obeys a very simple equation of change. Remember that Newton's second law for a particle can be re-written as

$$\underline{F} = \frac{d\underline{p}}{dt}$$

where $\underline{p}$ is the linear momentum of the particle and $\underline{F}$ the force acting on it. In Unit 2 we derived a similar result for a system of particles:

$$\underline{F} = \frac{d\underline{P}}{dt}$$

where $\underline{P}$ is the total linear momentum of the system and $\underline{F}$ the resultant of all the external forces acting on the system. In words,

$$\left.\begin{array}{l}\text{The sum of all the } \textbf{forces}\\ \text{acting on the system}\end{array}\right\} = \left\{\begin{array}{l}\text{rate of change of}\\ \textbf{linear momentum}\\ \text{of the system}\end{array}\right.$$

Now, we are going to prove a similar result for the angular momentum, namely

$$\left.\begin{array}{l}\text{The sum of the } \textbf{moments}\\ \text{of all the forces}\\ \text{acting on the system}\end{array}\right\} = \left\{\begin{array}{l}\text{rate of change of the}\\ \textbf{angular momentum}\\ \text{of the system}\end{array}\right.$$

(provided that the quantities on both sides are with respect to the same fixed point O!)

In order to prove this result, we shall simply calculate $d\underline{L}/dt$ for the system using the definition (9.2).

$$\begin{aligned}\frac{d\underline{L}}{dt} &= \frac{d}{dt}\left(\sum_{i=1}^{n} m_i \underline{r}_i \times \dot{\underline{r}}_i\right)\\ &= \sum_{i=1}^{n} m_i \left[\frac{d}{dt}\left(\underline{r}_i\right) \times \dot{\underline{r}}_i + \underline{r}_i \times \frac{d}{dt}\left(\dot{\underline{r}}_i\right)\right]\end{aligned}$$

(Note that we used here the product rule of differentiation, which holds for the vector product also!)

$$= \sum_{i=1}^{n} m_i \left[\dot{\underline{r}}_1 \times \dot{\underline{r}}_i + \underline{r}_i \times \ddot{\underline{r}}_i\right]$$

But $\dot{\underline{r}}_i \times \dot{\underline{r}}_i = \underline{0}$ according to (8.3), and so we have

$$\begin{aligned} \frac{d\underline{L}}{dt} &= \sum_{i=1}^{n} m_i \left(\underline{r}_i \times \ddot{\underline{r}}_i \right) \\ &= \sum_{i=1}^{n} \underline{r}_i \times \left(m_i \ddot{\underline{r}}_i \right). \end{aligned} \tag{9.3}$$

According to Newton's second law we have for particle number i

$$\underline{F}_i = m_i \ddot{\underline{r}}_i \tag{9.4}$$

where $\underline{F}_i$ is the resultant of all the forces on the particle. We shall assume that the moments of all the internal forces cancel out, in which case we may assume that $\underline{F}_i$ represents the sum of all the external forces acting on the ith particle. Using (9.4) we can rewrite (9.3) as

$$\frac{d\underline{L}}{dt} = \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i .$$

Result 9.3 The law of change of angular momentum
Let the point O be fixed, and let $\underline{L}$ be the angular momentum of the system about point O. Then

$$\frac{d\underline{L}}{dt} = \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i \tag{9.5}$$

where the right-hand side represents the sum of the moments of all the external forces about the chosen point O.

Note that this is not a new law of nature, but just a consequence of Newton's second law of motion — we have derived from that law an explanation of how the quantity called angular momentum changes, due to forces acting on a system.

Equation (9.5) is a very general result, which applies to any system of particles in arbitrary motion. This generality, however, also means that (9.5) is a bit difficult to understand. In this module we are, in any case, mainly interested in describing the rotation of a rigid body about an axis; and then equation (9.5) takes a much more concrete form!

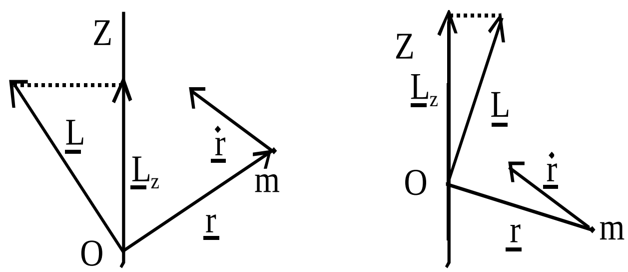
9.3 The angular momentum with respect to a fixed axis

Remember that $\underline{L}$ is a vector, with both direction and magnitude. Accordingly, its derivative (i.e. its change in time) $d\underline{L}/dt$ can involve change in both the magnitude and the direction of the vector $\underline{L}$. The situation would be simpler if we could assume that the direction of the vector $\underline{L}$ does not change, because then only a change in the magnitude and sign of $\underline{L}$ would be involved. The quantities $\underline{L}$ and $d\underline{L}/dt$ can then be interpreted in terms of rotation about an axis, as we will attempt to explain in the following.

Firstly, let us assume that we wish to investigate the "rotation" of the system about a given fixed axis. Remember that $\underline{L}$ was defined earlier in respect of a point O, with each of the particles having its own instantaneous "axis of rotation" in relation to this point. Assume now that we specify a fixed axis instead, that is, any line in the three-dimensional space, and wish to consider how the particles rotate around this axis. The angular momentum $\underline{L}$

can be used to make sense of this as well, by considering the component of the angular momentum which is parallel to the axis of rotation, as we will explain now.

First, note that we can always choose our coordinate system so that the fixed axis we are interested in is *parallel to the Z-axis*. Now, we could analyse the angular momentum of the system about any point O situated on the axis. The vector $\underline{L}$ obtained would then depend on the choice of the point O. However, the Z-component of $\underline{L}$ is always the same, and does not depend on which particular point O we choose.



We are therefore justified in simply calling the component of the angular momentum parallel to the chosen fixed axis the **angular momentum of the system with respect to the given axis**, denoted by $\underline{L}_z$. (Remember that we chose the axis to go in the Z-direction!). $\underline{L}_z$ will then be parallel to the Z-axis.

To calculate $\underline{L}_z$, we simply need to replace in (9.2) the position vectors $\underline{r}_i$ from O to particle number i, with position vectors from the **axis** to particle number i, perpendicular to the axis. The vector $\underline{r}_i$ will then be in the XY-plane. Similarly, $\dot{\underline{r}}_i$ should be replaced with its component parallel to the XY-plane. (This follows from the fact that in a vector product $\underline{a} \times \underline{b}$, only the X- and Y-components of $\underline{a}$ and $\underline{b}$ contribute to the Z-component of $\underline{a} \times \underline{b}$; the Z-component of $\underline{a} \times \underline{b}$ does not depend at all on the Z-components of $\underline{a}$ and $\underline{b}$!) The following definition can therefore be used:

Definition 9.4 (The angular momentum about an axis)
Let a system consist of n particles, and let an axis parallel to the Z-axis be given. Then the angular momentum of the system about the given axis is

$$\underline{L}_z = \sum_{i=1}^{n} m_i \underline{r}_i \times \dot{\underline{r}}_i$$

where m_i is the mass of particle i, $\underline{r}_i = x_i \underline{i} + y_i \underline{j}$ is the position vector from the axis to particle i, and $\dot{\underline{r}}_i = \dot{x}_i \underline{i} + \dot{y}_i \underline{j}$ is the velocity of particle i on the XY-plane.

Note that $\underline{r}_i$ and $\dot{\underline{r}}_i$ as defined above will be parallel to the XY-plane, and the vector $\underline{L}_z$ will be parallel to the Z-axis.

We can apply (9.5) to find a differential equation for $\underline{L}_z$ by simply considering the Z-component of

$$\sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i .$$

Again this means looking only at the XY-components of $\underline{r}_i$ and $\underline{F}_i$. It follows that

$$\frac{d\underline{L}_z}{dt} = \sum_{i=1}^{n} \underline{r}_i \times \hat{\underline{F}}_i$$

where $\underline{\hat{F}}_i$ is the projection of force $\underline{F}_i$ into the XY-plane.

Result 9.5 The law of change of angular momentum about an axis
Let an axis parallel to the Z-axis be given, and let $\underline{L}_Z$ be the angular momentum of the system of n particles about the axis. Then

$$\frac{d\underline{L}_z}{dt} = \sum_{i=1}^{n} \underline{r}_i \times \underline{\hat{F}}_i$$

where the summing is over all the external forces $\underline{F}_i$ acting on the system of particles, $\underline{\hat{F}}_i$ is the projection of force $\underline{F}_i$ into the XY-plane, and $\underline{r}_i$ is the position vector on the XY-plane from the axis of rotation to the point where the force $\underline{F}_i$ acts.

Of course, if $\underline{L}$ is always parallel to the Z-axis, then we automatically have $\underline{L}_z = L$. This holds if we assume that each particle in the system can only move parallel to the XY-plane. And this can only be true if we assume that all the forces acting on the system also act parallel to the XY-plane.

From now on we will assume that the following condition holds:

It is possible to introduce an XYZ coordinate system such that the velocities of all particles, and forces acting on them, will always be parallel to the XY-plane.

This assumption implies that each particle will always move on the same plane, parallel to the XY-plane. It follows that the angular momentum with respect to any axis parallel to the Z-axis is always well defined as

$$\underline{L} = \sum_{i=1}^{n} m_i \underline{r}_i \times \underline{\dot{r}}_i \tag{9.6}$$

where $\underline{r}_i$ is the distance from the axis to particle number i; and

$$\frac{d\underline{L}}{dt} = \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i \tag{9.7}$$

holds. We are justified to drop the subscript Z when it is assumed the forces and velocities are always parallel to the XY-plane, as in this case $\underline{L}$ and $\frac{d\underline{L}}{dt}$ are always parallel to the Z-axis. In particular, it follows that $\underline{L}$ will never change its direction; only its magnitude and sign may change.

The assumptions above are true in the special cases that we are most interested in, namely

- a rigid body constrained to rotate about a fixed axis, or
- a rigid body rotating about a moving axis when the direction of the axis does not change.

In the next unit we shall proceed to look at the case of rigid bodies, and find that in that case the expression for $\underline{L}$ and $d\underline{L}/dt$ can be replaced with something a lot more easy to understand!

CONCLUSION

In this unit you have learned

- what is meant by the angular momentum of a particle, and of a system of particles
- how the moment of a force acts to change the angular momentum

Unit 10 A RIGID BODY ROTATING ABOUT A FIXED AXIS

Key questions:

- *What is the promised simpler equation for rotation in the case of a rigid body?*
- *How do we use it to analyse rotational motion?*

Now it is time to move from the general system-of-particles case to the special case we are really most interested in, namely that of rigid bodies. In the rest of Learning Unit 3 we will consider the case of a rigid body rotating about a fixed axis. We assume, in particular, that the axis of rotation does not move, but that the axis can be considered to be at rest. Examples of this type of behaviour are doors or windows rotating about their hinges — or any objects which can be considered to be "nailed" to an unmoving floor, wall, etc. by a long nail, rotating about the nail! This motion is called **pure rotation**. (In Learning Unit 4 of the study guide we will consider a more general case, where the axis of rotation is also allowed to move — such as the door of a car, opening on its hinges, when the car is driving along a road!)

In this special case of pure rotation, we can re–write the angular momentum of the system in terms of the rotational velocity $\dot{\theta}$, and the derivative of the angular momentum in terms of the rotational acceleration $\ddot{\theta}$ and a quantity called the moment of inertia, which is specific to the particular object and axis. In this unit, we will derive the appropriate equation of motion, and explain how moments of inertia are found.

Contents of this unit:

10.1 Describing the rotation of a rigid body

10.2 The equation of motion of a rigid body rotating about a fixed axis

10.3 Moments of inertia

What you are expected know before working through this unit:

This section starts off with the result of the previous unit, and derives from there the equation of rotation for rigid bodies. So, it is assumed that you have gone through that unit.

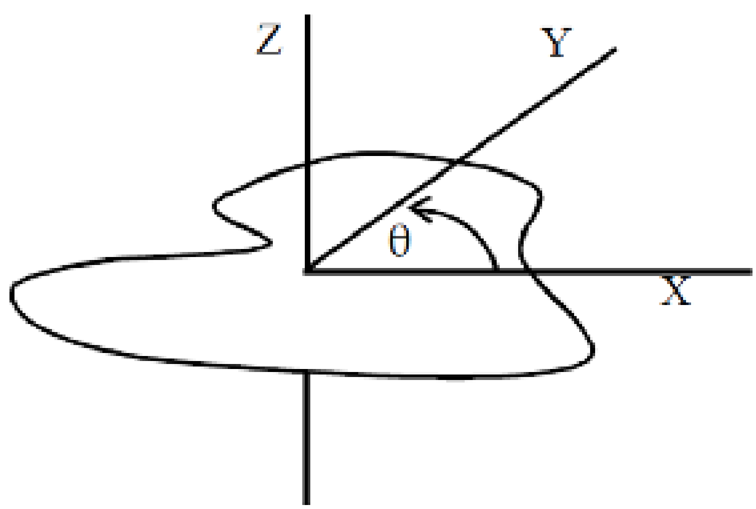
10.1 Describing the rotation of a rigid body

In what follows, we will always assume that the fixed axis of rotation is parallel to the Z-axis. In that case, since the rigid body rotates about an unmoving axis, it follows that each particle can only move parallel to the XY-plane, travelling in a circle around the axis of rotation. Also from the fact that the body is constrained in such a manner that it can only rotate about the axis, it follows that we may assume that all external forces acting on the body are also parallel to the XY-plane. (After all, we know that any external force trying to act on any of the particles e.g. in the Z-direction, cannot cause acceleration into the Z-direction since the particles are constrained to move on the XY-plane only — therefore any such force must be cancelled out by another force. For instance, trying to pull down on a door which is hanging on its hinges will not move the door downwards, but will just cause extra strain on the hinges!)

Remember that a rigid body is a system of particles where strong internal forces keep each particle in a fixed position with respect to all the other particles. When a rigid body rotates about a fixed axis, then every point of the body moves in a circle, the centre of which lies on the axis of rotation. Also, every point moves through the same angle during a particular time interval.

It follows that we can fully describe the motion of a rigid body rotating about a fixed axis in terms of a angular position, or **angle of rotation**, θ, which is specified as follows: We select one point of the body and use it as a reference point; any point will do — as long as it is not on the axis of rotation. The circular motion of this particle is then representative of the rotational motion of the entire rigid body, and the angular position of this particle is representative of the angular position of the entire body. To determine the angular position of the representative point P, and hence the angular position of the entire body, we determine the angle θ that the line from the axis to P makes with some fixed (non-rotating) reference axis, and we then say that the angle of rotation of the body is θ (with respect to our chosen reference system).

We will always measure the position angle θ in radians, not degrees, and we will always measure the angle of rotation θ **counterclockwise** on the XY-plane; see the figure below. (Remember that the axis of rotation is assumed to be parallel to the Z-axis!)



The reason for measuring the angle of rotation counterclockwise is that it will then match the right-hand rule for defining vector products. Once we have decided how we shall measure θ, then the behaviour of the rigid body as it rotates about the fixed axis is completely known at all times if we are given $\theta(t)$, the angle of rotation of the body as a function of time t. Remember that we consider only pure rotation here, so that the only motion is that of the body rotating about the axis; the axis itself is not moving.

Angular velocity, denoted by $\dot{\theta}$, is the rate of change of the angle of rotation:

$$\dot{\theta} = \frac{d\theta}{dt}.$$

Angular acceleration, $\ddot{\theta}$ is similarly defined:

$$\ddot{\theta} = \frac{d^2\theta}{dt^2}.$$

You may have come across the notations ω and α for the angular velocity and acceleration elsewhere; we will mostly to use the notations $\dot{\theta}$ and $\ddot{\theta}$ here as a reminder of how these quantities are linked to each other.

If the rotating body has an angular velocity $\dot{\theta}$, then **every** point of the body is moving with the same angular velocity. Since we have decided to measure θ counterclockwise, we see that a positive angular velocity means counterclockwise rotation, and a negative angular velocity means clockwise rotation on the XY-plane.

If the angular acceleration $\ddot{\theta} = \alpha$ is constant, then integration gives us the familiar looking formulas

$$\dot{\theta}(t) = \dot{\theta}(0) + \alpha t$$

$$\theta(t) = \theta(0) + \omega(0)\,t + \frac{1}{2}\alpha t^2$$

$$\left(\dot{\theta}(t)\right)^2 - \left(\dot{\theta}(0)\right)^2 = 2\alpha\left(\theta(t) - \theta(0)\right).$$

Compare these with the similar formulas for linear position, speed and acceleration in kinematics!

In general, the angular velocity and acceleration can be functions of time; in the following example they are specified as given functions of time. Note that in the rest of the study guide, this will very rarely be the case as instead our job will be to figure out (usually) the value of the angular acceleration from the equations of motion, or the angular velocity through energy considerations, in a given situation! However, you must be aware of the meanings of the angle of rotation (angular position) , the angular velocity and the angular acceleration, hence the inclusion of the following example and the activity after it!

Example 10.1

The angular position of a reference point on a spinning wheel at time t is given by

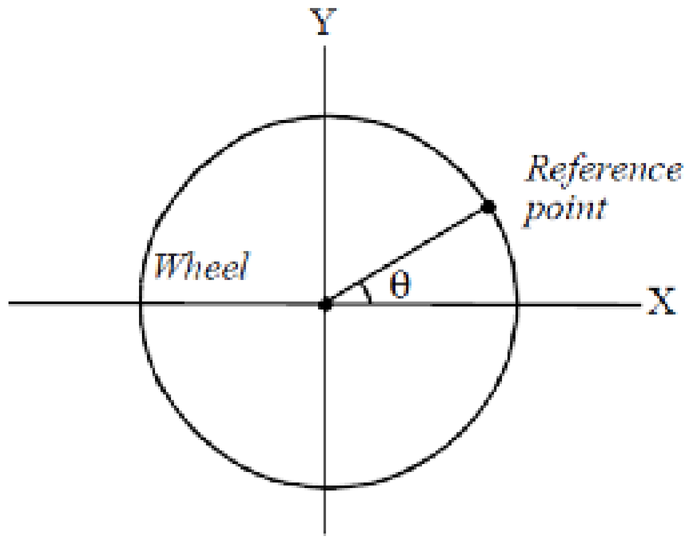
$$\theta = t^3 - 27t + 4.$$

(a) Find the angular velocity $\dot{\theta}$ and the angular acceleration $\ddot{\theta}$. Is there ever a time when $\omega = 0$?

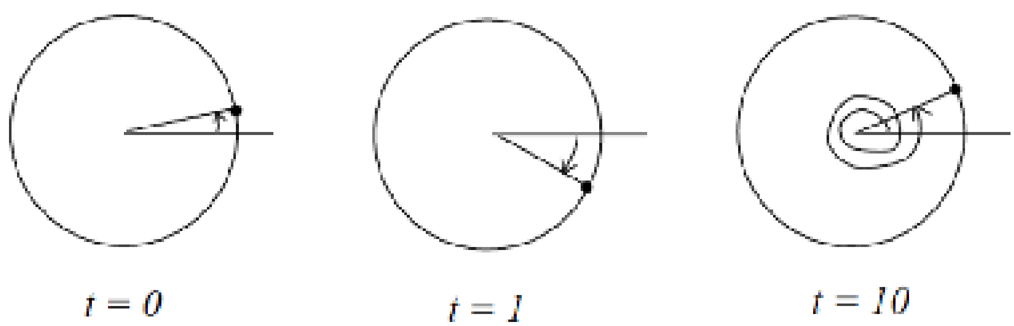
(b) Describe the wheel's motion for $t \geq 0$.

Solution:

Let us assume that the wheel lies on the XY-plane, and that the angular position is measured, as usual, as the angle θ from the positive X-axis to a line from the origin to the reference point on the wheel. A positive angle is taken to be counterclockwise.



If, for convenience, the angle is measured in degrees and time t in seconds, then at time $t = 0$, the reference point has the angular position of 4 degrees; after 1 second it has the angular position of -22 degrees, and after 10 seconds it has an angular position $734 = 2 \times 360 + 14$ degrees.



(a) The angular velocity is the derivative of the angular position:

$$\dot{\theta}(t) = \frac{d\theta(t)}{dt} = 3t^2 - 27.$$

The angular acceleration is the derivative of the angular velocity:

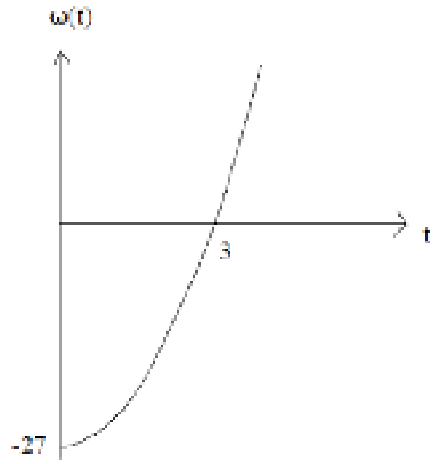
$$\ddot{\theta}(t) = \frac{d\omega(t)}{dt} = 6t.$$

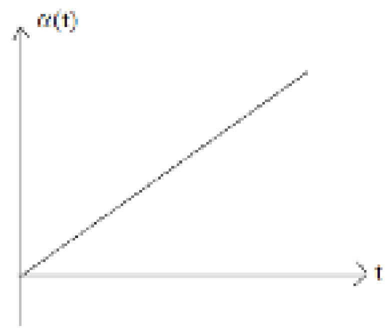
We see that $\dot{\theta} = 0$ when

$$\begin{aligned} & 3t^2 - 27 = 0 \\ \therefore\ & t^2 = 9 \\ \therefore\ & t = \pm 3, \end{aligned}$$

that is, $\dot{\theta} = 0$ at time $t = 3$ (and at time $t = -3$, three time units before the starting time).

(b) The motion of the wheel is easiest to figure out by following how its angular velocity changes as a function of time. Below are plots of the angular velocity $\dot{\theta}(t)$ and the angular acceleration $\ddot{\theta}(t)$ as functions of time, starting at $t = 0$:





At time $t = 0$, the wheel is at the positive angle $\theta\left(0\right) = 4$. The initial angular velocity is $\dot{\theta}\left(0\right) = -27$, meaning that the wheel is initially in motion, rotating clockwise. However, the velocity of the motion is decreasing, until at last the wheel comes to a momentary stop at time $t = 3$. The angular position of the wheel when this happens is at

$$\theta\left(3\right) = \left(3\right)^{3} - 27\left(3\right) + u = -50.$$

The wheel then starts to rotate counterclockwise, with ever–increasing angular velocity and ever–increasing angular acceleration.◄

Activity 10.1

The angle of rotation of a wheel varies in time according to the function

$$\theta\left(t\right) = -t + \frac{1}{2}t^{2}.$$

Find the angular velocity and angular acceleration as functions of time, and explain how the wheel moves for $t \geq 0$.

....

Feedback: The wheel will start with a zero angle of rotation, will rotate clockwise but slower and slower, will come to a standstill at $t = 0$, and after that will rotate counterclockwise, faster and faster. The angular velocity is constant, so the angular velocity will grow linearly.

10.2 The equation of motion of a rigid body rotating about a fixed axis

We are now ready to derive the equation of motion to describe pure rotation. We start off with the angular momentum $\underline{L}$ of a system of particles with respect to the fixed axis (9.6) and its equation of change (9.7), since these apply in our case of the pure rotation of a rigid body. Remember that according to (9.6), we have

$$\underline{L} = \sum_{i=1}^{n} m_i \underline{r}_i \times \underline{\dot{r}}_i$$

where $\underline{r}_i$ is the position vector from the axis of rotation to particle number i of the system. In terms of polar coordinates, we can write

$$\underline{r}_i = r_i\left(\cos\theta_i\,\underline{i} + \sin\theta_i\,\underline{j}\right)$$

(where $r_i = \left|\underline{r}_i\right|$ and θ_i is the angle $\underline{r}_i$ makes with the X-axis, measured counterclockwise). Since the particles form a rigid body, each r_i must be **constant** (each particle rotates about the axis along a circle). Hence we have

$$\begin{aligned}\underline{\dot{r}}_i &= \frac{d}{dt}\underline{r}_i = \frac{d}{dt}\left(r_i\left(\cos\theta_i\underline{i} + \sin\theta_i\underline{j}\right)\right)\\ &= r_i\left(-\dot{\theta}_i\sin\theta_i\underline{i} + \dot{\theta}_i\cos\theta_i\underline{j}\right),\end{aligned}$$

and thus

$$\begin{aligned}\underline{L}_z &= \sum_{i=1}^{n} m_i r_i\left(\cos\theta_i\underline{i} + \sin\theta_i\underline{j}\right) \times r_i\left(-\dot{\theta}_i\sin\theta_i\underline{i} + \dot{\theta}_i\cos\theta_i\underline{j}\right)\\ &= \sum_{i=1}^{n} m_i r_i^2\dot{\theta}_i\left(\sin^2\theta_i + \cos^2\theta_i\right)\underline{k}\\ &= \sum_{i=1}^{n} m_i r_i^2\dot{\theta}_i\underline{k}.\end{aligned}$$

But for a rigid body it is also true that all particles move at the same angular velocity: $\dot{\theta}_i = \dot{\theta}$, for all particles i. Therefore,

$$\underline{L}_z = \left(\sum_{i=1}^{n} m_i r_i^2\right)\dot{\theta}\underline{k}. \tag{10.1}$$

Note that we have managed to write the angular momentum as the product of two quantities, one of which (that is, $\dot{\theta}$) describes the rotational motion of the object about the axis, while the other (the quantity $\sum_{i=1}^{n} m_i r_i^2$) is an unchanging quantity describing the object itself. The expression in the brackets is called the **moment of inertia** of the system for the given axis, and is usually denoted by I.

Definition 10.1 (The moment of inertia)
Let a rigid body consist of n particles, with masses m_i and position vectors $\underline{r}_i$ from some given axis. Then the moment of inertia I of the system for rotation about the given axis is defined as

$$I = \sum_{i=1}^{n} m_i r_i^2$$

Using the concept of the moment of inertia, we can now write (10.1) as

$$\underline{L}_z = I\dot{\theta}\underline{k}. \tag{10.2}$$

The equation of motion (9.7) then becomes

$$\frac{d}{dt}\underline{L}_z = \frac{d}{dt}\left(I\dot{\theta}\underline{k}\right) = I\ddot{\theta}\underline{k} = \sum_{i=1}^{n}\underline{r}_i \times \underline{F}_i.$$

Result 10.2 (The equation of motion for pure rotation)
The rotation of a rigid body about a fixed axis is described by the equation

$$\sum_{i=1}^{n}\underline{r}_i \times \underline{F}_i = I\ddot{\theta}\underline{k} \tag{10.3}$$

when the axis of rotation is parallel to the Z-axis.

Here, $\ddot{\theta}$ is the angular acceleration of the rigid body, measured counterclockwise, I is its moment of inertia for rotation about the given axis, and the left hand side of the equation is the sum of the moments of all the external forces acting on the body. For a rigid body constrained to rotate about a fixed axis which is parallel to the Z-axis, these forces can be assumed to act parallel to the XY-plane. Therefore, the left-hand side of (10.3) will also be a vector, parallel to the Z-axis.

The sum on the right if the sum of the moments of all the external forces acting on the system; we could denote it by $\underline{M}$ and get a very neat equation: $\underline{M} = I\ddot{\theta}\underline{k}$. The reason we did not is because really, you do still have to calculate the vector products on the left hand side, and it is good to be reminded of that fact!

Note that (10.3) is again not a new result, but a reformulation of Newton's second law in a form very convenient to describe the rotation of any rigid body. The rotational moment of inertia I is a rotational equivalent of **mass** (translational inertia), in that it relates the moment of a force acting on a rigid body to the resulting angular acceleration. More generally, we have the following analogies between translational and rotational concepts:

Translation		**Rotation**	
Position	$\underline{x}$	Angular position	θ
Velocity	$\dot{\underline{x}}$	Angular velocity	$\dot{\theta}$
Acceleration	$\ddot{\underline{x}}$	Angular acceleration	$\ddot{\theta}$
Translational inertia (mass)	m	Rotational inertia (moment of inertia)	I
Force	$\underline{F}$	Moment of force	$\underline{M} = \underline{r} \times \underline{F}$
Linear momentum	$\underline{p} = m\dot{\underline{x}}$	Angular momentum	$\underline{L}_z = I\dot{\theta}\underline{k}$
Equation of motion	$\underline{F} = m\ddot{\underline{x}}$	Equation of rotation	$\underline{M} = I\ddot{\theta}\underline{k}$

A final reminder: Please remember that the result $\underline{M} = I\ddot{\theta}\underline{k}$ above only applies to a **rigid body** rotating about an axis, since that is the only case where we can summarize the angular momentum by means of I and the angular acceleration, $\ddot{\theta}$! In the more general case of an arbitrary system of particles, the best we can hope for is $M = \frac{dL}{dt}$, as given in equation (9.5).

The equation of motion for pure rotation, derived above, is a key equation which you will be using a lot in the rest of the study guide. However, before we start working with it, we will spend a bit of time discussing the newly introduced concept of *moment of inertia*, since clearly that will be very important in the applications of the equation of rotation. This is what we will do in the rest of this unit, and all of Unit 11. However, first we will look at a little example of a typical situation where we could apply the equation of rotation derived above!

Example 10.2

A door is roughly a rectangular lamina, rotating about its hinges which can be considered an axis of rotation which is parallel to one of the longer edges of the lamina. When opening the door, we apply a force on the doorhandle which is at the opposite side of the lamina from the axis of rotation. The moment of inertia for the rotation here is known to be

$$I = \frac{1}{3}Ma^2$$

where M is the mass of the door and a is the width of the door. (We will derive this result soon.) If the distance from the hinges to the doorhandle is also assumed to be a, and if we assume that the force F is applied optimally, that is, in a direction which forms a right angle with the position vector from the axis of rotation to the applying point (i.e. from the hinges to the door handle), then the angular acceleration of the rotation is $\ddot{\theta}$ where

$$aF = \frac{1}{3}Ma^2\ddot{\theta} \quad \therefore \quad \ddot{\theta} = \frac{3F}{Ma}.$$

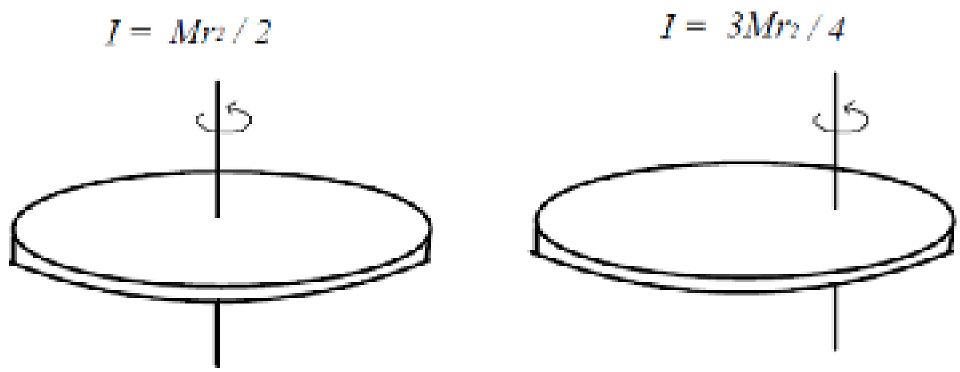
10.3 Moments of inertia

According to the definition, the moment of inertia I of a body made up of n particles held rigidly in their places in relation to an axis is given by

$$I = \sum_{i=1}^{n} m_i r_i^2 \tag{10.4}$$

where r_i is the distance from the axis of rotation to particle number i with mass m_i. (Remember that for a rigid body rotating about a fixed axis, each particle always stays at the same distance from the axis!)

The moment of inertia I of a system depends on the mass of the rigid body, but also on the distribution of the mass of the body in relation to the axis of rotation. If the axis changes, then the moment of inertia will also change.



It is important to remember that the moment of inertia of an object is defined for rotation **about some axis**! It makes no sense to talk about “the moment of inertia of a ring” unless we specify the axis of rotation as well.

Equation (10.4) tells us that the further away from the axis a particle is, the more its mass contributes to the moment of inertia. This is why, for instance, a ring will turn out to have a larger moment of inertia than a disk of the same mass and radius about an axis which goes perpendicularly through the centre of the object. In the ring, all the mass is on the outmost edge of the object, while in the disc some of the mass is closer to the axis. On the other hand, it is in principle possible to have an object of an arbitrarily large mass with a zero moment of inertia, simply by ensuring that all the mass is concentrated at the axis of rotation.

The definition of the moment of inertia allows us to also define the moment of inertia of a rigid body which rotates about an axis which does not necessarily go through the body. In this case, we must imagine the body to be connected rigidly to the axis of rotation with massless rods, or any other massless structure.

As the moment of inertia plays a vital Learning Unit in the dynamics of a rotating body, we shall spend some time on calculating it for various bodies, about various axes. The moments of inertia of some common objects about certain axes are given in the table on page 225. We will derive most of these results here, in this study guide!

10.3.1 The moment of inertia of a rigid systems of particles

For a rigid system of particles, we can use the definition (10.4) directly. Note that for a system of particles to form a rigid body, which rotates about a fixed axis, we must assume that the particles are rigidly joined to each other and to the axis. We can imagine this attachment to be by means of massless rods.

In particular, for a one-particle case we have

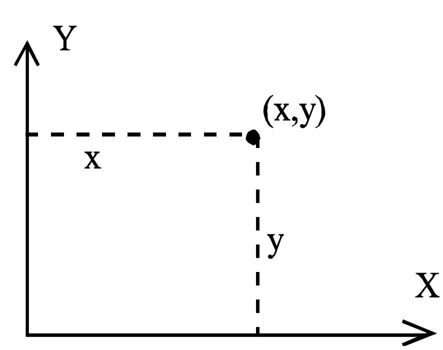
$$I = mr^2. \tag{10.5}$$

Again, for one particle to form a rigid body rotating about a fixed axis, the particle would have to be held rigidly at the distance r from the axis of rotation! In this one-particle case, we can assume that particle is attached to the axis with a massless rod, or the particle is attached to some other rigid body rotating about the axis.

Example 10.3
Four particles, each of mass M, are held rigidly together by massless rods. The system is placed on the XY-plane, in such a manner that the particles have the coordinates $P = (3a, 2a)$, $Q = (7a, 2a)$, $R = (7a, 6a)$ and $S = (3a, 6a)$, respectively. Find the following moments of inertia: I_X for rotation about the X-axis, I_Y for rotation about the Y-axis and I_z for rotation about the Z-axis. In the case of each rotation, we assume that the system is rigidly attached to the axis of rotation.

Solution:
Here, each particle has the same mass M and their initial positions are $P = (3a, 2a)$, $Q = (7a, 2a)$, $R = (7a, 6a)$ and $S = (3a, 6a)$ on the XY-plane. To calculate the requested moments of inertia, we must find the (shortest) distance from each particle to the axis in question.

Let the axis of rotation be the X-axis. If a particle has the coordinates (x, y), then its distance to the X-axis is given by the Y-coordinate, y.



Therefore, the moment of inertia about the X-axis is

$$\begin{aligned} I_X &= M(2a)^2 + M(2a)^2 + M(6a)^2 + M(6a)^2 \\ &= 80Ma^2. \end{aligned}$$

If the axis of rotation is the Y-axis, then the distance from a particle to the axis is given by its X-coordinate. Therefore, the moment of inertia of the system about the Y-axis is

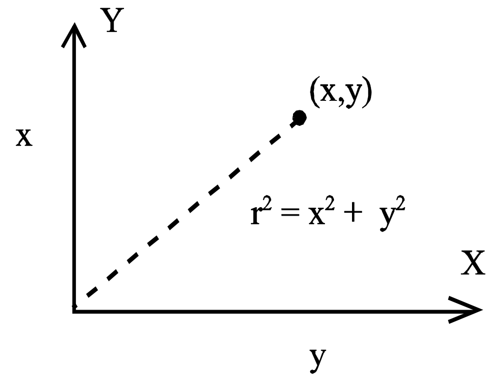
$$\begin{aligned} I_Y &= M(3a)^2 + M(7a)^2 + M(7a)^2 + M(3a)^2 \\ &= 116Ma^2. \end{aligned}$$

Let the axis of rotation be the Z-axis. The Z-axis goes through the origin of the XY-plane, perpendicular to that plane, so the distance r from a particle at point (x, y) to the Z-axis equals the distance to the origin, that is,

$$r = \sqrt{x^2 + y^2},$$

and therefore

$$r^2 = x^2 + y^2.$$



Hence, the moment of inertia for rotation about the Z-axis is

$$\begin{aligned} I_Z &= M\left((3a)^2 + (2a)^2\right) + M\left((7a)^2 + (2a)^2\right) \\ &\quad + M\left((7a)^2 + (6a)^2\right) + M\left((3a)^2 + (6a)^2\right) \\ &= 196Ma^2. \end{aligned}$$

Activity 10.2

The masses and coordinates of four particles on the XY-plane are as follows:

- $m_1 = 50m$, at $(x_1, y_1) = (2, 2)$;
- $m_2 = 25m$, at $(x_2, y_2) = (0, 4)$;
- $m_3 = 25m$, at $(x_3, y_3) = (-3, -3)$;
- $m_4 = 30m$, at $(x_4, y_4) = (-2, 4)$.

What is the moment of inertia of this system about the (a) X-axis, (b) Y-axis, and (c) Z-axis?

....

Feedback: $1305m$; $545m$; $1850m$.

Activity 10.3

On the XY-plane, a particle of mass m is situated at the point $(x, y) = (0, a)$, and particles with masses $2m$ are situated at points $(0, -a)$, $(-a, 0)$ and $(0, 0)$. What should the mass of a particle situated at point $(2a, 0)$ be, so that the moment of inertia of the whole system about the Z-axis equals $I_Z = 7a^2m$?

....

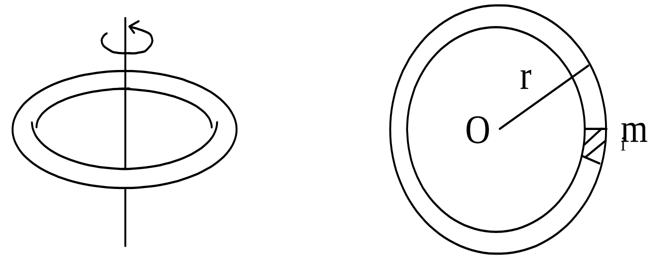
Feedback: the mass should be $m/2$.

10.3.2 The moments of inertia of rigid bodies

Once again the system of particles is only a stepping stone to what we are really interested in, namely rigid bodies with a continuous structure! In some very simple cases, the original definition of the moment of inertia for a system of particles can also be utilised for rigid bodies with a continuous structure. The next example deals with such a case.

Example 10.4
Calculate the moment of inertia of a uniform ring with mass M and radius r about an axis through its centre and perpendicular to its plane.

Solution:



For a thin ring, we can assume that all the mass of the ring is situated at the distance r from the axis. Let us assume that we divide the ring into n small pieces, each with a mass m_i. If n is quite large, then each small piece is approximately a particle at the distance r from the axis, and thus we have approximately

$$I \approx \sum_{i=1}^{n} m_i r^2 = \left(\sum_{i=1}^{n} m_i\right) r^2 = Mr^2$$

where $M = \sum_{i=1}^{n} m_i$ is the total mass. This is only an approximation, since the small pieces are not really particles. However, the end result did not depend on n, the number of pieces that we divided the ring into, and therefore we can take n as large as we like. Hence we can conclude that for a ring with radius r and mass M, we do have

$$I = Mr^2$$

for rotation about the described axis.◀

Note that although the ring in the previous example is a body with a continuous mass distribution, we could still use the original definition of I for a system of particles with great success. This was because for this particular object and for this particular axis of rotation, all the particles that the object consists of happened to be at the same distance from the axis. In most cases where the object has a continuous mass distribution this will not be the case, and different particles will be at different distances from the axis. In such a case, there are usually just too many particles (atoms) for us to be able to use equation (10.4) in practice. Fortunately, integration will again come to the rescue.

In the case of a solid object with **continuous mass distribution** we will start with the definition

$$I = \sum_{i=1}^{n} m_i r^2$$

which is valid for a system of n particles. To apply this to a body with a continuous mass structure, we shall proceed as we did when calculating the centres of mass by means of integration. We can divide the body into infinitely many very small, particle-like mass elements with mass dm, replace summation by integration and get the following result.

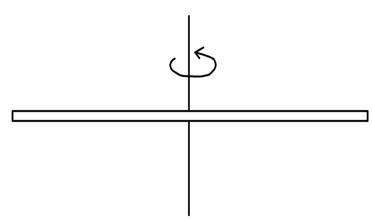
Result 10.3 (The moment of inertia calculated by means of integration)
For a rigid body with a continuous structure,

$$I = \int r^2 dm, \tag{10.6}$$

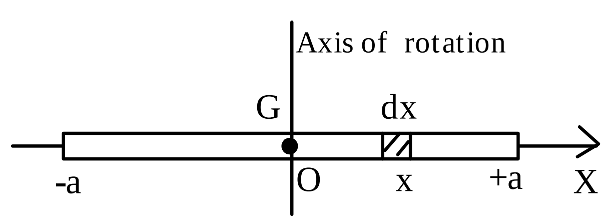
where r is the distance from the axis to a small particle-like mass element of mass dm and integration is over all the small particle-like mass elements.

We will discuss the finer points of applying this integration formula later on, in the next unit. Meanwhile, here is an example of how it could be applied!

Example 10.5
Find the moment of inertia of a uniform rod of length $2a$ and mass M about an axis perpendicular to the rod and through its centre of mass G.



Solution:
We will assume that the rod lies along the X-axis as shown, and that the Y-axis coincides with the axis of rotation.



Let ρ be the linear density of the rod. We divide the rod into small mass elements, each consisting of a small segment of the rod. The one situated at position x, $-a \leq x \leq a$, of the rod has length dx and thus mass $dm = \rho dx$, and it is at a distance $|x|$ from the axis of rotation. Thus, (10.6) gives

$$I = \int r^2 dm = \int_{-a}^{a} x^2 \rho dx = \rho \frac{x^3}{3}\Big]_{-a}^{a} = \frac{2}{3}\rho a^3.$$

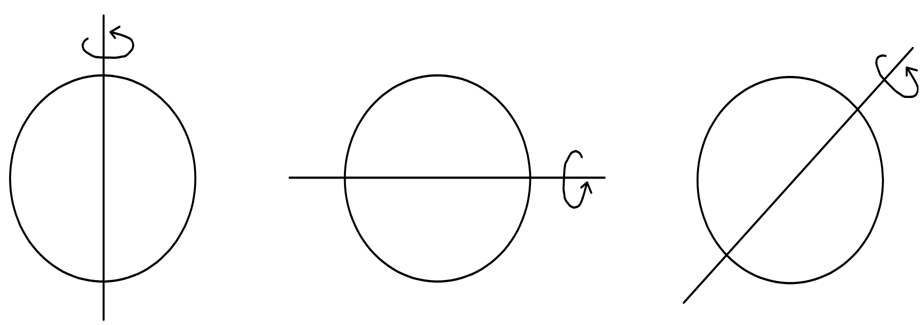
We have found an expression for I in terms of the density ρ and the length of the rod. We would rather wish to express I in terms of the mass M, since most of the time we know the mass and the length of a rod, rather than its density and length. However, we can of course convert easily from density to mass, since there is a link between M and ρ when the length of the rod is known: $M = 2\rho a$ and therefore $\rho = \frac{M}{2a}$. When we substitute this into the last equation above, we finally get

$$I = \frac{Ma^2}{3}.$$

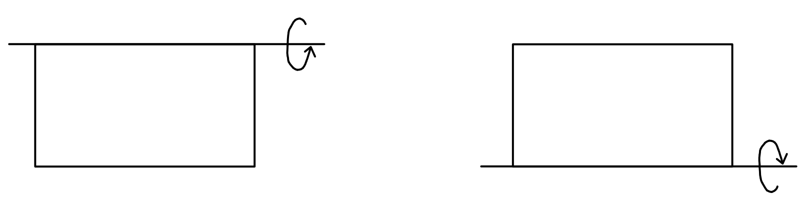
10.3.3 General simplifying rules for calculating moments of inertia

Before we consider more examples of using integration to find moments of inertia, we shall first introduce some results which will often help to shorten some of the calculations. These results include two theorems which enable us to move from one axis of rotation to another for the same body; and we will also look at how to find the moment of inertia for composite bodies. As happened with the calculation of centres of mass, this last result will lead to a more general way of finding the moments of inertia by integration! But let us start off with a very trivial, but still useful observation.

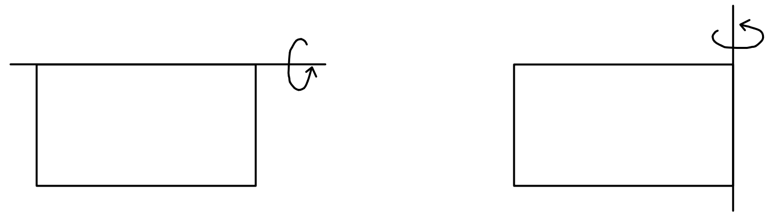
For many objects (e.g. discs, spheres and squares) it is possible to choose several axes which travel through the object in an identical way. Then, clearly, the moment of inertia is the same for all these axes. As an example, the moment of inertia of a disc is the same for all of the axes of rotation shown below:



For a rectangle, the following axes are identical:



but the following two are not:



Since the moment of inertia always depends not only on the object, but also the position of the axis of rotation in relation to the body, it should be clear that life would be a lot simpler if we had a way of moving easily from one axis to another. The parallel and perpendicular axes theorems, which we discuss next, do just this.

Result 10.4 — The Parallel Axes Theorem

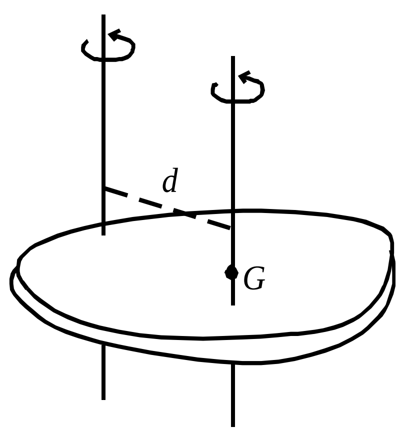
If I_G is the moment of inertia about an axis through the **centre of mass** of the body and I is the moment of inertia about any axis **parallel** to the first axis, then

$$I = I_G + Md^2 \tag{10.7}$$

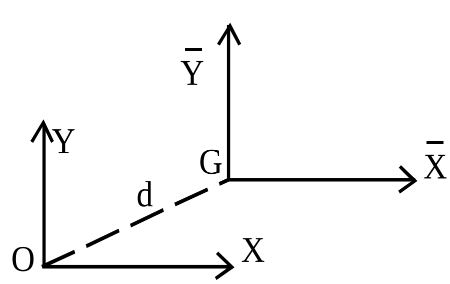
where

$M \quad = \quad$ total mass of the body

$d \quad = \quad$ distance between the two axes.



Proof: Let the centre of mass G be the origin of the $\overline{XY}$-plane and let any other point O be the origin of the XY-plane. In the sketch above, we assume that both the $\overline{XY}$- and the XY-planes are parallel to this page, with the corresponding $\overline{Z}$- and Z-axes perpendicular to the page, towards the reader.



Let I_G be the moment of inertia of the body about an axis through G parallel to the $-$axis (perpendicular to the page), while I is the moment of inertia about a parallel axis through O.

Let the coordinates of a particle of mass m_i at P be (x_i, y_i) in the XY system and let G have coordinates $(\overline{x}, \overline{y})$ in the XY system.

Suppose that the coordinates of P are $\left(x_i', y_i'\right)$ in the $\overline{XY}$-system. Then

$$\begin{aligned} x_i &= \overline{x} + x_i' \\ y_i &= \overline{y} + y_i'. \end{aligned}$$

The moment of inertia about O is

$$\begin{aligned} I &= \sum_{i=1}^{n} m_i \left(x_i^2 + y_i^2\right) \\ &= \sum_{i=1}^{n} m_i \left[\left(\overline{x} + x_i'\right)^2 + \left(\overline{y} + y_i'\right)^2\right] \\ &= \sum_{i=1}^{n} m_i \left(\overline{x}^2 + \overline{y}^2\right) + \sum_{i=1}^{n} m_i \left(x_i'^2 + y_i'^2\right) + \sum_{i=1}^{n} 2m_i \overline{x} x_i' + \sum_{i=1}^{n} 2m_i \overline{y} y_i'. \end{aligned}$$

Now consider the term

$$\sum_{i=1}^{n} 2m_i \overline{x} x_i' = 2\overline{x} \sum_{i=1}^{n} 2m_i x_i'.$$

If you refer back to equation (3.2), you will see that

$$\left(\sum_{i=1}^{n} m_i\right)\overline{x} = \sum_{i=1}^{n} m_i \left(\overline{x} + x_i'\right)$$

so that

$$\sum_{i=1}^{n} m_i x_i' = 0$$

and hence

$$2\overline{x} \sum_{i=1}^{n} m_i x_i' = 0.$$

Similarly

$$2\overline{y} \sum_{i=1}^{n} m_i y_i' = 0.$$

Since

$$\overline{x}^2 + \overline{y}^2 = d^2$$

and

$$I_G = \sum m_i \left(x_i'^2 + y_i'^2\right)$$

we have proved the theorem.

Note that the parallel axes theorem **only** applies when one of the two axes goes through the centre of mass of the object. The moment of inertia I about a second axis, parallel to the one going through the centre of mass, is now equal to the moment of inertia about the first axis, I_G, plus another quantity Md^2, where M is the mass of the object and d the distance between the axes.

Note that we have in fact decomposed the moment of inertia I into two parts, the first of which is the moment of inertia for the object rotating about its centre of mass, and the second of which is equal to the moment of inertia we would have if the object were replaced by a particle of mass M situated at the centre of mass, rotating about the given axis.

Remember also that I is found from I_G by adding a term md^2 into it. Alternatively, if I is known, then we can of course also find I_G from I by subtracting the term md^2. Make sure you understand which way around this goes, and why the axes in I and I_G are not equivalent!

The following example gives a typical application of the parallel axes theorem.

Example 10.6

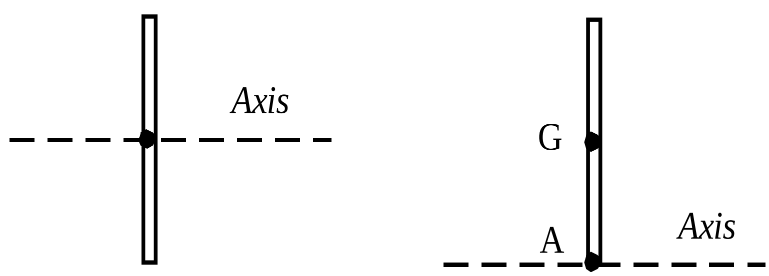
Find the moment of inertia of a uniform rod of length $2a$ and mass M about an axis perpendicular to the rod and through one of its endpoints, A.

Solution:

In Example 10.5, we have already found the moment of inertia of the rod when it rotates about any axis perpendicular to the rod and through its centre, and therefore through its centre of mass G :

$$I_G = \frac{1}{3}Ma^2.$$

The axis in this example is parallel to an axis such as the one in Example 10.5, and travels through the endpoint A which is a distance a from the centre of mass G (since for a uniform rod, the centre of mass is at the midpoint of the rod).



The parallel axes theorem tells us that the moment of inertia I_A for rotation about the described axis through the point A equals

$$I_A \quad = \quad I_G + Ma^2 \quad = \quad \frac{1}{3}Ma^2 + Ma^2 \quad = \quad \frac{4}{3}Ma^2.$$

Activity 10.4

Use the parallel axes theorem to prove that the moment of inertia of a rod of length 2ℓ and mass $3m$ about an axis perpendicular to the rod, through one end of the rod, is $4ma^2$.

Activity 10.5

A rigid body consists of two particles of mass m connected by a rod of length L and negligible mass.

(a) Find the moment of inertia of this body about an axis through its centre, perpendicular to the rod.

(b) Find the moment of inertia of this body about an axis through one end of the rod and parallel to the axis in (a),

(i) through direct calculation of the distance of the particles to the axis,

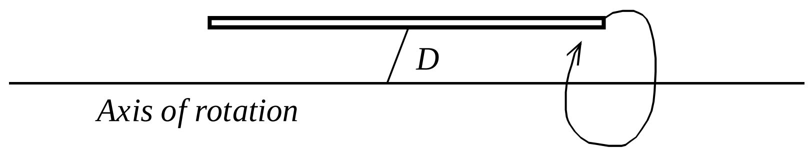
(ii) by using the result in (a) and the parallel axis theorem.

....

Feedback: (a) $\frac{1}{2}mL^2$, (b) mL^2. (Note that to apply the parallel axis theorem in (b) (ii), you must use the mass of the whole object, which is $2m$!)

Example 10.7
Find the moment of inertia of a uniform rod of length $2a$ and mass M about all possible axes which are parallel to the rod.

Solution:
We will have to find the moment of inertia when the rod rotates about an axis parallel to the rod, at a distance D from the rod, for all possible values of D ($D \geq 0$). We will first consider the case $D = 0$, and then use the parallel axes theorem to find the moment of inertia for all other values of D.



In the case $D = 0$, the rod lies on the axis of rotation. If the rod is very thin, then all the particles of the rod can be assumed to lie on the axis of rotation. But then the distance from each particle to the axis is zero, and it follows that the moment of inertia is zero.

Let us then consider the general case, where the axis of rotation is parallel to the rod and lies at a distance D from it. Let I_D denote the moment of inertia of the rod for rotation about this axis. Now, the axis which coincides with the rod goes through the centre of mass of the rod, and therefore by the parallel axes theorem, we must have

$$I_D = 0 + MD^2 = MD^2.$$

(We could alternatively have reasoned as follows: In this case, all the particles of the rod lie at a distance D from the axis of rotation, so that a particle i with mass m_i will have the moment of inertia $m_i D^2$ about the axis. Summing over all the particles gives MD^2 as the moment of inertia for the whole rod.)

Thus we have proved that the moment of inertia of the rod for rotation about any axis parallel to it is given by

$$I = MD^2$$

where D is the distance to the parallel axis. Note that the result does not depend on the length of the rod!◀

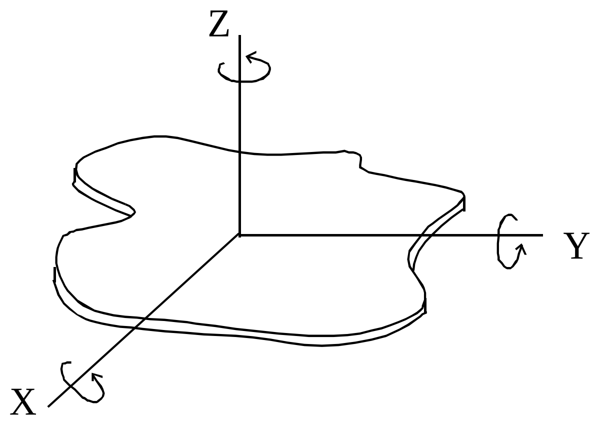
Activity 10.6

Use the parallel axis theorem to prove that the axis about which a given rigid body has its smallest moment of inertia must pass through its centre of mass. *(Hint: prove that if it does not pass through the centre of mass, then the moment of inertia about an axis which does go through the centre of mass will be smaller.)*

The parallel axes theorem enables us to start with one result of a moment of inertia I_G of an object about an axis, and to generate from that all moments of inertia for the same body about all axes parallel to the original one. Next, we will discuss the perpendicular axes theorem, which similarly links moments of inertia for rotation with respect to mutually perpendicular axes.

Consider a flat object (lamina) which lies on the XY-plane. The perpendicular axes theorem links the moments of inertia for the following three types of rotation: I_X about the

X-axis, I_Y about the Y-axis, and I_Z about the Z-axis.



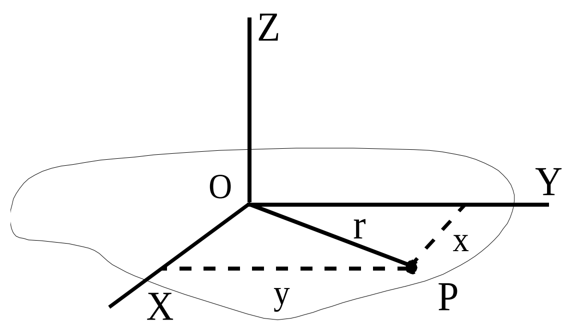
Result 10.5 — The Perpendicular Axes Theorem

Let X, Y and Z be mutually perpendicular axes of rotation. Assume that a rigid body forms a lamina which lies on the XY-plane. Then

$$I_Z = I_X + I_Y \tag{10.8}$$

where I_X, I_Y and I_Z are the moments of inertia for rotation about the X-, Y- and Z-axes respectively.

Proof of the perpendicular axis theorem:



Let a particle P of mass m_i be situated at point (x_i, y_i) on the XY-plane. Then, if r_i is the distance from this particle to the Z-axis we have

$$r_i^2 = x_i^2 + y_i^2$$

since the distance to the Z-axis is the same as the distance to the origin of the XY-plane for all particles on the XY-plane. But then, summing over all the particles to find the moment of inertia about the Z-axis, we get

$$\begin{aligned} I_Z &= \sum_{i=1}^{n} m_i r_i^2 = \sum_{i=1}^{n} m_i x_i^2 + \sum_{i=1}^{n} m_i y_i^2 \\ &= I_Y + I_X. \end{aligned}$$

Remarks:

- The axes of rotation could, for instance, be the coordinate axes of the XYZ coordinate system. Here, it does not matter whether they form a right-handed system or not!
- Note that when we say that the axes are mutually perpendicular, we assume that they all intersect at one point — as you see, this assumption is important in the proof of the perpendicular axes theorem! (With a bit of experimentation, you'll easily see that it is quite possible to have an axis parallel to the Z-axis, another one parallel to the Y-axis, and another one parallel to the X-axis, such that the three axes never intersect each other! Try it, using, for instance, pencils for the axes.)
- Remember that this result only holds if the entire object lies on the XY-plane! The rigid body must be a lamina. To see why, consider a non-flat object — if a particle has the coordinates (x, y, z) in the XYZ system, then the distance to the Z-axis is still given by r, with $r^2 = x^2 + y^2$, yet y is no longer the distance to the X-axis, but rather the distance to the XZ-plane, and similarly x is no longer the distance to the Y-axis.

As a first application of the perpendicular axes theorem, we shall re-calculate I_Z in Example 10.3, with four particles situated on the XY-plane. By finding the distance of each particle to the X-axis, we found that $I_X = 80Ma^2$; by finding the distance of each particle to the Y-axis, we found that $I_Y = 116Ma^2$; and finally by finding the distance of each particle to the origin of the XY-plane, we found that $I_Z = 196Ma^2$, as this gives the distance to the Z-axis. But the perpendicular axes theorem applies here, so as soon as we have found I_X and I_Y, we can calculate I_Z directly from

$$I_Z = I_X + I_Y = 80Ma^2 + 116Ma^2 = 196Ma^2.$$

Example 10.8

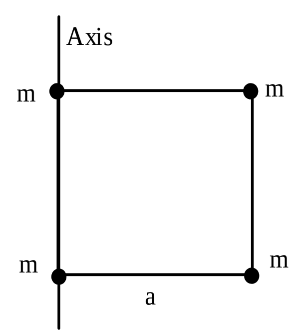
Four particles of mass m are attached to the corners of a thin massless square with sides of length a. Find the moment of inertia for rotation,

(a) about an axis along one of the sides of the square,

(b) about an axis through the middle of the square, perpendicular to it.

Solution:

(a)

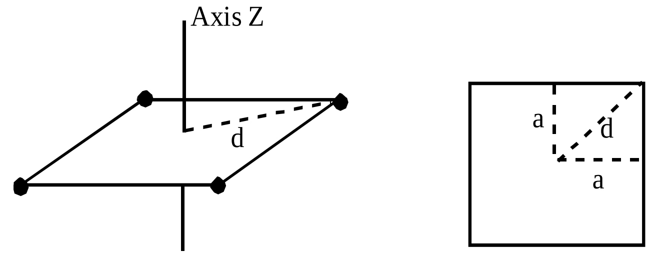


The moment of inertia about the axis A described in this question is

$$I_A = m\,(0)^2 + m\,(0)^2 + m\,(a)^2 = 2ma^2.$$

(The moment of inertia of each particle is calculated from md^2, where m is the mass of the particle and d is the distance from the particle to the axis of rotation.).

(b)



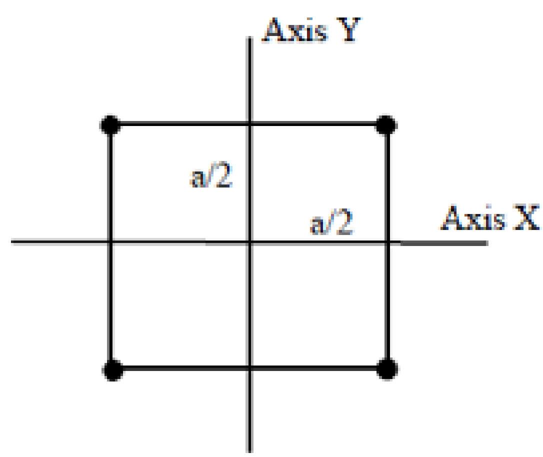
Since the distance from each corner to the middle equals

$$d = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2},$$

we get for the moment of inertia for rotation about the Z-axis

$$\begin{aligned} I_Z &= 4 \times m\,(d)^2 = 4m\left(\left(\frac{a}{2}\right)^2 + \left(\frac{a}{2}\right)^2\right) \\ &= 2ma^2. \end{aligned}$$

Alternatively, we could calculate I_Z as follows: Introduce the two axes of rotation X and Y, which are on the plane of the square, through the centre of the square and parallel to its edges, as shown below. The Z-axis is then perpendicular to the X- and Y-axes.



Then the moments of inertia I_X and I_Y are

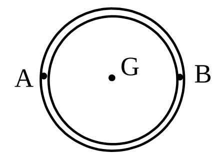
$$I_X = I_Y = 4 \times m\left(\frac{a}{2}\right)^2 = ma^2$$

and by applying the perpendicular axes theorem, $I_Z = I_X + I_Y = 2ma^2$. ◄

The perpendicular axes theorem can also used in the other direction, to find out about I_X and I_Y when I_Z is known. The following example is a typical application.

Example 10.9

The figure below shows a ring with radius R and mass M.



Point G is the midpoint of the ring, A and B are points on its rim and AGB is a diameter of the ring. Then the moment of inertia of the disc for rotation about an axis perpendicular to the disc, through G, is $I_G = MR^2$. Find

(a) I_A, the moment of inertia for rotation about an axis perpendicular to the ring and through point A.

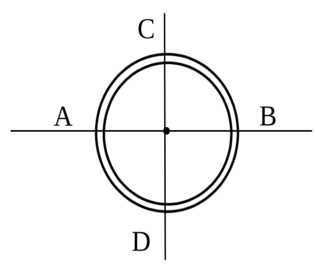
(b) I_{AB}, the moment of inertia for rotation about the diameter AB.

Solution:

(a) The two axes of rotation in I_G and I_A are parallel, with a distance R between them. Therefore, according to the parallel axes theorem,

$$I_A = I_G + MR^2 = 2MR^2.$$

(b) Let CD be another diameter of the ring, which is perpendicular to the diameter AB, as shown below.



Then, according to the perpendicular axes theorem,

$$I_G = I_{AB} + I_{CD}$$

where I_{AB} and I_{CD} are the moments of inertia for rotation about the diameters AB and CD, respectively. But clearly the rotations about these diameters are identical, so that $I_{AB} = I_{CD}$, and therefore in fact

$$\begin{aligned} I_G &= I_{AB} + I_{AB} = 2I_{AB} \\ I_{AB} &= \frac{1}{2} I_G = \frac{1}{2}\left(MR^2\right) \\ I_{AB} &= \frac{1}{2} MR^2. \end{aligned}$$

Activity 10.7

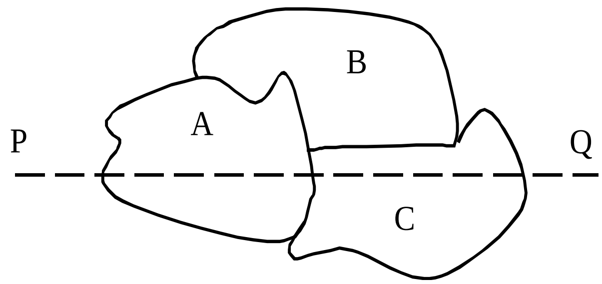
The moment of inertia of a disc with mass M and radius R, when it rotates about an axis through the centre of the disc, perpendicular to the disc, is $I = \frac{1}{2}MR^2$. Use this result and the perpendicular axis theorem, to find the moment of inertia of the disc when it rotates about its diameter.

....

Feedback: You should get $I = \frac{1}{4}MR^2$

Composite bodies

If a rigid body consists of two or more bodies joined together, then it is called a composite body. The moment of inertia of such an object about an axis can be found by simply adding up the moments of inertia of the separate parts that make up the object.



Result 10.6 — The moment of inertia of a composite body

If a rigid body is made up of separate parts $A, B, C, \ldots$ then the moment of inertia of the whole body about an axis PQ is

$$I = I_A + I_B + I_C + ...$$

where I_A, I_B, $I_C \ldots$ are the moments of inertia of $A, B, C \ldots$ about the axis PQ.

Proof:

Let us assume that the total number of particles in the body is n. Then the moment of inertia of the whole body is given by

$$I = \sum_{i=1}^{n} m_i r_i^2.$$

where m_i is the mass of particle i and r_i the distance from that particle to the axis of rotation. Now, each particle i will belong to one and only one of the objects $A, B, C, \ldots,$ and thus the sum can be re-arranged so that the particles of each body are summed up together:

$$\sum_{i=1}^{n} m_i r_i^2 = \sum_{particles\ i\ in\ A} m_i r_i^2 + \sum_{particles\ j\ in\ B} m_j r_j^2 + \cdots,$$

but the right-hand side of this is then just the sum $I_A + I_B + \ldots$

Note that to apply the rule for composite bodies, we may have to calculate the moments of inertia about an axis for rigid bodies where the axis does not actually go through the body itself! See, for instance, Learning Unit B in the sketch above. For the concept of the moment of inertia to make sense, we must assume that the body is rigidly attached to the axis. If the axis is outside the body, we can imagine that the body and the axis are connected by massless rigid rods.

Example 10.10

Find the moment of inertia of a body formed from a ring A of mass M, radius a and a particle P of mass m attached to a point on the circumference of the ring, for rotation about an axis through the centre of the ring and perpendicular to the plane of the ring.

Solution:
The moment of inertia of the ring about the given axis is Ma^2. The particle is attached to the ring at its circumference, so that it lies the distance a from the axis. Therefore the moment of inertia of the particle about the axis is ma^2. (See equation (10.5). According to the rule for compound bodies, the moment of inertia of the entire body is

$$I = I_A + I_P = Ma^2 + ma^2.$$

Activity 10.8

Find the following moments of inertia:

(a) A rod AB of length $2a$ and mass M, rotating about an axis perpendicular to the rod, through end point A.

(b) A rod AB of length $2a$ and mass M, with a particle of mass M attached at its centre and a particle of mass $2M$ attached at end B, rotating about an axis perpendicular to the rod, through end point A. (hint: $I_A = I_A^{\text{rod}} + I_A^{\text{particle 1}} + I_A^{\text{particle 2}}$ where for the particles you must use $I = mr^2$!

(c) A rod AB of length $2a$ and and negligible mass, with a particle of mass M attached at its centre and a particle of mass $2M$ attached at end B, rotating about an axis perpendicular to the rod, through end point A. (Hint: here the mass of the rod is zero!)

....

Feedback: (a) $\frac{4}{3}Ma^2$, (b) $\frac{4}{3}Ma^2 + Ma^2 + (2M)\,(2a)^2 = 10\frac{1}{3}Ma^2$, (c) $9Ma^2$.

Warning: Integration cannot handle particle masses! This is why in the previous activity, in point (b) you could not just pretend that the object was a rod with mass $M+M+2M$ — this would not give you the same result! For a particles we must use the expression md^2. Integration can only be used for bodies with a continuous structure.

Activity 10.9

A rod AB of length 2ℓ and mass M has a particle of mass $2M$ attached at point B and a particle of mass $4M$ attached at point A. Find the moment of inertia for rotation about an axis perpendicular to the rod, (a) through its centre, (b) through point A.

....

Feedback: (a) $\frac{1}{3}M\ell^2 + (2M)\,\ell^2 + (4M)\,\ell^2 = \frac{19}{3}M\ell^2$, (b) $\frac{4}{3}M\ell^2 + (2M)\,(2\ell)^2 + (4M)\,0^2 = \frac{28}{3}M\ell^2$.

Remark: In the activity above, you might have thought of using the parallel axis theorem to answer (b) by using the result on (a) and adding a "Md^2" term to it. However, please note that the answer in (a) is NOT the moment of inertia of the entire object (rod plus particles) for an axis of rotation which goes through its centre of mass: the centre of the rod is the centre of mass for the rod, true, but not for the entire object.

Indeed if you do want to use the parallel axis theorem $I = I_G + Md^2$ for composite bodies, you must make sure of the following points:

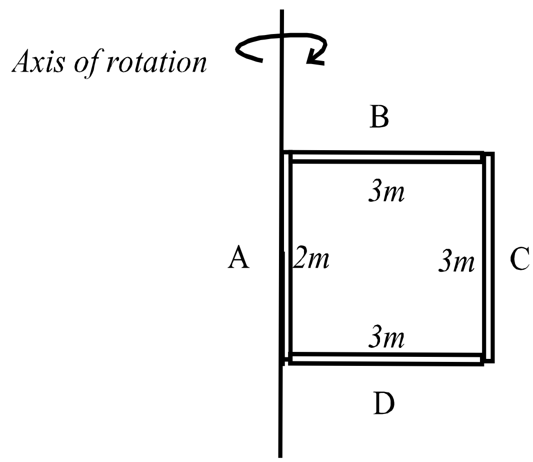
- That the value you use as I_G is indeed for an axis which goes through the centre of mass of the entire object
- That you use as the value of mass, M, the mass of the entire object.

Example 10.11

Four thin, uniform rods with masses $2m$, $3m$, $3m$ and $3m$ all of the same length a are joined together to form the four edges of a square. What is the moment of inertial when the object rotates about an axis which travels along the lightest rod?

Solution

A sketch of the object, and of the axis of rotation, looks as follows.



We have labelled the rods A (with mass $2m$), B, C and D (with masses $3m$). The moment of inertia of the entire object is the sum of the moments of inertia of the rods, for rotation around the given axis:

$$I = I_A + I_B + I_C + I_D.$$

Two of the rods (B and D) are perpendicular to the axis of rotation, and two of them (A and C) are parallel to the axis of rotation (A goes along the axis of rotation, and C the distance a away from it). For rods B and D, since the axis of rotation is perpendicular to the rods and goes through their endpoints, we can use the result in Example 10.6 which deals with just such a case. In that result, we found that for a rod with mass M and length $2a$, the moment of inertia when the rod rotates about an axis perpendicular to the rod, through its end point is

$$I = \frac{4}{3} Ma^2.$$

Here of course the length of the rods is a instead of $2a$, and the mass for both B and D is $3m$, so we get

$$I_B = I_D = \frac{4}{3}(3m)\left(\frac{a}{2}\right)^2 = ma^2.$$

You should really also be able to derive this directly from the moment of inertia of a rod rotating about a perpendicular axis through its midpoint, using the parallel axis theorem!

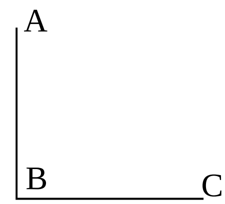
For rods A and C, we can reason as in Example 10.7: Rod A lies along the axis of rotation so its moment of inertia is zero: $I_A = 0$; and Rod C lies at the distance a from the rod, so its moment of inertia is $I_C = (3m)\, a^2$.

Combining all these calculations, we find that the moment of inertia of the whole object is

$$\begin{aligned} I &= I_A + I_B + I_C + I_D \\ &= 0 + ma^2 + 3ma^2 + ma^2 \\ &= 5ma^2. \end{aligned}$$

Activity 10.10

Two identical rods, both with mass m and length $2a$ are joined together at right angles to form an L-shaped object ABC, as shown below.



Find the moment of inertia of the object for the following axes of rotation:

(a) An axis along rod AB.

(b) An axis perpendicular to both rods, through point B.

(c) An axis perpendicular to both rods, through point A.

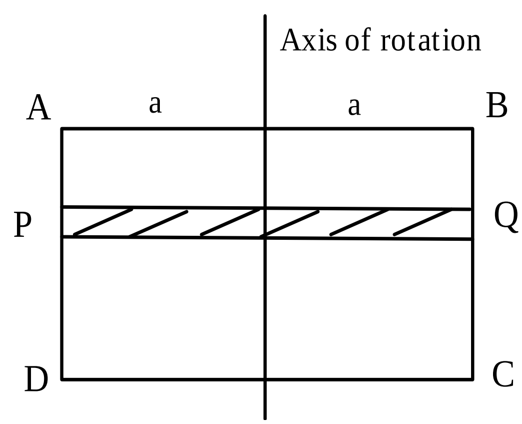
....

Feedback: (a) $\frac{4}{3}ma^2$, (b) $\frac{8}{3}ma^2$, (c) $\frac{20}{3}ma^2$.

In the following example, we apply the rule for finding moments of inertia for composite bodies, to find the moment of inertia of a rectangle from the moments of inertia of rods, which we already know.

Example 10.12
Find the moment of inertia of a rectangular, uniform lamina $ABCD$ with sides AB of length $2a$ and BC of length $2b$, about an axis passing through the midpoints of AB and CD.

Solution:



If the lamina is divided into n thin strips of length $2a$ parallel to AB, then for n large enough, each strip is approximately a rod of length $2a$ and mass m_i. As the thin strips do not overlap, their masses add up to the mass of the entire lamina:

$$M = \sum_{i=1}^{n} m_i .$$

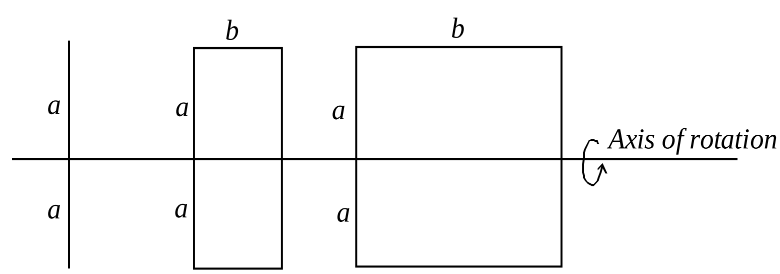
From Example 10.5 we know that the moment of inertia of rod PQ is $m_i \frac{a^2}{3}$. For the whole lamina we sum over all the thin strips and find that

$$\begin{aligned} I &= \sum_{i=1}^{n} m_i \frac{a^2}{3} = \frac{a^2}{3}\left(\sum_{i=1}^{n} m_i\right) \\ &= \frac{Ma^2}{3}. \end{aligned}$$

Remark: compare this with the way we found the moment of inertia for the ring! By slicing as we do, we are of course approaching integration, and indeed we will show you later on how to do this by applying a slicing and integrating technique. Remember that in slicing and integration, we need to go to the limit: we will divide the object into infinitely small, infinitely thing objects. Here we just slice it into very many objects. Things will work out since the moment of inertia of each thin slice only depends on its mass and length, and not really on its position or its width. ◄

Remark: Note that, according to this result, the moment of inertia of the rectangular lamina does not depend on its length along the axis of rotation (b in the rectangle above), but only on its width perpendicular to the axis (the length a, here). Accordingly, provided that the following objects all have the same mass M, they all have the same moment of inertia about the shown axis, namely

$$I_X = \frac{1}{3}Ma^2 :$$



Rod of length 2a (b=0)

Rectangles of length 2a, with varying widths b

Activity 10.11

Here you need to use both parallel and perpendicular axis theorems! Include in your answer sketches of each of the four situations, with the axis of rotation indicated in the sketch.

Consider a rectangular lamina $ABCD$ with $AB = 2a$ and $BC = 2b$. If the mass of the lamina is M, find the moment of inertia when the lamina rotates about the following axes:

(a) an axis through the midpoints of sides AB and CD,

(b) an axis along the side BC,

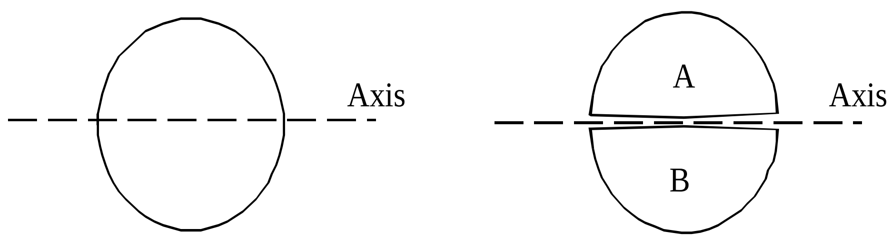
(c) an axis perpendicular to the lamina and through its centre,

(d) an axis perpendicular to the lamina, through corner A.

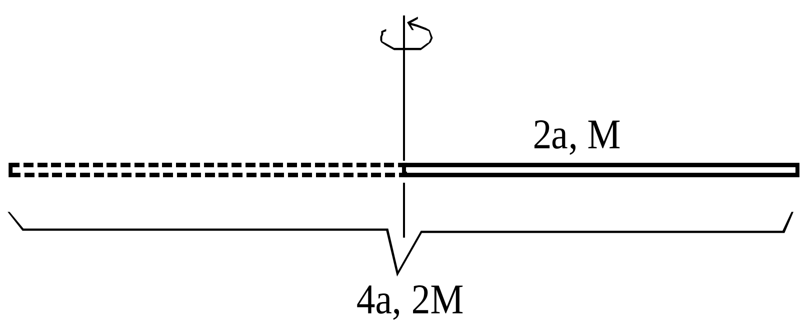
....

Feedback: (a) $\frac{1}{3}Ma^2$, (b) $\frac{4}{3}Ma^2$, (c) $\frac{1}{3}M\left(a^2+b^2\right)$, (d) $\frac{4}{3}M\left(a^2+b^2\right)$.

The rule for the moment of inertia of composite bodies leads to yet another useful general result. Let us assume that the object is a lamina and has the moment of inertia I about a given axis. If the axis of rotation is an axis of symmetry, then the axis of rotation divides the object into two parts. The moment of inertia of each of these parts is now equal to $\frac{1}{2}I$. This follows, because the two parts are each other's mirror images, and hence must have the same moment of inertia about the axis while, on the other hand, their moments of inertia must sum up to I.



Example 10.13
As an example, we will re-calculate the moment of inertia I of a rod of length $2a$ and mass M when it rotates about an axis perpendicular to the rod, through one of its endpoints, by using Example 10.5 and the result above. The rod of length $2a$ and mass M, rotating about the axis described above, can be imagined to form one of two identical halves of a rod of length $4a$ and mass $2M$ rotating about an axis through its middle.



But by Example 10.5 we know what the moment of inertia of the rod of length $4a$ as it rotates about the given axis is: It has a mass of $2M$ and a length of $2(2a)$, and therefore

$$I_{long\ rod} = \frac{1}{3}(2M)(2a)^2 = \frac{8}{3}Ma^2.$$

But dividing the long rod into half along the axis of rotation gives us two identical copies of our rod, and hence

$$I_{long\ rod} = 2I$$

and accordingly,

$$I = \frac{1}{2}I_{long\ rod} = \frac{1}{2}\cdot\frac{8}{3}Ma^2 = \frac{4}{3}Ma^2.$$

Activity 10.12

Find the moment of inertia of a thin, uniform rod with mass M, bent into the shape of a semicircle of radius R, rotating about its diameter.

....

Feedback: $I = \frac{1}{2}MR^2$. *(Hint: This object is obviously half of a ring rotating about its diameter – but what is the mass of the whole ring?)*

CONCLUSION

In this unit you have learned:

- what is meant by the pure rotation of a rigid body, and how it can be fully described by the angle of rotation
- the equation of motion describing pure rotation, and how it can be applied
- why moments of inertia are important in rotation
- how to calculate moments of inertia for systems of particles, and (at least in principle) for rigid bodies
- how to apply symmetry, identical axes, the perpendicular and parallel axes theorems and the rule for composite bodies to find moments of inertia

Remember to add the following tools to your toolbox:

- the principle that moments of forces cause angular acceleration in the pure rotation of a rigid body
- using integration to find the moment of inertia of a rigid body
- simplifying calculations by using symmetries and identical axes of rotation
- the definition of the moment of inertia
- the equation of motion for a rigid body in pure rotation
- the parallel and perpendicular axes theorems
- the rule for finding the moment of inertia for compound bodies

Unit 11 MORE CALCULATIONS OF MOMENTS OF INERTIA

Key questions:

- *How do we find the moments of inertia of more complicated bodies, such as a cylinder?*

In this unit, we will find out how to calculate moments of inertia for more complicated rigid bodies and systems. The first idea is that of "slicing and integrating", similar to the idea we have already used to find centres of mass in Learning Unit 2. Later, we will look at more general cases, and will introduce a toolbox which should cover almost all cases we come across in this module!

Contents of this unit:

11.1 Calculating moments of inertia by slicing and integrating

11.2 More complicated examples of moments of inertia

What you are expected know before working through this unit:

The results derived here are all based on the more basic results in Unit 10. In addition, you will here and there need to refer to centres of mass of objects.

11.1 Calculating moments of inertia by slicing and integrating

As in the calculation of centres of mass, we can use the rule for composite bodies to justify integrating by dividing the body into any small mass elements, not just particle-like ones. More specifically, the moment of inertia can often be calculated by dividing the object into thin slices (small mass elements), for which the moment of inertia ΔI_i is known or can be calculated easily. The rule for compound bodies tells us that the total moment of inertia is the sum of the moments of inertia of the components,

$$I = \sum \Delta I_i$$

even if the small parts are not particle-like. As the masses of the components get smaller and smaller, and their number larger and larger, summation is replaced by integration, leading to

$$I = \int dI$$

where dI denotes the moment of inertia of one of the small mass elements.

Result 11.1 (Calculating the moment of inertia by slicing and integrating)
For a rigid body with continuous structure,

$$I = \int dI$$

where dI is the moment of inertia of a small mass element, and integration is over all the disjoint mass elements that the object is composed of.

To apply this process of slicing and integrating, we need to have a way of referring to each mass component such that we can integrate over all of them, and we must be able to find the moment of inertia of all of them. As in the case of calculating centres of mass, we will often slice the object into thin components ("slices"), whose moments of inertia about the given axis we can find more easily than if we were to calculate the moment of inertia of the entire body directly. For instance, if each small mass element is particle-like, that is, very small, then we may assume that it is approximately a particle with a moment of inertia given by the formula $I = mr^2$. In this case, the calculation of the moments of inertia for each of the smaller elements is easy if the distance from them to the axis is easy to find. Alternatively, we might select the small mass elements in such a manner that we can apply some already established result to find their moments of inertia about the given axis (for instance, at the moment we already know how to find the moments of inertia of simpler objects such as rods or rings).

TOOLBOX FOR FINDING MOMENTS OF INERTIA BY SLICING AND INTEGRATING

1. Understanding the problem: Make sure you know what the body is like, and where the axis of rotation lies! Draw a sketch of the body and the axis.

2. Check whether you can apply any of the simplifying tricks:
   - symmetry, identical axes
   - interpreting the object as a composite body – in which case you should proceed to slice and integrate the components first

3. Decide on the best way to slice the object. Remember that you wish the slices to be such that you can find their moments of inertia dI easily! Also make sure that you know what kind of an object you get when slicing!

4. Select a coordinate system. The moment of inertia is an absolute quantity, which does not need to be referred to in terms of a specified coordinate system. However, if we are going to apply integration to find it, we shall have to use an integration variable, which means that we will need at least one coordinate axis! If the slicing is done perpendicularly to the axis of rotation, then sometimes we can take that axis as one of our coordinate axes, for example the X-axis, and integration will then be over the variables x which denote the position on the X-axis of each slice. In general, the choice of the coordinate system is closely linked to the decision of how you will "slice" the object! Add the coordinate system to your sketch.

5. Identify your integration variable. Find the value of the moment of inertia dI for all the small mass elements, in terms of the integration variable. This will usually involve the mass of the small mass element, which may be found by applying the concept of density and the volume, area or length of the small element!. Identify the upper and lower limits of integration.

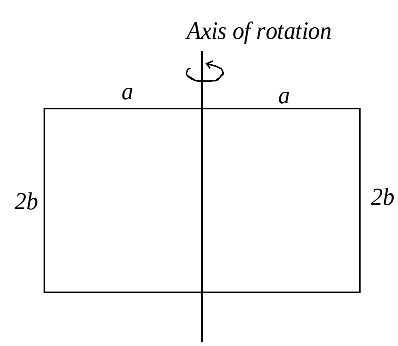
6. Evaluate the integral. The end result may be in terms of the density ρ, in which case we also have to apply the link between the total mass M, the density ρ and the dimensions of the body, to express the result in terms of M instead.

7. Check the solution.

As a first simple example, we will re-calculate the moment of inertia for the rectangular lamina in Example **10.12** by straightforward slicing-and-integrating.

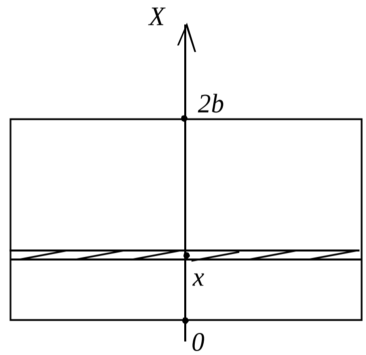
Example 11.1
Example 10.12 revisited — finding the solution using integration
Find the moment of inertia of a rectangular, uniform lamina $ABCD$ with sides AB of length $2a$ and BC of length $2b$, about an axis passing through the midpoints of AB and CD.

Solution:
The rigid body and the axis of rotation are shown below.



The simplest way to slice a rectangle is into thin strips, and we already know how to find I for a rod rotating about an axis perpendicular to the rod, through its midpoint. Therefore, the natural choice here is to divide the rectangle into thin strips, perpendicular to the axis of rotation.

We will take the X-axis to go along the axis of rotation. We slice the lamina into thin strips perpendicular to the X-axis, so that the strip which lies at position x of the X-axis has a width dx and length $2a$.



The mass of such a strip is then

$$\begin{aligned} dm &= \rho \times \mathrm{width} \times \mathrm{length} \\ &= 2a\rho dx. \end{aligned}$$

The strip is approximately a rod of length $2a$ and mass dm, and therefore, as in Example 10.5, it has the moment of inertia

$$dI = \frac{1}{3}(dm)\, a^2$$

for rotation about the given axis (which, of course, is perpendicular to each strip and through its centre of mass). To integrate over all the strips, we need to integrate from $x = 0$ to $x = b$. Thus, the moment of inertia of the whole lamina is

$$\begin{aligned} I &= \int dI = \int \frac{1}{3}(dm)\, a^2 = \int_0^{2b} \frac{2}{3} a^3 \rho dx. \\ &= \frac{2}{3} a^3 \rho \int_0^{2b} dx = \frac{2}{3} a^3 \rho \cdot (2b) \\ &= \frac{4}{3} a^3 b \rho. \end{aligned} \tag{11.1}$$

We would like to express the moment of inertia in terms of the total mass of the body, rather than in terms of the density. To this end, we note that if the mass of the entire body is M, then the mass and density are linked by

$$M = \rho\,(2a)\,(2b) \quad \therefore \quad \rho = \frac{M}{4ab.}$$

If we apply this in (11.1), we finally get the result that

$$I = \frac{1}{3} M a^2.$$

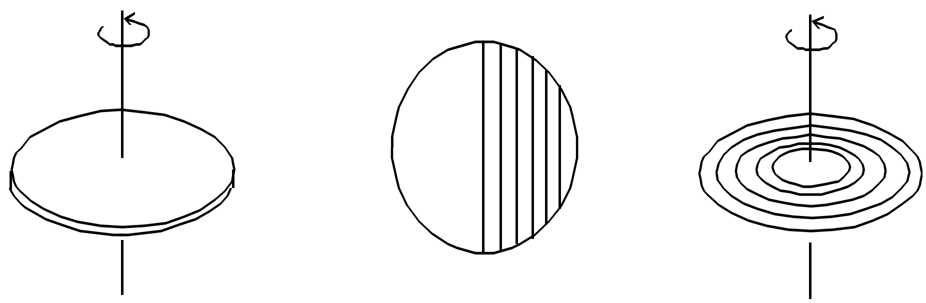
The following example uses the already know moment of inertia for a ring, to find the moment of inertia for a disc.

Example 11.2

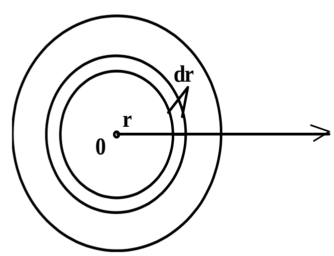
Find the moment of inertia I of a uniform disc of mass M and with radius a about an axis perpendicular to the disc, through its centre.

Solution:

The disc and the axis of rotation are sketched below, on the left. How should be slice the disc into small mass elements? Slicing it into thin strips parallel to the plane of the disc, as suggested in the sketch in the middle, is clearly not going to be very helpful – for one thing, the lengths of the thin slices will be very difficult to find, and besides, even if we assume the slices to be approximately rods, we have no easy way to find the moments of inertia of these rods about the axis indicated! On the other hand, we do know how to find the moment of inertia of a ring about just this kind of an axis, and therefore we will divide the disc into thin concentric strips, approximately rings, as shown on the right. Each of the rings is uniquely defined by its radius, and therefore we can use the radius as the integration variable.



Consider the disc to be divided into concentric strips, which will be approximately rings. One such ring with radius r and thickness dr is shown below.



If ρ is the area density, then the mass of the ring is

$$\begin{aligned} dm &= \text{density} \times \text{area} \\ &= \text{density} \times \text{circumference} \times \text{width} \\ &= \rho 2\pi r dr, \end{aligned}$$

so that the moment of inertia of each ring about the given axis is, according to Example 10.4,

$$\begin{aligned} dI_r &= \text{mass} \times \text{radius}^2 \\ &= \rho 2\pi r dr \times r^2 = 2\pi r^3 \rho dr. \end{aligned}$$

For the disc, we integrate over all the rings which make up the disc, which means integrating from $r = 0$ to $r = a$.

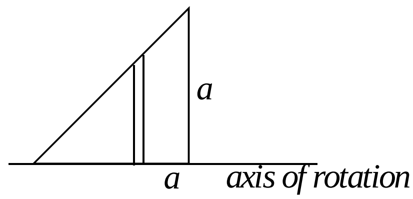
$$\begin{aligned} I &= \int dI_r = \int_0^a 2\pi \rho r^3 dr \\ &= 2\pi\rho \int_0^a r^3 dr = 2\pi\, p \frac{r^4}{4}\Big]_0^a = \frac{\pi\rho a^4}{2}. \end{aligned}$$

Now $\rho = M/\pi a^2$, so that $M = \rho\pi a^2$ and hence

$$I = \frac{1}{2} M a^2.$$

Activity 11.1

Use slicing and integrating to find the moment of inertia of a triangular lamina with sides a, a and $a\sqrt{2}$, and mass M, when it rotates about an axis along one of the sides of length a.



Hint: slice the triangle into thin vertical slices which are approximately rods, as shown.

....

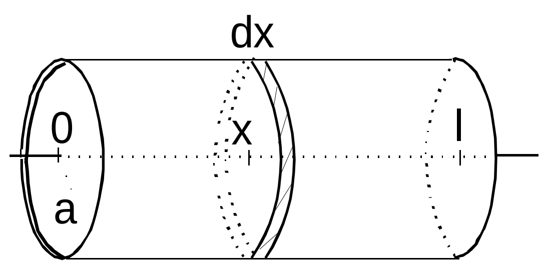
Feedback: $I = \int_0^a \frac{4}{3}\left(\frac{x}{2}\right)^2 \rho x dx = \frac{1}{12}a^4\rho$ where $M = \rho\frac{1}{2}a^2$; therefore we get $I = \frac{1}{6}Ma^2$.

Example 11.3
Show that the moment of inertia of a hollow cylinder of mass M and with radius a about its axis is Ma^2.

Solution:
A hollow cylinder could be a piece of a pipe, or a can (with ends cut off). Note that by the axis of a cylinder, we mean the axis which goes through the middle of the cylinder, along its length. There is no mention here about the length of the cylinder, and indeed, it will turn out that the moment of inertia does not depend on the length of the cylinder. We will assume that the length is l.

We will prove the result here in three different ways, to illustrate the possible approaches.

Method 1 (Assuming that we know the moment of inertia of a uniform ring about an axis through its centre.) We can slice the cylinder perpendicularly to the axis of rotation, in which case we get thin bands, as shown. Each band is approximately a ring, and we do certainly know how to find its moment of inertia about the axis shown. We will assume that the axis of rotation coincides with the X-axis, and that the cylinder is situated as shown. Then each slice is uniquely determined by its position on the X-axis, so that x is the integrating variable, ranging from 0 to l.



If one of the thin slices is situated at position x, $0 \leq x \leq \ell$ along the axis, then it has the width dx and if ρ denotes the surface density of the cylinder, then the mass of the thin band is

$$\begin{aligned} dm &= \rho \times \text{area} \\ &= \rho \times \text{width} \times \text{circumference} \\ &= \rho \cdot dx \cdot 2\pi a. \end{aligned}$$

But each of these bands is approximately a ring with radius a, so its moment of inertia about the given axis is

$$\begin{aligned} dI &= \text{mass} \times \text{radius}^2 \\ &= (dm)a^2 = 2\pi\rho a^3 \cdot dx. \end{aligned}$$

The moment of inertia of the whole cylinder is obtained by integrating over all the bands:

$$I = \int dI = \int_0^\ell 2\pi\rho a^3 dx = 2\pi\rho a^3\ell.$$

But the mass of the whole cylinder is

$$\begin{aligned} M &= \rho \times \text{length} \times \text{circumference} \\ &= \rho\ell 2\pi a. \end{aligned}$$

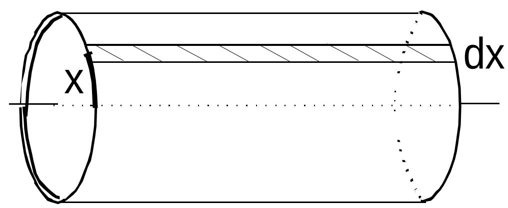
This gives

$$\rho = \frac{M}{2\pi a\ell},$$

so that

$$I = Ma^2.$$

Method 2 (Assuming we know the moment of inertia of a rod, about an axis parallel to it.) This time, let's divide the cylinder into thin strips, parallel to the axis.



Then each strip is approximately a rod, parallel to the axis of rotation, at a distance a from the axis. Again, we already know how to find the moments of inertia of the rods! But how do we refer to the thin strips? Each of them is uniquely determined by its position along the circumference of the cylinder, so we can use as the integrating variable a value x, measured around the circumference from some point of reference. The values of x range from 0 to $2\pi a$. (Note that the exact position of the strip along the circumference is not really important, but we do need to make sure that we use the correct upper and lower limits for the integration!) Now, if one of the strips is at position x, measured along the circumference of the cylinder, and if it has a width of dx, then the mass of this strip is

$$dm = \rho \cdot \text{area} = \rho \times \text{width} \times \text{length} = \rho \cdot dx \cdot \ell.$$

Each strip is approximately a rod, parallel to the axis of the cylinder and at a distance a from it, so its moment of inertia is given by

$$dI = (dm)a^2 = \rho\ell a^2 dx.$$

For the whole cylinder,

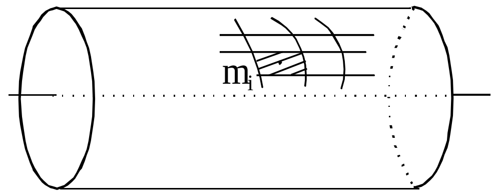
$$I = \int dI = \int_0^{2\pi a} \rho\ell a^2 dx = 2\pi\rho a^3\ell = Ma^2.$$

Method 3 (Compare this with Example 10.4) In fact, if we divide the surface of the cylinder into *any* kind of small (particle-like) elements, the ith one with mass m_i, then each of them is in fact situated at the same distance, a, from the axis. So, each element has a moment of inertia given by

$$I_i = m_i a^2$$

and for the whole object,

$$\begin{aligned} I &= \sum_{i=1}^{n} I_i = \sum_{i=1}^{n} m_i a^2 \\ &= \left[\sum_{i=1}^{n} m_i\right] a^2 = Ma^2. \end{aligned}$$



What about a solid cylinder then? You'll have a chance to deal that yourself, in the next activity!

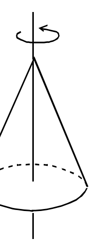
Activity 11.2

Show that the moment of inertia of a solid cylinder of mass M and radius a about its axis is $\frac{1}{2}Ma^2$.

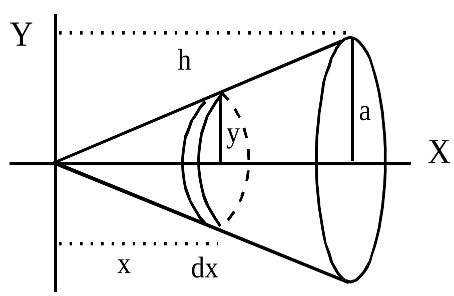
Hint: you must slice the cylinder into objects for which you can find the moment of inertia. One way is to slice it perpendicularly to the axis, which will give you thin discs; another one is to slice it similarly to when we sliced the disc into rings, in Example 11.2.

Example 11.4
Find the moment of inertia of a uniform, solid cone with radius a, height h and mass M, about its axis.

Solution
The cone and the axis of rotation look like this:



Again the most sensible way to slice the object would be perpendicular to the axis of rotation. The slices will then be thin discs, and we know how to find their moment of inertia about the given axis. We will assume that the X-axis coincides with the axis of rotation, so that integration is over the x-variable.



Assume that the cone is divided into cylindrical slices each with a width of dx, and each approximately a disc. Consider a disc at position x on the X-axis. If the radius of this disc is y, then its volume is $\pi y^2 dx$ (volume = area × thickness = π(radius)^2 × thickness).

If ρ is the density of the cone, then the mass of this disc is

$$dm = \pi \rho y^2 dx.$$

The moment of inertia of the disc about the X-axis is

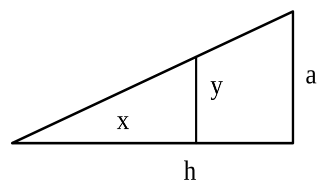
$$dI = \frac{1}{2}\,(\text{mass})\,\text{radius}^2 = \frac{1}{2}\left(\pi y^2 \rho dx\right) y^2$$

(according to Example 11.2). Hence, for the whole cone we have

$$I = \int dI = \frac{\pi}{2}\rho \int_0^h y^4 dx.$$

We still have to calculate y in terms of x. From similar triangles we have that

$$y = \frac{ax}{h}.$$



Therefore

$$\begin{aligned} I &= \frac{\pi}{2}\rho \int_0^h \left(\frac{ax}{h}\right)^4 dx \\ &= \frac{\pi \rho a^4 h}{10} \\ &= \frac{3Ma^2}{10} \quad \text{since} \quad \rho = \frac{3M}{\pi a^2 h}. \end{aligned}$$

The moment of inertia is

$$I = \frac{3}{10} M a^2.$$

Activity 11.3

Find the moment of inertia of a pyramid with mass M, height h and a base which is a $2a \times 2a$ square, when it rotates about its axis of symmetry (that is, the line from the centre of the base to the apex of the pyramid).

....

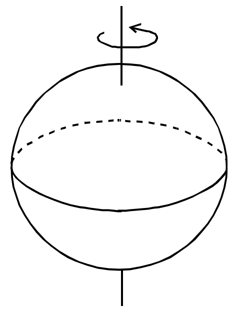
Feedback: $I = \frac{1}{5}Ma^2$.

The following two examples involve much more complicated integration. They are only included since we wish to derive all the moments of inertia we use in this module; however, you will not be expected to reproduce the proofs on your own, or indeed to derive any moments of inertia which involve integration at this level!

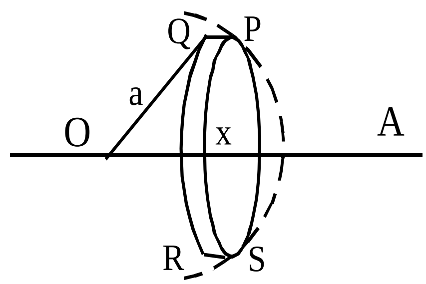
Example 11.5

Find the moment of inertia of a solid, uniform sphere with mass M and with radius a about its diameter.

Solution:
The object and the axis of rotation look like this:



Assume that the X-axis coincides with the axis of rotation (a diameter of the sphere). We will slice the solid sphere into thin slices perpendicularly to the X-axis, so that integration is over the positions of the slices on the X-axis. The slices will be thin discs. Assume that the sphere is divided into discs such as $PQRS$.



The moment of inertia of $PQRS$ about OA is, as in Example 11.2,

$$\begin{aligned} dI &= \frac{1}{2}\,\text{mass} \times\ (\text{radius})^2 \\ &= \frac{1}{2}\rho\pi\left(a^2 - x^2\right)dx \times \left(a^2 - x^2\right), \end{aligned}$$

where ρ is the density (mass per unit volume). The moment of inertia of the sphere is then given by

$$\begin{aligned} I &= \int dI = \int_{-a}^{a} \frac{\rho\pi}{2}\left(a^2 - x^2\right)^2 dx \\ &= \frac{\rho\pi}{2}\int_{-a}^{a}\left(a^4 - 2a^2x^2 + x^4\right)dx \\ &= \frac{\rho\pi}{2}\left[a^4x - \frac{2}{3}a^2x^3 + \frac{x^5}{5}\right]_{-a}^{a} \\ &= \frac{\rho\pi a^5}{2}\left[1 - \frac{2}{3} + \frac{1}{5} - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right] \\ &= \frac{8}{15}\pi\rho a^5. \end{aligned}$$

However, the total mass M and density are linked by

$$M = \rho\frac{4}{3}\pi a^3$$

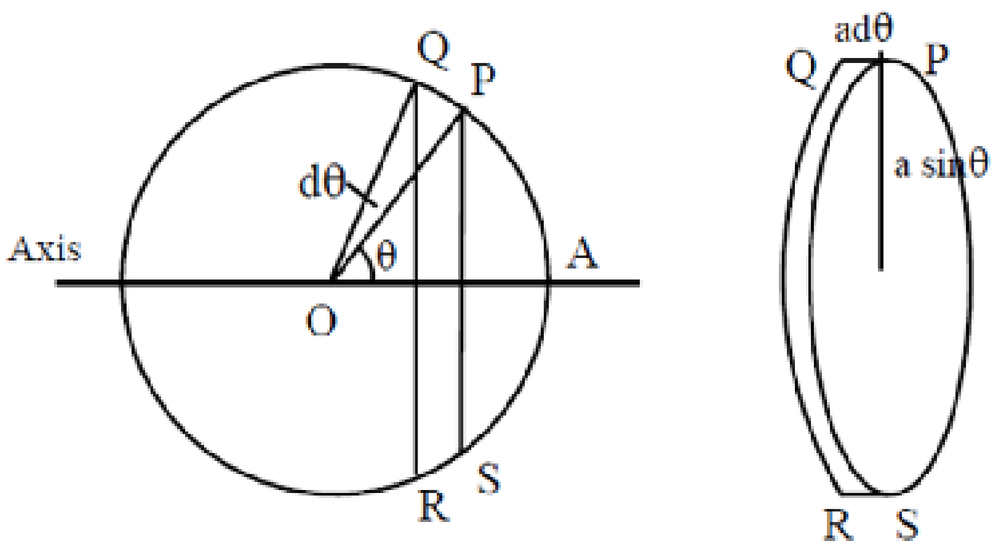
and therefore we get

$$I = \frac{2}{5}Ma^2.$$

Polar coordinates can be used to simplify the integration to find moments of inertia, just like in the calculation of centres of mass by integration. The following example gives a typical case.

Example 11.6
Find the moment of inertia of a thin, uniform, hollow sphere with mass M and radius a about an axis along its diameter.

Solution:
Again, slicing at right angles to the axis seems to be the sensible thing to do. This time, since the sphere is hollow, the slices will be thin bands, approximately rings. As when we used polar coordinates to find centres of mass, calculations are easiest if we slice the object to form thin angles. Consider a band of the sphere cut off by two planes perpendicular to the diameter. Let angle $POA = \theta$ and angle $QOP = d\theta$. Then $QP = ad\theta$. The circumference of circle PS is $2\pi a \sin\theta$.



$PRSQ$ forms a thin hollow cylinder with height $ad\theta$. The surface area of $PRSQ$ is $2\pi a \sin\theta \; a \; d\theta$ and if its surface density is ρ, then its mass is given by

$$dm = 2\pi \left(a^2 \sin\theta\right) \rho d\theta.$$

The band $PRSQ$ is approximately a ring, and according to Example 10.4, the moment of inertia of $PRSQ$ about the axis OA is given by

$$dI = 2\pi \rho a^2 \sin\theta \cdot (a \sin\theta)^2 \cdot d\theta.$$

For the whole hollow sphere, we then have

$$\begin{aligned}
I &= \int dI = \int_0^\pi 2\pi\rho a^4 \sin^3\theta d\theta \\
&= 2\pi\rho a^4 \int_0^\pi \sin\theta \left(\sin^2\theta\right) d\theta \\
&= 2\pi\rho a^4 \int_0^\pi \sin\theta \left(1-\cos^2\theta\right) d\theta \\
&= 2\pi\rho a^4 \left\{\int_0^\pi \sin\theta d\theta - \int_0^\pi \sin\theta\cos^2\theta d\theta\right\} \\
&= 2\pi\rho a^4 \left\{[-\cos]_0^\pi + \left[\frac{\cos^3\theta}{3}\right]_0^\pi\right\} \\
&= 2\pi\rho a^4 \left(-\cos\pi + \cos 0 + \frac{\cos^3\pi}{3} - \frac{\cos^3 0}{3}\right) \\
&= 2\pi\rho a^4 \left(2 - \frac{2}{3}\right) \\
&= \frac{8\pi\rho a^4}{3}
\end{aligned}$$

and since

$$\rho = \frac{M}{4\pi a^2},$$

we get

$$I = \frac{2}{3}Ma^2.$$

11.2 More complicated examples of moments of inertia

We have explained, and illustrated with many examples, how to find the moment of inertia of various rigid bodies and systems. We will now consider a bit more complicated examples, and to be able to deal with them, we will introduce a systematic way of looking at the problem of finding the moment of inertia of any object.

TOOLBOX FOR THE TASK OF FINDING THE MOMENT OF INERTIA OF A RIGID OBJECT

1. UNDERSTAND THE PROBLEM

   Make sure that you understand, firstly, what the object is like; and secondly, where the axis of rotation lies in relation to the body. Some of the following tactics may help you to make sure you achieve this!

   - Draw a sketch of the object.
   - Think of a real-life example of the situation.

2. PLANNING THE SOLUTION

   We have the following ways of finding moments of inertia:

- For systems of particles, $I = \sum m_i r_i^2$.
- For objects with a continuous structure, slicing and integrating: $I = \int dI$.
  We have also introduced several simplifying tools:
  - symmetries, identical axes of rotation
  - the parallel and perpendicular axes theorems
  - the rule for compound bodies

  Finally, you usually have at your disposal a set of basic or previously calculated moments of inertia for certain objects: rods, rings, discs, etc...
  You will need to decide which of these tools apply for the particular object or its components, and in which order you should apply them.

3. EXECUTING THE PLAN

   You will now have to do the calculations you have decided on. The following points should help you here:

- Introduce notation for the axes, objects etc. involved.
- If you have to integrate, you will also need to decide on the variable of integration.
- The link between density and mass will help you express the end result in terms of the mass of the object, where necessary.

4. ANALYSING THE SOLUTION

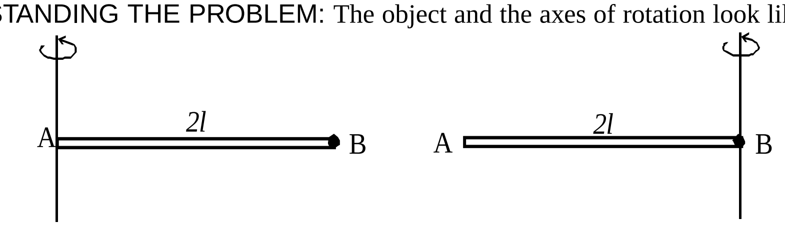
- Do basic checks for correctness: The moment of inertia should be positiveness, increase when mass increases, and so on.
- Re-calculate, using another method.
- Compare with other results for the same object with different axes, or different objects with the same axis.

The following examples deal with finding moments of inertia in a wide range of examples.

Example 11.7
A rod AB of length 2ℓ and with mass M has a particle with a mass of $2M$ attached at point B. Find the moment of inertia for rotation about an axis perpendicular to the rod, (a) through point A, and (b) through point B.

Solution:

1. UNDERSTANDING THE PROBLEM: The object and the axes of rotation look like this:



Axis in (a) *Axis in (b)*

2. PLANNING THE SOLUTION: Since in this example the rigid body is constituted of an object with a continuous structure, plus a point mass, we must use the rule for

compound bodies. (We cannot integrate over the whole object, nor can we consider the whole object as consisting of a system of particles.) So, to find the moment of inertia of the whole object (the rod plus the particle) about any given axis, we add together the moment of inertia of the rod and the moment of inertia of the particle about that axis. We already know the moment of inertia for a rod rotating about a perpendicular axis through its centre. To find the moment of inertia for the rod rotating about a perpendicular axis through any other point, we can use the parallel axes theorem. From Example 10.5 we know that the moment of inertia for rotation of the rod about an axis which is perpendicular to the rod, through its centre of mass G, is

$$I_G^{\text{rod}} = \frac{1}{3}Ml^2.$$

The axes of rotation described in (a) and (b) are both parallel to the axis in Example 10.5, and therefore the parallel axes theorem applies: The moment of inertia of the rod for an axis which goes through a point X is given by

$$I_X^{\text{rod}} = I_G^{\text{rod}} + Md^2$$

where d is the distance between X and G. On the other hand, for the particle with mass $2M$, the moment of inertia is

$$I_X^{\text{particle}} = (2M)\,r^2$$

where r denotes the distance from the particle to the axis through point X on the rod.

(a) For an axis perpendicular to the rod, through A:

$$\begin{aligned} I_A^{\text{total}} &= I_A^{\text{rod}} + I_A^{\text{particle}} \\ &= I_G^{\text{rod}} + Ml^2 + (2M)\,(2l)^2 \\ &= \frac{1}{3}Ml^2 + Ml^2 + (2M)\,(2l)^2 \\ &= 9\frac{1}{3}Ml^2 \end{aligned}$$

(b) For an axis perpendicular to the rod, through B:

$$\begin{aligned} I_B^{\text{total}} &= I_B^{\text{rod}} + I_B^{\text{particle}} \\ &= I_G^{\text{rod}} + Ml^2 + (2M)\,(O)^2 \\ &= \frac{1}{3}Ml^2 + Ml^2 \\ &= \frac{4}{3}Ml^2 \end{aligned}$$

Example 11.8

Calculate the moment of inertia of a uniform disc of mass M, with radius a,

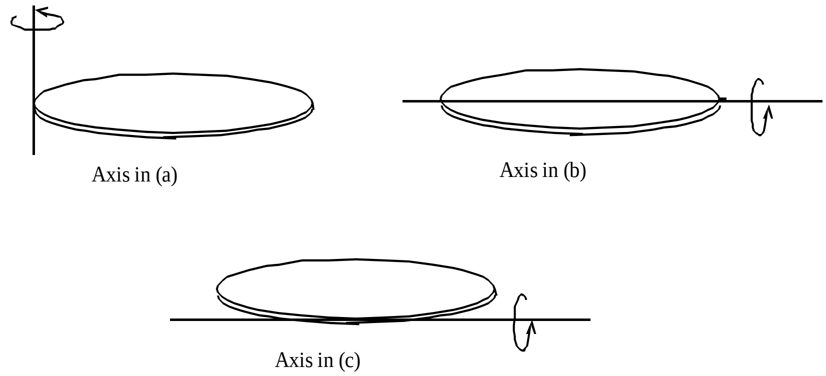
(a) about an axis perpendicular to the disc, through a point A on its rim

(b) about an axis through its centre, parallel to the plane of the disc

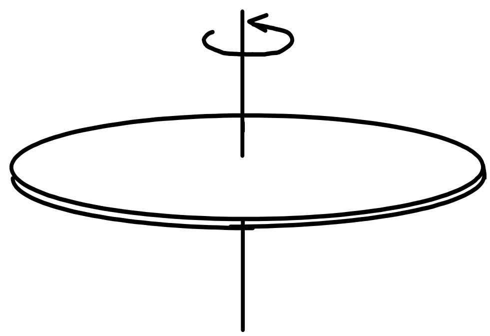
(c) about an axis through a point A on its rim, parallel to the plane of the disc and tangential to the disc.

Solution:

The rotations in the three cases are indicated in the sketches below.



What results do we already have regarding moments of inertia of discs? From Example 11.2 we know the moment of inertia for rotation about an axis perpendicular to the disc, through its centre, as shown below:



Before we start to think about integration, we should check whether we can use any of the simplifying tactics to utilise this known result. Firstly, we note that the axis of rotation in (a) is parallel to the axis shown above. Therefore, we can use the parallel axes theorem and the known result for the axis shown above, to deal with the axis in (a). Similarly, the axes of rotation in (b) and (c) are parallel, so the parallel axes theorem will help us to solve one from the other. Finally, the axis shown above and the one described in (b) are perpendicular, so with help from a symmetry argument, we should also be able to solve (b) from (a). Thus we should be able to answer all the questions without any integration!

(a) The axis here is parallel to the axis in Example 11.2 . There, the axis was through G (the centre of mass of the disc) and here, through a point A on the circumference of the disc. Thus, we can use the parallel axes theorem:

$$I_A = I_G + Ma^2$$

where according to Example 11.2 ,

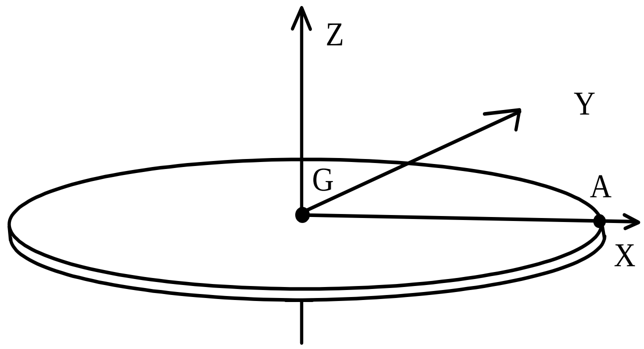
$$I_G = \frac{1}{2}Ma^2.$$

So,

$$I_A = \frac{1}{2}Ma^2 + Ma^2 = \frac{3}{2}Ma^2.$$

Note that due to symmetry, it obviously does not matter which point on the circumference is chosen as A.

(b) For simplicity, we shall assume the following coordinates: the disc is on the XY-plane; the Z-axis is perpendicular to the disc.



In Example 11.2 , we calculated $I_G^Z = Ma^2/2$ for the moment of inertia about an axis through G, in the Z-direction. In (a) above, we calculated $I_A^Z = 3Ma^2/2$ for the moment of inertia about an axis through A, in the Z-direction. Now, in (b) and (c) we want to calculate I_G^Y, the moment of inertia about an axis through point G, parallel to the disc, in the Y-direction, and I_A^Y about an axis through point A, tangential to the disc, in the Y-direction. Firstly, we calculate I_G^Y from I_G^Z, using the perpendicular axes theorem. The disc is on the XY-plane, so

$$I_G^Z = I_G^X + I_G^Y.$$

But, due to symmetry, $I_G^X = I_G^Y$. So,

$$I_G^Y = \frac{1}{2} I_G^Z = \frac{1}{2} \cdot \frac{1}{2} Ma^2 = \frac{1}{4} Ma^2.$$

(c) Next, we use the parallel axes theorem to calculate I_A^Y from I_G^Y :

$$I_A^Y = I_G^Y + Ma^2 = \frac{1}{4} Ma^2 + Ma^2,$$

so

$$I_A^Y = \frac{5}{4} Ma^2.$$

(Of course, it is not necessary for A to be along one of the coordinate axes — it could be any point on the rim of the disc!)

Note that we cannot calculate I_A^Y directly from I_A^Z using the perpendicular axes theorem, as in (c). This is because there will no longer be symmetry of the X- and Y-axes: rotations about axes through A in the X- and Y-directions, respectively, are very different. Instead, we can use the perpendicular axes theorem as follows: $I_A^X + I_A^Y = I_A^Z$, so $I_A^Y = I_A^Z - I_A^X$; we know I_A^Z from (a); and $I_A^X = I_G^X$, which we have found in (b), since the X-axis goes through both points G and A. This again gives

$$I_A^Y = I_A^Z - I_A^X = \frac{3}{2} Ma^2 - \frac{1}{4} Ma^2 = \frac{5}{4} Ma^2.$$

Activity 11.4

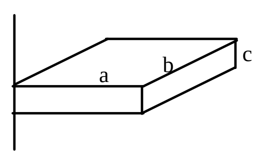
A ring with mass m and radius a performs small oscillations about a horizontal axis which is *tangential* to the disc at a point A on the rim of the disc. Find the moment of inertia about A.

....

Feedback: As in the previous example, note that you have to use both the perpendicular and the parallel axis theorems! Make sure you can write down all the details of the proof. You should get

$$I_A = \frac{3}{2}ma^2.$$

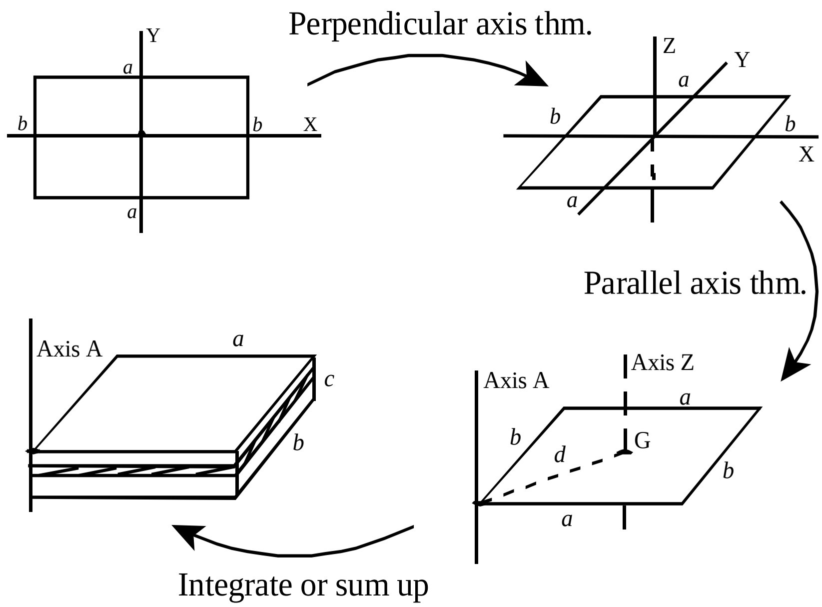
Example 11.9
The figure below shows a uniform, solid block of mass M and edge lengths a, b and c. Calculate its moment of inertia about an axis through one corner and parallel to an edge of length c.



Solution:
To calculate the moment of inertia about a given axis of a body with a continuous structure, like here, we usually try to divide the object into small parts, such that the moments of inertia of these are easy to calculate. The moment of inertia of the whole object is the obtained by "summing" (integrating) over all these small mass elements. The calculation of the moments of inertia of each of the smaller elements is, in turn, easy if the small mass element is one to which we can apply some already established result.

In this particular exercise, clearly, slicing the block into smaller objects produces rectangles. Let us choose to slice the block into thin rectangles, parallel to the ab–plane. We are then faced with the problem of finding the moment of inertia of a rectangle with edges a and b, about an axis which is perpendicular to the rectangle, through one of its corners. But so far, the only result we have derived for rectangles is the moment of inertia for rotation about an axis parallel to the rectangle, through the midpoints of two opposite sides (Example 10.12). However, applying the perpendicular and then the parallel axes theorems, we will be able to derive the required moment of inertia.

The following picture illustrates the order of calculations, and specifies the various axes we shall refer to.



We divide the block horizontally into n thin slices, as shown. Then each slice is approximately a thin rectangle. Let rectangle number i have mass m_i. To calculate the moment of inertia I_i^A of this rectangle about axis A, we first note that according to Example 10.12, we have

$$I_i^X = \frac{1}{3}m_i\left(\frac{b}{2}\right)^2 = \frac{1}{12}m_i b^2$$

and similarly,

$$I_i^Y = \frac{1}{3}m_i\left(\frac{a}{2}\right)^2 = \frac{1}{12}m_i a^2,$$

if the X- and Y-axes are as shown above.

Next, since the rectangle lies on XY-plane, we can apply the perpendicular axes theorem, which gives us

$$I_i^Z = I_i^X + I_i^Y = \frac{1}{12}m_i(a^2+b^2).$$

Finally, we can apply the parallel axes theorem to calculate I_i^A from I_i^Z : The axis A is parallel to axis Z, Z goes through the centre of mass of the rectangle and the distance between the two axes is

$$d = \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2} = \sqrt{\left(\frac{a^2+b^2}{4}\right)}.$$

Therefore

$$\begin{aligned} I_i^A &= I_i^Z + m_i d^2 = \frac{1}{12}m_i(a^2+b^2) + \frac{1}{4}m_i(a^2+b^2) \\ &= \frac{1}{3}m_i(a^2+b^2). \end{aligned}$$

Finally, to find the moment of inertia I^A of the whole block, we add up the moments of inertia I_i^A of all the rectangles:

$$\begin{aligned} I^A &= \sum_{i=1}^{n} I_i^A = \sum_{i=1}^{n} \frac{1}{3} m_i (a^2 + b^2) \\ &= \frac{1}{3} \left(\sum_{i=1}^{n} m_i \right) (a^2 + b^2). \end{aligned}$$

But the sum

$$\sum_{i=1}^{n} m_i$$

equals M, the total mass of the whole block. Hence, we have

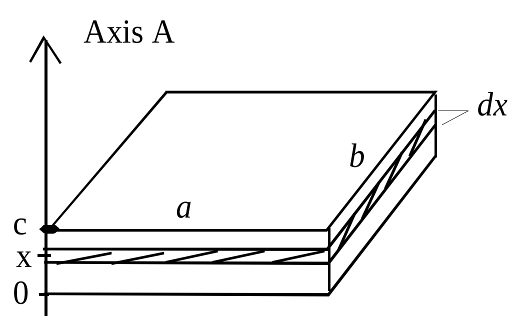
$$I^A = \frac{1}{3} M (a^2 + b^2).$$

Note that we did not have to integrate here, since all the rectangles had a similar moment of inertia for rotation about the A-axis, namely

$$I_i^A = \frac{1}{3} m_i (a^2 + b^2)$$

where m_i is the mass of the rectangle! That is, the moments of inertia of the rectangles do not depend on their position on the A-axis.

Of course, using integration would give us the same result as obtained above. Here is how we would solve the problem, using integral notation:



We divide the block into thin slices, as shown. The slices are then thin rectangles, and if the rectangle situated at position x, $0 \leq x \leq c$ along the A-axis has a width of dx and if ρ denotes the density of the block (mass per unit volume), then the mass of this rectangle is

$$\begin{aligned} dm &= \rho \cdot \text{volume} \\ &= \rho \cdot dx \cdot a \cdot b \\ &= \rho a b dx. \end{aligned}$$

Now, from the calculations above, we know that the moment of inertia of a thin $a \times b$ rectangle with mass dm about axis A is equal to

$$dI = \frac{1}{3} dm (a^2 + b^2) = \frac{1}{3} (a^2 + b^2) \rho a b dx.$$

The moment of inertia of the whole block is found by integrating over all the rectangles

$(x = 0 \text{ to } x = c)$:

$$I^A = \int dI = \int_0^c \frac{1}{3}(a^2 + b^2)\rho ab dx$$
$$= \frac{1}{3}(a^2 + b^2)\rho ab \int_0^c dx = \frac{1}{3}(a^2 + b^2)\rho abc.$$

Finally, we use the fact that the total mass of the block is

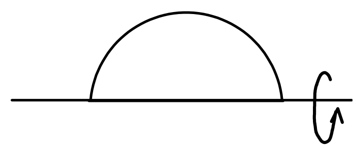
$$M = \rho abc.$$

Thus, we have

$$I^A = \frac{1}{3}(a^2 + b^2)M$$

as before.

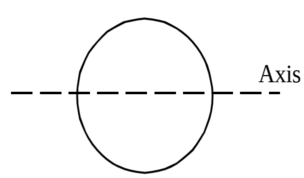
Example 11.10
Calculate the moment of inertia of a thin, uniform lamina shaped like a semicircle with mass M, about an axis along its diameter.



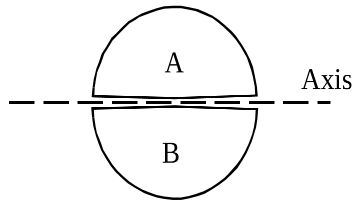
Solution:
When asked to find the moment of inertia of an object, it is always a good idea to see whether one cannot calculate the moment of inertia using an already established result, rather than using direct integration. In this particular case, it turns out that we can calculate the required moment of inertia fairly easily from the results of Example 11.8. In the following, we give this method, as well as a way of calculating the moment of inertia by direct integration. We shall assume that the radius of the semicircle equals a and its mass is M.

Method 1
In Example 11.8, Learning Unit (b), we calculated the moment of inertia of a uniform disc about an axis along its diameter: if the disc has a mass of $2M$ and a radius a, then the moment of inertia is

$$I_{\text{disc}} = \frac{1}{2}Ma^2.$$



But the disc consists of two identical semicircles of the type we are interested in here, each with a mass M; let us denote them by A and B.



In terms of the rule for calculating moments of inertia of compound bodies,

$$I_{\text{disc}} = I_A + I_B.$$

On the other hand, the semicircles A and B are situated identically in relation to the axis, and therefore $I_A = I_B$. Therefore,

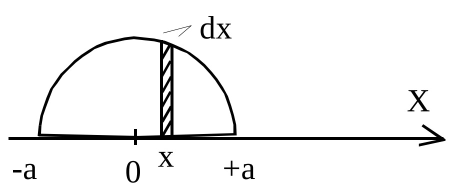
$$2I_A = I_{\text{disc}}$$

$$\therefore \quad I_A = \frac{1}{2} I_{\text{disc}} = \frac{1}{2} \cdot \frac{1}{2} M a^2$$

$$\therefore \quad I_A = \frac{1}{4} M a^2.$$

Method 2

Take the X-axis to go along the axis of rotation, and let us divide the semicircle into thin slices, as shown below:



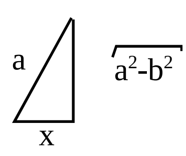
The slice situated at position x on the X-axis has the length

$$\ell = \sqrt{a^2 - x^2}.$$

(This can be seen from the fact that the semicircle is described by the function

$$y = \sqrt{a^2 - x^2};$$

or alternatively, by using elementary trigonometry as in the figure below.)



If this slice has a thickness dx, then its mass is

$$dm = \rho \sqrt{a^2 - x^2} dx$$

where ρ is the density (mass per unit area).

The slice is approximately a rod of mass dm and length

$$\ell = \sqrt{a^2 - x^2},$$

and so we can find its moment of inertia, which we will denote by dI_x, about the X-axis by applying the results for rods.

First, we will find the moment of inertia of a general rod rotating about its endpoint – this is a very useful result!

According to Example 10.5, the moment of inertia of a uniform rod of length ℓ and mass m about an axis perpendicular to the rod, through its centre G, is

$$I_G = \frac{1}{3}m\left(\frac{\ell}{2}\right)^2 = \frac{1}{12}m\ell^2.$$

To find its moment of inertia I_A about an axis perpendicular to the rod, through the end A of the rod, we can apply the parallel axes theorem. This distance between A and G is $\frac{\ell}{2}$, and therefore, according to the parallel axes theorem,

$$\begin{aligned} I_A &= I_G + m\left(\frac{\ell}{2}\right)^2 \\ &= \frac{1}{12}m\ell^2 + \frac{1}{4}m\ell^2 \\ \therefore\ I_A &= \frac{1}{3}m\ell^2. \end{aligned}$$

Now, we apply this result to the rod of mass dm and length

$$\ell = \sqrt{a^2 - x^2},$$

to find that

$$dI_x = \frac{1}{3}dm(a^2 - x^2) = \frac{1}{3}\rho\sqrt{a^2 - x^2}(a^2 - x^2)dx$$

Finally, to find the moment of inertia of the entire semicircle, we have to integrate over all the slices, that is, from $x = -a$ to $x = a$. Then the total inertia is

$$\begin{aligned} I &= \int dI_x = \int_{-a}^{a} \frac{1}{3}\rho\sqrt{a^2 - x^2}(a^2 - x^2)dx \\ \therefore I &= \frac{1}{3}\int_{-a}^{a} \rho(a^2 - x^2)^{\frac{3}{2}}dx. \end{aligned} \tag{11.2}$$

So, all we have to do is to calculate the value of this integral – but this is easier said than done! This integration is beyond the scope of a first–year module so you are not expected to be able to do it yourself. We will, however, show below how this can be done. The complexity of this integration should again emphasise how much easier Method 1 was!

To calculate the value of the integral (11.2), we can first make a change of variables by introducing a new variable u, where

$$x = a\sin(u);$$

then we get

$$\begin{aligned} dx &= a\cos(u)du, \\ a^2 - x^2 &= (a\cos(u))^2, \end{aligned}$$

integration is now from $u = -\frac{\pi}{2}$ to $u = \frac{\pi}{2}$, and thus the integral (11.2) becomes

$$\begin{aligned} I &= \rho\frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (a\cos(u))^3 du = \rho\frac{a^4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4 u du \\ &= \rho\frac{a^4}{3}\left[\frac{3u}{8} + \frac{\sin 2u}{4} + \frac{\sin 4u}{32}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \rho\frac{a^4}{3}\left(\frac{3}{8}\pi\right) = \delta\frac{a^4}{8}\pi\,. \end{aligned}$$

On the other hand, the mass of the entire semicircle must be equal to

$$M = \rho\cdot\frac{1}{2}\pi a^2;$$

therefore, we have

$$\rho = \frac{2M}{\pi a^2},$$

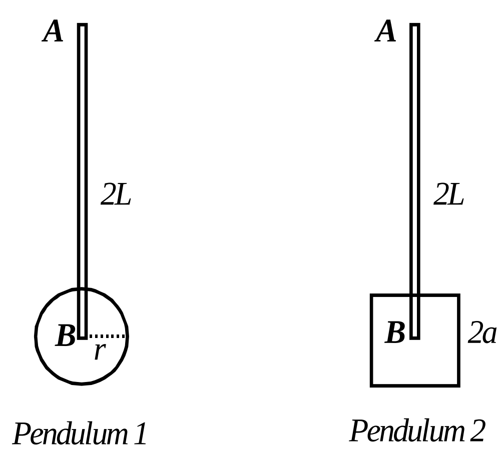
and hence we again get

$$I = \frac{1}{4}Ma^2.$$

Example 11.11

Pendulum 1 consists of a rod AB of length $2L$ and mass m with a thin disc of mass M and radius r attached rigidly at its centre to the rod's end point B, and pendulum 2 consists of a similar rod with a thin square of mass M and sides of length $2a$ attached rigidly at its centre to the rod's end point B. Find the moments of inertia of the two pendulums, when rotation is for an axis through point A of the rod, perpendicular to the plane of the disc / the square.

Solution:
The pendulums look like this:



The moment of inertia of each pendulum can be found by adding together the moment of inertia of rod AB rotating about point A, and the moment of inertia of the disc/square for rotation about point A; the latter can in turn be found by parallel axis theorem from the moment of inertia of a disc/square rotating about an axis perpendicular to it, through its centre.

For rotation about an axis through the centre of a disc/square, perpendicular to its plane:

For the disc we have

$$I_G^{\text{disc}} = \frac{1}{2}Mr^2,$$

and for the square, using Example 10.12 and the perpendicular axis theorem,

$$I_G^{\text{square}} = \frac{1}{3}Ma^2 + \frac{1}{3}Ma^2 = \frac{2}{3}Ma^2.$$

Since the distance from B to A is $2L$, we further get

$$\begin{aligned} I_A^{\text{disc}} &= I_G^{\text{disc}} + M(2L)^2 = \frac{1}{2}Mr^2 + 4ML^2, \\ I_A^{\text{square}} &= I_G^{\text{square}} + M(2L)^2 = \frac{2}{3}Mr^2 + 4ML^2. \end{aligned}$$

The moments of inertia for the pendulums are therefore: For pendulum 1,

$$\begin{aligned} I_A^1 &= I_A^{\text{rod}} + I_A^{\text{disc}} \\ &= \frac{4}{3}mL^2 + 4ML^2 + \frac{1}{2}Mr^2. \end{aligned}$$

and for pendulum 2,

$$\begin{aligned} I_A^2 &= I_A^{\text{rod}} + I_A^{\text{square}} \\ &= \frac{4}{3}mL^2 + 4ML^2 + \frac{2}{3}Mr^2. \end{aligned}$$

The following activities involve you finding the centres of mass of various objects, for various axes of rotation. Please do use the toolbox, to make the job of finding the result as easy as possible!

Activity 11.5

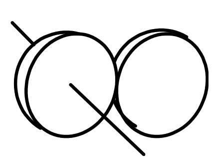
A pendulum is made of two discs (each with a mass M and radius R), which are separated by a massless rod. The discs lie on the same plane and their centres are a distance ℓ apart. One of the discs is pivoted through its centre by a small pin. Find the moment of inertia for the pendulum, for rotation about the pin.

....

Feedback: $I = M\left(R^2 + \ell^2\right)$.

Activity 11.6

Two thin discs, both with mass m and radius R, are attached as shown below to form a rigid body.



What is the moment of inertia of the system about an axis perpendicular to the plane of the discs, through the centre of one of them?

....

Feedback: You will get $5Mr^2$.

Activity 11.7

Find the moment of inertia of a square lamina, with sides of length $2a$ and mass m, about an axis passing diagonally through the square. *Hint: the perpendicular axis theorem can also be used to find the moments of inertia for rotation about the two diagonals of the square from the moment of inertia for rotation about an axis which goes perpendicularly through the centre of the square!*

....

Feedback: you should get $\frac{1}{3}ma^2$.

Activity 11.8

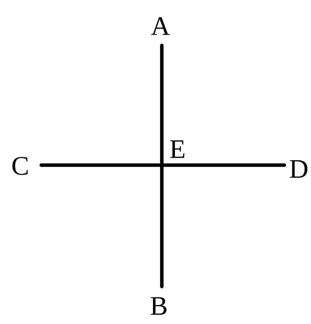
Four identical particles of mass m are placed at the corners of an $a \times a$ square and held there by four massless rods, which form the sides of the square. What is the moment of inertia of this rigid body about an axis (a) that passes through the midpoints of opposite sides and lies in the plane of the square, (b) that passes through the midpoint of one of the sides and is perpendicular to the plane of the square, and (c) that lies on the plane of the square and passes through two diagonally opposite particles?

....

Feedback: Throughout, since you have here just four particles, you will be able to find the moment of inertia from $I = r_1^2 + mr_2^2 + mr_3^2 + mr_4^2$ where r_i is the distance from particle number i to the axis of rotation. You will get the following: (a): ma^2; (b) $\frac{9}{2}ma^2$; (c) ma^2.

Activity 11.9

The object shown below consists of two identical thin rods AB and CD, of mass M and length L, joined together at their midpoints to form a cross. Let E be the point where the two rods intersect.



Find the moment of inertia of the object when it rotates about the following axes of rotation:

(a) An axis along rod AB.

(b) An axis through point E, perpendicular to the cross.

(c) An axis through point A, perpendicular to the cross.

(d) An axis which goes through points A and D.

....

Feedback: (a) $\frac{1}{12}ML^2$,(b) $\frac{1}{6}ML^2$ (from (a) and perpendicular axis theorem), (c) $\frac{2}{3}ML^2$ (use (b) and parallel axis theorem), (d) $\frac{1}{3}ML^2$ (there are many ways to do this, for instance, you can find from (b) the moment of inertia for rotation about a diagonal to the object, and then use parallel axis theorem - but remember to use the mass of the entire object in the formula in the parallel axis theorem!) Please see the workbook for a full worked-out solution.

Activity 11.10

Find the moment of inertia of a thick hollow sphere, with an inner radius R_1 and an outer radius R_2 and mass M, about an axis through its centre.

. . . .

Feedback: You will get $\frac{2}{5}M\frac{(R_1^5-R_2^5)}{R_1^3-R_2^3}$. You can find the moment of inertia by slicing the object into thin, concentric hollow spheres (you'll need to find the volume of the object to change from ρ to M), or you can use the fact that the object consists of a big sphere minus a small sphere, so the moment of inertia of the object must be the moment of inertia of the big sphere minus the small sphere (but make sure you use the right masses for both!)

CONCLUSION

In this unit you have learned how to calculate the moments of inertia

- by slicing and integrating
- by applying all the tricks we have come across so far

Remember to add the following tools to your toolbox:

- using slicing and integrating to find the moment of inertia of a rigid body
- The results for the moments of inertia of various objects
- the toolbox for finding moments of inertia by slicing and integrating
- the toolbox for the task of finding the moment of inertia of a rigid object
- The table of moments of inertia, on the next page

Some moments of inertia

Uniform Body of Mass M	**Axis**	**Moment of Inertia**
Rod of length 2ℓ	Perpendicular to the rod through the centre	$M\ell^2/3$
Rod	Parallel to the rod and a distance d from it	Md^2
Rectangle of length 2ℓ and width $2d$	Perpendicular to the sides of length 2ℓ and passing through their midpoints	$M\ell^2/3$
Ring with radius a	Perpendicular to the ring and through its centre	Ma^2
Disc with radius a	Perpendicular to the disc and through its centre	$Ma^2/2$
Solid sphere with radius a	A diameter	$2Ma^2/5$
Hollow sphere with radius a	A diameter	$2Ma^2/3$
Solid cylinder with radius a	The axis	$Ma^2/2$
Hollow cylinder with radius a	The axis	Ma^2

Unit 12 APPLICATIONS OF THE EQUATION FOR PURE ROTATION

Key questions:

- *How do we apply the equation of motion describing pure rotation?*

In this unit you will learn how to solve problems using the equation for pure rotation. We shall also introduce the concepts of pulleys and compound pendulums.

Contents of this unit:

12.1 How to apply the equation for pure rotation

12.2 Problem-solving strategies

What you are expected know before working through this unit:

In this unit, we combine the knowledge and skills you have gained in Units 8, 10 and 11. You will also occasionally need to use the concept of centre of mass in the problem solving, sometimes to apply the parallel axis theorem, and sometimes to find where the force of gravity acts!

12.1 How to apply the equation for pure rotation

We are now ready to start applying Result 10.2, and the equation of motion for pure rotation:

$$\sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i = I\ddot{\theta}\underline{k} \tag{12.1}$$

to solve problems. In this equation,

- $\underline{F}_i$ are the external forces acting on the object
- for each i, vector $\underline{r}_i$ is the position vector from the axis of rotation to the point where the particular external force $\underline{F}_i$ acts
- the sum on the left is therefore the sum of the moments about the given axis of all the external forces acting on the system
- I is the moment of inertia of the object for rotation about the fixed axis
- $\ddot{\theta}$ is the angular acceleration, measured counterclockwise

Remember that (12.1) assumes that we are considering an object which is a solid body rotating about a fixed, unmoving axis, and that the axis of rotation is parallel to the Z-axis. Thus, all parts of the body move parallel to the XY-plane. We can assume that all the external forces acting on the body also lie on the XY-plane; forces which act in the direction of the axis of rotation do not contribute to the rotational motion. The position vectors $\underline{r}_i$ will also be parallel to the XY-plane, and it follows that the left-hand side of the equation will give a vector which is a multiple of the $\underline{k}$ unit vector.

To apply the equation of rotation it is necessary to:

- identify the axis of rotation
- find the moment of inertia of the body for rotation about the given axis
- identify all the forces acting on the body, with their magnitudes, directions and points of action
- find the position vectors from the axis to the points of action of the forces
- calculate the vector products on the left.

The evaluation of the vector product in the equation of rotation is easiest if we use the XYZ unit vectors. To use the equation of rotation (12.1) in the first place, we must have selected a coordinate system so that the axis of rotation is parallel to the Z-axis, and indeed, since the axis is unmoving here, it is easiest to select the axis of rotation to coincide with the Z-axis, so that the origin of the XYZ coordinate system lies on the axis of rotation. However, we still have the option to decide how the X- and Y-axes should go, and once again some choices are better than others. When choosing the coordinate system, it should be done so as to make it as easy as possible to express the forces $\underline{F}_i$ and the position vectors $\underline{r}_i$ (from the axis of rotation to the points of action of the forces) in terms of the $\underline{i}$ and $\underline{j}$ unit vectors.

Note that since we already know that both sides of the equation (12.1) are vectors in the Z-direction (parallel to the axis of rotation), only the magnitudes of the right- and left-hand sides are of interest. In fact once the calculations are done, we usually drop the unit vector $\underline{k}$ from both sides, and obtain an equation of rotation which in effect gives us an expression for $\ddot{\theta}$.

The following examples and activities involve straightforward applications of the equation of rotation.

Example 12.1
Sphere 1 has mass M and radius R, while Sphere 2 has mass $2M$ and radius $\frac{1}{2}R$. Each sphere rotates about an axis through its centre.

(a) For each of the spheres, find the moment of the force required to give the sphere the angular acceleration α.

(b) For each sphere, what force applied tangentially at the equator of the sphere would provide the required moment of force?

Solution:
For a rigid body rotating about a fixed axis, the moment of a force $\underline{M}$ about the given axis is linked to angular acceleration through the equation of rotation,

$$\underline{M} = I\ddot{\theta}\underline{k}.$$

We wish to have $\ddot{\theta} = \alpha$. Finally we need to find the values of the moment of inertia, I. The objects here are spheres; and from the table on moments of inertia, we find that the moment of inertia for a sphere with radius a and mass M about an axis through its centre is $2Ma^2/5$. Applying this in the situation here, we get

$$I_1 = \frac{2}{5}MR^2, \quad I_2 = \frac{2}{5}(2M)\left(\frac{R}{2}\right)^2 = \frac{1}{5}MR^2$$

for spheres 1 and 2.

(a) Thus, we find that the moments of force required are

$$\underline{M}_1 = \frac{2}{5}MR^2\,\frac{\omega}{t}\,\underline{k}$$

for sphere 1 and

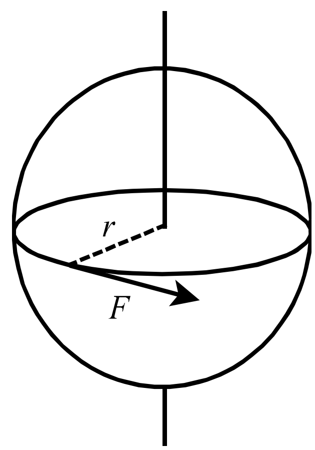
$$\underline{M}_2 = \frac{1}{5}MR^2\,\frac{\omega}{t}\,\underline{k}$$

for sphere 2.

(b) A force $\underline{F}$ with magnitude F, applied tangentially at the equator of the sphere, would have the moment

$$\underline{M} = rF\underline{k}$$

where $\underline{k}$ is parallel to the axis of rotation. Here, the radius of the sphere is r, and we assume that F is applied as shown below, so that the turning effect is counterclockwise!



Applying this for sphere 1, we find that to obtain the required $\underline{M}_1$ value, the required force F_1 must satisfy the condition

$$RF_1\underline{k} = \frac{2}{5}MR^2\frac{\omega}{t}\underline{k}.$$

Similarly, for sphere 2, we need force F_2 such that

$$\frac{R}{2}F_2\underline{k} = \frac{1}{5}MR^2\frac{\omega}{t}\underline{k}.$$

Solving for F_1 and F_2 from these, we get

$$\begin{aligned} F_1 &= \frac{2}{5}MR\frac{\omega}{t}, \\ F_2 &= \frac{2}{5}MR\frac{\omega}{t}. \end{aligned}$$

The forces that need to be applied are therefore the same. *This should be clear from the fact that for sphere 1 we need twice as large a moment of the force since I is two times larger there; but then again the radius is twice as long, and the same force applied at a twice the distance gives twice as large a moment!*◀

Activity 12.1

A disc–shaped object is made of a non–uniform material. Its radius is r, and it is free to rotate about an axis through its centre. If a force $\underline{F}$ of magnitude F applied tangentially at the edge of the object produces the angular acceleration α, what is its moment of inertia for rotation about that axis?

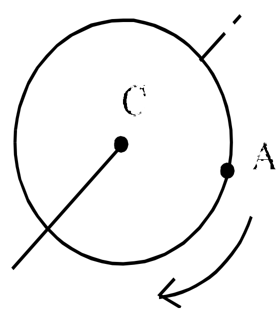
....

Feedback: rF/α, of course!

Quite often it happens that the directions of the forces acting on the object or their points of action, and hence the moments of the forces, depend on the position of the object. Since in pure rotation the object can only rotate about the axis, its position is, of course, fully described by its angle of rotation, which we have agreed will be measured counterclockwise. The exact way we measure the angle of rotation must, if necessary, be specified. The following example illustrates this.

Example 12.2

An object consists of a massless disc of radius R, at the rim of which a particle of mass M is attached. Let A denote the point on the rim where the particle is attached, and let C denote the centre of the disc. The object is free to rotate in the vertical plane about a horizontal axis through point C.



We will find the equation of motion that describes the rotation of the object about the given axis.

Solution: The forces acting on the disc are:

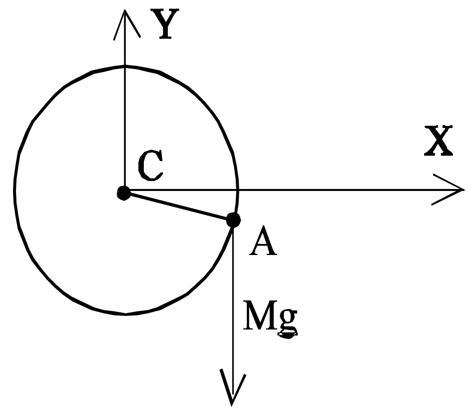
1. The force of gravity $M\underline{g}$, acting at the particle, which is at point A. (Remember that the disc itself is massless, so there is no force of gravity acting at the centre of the disc!) The force $M\underline{g}$ is directed downwards.

2. The force of reaction $\underline{F}$ which fixes the disc at point C. Since the axis of rotation goes through point C, this force has a zero moment and therefore we can ignore it when writing out the equation of rotation.

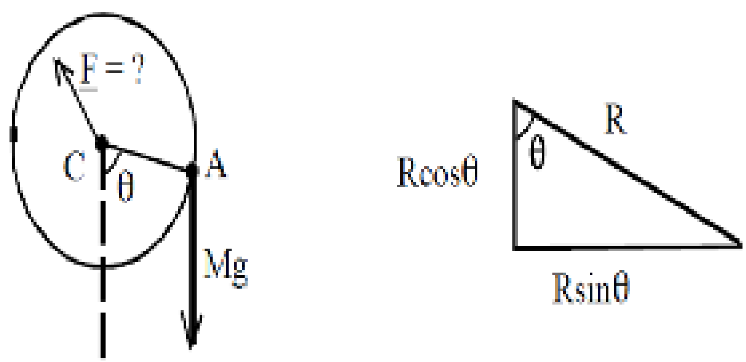
The equation of rotation then gives

$$\underline{CA} \times M\underline{g} = I\,\ddot{\theta}\,\underline{k},$$

where $I = MR^2$ (this is the moment of inertia of a single particle of mass M at the distance R from C). Let us take the XY coordinates as shown below, with the origin at point C in the middle of the disc.



Now, it is important to notice here that while the force $M\underline{g}$ is always directed downwards, with the magnitude Mg (so that $M\underline{g} = -Mg\underline{j}$ in our coordinate system), the position vector from C to A varies according to the position of the point A — remember that the disc rotates about point C and point A rotates with it! It follows that the moment of the force $M\underline{g}$, and accordingly the angular acceleration $\ddot{\theta}$ at each moment depend on the current position of point A. But we know that this position can be described by the angle of rotation. To introduce a way of measuring the angle of rotation, we will assume that the line CA forms the angle θ with the vertical line downwards from point C, measured counterclockwise, as shown in the sketch below.



Note that the positions where A is to the left of C can then be referred to either by negative angles θ, or by angles θ between π and 2π.

With this agreement on the angle of rotation, we see that if the angle is θ, then

$$\underline{CA} = R\,\sin\theta\,\underline{i} - R\cos\theta\,\underline{j}.$$

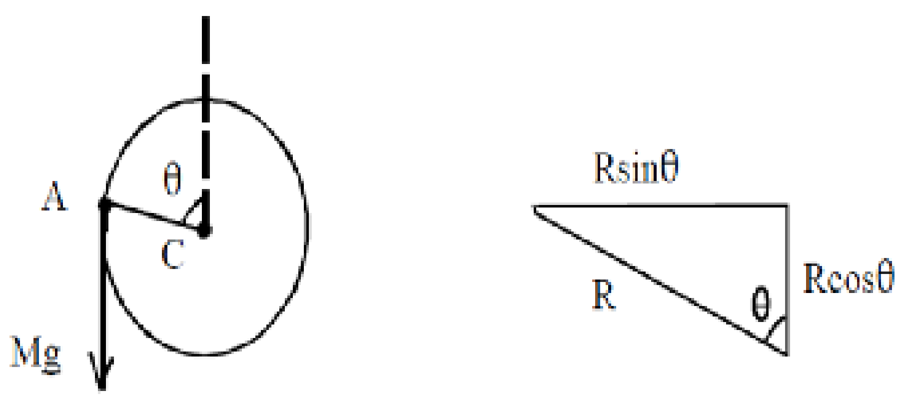
(This follows from a bit of trigonometric reasoning!) Thus, we get the equation of rotation

$$\begin{aligned} \left(R\sin\theta\,\underline{i} - R\cos\theta\,\underline{j}\right) \times \left(-Mg\,\underline{j}\right) &= \left(MR^2\right)\ddot{\theta}\,\underline{k} \\ \therefore\ -R\,Mg\sin\theta &= MR^2\ddot{\theta} \\ \therefore\ \ddot{\theta} &= -\frac{g}{R}\sin\theta. \end{aligned} \tag{12.2}$$

Note that we have arrived at an equation linking the angle of rotation θ and its second derivative, $\ddot{\theta}$. In principle at least, this fully describes the motion of the disc. The solution $\theta(t)$, $t \geq 0$ to this differential equation, with given initial values (initial angle and initial angular velocity), gives the angle of rotation at all times t!

Even without finding the solution to the differential equation, we can draw conclusions from equation (12.2) on how the disc will rotate. For instance, we see that when the disc has rotated counterclockwise such that A is to the left of point C (as in the sketch), the value of $\sin(\theta)$ is positive and hence the angular acceleration $\ddot{\theta}$ is negative, which means that the disc has clockwise acceleration. This is caused by the gravity pulling down at the particle at A , and braking down the counterclockwise rotation of the disc. Similarly, when the disc has rotated clockwise, and A is to the right of point C, the angular acceleration is counterclockwise. The disc will behave like a pendulum, swinging endlessly, first clockwise and then counterclockwise.

Does the end result depend on how we chose the coordinate system, or on how we decided to measure the angle of rotation θ? Firstly, we see that the final result (12.2) does not depend on the unit vectors $\underline{i}$ and $\underline{j}$ — which makes sense, since we only introduced those in order to calculate the vector products. The final equation of rotation does not depend on how the coordinates X and Y were chosen. How about the choice of the angle of rotation θ? We could also have decided to measure the angle of rotation by taking θ to be the counterclockwise angle that the line AC forms with the vertical line *upwards* from point C, as shown in the following sketch:



In this case, we see that when the angle of rotation is $\hat{\theta}$ according to this new way of measuring the angle, the position vector from C to A is $\underline{CA} = -R\,\sin\hat{\theta}\,\underline{i} + R\cos\hat{\theta}\,\underline{j}$. The equation of rotation will then be

$$
\begin{aligned}
\left(-R\sin\hat{\theta}\,\underline{i} + R\cos\hat{\theta}\,\underline{j}\right) \times \left(-Mg\,\underline{j}\right) &= \left(MR^2\right)\ddot{\theta}\,\underline{k} \\
\therefore\ \ R\,Mg\sin\hat{\theta} &= MR^2\,\ddot{\theta} \\
\therefore\ \ \ddot{\theta} &= \frac{g}{R}\sin\hat{\theta}. \qquad (12.3)
\end{aligned}
$$

We see that the sign of the right-hand term has changed from the previous equation of motion, (12.2). This is of course as expected, since the two different methods we have discussed for measuring the angle of rotation are linked by $\hat{\theta} = \pi + \theta$, which means that $\hat{\theta} = -\theta$ and therefore $\sin\hat{\theta} = -\sin\theta$. The lesson from this is that the exact way of measuring the angle of rotation must be specified if that angle appears in the equation of rotation!◀

Note that in the solution above, the only time we really needed the X- and Y-coordinates was when we had to calculate the vectors products in the moments of the forces by utilising the unit vectors. We could alternatively have found the magnitudes of the moments directly, by taking moments. However, introducing the coordinate system does reduce errors. Therefore, unless you are quite sure that you know what you are doing, it is always best to proceed as systematically as possible!

Activity 12.2

A uniform rod of length L and mass m is placed on top of a pivot, which is situated at a point at the length $L/4$ from one end of the rod.

The rod is held horizontally, and is then released. Write down the equation of rotation, and calculate the angular acceleration of the rod, in the position shown.

....

Feedback: The force of gravity acts at the centre of the rod, and you will need to find the position vector from the pivot point to the centre of the rod; and you will also need to find the moment of inertia of the rod when it rotates about the pivot. Note that the length of the rod is L, not $2L$! The equation of rotation $-\frac{L}{4}mg = \frac{7}{48}mL^2\ddot{\theta}$ holds at the initial moment, and the angular acceleration $\ddot{\theta}$ can be found from this..

In the activity above, the position of the rod was fixed. In the next activity, you will have to take into account the fact that the pendulum could be at any angle!.

Activity 12.3

A pendulum consists of a massless rod of length ℓ, with a disc with mass M and radius r attached from its centre to one end of the rod. The other end of the rod is pivoted so that the pendulum swings freely, such that the plane of the disc is parallel to the plane in which the pendulum swings. Find the moment of inertia of the pendulum, and write down the equation of rotation of the pendulum. *(Note that you have to introduce a variable for the angle of rotation for the pendulum. You have to define this very clearly in your answer! The equation of rotation will be an expression that links the angular acceleration $\ddot{\theta}$ and the angle of rotation, θ. Remember also that the gravity acts at the centre of mass of the pendulum - you will need to find that!)*

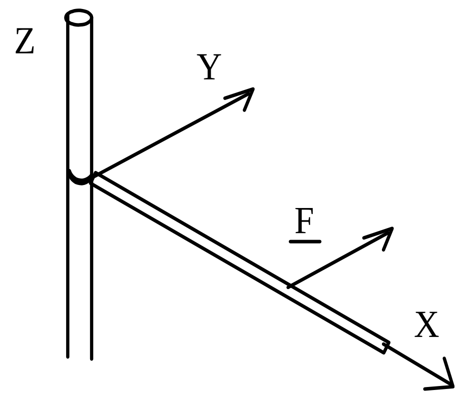
....

Feedback: $I = M\left(\frac{1}{2}r^2 + \ell^2\right)$. Remember that the rod itself is massless, so you will just need the moment of inertia of the disc, and the parallel axis theorem! The equation of motion will be $-\ell\sin\theta Mg\underline{k} = I\,\ddot{\theta}\underline{k}$.

Example 12.3

A rod of mass M and length $2a$ is pivoted at one end to a vertical pole so that it can move freely in a horizontal plane. Find the angular acceleration of the rod if a force $\underline{F}$ is applied at a point of the rod at a distance (a) $a/2$, and (b) $3a/2$ from the pole. The force is applied in a direction perpendicular to the rod, on its plane of motion.

Solution:



Let the XYZ coordinate system be as shown in the sketch above. Then the force applied can be written as $\underline{F} = F\underline{j}$, in the direction of Y-axis. We can calculate I_Z, the moment of inertia of the rod for rotation about the Z-axis by applying the parallel axes theorem: We get

$$I_Z = I_G + Md^2$$

where I_G is the inertia of the rod about an axis parallel to the Z-axis, through the centre of mass of the rod, and d is the distance between the two axes. But the centre of mass of the rod is at its midpoint, and therefore $d = a$; and as in Example 10.5,

$$I_G = \frac{1}{3}Ma^2.$$

Thus,

$$I_Z = \frac{1}{3}Ma^2 + Ma^2 = \frac{4}{3}Ma^2.$$

According to the equation of rotation,

$$\underline{r} \times \underline{F} = I_Z\ddot{\theta}\underline{k} \tag{12.4}$$

where $\ddot{\theta}$ is the angular acceleration (measured counterclockwise on the XY-plane) and $\underline{r}$ is the position vector from the axis to the point where the force $\underline{F}$ is applied.

(a) Here, $\underline{r} = \frac{a}{2}\underline{i}$. Thus,

$$\underline{r} \times \underline{F} = \frac{a}{2}\underline{i} \times F\underline{j} = \frac{1}{2}aF\underline{k}$$

and the equation of rotation (12.4) gives

$$\begin{aligned} \frac{1}{2}aF\underline{k} &= \frac{4}{3}Ma^2\ddot{\theta}\underline{k} \\ \therefore \quad \frac{1}{2}aF &= \frac{4}{3}Ma^2\ddot{\theta} \\ \therefore \quad \ddot{\theta} &= \frac{3}{8}\frac{F}{Ma}. \end{aligned}$$

(b) Now, $\underline{r} = \frac{3a}{2}\underline{i}$,

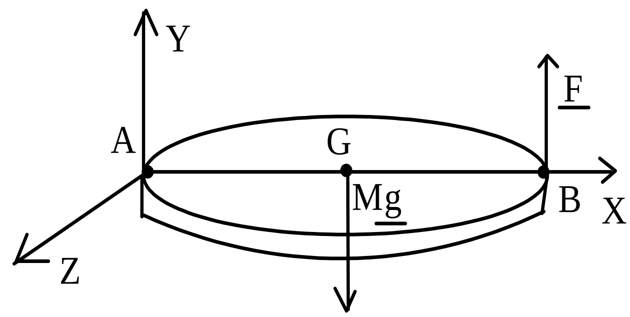
$$\underline{r} \times \underline{F} = \frac{3a}{2}\underline{i} \times F\underline{j} = \frac{3}{2}aF\underline{k}$$

and (12.4) gives

$$\begin{aligned} \frac{3}{2}aF\underline{k} &= \frac{4}{3}Ma^2\ddot{\theta}k \\ \therefore \quad \ddot{\theta} &= \frac{9}{8}\frac{F}{Ma}. \end{aligned}$$

Activity 12.4

A disc with mass M and radius r lies flat on a rough, horizontal surface. We wish to lift it up by applying an upwards force F to one edge, so that the disc pivots on the opposite edge.



Write down the equation of rotation at the position shown, when the lifting starts. How much initial force is needed to lift the disc with constant angular acceleration α?

....

Feedback: the equation of motion will be $2rF - rMg = I\ddot{\theta}$ where $I = \frac{5}{4}Mr^2$. Thus to get $\ddot{\theta} = \alpha$, we need $F = \frac{1}{2}M\left(g + \frac{5}{4}r\alpha\right)$.

12.2 Problem-solving strategies

In the rest of the chapter we will look at examples of general problems. To solve these problems, we will need to use all the concepts and results we have come across so far. In this case, our four-step problem-solving strategy could have the following tools in it (please add any extra tools you can think of to the list!)

TOOLBOX FOR SOLVING PROBLEMS INVOLVING PURE ROTATION

1. UNDERSTANDING THE PROBLEM

   Here, you must understand what the rotating object and any other components of the system are like, and what are you asked to do. To make sure that you achieve this, you might make use of the following tools:

   - Look for keywords for hints about
     - the position of the rotating object: horizontal, vertical, tangential etc.
     - the position of the axis of rotation
     - the shape and composition of the rotating object (disc, rod; uniform, composite etc.)
     - any other objects which form Learning Unit of the system, their way of motion, their links with the rotating object: pulleys, ropes, etc.
   - Use sketches and diagrams of the whole system and its components.
   - Use real-life examples and experiments.
   - Try to rephrase the problem in your own words.
   - Use standard mathematical notation for the known and unknown quantities.

2. PLANNING WHAT TO DO

   Review the available principles, results and definitions:

   - The equation of pure rotation, the moment of inertia, angular position and acceleration
   - From previous parts: Newton's equations for the motion of particles, the centre of mass, the equation for the motion of the centre of mass, and all the tools listed in Learning Unit 1

   To decide on which principles you should apply to the system and/or its various components, you may wish to try the following tools:

   - Know when the principles apply and when not.
   - Find similar problems and examples.
   - Look for principles dealing with the types of variables which are given and wanted.

   To make sure that a plan will work, also check the following:

   - Do you have all the information necessary to apply the definitions, principles and results decided on? If not, can you find the information from what is given? Alternatively, can one introduce the information as another unknown? Which definitions, principles or results deal with the new unknown?
   - Have you used all the given facts and all the conditions in the problem statement?

3. EXECUTING THE PLAN

   This is where you will have to set up the equations and solve them. You may need:

   - mathematical notation, symbols for variables, coordinate systems
   - mathematical tools (integration, solving equations etc.)
   - sketches and diagrams
   - already calculated results, tables of moments of inertia

4. ANALYSING THE SOLUTION

   To check the correctness of the solution, you can do the following:

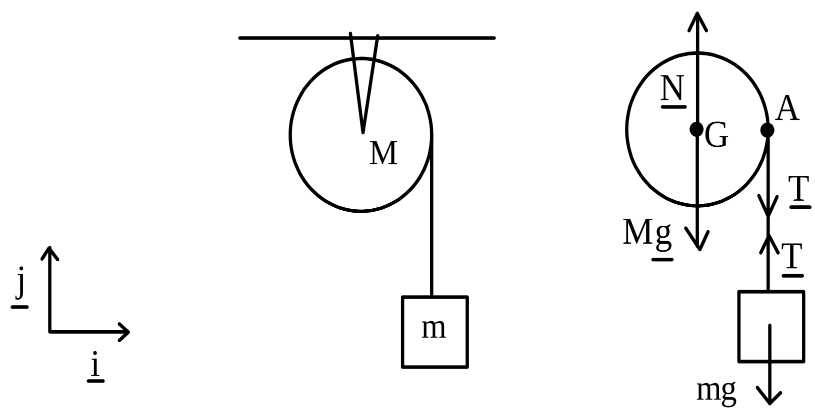
   - See whether the solution makes sense. Compare the end result to other known, similar results.
   - Try to think of other ways to solve the same problem.
   - Compare it with experiments and guesses based on real-life objects.
   - Work in a group and compare your results with those of others.

   To reflect on and learn from the solution, you can think of other systems where a similar approach would work. Try to generalise the result. Compare this problem with other systems that you have come across: what are the differences and similarities?

The following example illustrates how this toolbox could be applied.

Example 12.4
A disk of mass M and radius a is free to rotate about a fixed horizontal axis through its centre. A string is wrapped around the rim of this disk and a mass m is attached to the string. What is the downward acceleration of the mass?

Solution:
UNDERSTANDING THE PROBLEM: A sketch of the system is shown below. What we would expect to happen is this: The mass will start to drop downwards, but it is not dropping freely, since it is attached to the end of the string which is wrapped around the disc. As the mass falls, it will be unravelling the string from the disc, causing the disc to rotate. It follows that we would expect the acceleration of the mass to be less than its acceleration would be in free fall (i.e. equal to g). We are supposed to find the value of this acceleration.



PLANNING THE SOLUTION: The motion of the mass is translation, described by Newton's laws of motion. In particular, the **acceleration of the mass** is fully determined once we know the forces acting on the mass. There are two forces acting on it: gravity downwards and the **tension** of the string, upwards. The force of gravity is known, of course, but the tension on the rope is not known! So, we have two unknowns but, so far, only one equation (the equation for the motion of the mass). So, we have to find at least one other equation where the unknown tension on the rope features. One possibility is to consider what the tension of the rope does to the pulley, by applying the equation of pure rotation to the pulley. For this we have to introduce another variable, namely the **angular acceleration** of the pulley. By now we have three unknowns, but still only two equations, so we will need yet another condition linking some of the unknowns. This link will be provided by the string: since the string is wrapped around the disc, the mass can move downwards if, and only if the string unravels from the disc, and that means that the disc must turn — there must be a one-to-one connection between how the mass moves and how the disc unravels. In particular, we should be able to link the angular acceleration of the disc to the downwards acceleration of the mass.

In conclusion, we should be applying Newton's second law for the motion of the mass, and the equation of pure rotation for the motion of the disc; and we should find the link between the two accelerations. (We could have arrived at the same conclusion simply by looking at both the components and how they move, and by analysing how the motion of one is linked to the motion of the other.) Do we have all the information necessary to apply these? We think so; what remains to be found is the link between the angular acceleration of the disc and the linear acceleration of the mass. Also, we'll need the moment of inertia of the disc, but that we can look up from a table. Is the number of equations equal to the number of unknowns? Yes, we will have three unknowns and three equations.

EXECUTING THE PLANWe will apply the equation of rotation (12.1) to describe the rotation of the disc, and Newton's second law (7.1), or (2.2) if we assume the mass m to be a particle) to describe the translation of the mass m. The forces acting on the disc and the mass m are indicated in the figure above.

The disc: The forces acting on the disc are gravity $M\underline{g}$ and $\underline{N}$, the force fixing the disc in its place, both acting at G, the centre of the disc; and the tension of the string, $\underline{T}$, acting at the point A where the string leaves the disc. If we take the origin of the XY-plane to be at G, with $\underline{i}$ and $\underline{j}$ unit vectors as shown in the figure, then the forces and the position vectors of their action points are as follows: $M\underline{g} = -Mg\underline{j}$ and $\underline{N} = N\underline{j}$ act at $\underline{r}_G = \underline{0}$; $\underline{T} = -T\underline{j}$ acts at point $\underline{r}_A = a\underline{i}$. Equation (12.1) then gives

$$\begin{aligned}
\underline{r}_G \times M\underline{g} + \underline{r}_G \times \underline{N} + \underline{r}_A \times \underline{T} &= I_G\ddot{\theta}\underline{k} \\
\therefore\ \underline{0} + \underline{0} + \left(a\underline{i}\right) \times \left(-T\underline{j}\right) &= I_G\ddot{\theta}\underline{k} \\
\therefore \qquad -aT\underline{k} &= I_G\ddot{\theta}\underline{k} \\
\therefore \qquad -aT &= I_G\ddot{\theta}
\end{aligned}$$

where $\ddot{\theta}$ is the angular acceleration of the disc. For a disc rotating about an axis through its centre, the moment of inertia is

$$I_G = \frac{1}{2}Ma^2.$$

Thus, the equation for the rotation of the disc is

$$\begin{aligned}
-aT &= \frac{1}{2}Ma^2\ddot{\theta} \\
\therefore \quad T &= -\frac{1}{2}Ma\ddot{\theta} \qquad (12.5)
\end{aligned}$$

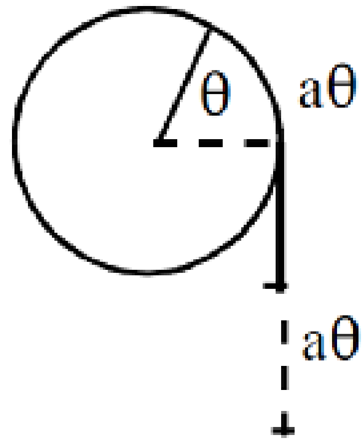
The mass m : The forces acting on the mass m are gravity $m\underline{g}$, downwards, and the tension of the rope, $\underline{T}$, upwards. In terms of our coordinate system, $m\underline{g} = -mg\underline{j}$ and $\underline{T} = T\underline{j}$. Newton's second law now gives

$$\begin{aligned}
m\underline{g} + \underline{T} &= m\underline{\ddot{x}} \\
\therefore \quad (-mg + T)\,\underline{j} &= m\ddot{x}\,\underline{j} \\
\therefore \quad -mg + T &= m\ddot{x} \qquad (12.6)
\end{aligned}$$

where $\underline{\ddot{x}} = \ddot{x}\,\underline{j}$ is the acceleration of the mass m. Combining the two equations of motion, (12.5) and (12.6), we have a set of two equations:

$$\begin{cases} T = -\frac{1}{2}Ma\ddot{\theta} \\ -mg + T = m\ddot{x} \end{cases} \qquad (12.7)$$

but there seems to be three unknowns — T, $\ddot{\theta}$ and $\ddot{x}$. However, there is clearly a link between $\ddot{\theta}$ and $\ddot{x}$: when the disc rotates through an angle θ counterclockwise, then the mass m goes up the distance $a\theta$ (which is the length of the arc of the disc, corresponding to the angle θ).



So, $x = a\theta$, $\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$ when x describes the vertical position of the mass m and θ describes the angular position of the rotating disc (we can, for instance, assume $x = 0$ and $\theta = 0$ to be at the initial position). Substituting $\ddot{\theta} = \frac{1}{a}\ddot{x}$ into (12.7), we get two equations with two unknowns:

$$\begin{cases} T = -\frac{1}{2}M\ddot{x} \\ \\ -mg + T = m\ddot{x}. \end{cases}$$

Subtracting the second equation from the first, we get the requested acceleration of the mass m:

$$\begin{aligned} mg &= \left(-\frac{1}{2}M - m\right)\ddot{x} \\ \therefore \quad \ddot{x} &= -\frac{m}{\frac{1}{2}M + m}g. \end{aligned}$$

ANALYSING THE SOLUTION: Does the solution makes sense? Compare the end result to other known, similar results. The negative sign of $\ddot{x}$ confirms that the mass accelerates **downwards** (remember that we wrote the acceleration vector of the mass m as $\underline{\ddot{x}} = \ddot{x}\,\underline{j}$ in our chosen coordinate system!) Also the acceleration, which we can write as

$$\ddot{x} = -g\left(1 - \frac{1}{1 + \frac{2m}{M}}\right)$$

is less that g, the acceleration of a free-falling particle; this is exactly what we would expect, since Learning Unit of the gravitational force $m\underline{g}$ is expended in making the disc turn. Try to think of other ways to solve the same problem – later on in this module, you will learn how to solve problems like this using the energy conservation method. Think of other systems where a similar approach would work. Try to generalise the result. It would be easy to re-do the calculations with the disc replaced by any other circular object with radius a but with an arbitrary moment of inertia I. An interesting thing about the result is that the value of $\ddot{x}$ does not depend on a, the radius of the disc, but only on its mass M. Will this always hold in similar kinds of situations?◀

About pulleys

The disc in the previous example was an example of a **pulley**. Pulleys were briefly mentioned earlier, but there a rope just passed over the pulley without friction, and therefore the pulley merely changed the direction of the rope, the tensions on both sides of the rope being equal. However, in this and the next unit we will investigate several examples where a rope passes over a rough pulley with mass (a non-smooth, non-massless pulley) consisting of a disc, ring, cylinder etc. In such a case, the pulley is an integral Learning Unit of the system and cannot be ignored.

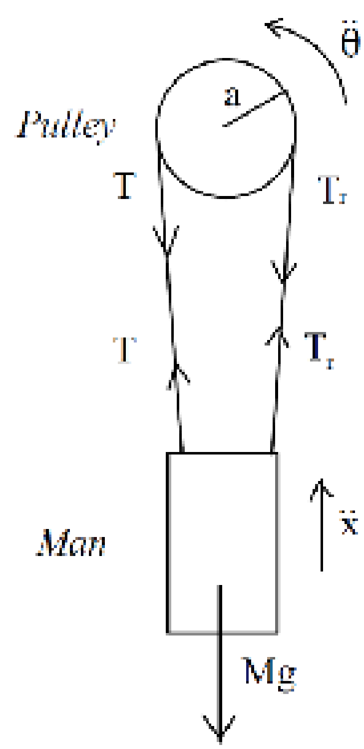
The pulley is usually fixed so that it is free to turn about an axis through its centre. Even if the pulley itself turns on its axis without any friction, for a pulley which is not smooth and which has a non-zero moment of inertia when it rotates about its axis, the rope does not just pass frictionlessly over it. It follows that the tensions on the rope on each side of the pulley in such a situation are **not equal** — Learning Unit of the tension of the rope is used to make the pulley turn. The bigger the moment of inertia of the pulley, the bigger the difference between the tensions on both sides of the pulley.

When investigating a system which involves such a pulley, we shall have to include the equation of pure rotation for the pulley, as we did in the last example. In the following, we will look at several different situations, to give you an idea of the things to take into account!

Example 12.5

A man of mass M has one end of a rope tied around his waist. The rope passes (without slipping) over a pulley. The pulley is fixed so that it can only rotate about its centre. The pulley has a radius a and a moment of inertia $\frac{1}{2}ma^2$. The man now pulls the other end of the rope with a tension T. Calculate the upward acceleration $\ddot{x}$ the man gives himself.

Solution:



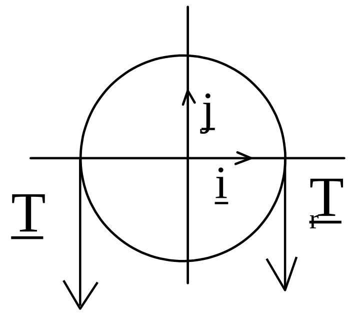
In the picture, we take the X-direction to be up (the direction of movement of the man), and θ, the angle of rotation of the pulley, is measured counterclockwise.

T is the tension with which the man pulls at the free end of the rope;
T_r is the tension on the rope tied around his waist;
$\ddot{x}$ is the upward acceleration of the man.

We write down the equations of motion, for:

Rotation of the pulley:

$$\begin{aligned} I\ddot{\theta}\underline{k} &= (-a\underline{i}) \times (-T\,\underline{j}) \\ &\quad + (a\underline{i}) \times (-T_r\,\underline{j}) \\ &= (aT - aT_r)\underline{k} \end{aligned}$$



so

$$I\,\ddot{\theta} = aT - aT_r.$$

But we assume that the rope does not slip over the pulley; therefore $\ddot{x}$ and $\ddot{\theta}$ are connected by

$$\ddot{x} = a\ddot{\theta}.$$

We substitute this, and the value $I = \frac{1}{2}ma^2$, into the equation above to get

$$\begin{aligned} \frac{1}{2}ma^2 \cdot \left(\frac{\ddot{x}}{a}\right) &= aT - aT_r \\ \therefore \quad \frac{1}{2}m\ddot{x} &= T - T_r. \end{aligned} \tag{12.8}$$

Translation of the man:

$$M\ddot{x} = T_r + T - Mg \tag{12.9}$$

(all of these forces are in the X-direction, so we don't need the vector notation). We can now proceed to solve $\ddot{x}$ from (12.8) and (12.9). Adding (12.8) and (12.8) together, we get

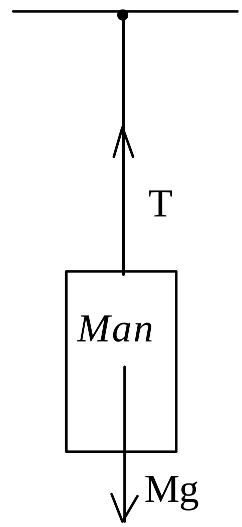
$$\begin{aligned} \ddot{x}\left(\frac{1}{2}m + M\right) &= 2T - Mg, \text{ so} \\ \ddot{x} &= \frac{2(2T - Mg)}{m + 2M} \end{aligned}$$

Note that

$$T \neq T_r \quad \text{and} \quad T_r \neq Mg!$$

There are two forces pulling the man up: firstly, the tension T of the free end of the rope, when the man is pulling himself up along it; secondly, the tension on the end of the rope tied around his waist. Compare this situation with the following two:

Here the man pulls himself up a fixed rope, with tension T. (The other end of the rope is not passed through a pulley and tied around his waist.) The upward acceleration here is simply $\ddot{x} = T - Mg$.

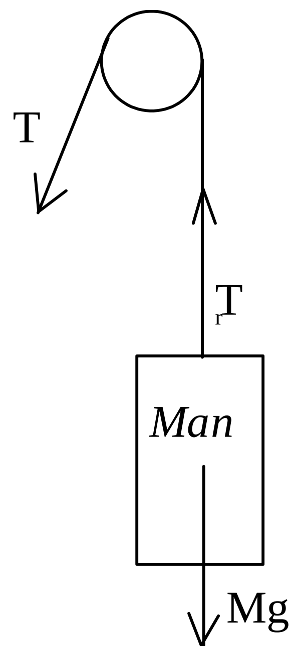


Here, someone else is pulling at the free end of the rope with a tension T.
The motions of equation are now

$$\begin{cases} \frac{1}{2} m\ddot{x} = T - T_r \\ M\ddot{x} = T_r - Mg \end{cases}$$

which give as the upward acceleration

$$\ddot{x} = \frac{2\,(T - Mg)}{m + 2M}.$$



Activity 12.5

Solve the following problem. Write down you solution in your exercise book, in complete detail, with sketches!

A pulley which is shaped like a wheel with spokes for negligible mass and with mass M and radius R is mounted on a fixed horizontal axle through its centre. You may assume that the moment of inertia of the pulley is that of a ring. A block with mass M hangs from a massless cord that is wrapped around the rim of the pulley. Find the angular acceleration of the pulley, the acceleration of the falling block, and the tension in the cord. The cord does not slip, and there is no friction at the axle.

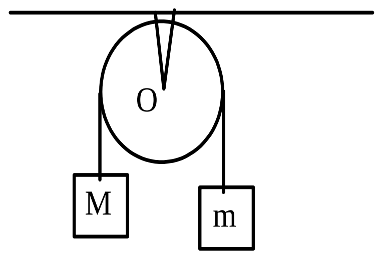
....

Feedback: $\ddot{\theta} = -\frac{g}{2R}$, $\ddot{x} = -\frac{g}{2}$, $T = \frac{1}{2}Mg$.

Activity 12.6

Solve the following problem. Note that you will need to introduce three equations of motion! You will also need *to find the link between $\ddot{x}$ and the angular acceleration of the pulley – make sure that you get the sign correct there!*

Two masses m and M $(M > m)$ are connected by a light string over a pulley with a mass m_p and a radius r. The pulley has the form of a disc fixed so that it can only rotate about its centre O. The system starts from rest. Find $\ddot{x}$, the upward acceleration of m and the downward acceleration of M.



....

Feedback: you should get $\ddot{x} = \frac{(M-m)}{\left(\frac{1}{2}m_p+M+m\right)}g$.

Small oscillations

When a body supported by a smooth horizontal axis hangs at rest, it is said to be in its stable equilibrium position. Suppose that it is shifted slightly from this position, so that it swings through a very small angle on either side of its equilibrium position. It is said to perform **small oscillations**.

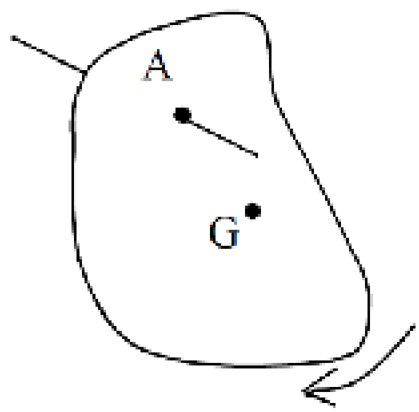
The reason why small oscillations are important is that for small angles

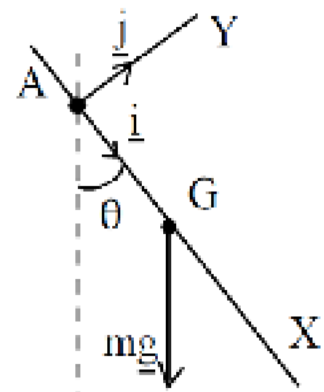
$$\sin\theta \simeq \theta$$

where $\simeq$ stands for "is approximately equal to". You might have noticed in a few previous examples that in a case involving a pendulum, we have ended up with an equation for rotation involving the terms $\sin\theta$ and $\ddot{\theta}$. The kind of a second order differential equation we obtain is difficult to solve, whereas if we can replace $\sin\theta$ by θ in the equation of rotation, the differential equation becomes immediately a very simple case of harmonic motion!

The following example illustrates this.

Example 12.6
The Pendulum
Any heavy rigid body that turns freely about a horizontal axis and is acted upon by no exterior force except gravity (ignoring the normal force at the pivot point) is called a pendulum. Consider a body of mass m which can rotate about a horizontal axis through a point A in the body.





Suppose that it performs small oscillations about the equilibrium position. Obtain the equations of motion.

Solution:
Using equation (12.1), we get

$$\underline{R} \times m\underline{g} = I_A \ddot{\theta} \underline{k}$$

where I_A is the moment of inertia about the axis at A and $\underline{R}$ is the position vector from A to G.. Let us choose X- and Y-axes as shown in the figure above. In terms of components along the X- and Y-axes we get

$$R\underline{i} \times mg \left(\cos\theta \underline{i} - \sin\theta \underline{j}\right) = I_A \ddot{\theta} \underline{k}$$

so that

$$-Rmg \sin\theta \underline{k} = I_A \ddot{\theta} \underline{k}$$

which we can express as

$$\ddot{\theta} = -\frac{Rmg \sin\theta}{I_A}.$$

This differential equation is very difficult to solve. However, if the pendulum performs small oscillations, then the angle θ is very small. In this case we can use the approximation $\sin\theta \simeq \theta$ and hence get

$$\ddot{\theta} = -\frac{Rmg\theta}{I_A}. \tag{2.26}$$

This is the equation of simple harmonic motion about the axis through A with

$$\text{period} = 2\pi \sqrt{\frac{I_A}{mgR}}.$$

Here is how this comes about: The solution to the differential equation

$$\ddot{x} = -B^2 x$$

is given by

$$x(t) = \cos(Bt) + \sin(Bt),$$

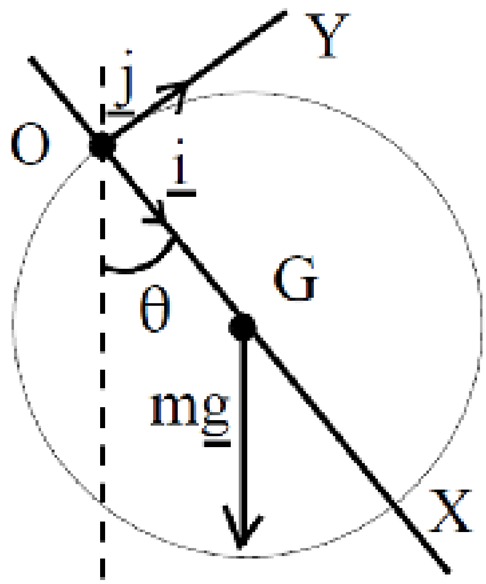
as can be seen by differentiating this function twice with respect to time t. But

$$x(t) = \cos(Bt) + \sin(Bt)$$

clearly describes periodic motion; and since the period of $\cos(t)$ *and* $\sin(t)$ *is* 2π, *(meaning that they return back to their original values after that time), the period of* $\cos(Bt)$

and $\sin(Bt)$ *is* $2\pi \cdot \frac{1}{B}$. ◄

Example 12.7
A wire circle with radius a rotates about an axis perpendicular to the plane of the circle, through a point O on the rim of the circle. The circle is in a position of stable equilibrium when its centre of mass G is vertically below O. If it is moved slightly from this position so that it performs small oscillations, calculate the period of these oscillations.



Solution:
Using equation (12.1), we get

$$a\underline{i} \times mg\left(\cos\theta\,\underline{i} - \sin\theta\,\underline{j}\right) = I_O\ddot{\theta}\underline{k}.$$

Here, using the parallel axes theorem, we get

$$I_O = I_G + ma^2 = ma^2 + ma^2 = 2ma^2.$$

Thus, we get

$$-mag\sin\theta = 2ma^2\ddot{\theta}$$

so that

$$\ddot{\theta} = -\frac{g\sin\theta}{2a}.$$

For very small oscillations $\sin\theta \simeq \theta$, and then approximately

$$\ddot{\theta} = -\frac{g\theta}{2a}$$

so that the approximate period of oscillation is

$$\text{period} = 2\pi\sqrt{\frac{2a}{g}}.$$

Activity 12.7

A rod of length $2a$ and mass m rotates about a horizontal axis through a point O at its one end.

(a) Find the equation of rotation for the rod.

(b) Suppose that the rod performs small oscillations about the equilibrium position. Find the approximate equation of rotation.

....

Feedback: (a) $\ddot{\theta} = -\frac{3}{4}\frac{g}{a}\sin\theta$, (b) $\ddot{\theta} \simeq -\frac{3}{4}\frac{g}{a}\theta$.

Activity 12.8

A uniform rod AB of mass m and length a rotates in a vertical plane about an axis which goes through a point P at a distance $a/3$ from A. If it performs small oscillations, find the period of oscillations.

....

Feedback: $\ddot{\theta} = -\frac{3}{2}\frac{g}{a}\sin\theta$, period $= 2\pi\sqrt{\frac{2a}{3g}}$.

Activity 12.9

A circular disc with mass m and radius a performs small oscillations about a horizontal axis which is *tangential* to the disc at a point A on the rim of the disc.

(a) Show that the moment of inertia about A is given by

$$I_A = \frac{5ma^2}{4}.$$

(b) Show that the period of oscillations is $2\pi\sqrt{\frac{5a}{4g}}$.

CONCLUSION

In this unit have you learned:

- how to solve problems involving pure rotation
- how to deal with pulleys, compound pendulums and small oscillations

Remember to add the following tools to your toolbox:

- the toolbox for solving problems involving pure rotation

LEARNING UNIT 4

ROTATION AND TRANSLATION

CONTENTS OF LEARNING UNIT 4

Study unit 13 The general motion of a rigid body moving in two dimensions

Introduction

We shall now deal with a rigid body which only moves in a plane and undergoes **both translation and rotation**. In order to deal with this situation we need to combine all the results we have derived so far.

It can be proven that the motion of such a body can be fully described as a combination of (1) translation of the centre of mass parallel to the plane, plus (2) rotation about an axis passing through the centre of mass and perpendicular to the plane.

In this Learning Unit 4, which consists of just one study unit, we will first discuss the problem of describing general motion and how it can be reduced to a combination of rotation and translation. After that, we shall solve many problems involving such combined motion.

The outcomes of Learning Unit 4

When you have worked through this Learning Unit of the study guide, you should be able to

- understand the differences between pure rotation, pure translation, and a combination of them
- apply the equation for the translation of the centre of mass and the equation for rotation about the centre of mass to analyse the motion of a rigid body undergoing general motion in two dimensions, for which you must be able to do, amongst other things, the following:
  - draw a sketch of a system and mark in it all relevant forces, with their directions and points of action
  - choose a suitable coordinate system
  - apply the relevant equations of motion and rotation correctly
  - understand what is meant by the rolling condition and when it applies

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LEARNING UNIT 4

ROTATION AND TRANSLATION

CONTENTS OF LEARNING UNIT 4

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Unit 13 THE GENERAL MOTION OF A RIGID BODY MOVING IN TWO DIMENSIONS

Key questions:

- *What if a rigid body does rotate, but not about a fixed axis? How do we then describe its motion?*

In this unit, we finally explain how we can analyse the motion of a rigid body by combining the equation of translation of its centre of mass, and the equation of rotation as it rotates about its centre of mass. The equations are then used to solve many problems in mechanics.

Contents of this unit:

13.1 About rotation, translation and combinations of both

13.2 Applications

13.3 Rolling without slipping

13.4 More applications

What you are expected know before working through this unit:

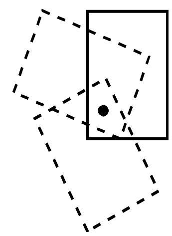
You will now need all the concepts you have come across up to now in Parts 2 and 3, as the approaches in this unit combine centres of mass and the motion of the centre of mass, plus the equation of rotation about a (moving) axis.

13.1 About rotation, translation and combinations of both

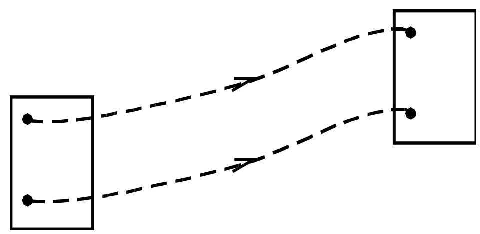
In Learning Unit 2 of the study guide we concentrated on how the centre of mass of an object moves, and in Learning Unit 3 we discussed pure rotation, where an object rotates about a fixed axis. In this study unit, we will make the very dramatic and useful discovery that being able to describe these two types of motion is in fact all that we need: combining these two types of motion, we can describe all possible types of motions of an object!

This amazing fact is actually easy to believe if you think about it a bit. We will here restrict ourselves to an object (two- or three-dimensional) which can only move in a two-dimensional plane, but you can easily check that the following reasoning works just as well in more general three-dimensional motion.

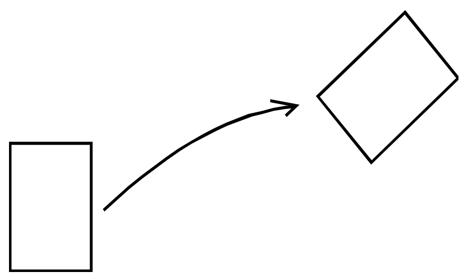
Let us start by looking at pure translation and pure rotation. Pure rotation can be defined as motion where one point in the object is motionless, but the rest of the object rotates about that point.



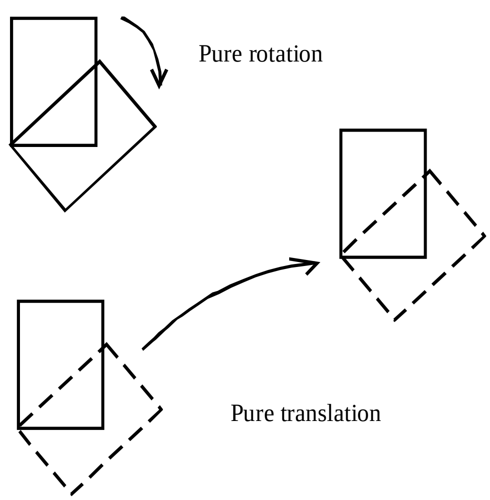
Pure translation, on the other hand, can be defined as motion where all parts of the body always move at exactly the same velocity, which implies that the object cannot change it orientation.



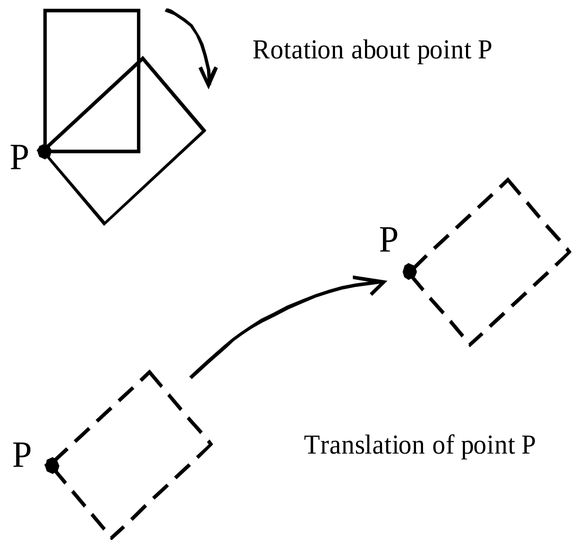
Now, in general motion the object can turn while it moves.



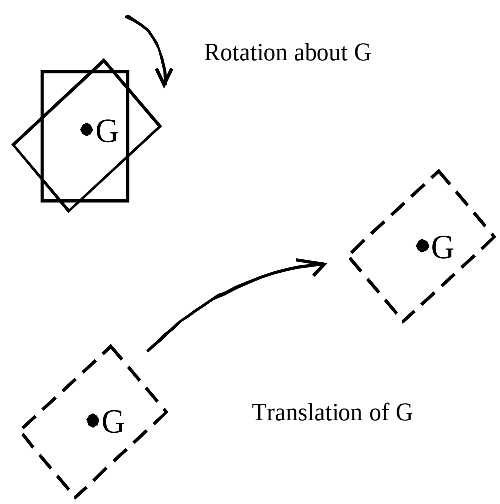
However, we can divide the general motion into separate actions of pure rotation and pure translation, which can happen simultaneously, but independently of each other.



We can select any point P on the object to do the pure rotation about. The pure translation can then be defined uniquely as the translational motion of point P, as in the pure translation of an object the rest of the body follows the same path as any fixed point of the body.



Any point of the body can be selected as the reference point about which the pure rotation is defined, and whose translation fully describes the pure translation of the body. However, it turns out that calculations of various kinds are easiest if we select the centre of mass G of the object as the reference point. In the rest of this unit, we shall therefore describe general motion as a combination of pure rotation about the centre of mass, plus translation of the centre of mass.



Consider a rigid body moving arbitrarily on a plane and acted on by n external forces, $\underline{F}_i$, $i = 1, \ldots, n$. Suppose that the body has a mass M and centre of mass G, with position vector $\underline{R}$. Then the motion of the rigid body is fully described by the following two equations we have derived in part 2 and part 3:

$$\sum_{i=1}^{n} \underline{F}_i = M\underline{\ddot{R}} \tag{13.1}$$

and

$$\sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i = I\ddot{\theta}\underline{k} \tag{13.2}$$

where $I = I_G$ and the position vectors $\underline{r}_i$ are from G. The first equation, (13.1), describes the translation of the centre of mass, and the second equation, (13.2), describes the rotation of the body about its centre of mass. That these two equations give a satisfactory description of the motion of a rigid body has been confirmed by experiment.

Remember from Learning Unit 3 that the equation of rotation (13.2) can only be applied to describe the rotation of a rigid body whenever the direction of the axis of rotation does not change; however, the axis itself may translate. This condition is guaranteed if we assume, as we shall do here, that the object moves parallel to some plane, which we can assume to be the XY-plane; the axis of rotation will then always be in Z-direction.

13.2 Applications

We will now proceed to solve various problems, using these two equations of motion. First, we will re-state our general toolbox, as it applies in this particular case.

TOOLBOX FOR SOLVING PROBLEMS BY MEANS OF EQUATIONS OF TRANSLATION AND ROTATION

1. UNDERSTANDING THE PROBLEM

   Here, you must understand what the object/system/situation is like, and what you are asked to do.

   To make sure that you have understood the problem, make sure that you can answer the following questions:

   - What is given and what is wanted? What conditions hold?
   - Can you describe the situation in your own words?

   You might make use of the following tools:

   - your knowledge of the language of mechanics problems, and using keywords for clues about objects and their properties, about positions, types of motion, links between the different components etc.
   - sketches and diagrams of the whole system and its components
   - real-life examples and experiments
   - listing in standard mathematical notation the known and unknown quantities

2. PLANNING THE SOLUTION

   Analyse the motion of the different components: what type of motion does each undergo? What are the connections between the different components and their motions?

   The three different types of motion each have their related principles, results and definitions:

   - the equation of motion for pure translation
   - the equation of motion for pure rotation
   - the equation of motion for a combination of rotation and translation

   Check the following:

   - Do we have all the information necessary to apply the equations of motion decided on? If not, can we find/calculate the information from what is given? Alternatively, can we introduce the information as another unknown? Which definitions, principles, results deal with the new unknown?
   - Is the number of equations equal to the number of unknowns?

If something seems to be missing,

- have you used all the given facts and all the conditions in the problem statement?

3. EXECUTING THE PLAN

   This is where we shall set up the equations and solve them. You may need

   - mathematical notation, symbols for variables, coordinate systems
   - mathematical tools (integration, solving equations etc.)
   - sketches and diagrams
   - already calculated results, tables of moments of inertia

4. ANALYSING THE SOLUTION

   To check the correctness of the solution you can do the following:

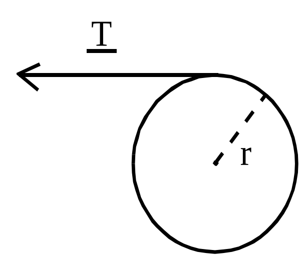
   - See whether the solution makes sense. Compare the end result to other known, similar results.
   - Try to think of other ways to solve the same problem.
   - Compare the solution with experiments and guesses based on real-life objects.
   - Work in a group and compare your results with those of others.

   To reflect on and learn from the solution, you can do the following:

   - Think of other systems where a similar approach would work. Try to generalise the result. Compare this problem with other systems that you have come across: what are the differences and similarities?

Example 13.1
A uniform disc with radius r and mass M rests flat on a smooth surface, and a thin thread is wrapped around its circumference. The thread is pulled with a constant force $\underline{T}$ along a line tangential to the disc. Calculate the acceleration of the centre of the disc, the angular acceleration of the disc, and the acceleration of the free end of the thread.

Solution:
UNDERSTANDING THE PROBLEM: Below is a sketch of the system.



Note that the sketch is shown from above. The disc rests on a smooth surface, which means that there is no friction between the surface and the disc, and the disc will slide freely along the surface. The thread is wrapped around the disc and pulled with at tension which is tangential to the disc. (Actually, whichever way we pull at the end of the thread, the ensuing tension will always be tangential to the disc, since the thread can only leave the disc tangentially to it!) What happens when the thread is pulled? The thread will start to unwind from the disc, and the disc rotates. On the other hand, the disc may also start to slide towards the left. However, there might not be any connection between the motion of the centre of the disc towards the left on the one hand, and its rotation on the other hand. We are told to find the acceleration of the centre of the disc, the angular acceleration of the disc and the acceleration of the free end of the thread.

PLANNING THE SOLUTION: The movement of the disc is on the plane, so we already know that the forces perpendicular to the plane (gravity and the normal reaction of the plane) will cancel each other out, and thus we can ignore them. The only other force acting on the disc, the tension $\underline{T}$, causes a combination of rotation and translation of the disc, which we can describe by applying the equation for the rotation of the disc about its centre, and the equation of motion for the translation of the centre of mass. The equation of rotation links the angular acceleration of the disc to the moment of the force T, and the equation for the translation of the centre of the disc links the acceleration of the centre of the disc to the force T. Thus we can easily find the acceleration of the centre of the disc and the angular acceleration of the disc. It should be possible to find the acceleration of the free end of the thread by combining these two accelerations. Do we have all the information necessary to apply the equations of motion decided on? Yes, all we need is the moment of inertia of the disc, and we also have to find an expression for the acceleration of the free end of the thread, in terms of the other two accelerations.

EXECUTING THE PLAN: We shall write down the two equations of motion.
Translation: (Equation (13.1))

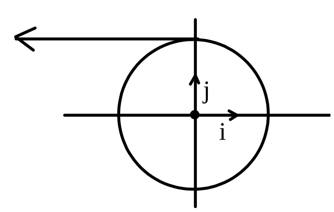
$$M\ddot{\underline{R}} = \underline{T}$$

where $\ddot{\underline{R}}$ is the acceleration of the centre of the disc (which is the centre of mass of the disc). This gives

$$\ddot{\underline{R}} = \frac{1}{M}\underline{T}$$

as the acceleration of the centre of the disc. The direction of the acceleration is the direction of the force $\underline{T}$.
Rotation: (Equation (13.2)) Let I be the moment of inertia of the disc about an axis perpendicular to the disc and through its centre:

$$I = \frac{1}{2}Mr^2.$$



Using coordinates as in the picture, the force $\underline{T} = -T\underline{i}$ acts in position $r\underline{j}$. Therefore its moment about the centre of the disc is

$$r\underline{j} \times (-T\underline{i}) = -Tr \cdot \underline{j} \times \underline{i} = Tr\underline{k},$$

so (13.2) gives

$$Tr\underline{k} = I\ddot{\theta}\underline{k}$$

when $\ddot{\theta}$ is the angular acceleration of the disc, measured counterclockwise. So, the angular acceleration is

$$\ddot{\theta} = \frac{rT}{I} = \frac{rT}{\frac{1}{2}Mr^2} = \frac{2T}{Mr}.$$

Finally, the acceleration of the free end of the thread must be equal to the tangential acceleration of the point P on the circumference where the thread leaves the disc. The tangential acceleration of P is the sum of the acceleration of P with respect to the centre of the disc, and the acceleration of the centre of the disc. The first term equals

$$\ddot{\theta}r = \frac{2T}{Mr} \cdot r = \frac{2T}{M}$$

(this is the tangential acceleration of rotational motion) and the second term is

$$\ddot{R} = \frac{T}{M}.$$

So, the acceleration of the free end of the thread equals

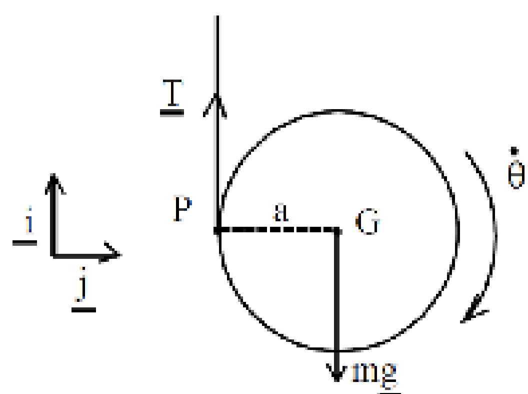
$$\ddot{R} + \ddot{\theta}r = \frac{T}{M} + \frac{2T}{M} = \frac{3T}{M},$$

in the direction of the force $\underline{T}$.

ANALYSING THE SOLUTION: The solution seems credible; the three accelerations are all in the correct direction, they are directly proportional to the force T and they are inversely proportional to the mass of the disc M.

Example 13.2

A light string is wound around a hollow cylinder of mass m, with radius a. The end of the string is fixed and the cylinder is allowed to fall so that the string unwinds. Find the acceleration of the centre of mass, G, and the tension in the string, T, if no slipping takes place.



Solution:

Again, the motion of the cylinder will be a combination of rotation and translation. However, this time there will definitely be a link between the two different types of motion: the rate at which the cylinder drops is directly linked to the way that the string unwinds and hence the way that the cylinder rotates, as there will never be any slack in the string. Let us choose our coordinate system as indicated by the $\underline{i}$ and $\underline{j}$ unit vectors in the figure above; we can assume that the origin is at the centre of the cylinder. Let $\underline{R}$ denote the position of the centre of mass G, and let θ be the angle of rotation of the cylinder, measured counterclockwise as usual. (Note that in our figure the cylinder rotates clockwise, so we should expect negative values for θ, $\dot{\theta}$ and $\ddot{\theta}$!) The two forces acting on the cylinder are the tension of the string, $\underline{T}$, upwards and acting at point P where the string leaves the cylinder; and gravity $m\underline{g}$, acting downwards at G, the centre of mass of the cylinder. In terms of our coordinate system, we can write

$$\underline{T} = T\,\underline{j}, \quad m\underline{g} = -mg\,\underline{j}$$

and the position vector from G to P is

$$\underline{r} = -a\underline{i}.$$

The equation for the translation of the centre of mass (13.1) now gives

$$\underline{T} + m\underline{g} = m\underline{\ddot{R}} \quad \therefore \quad T\,\underline{j} - mg\,\underline{j} = m\underline{\ddot{R}}.$$

If $\underline{R} = x\underline{i} + y\,\underline{j}$ are the X- and Y-components of the centre of mass, this tells us that

$$0\underline{i} + (T - mg)\,\underline{j} = m\ddot{x}\underline{i} + m\ddot{y}\,\underline{j}$$

$$\therefore \quad \begin{cases} \ddot{x} = 0 \\ m\ddot{y} = T - mg \end{cases} \tag{13.3}$$

The equation for rotation, (13.2), gives

$$\underline{r} \times \underline{T} = I_G\ddot{\theta}\underline{k}.$$

For a cylinder rotating about its centre,

$$I_G = ma^2.$$

Hence we get

$$\begin{aligned} -a\underline{i} \times T\,\underline{j} &= ma^2\ddot{\theta}\underline{k} \\ -aT\underline{k} &= ma^2\ddot{\theta}\underline{k} \\ -aT &= ma^2\ddot{\theta}. \end{aligned} \tag{13.4}$$

We have two equations, (13.3) and (13.4) but three unknowns, $\ddot{y}$, $\ddot{\theta}$ and T; however, there is a link between y and θ: when the cylinder turns counterclockwise through the angle θ, then the centre of mass goes up by the distance $y = a\theta$. Thus, $\ddot{\theta} = \frac{1}{a}\ddot{y}$. If we substitute this into (13.4), it becomes

$$\begin{aligned} -aT &= ma^2\frac{1}{a}\ddot{y} \\ \therefore \quad -T &= m\ddot{y}. \end{aligned}$$

The two equations are now

$$\begin{cases} -T = m\ddot{y} \\ T - mg = m\ddot{y}; \end{cases} \tag{13.5}$$

adding them together gives

$$\begin{aligned} -mg &= 2m\ddot{y} \\ \therefore \quad \ddot{y} &= -\frac{1}{2}g. \end{aligned}$$

The centre of mass moves straight down with an acceleration $g/2$. The tension of the rope is found when we substitute the value of $\ddot{y}$ into (13.5):

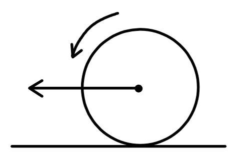
$$T = -m\ddot{y} = \frac{1}{2}mg.$$

13.3 Rolling without slipping

An important type of motion we will come across frequently when investigating combined rotation and translation is **rolling without slipping**. A typical example is a disc rolling along a rough plane. If a suitable amount of friction acts at the point of contact between the disc and the plane, then it will roll without slipping.

In the case of an object rolling without slipping, the translational and rotational motions are directly linked to each other.

As an illustration, let us look at the case of a disc with radius r rolling without slipping along a plane. (The principle can easily be applied in many other situations too, of course.) The disc will rotate about its centre, while at the same time the disc moves along the plane.

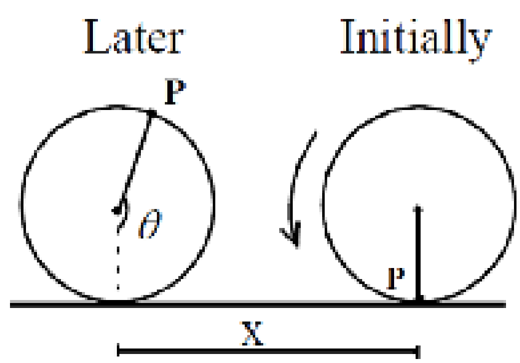


Then the following equations, called the rolling conditions, hold between θ, the angle of rotation, and x, the linear position of the centre of the disc:

$$x = r\theta, \quad \dot{x} = r\dot{\theta}, \quad \ddot{x} = r\ddot{\theta}.$$

We will explain why this is so in two different ways.

Argument 1 Let us assume that the disc has rotated through the angle θ, and let x denote the distance that the disc (its centre) has travelled along the plane during the time it took to rotate through the angle θ.



Looking at the picture above, you can see that if there was no slipping, then the distance x **equals the length of the arc corresponding to the angle** θ. But, when θ is measured in radians, the length of the arc equals $r \cdot \theta$, so therefore $x = r\theta$.

Further, if θ and x are functions of time, we can differentiate the equation above to get

$$\dot{x} = r\dot{\theta}$$

which links $\dot{x}$ (the speed of the centre of the disc along the plane) and $\dot{\theta}$ (the angular velocity of rotation). Differentiating one more time, we get

$$\ddot{x} = r\ddot{\theta}$$

which links $\ddot{x}$ (the acceleration of the disc along the surface) and $\ddot{\theta}$ (the angular acceleration of rotation).

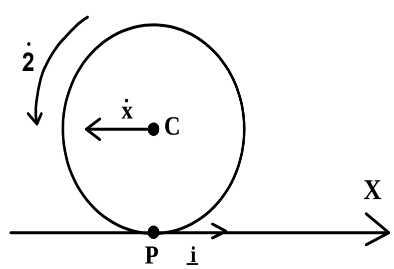
(Above, x is **distance**; if you use a coordinate system, i.e. choose to measure x along some axis parallel to the plane, then the signs you should choose for the left- and right-hand sides of these 3 equations will depend on how you choose to measure x.)

Argument 2 An alternative way to derive the link

$$\dot{x} = r\dot{\theta},$$

between the angular and linear velocities, $\dot{x}$ and $\dot{\theta}$, in rolling without slipping is as follows:

Let us assume that the disc is rolling without slipping along the plane, with angular velocity $\dot{\theta}$. This time we will be dealing with velocities (vectors), so we must carefully specify the **directions** in which things are measured. So, let θ and $\dot{\theta}$ be measured **counterclockwise** and choose the X-axis to lie along the plane, so that X increases towards the right, as usual. Let $\underline{i}$ denote the unit vector in the X-direction (see the figure below), and let $\underline{\dot{x}} = \dot{x}\underline{i}$ be the velocity of the centre of the disc in the X-direction.

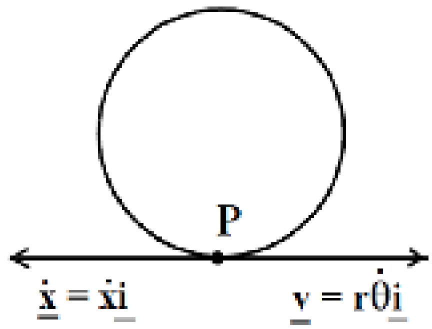


Let C denote the centre of the disc, and P the point of the disc which is currently touching the plane.

The condition of rolling without slipping means that P **cannot move horizontally along the plane** – if it did, then it would be slipping along the plane! That is, the velocity of P along the plane is $0\underline{i}$ (a zero vector). But, on the other hand, the velocity of P along the plane can be written as:

$$\begin{array}{cl} & \text{(velocity of } P \text{ with respect to } C\text{)} \\ + & \text{(velocity of } C \text{ along the plane).} \end{array}$$

The velocity of C along the plane we have agreed to denote by $\dot{x}\underline{i}$. As for the velocity of P with respect to C, which we will denote by $\underline{v}$: P is of course rotating around C on a circular path with radius r, so $\underline{v}$ is the tangential velocity of circular motion. The size of this tangential velocity is $r\dot{\theta}$, and it is tangential to the disc, in the direction of the rotation. So, in particular, at point P it is in the positive $\underline{i}$ direction: $\underline{v} = +r\dot{\theta}\underline{i}$. (See the picture below!)



In conclusion, we see that we must have

$$0\underline{i} = +r\dot{\theta}\underline{i} + \dot{x}\underline{i}$$

which gives

$$\dot{x} = -r\dot{\theta}.$$

Again, by differentiating this with respect to time, we get

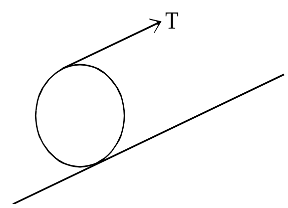
$$\ddot{x} = -r\ddot{\theta}.$$

Finally, please note that a similar kind of condition applies when a string or rope unravels from a circular object or passes without slipping over a pulley: there is a link between the distance that a point on the string/rope has moved, and the angle by which the object/pulley has rotated.

A final reminder then about rolling without slipping and ropes unwinding from pulleys: It is very important to get the sign right, or your answers to many other questions will be wrong! You must decide which way is positive for the angular variable, and which way is positive for the linear variable, and then whether they increase in the same circumstances (i.e. does x increase whenever θ increases?) in which case the sign in the equation $\ddot{x} = r\ddot{\theta}$, or whatever, will be positive. If on the other hand x increases whenever θ increases then the sign is negative and you have $\ddot{x} = -r\ddot{\theta}$ instead!

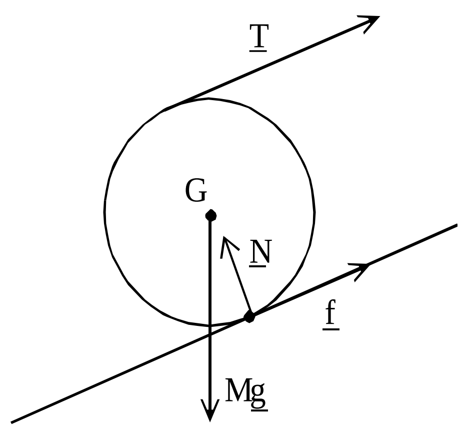
Example 13.3

A disc with radius R and mass M is placed upright on a rough slope which forms the angle α with horizontal. A thin string is wound around the disc and pulled upwards parallel to the slope with tension T, as shown below. The string does not slip on the disc, and the disc rolls without slipping along the slope.



Write down the equations of motion for the translation and rotation of the disc. What should the value of T be, so that the disc does not move or rotate?

Solution



The forces acting on the disc are: Tension T, acting along the string; force of gravity, Mg, acting downwards, at the centre of the disc; friction f acting at the point where the disc touches the plane; and the normal force from the plane, N, acting at the point where the disc touches the plane. The points of action and directions of the forces are shown in the sketch above. *You must get both of these right when you draw the forces as vectors! The normal force will act perpendicular to the plane, and gravity downwards. Friction acts parallel to the plane.*

The equation of motion parallel to the plane:

$$T + f - Mg \sin \alpha = M\ddot{x}$$

where $\ddot{x}$ is the acceleration of the centre of the disc parallel to the plane, measured positive up the slope. *Note that whenever you introduce variables for acceleration etc., you must* *explain what your coordinate system is, or alternatively, how you decide to measure them! And after you have done that, make sure that the signs of the forces match with the way you measure acceleration: If as here $\ddot{x}$ is positive up the slope then all forces acting up the slope must also be positive, and all forces down the slope must be negative!*

The equation of motion perpendicular to the plane:

$$N - Mg\cos\alpha = M\ddot{y}$$

where $\ddot{y}$ is the acceleration of the centre of the disc perpendicular to the plane, measured positive upwards.

Equation of rotation:

$$Rf - RT = \left(\frac{1}{2}MR^2\right)\ddot{\theta}$$

where $\ddot{\theta}$ is the angular acceleration, taken to be positive counterclockwise.

The translation and rotation of the disc are linked together by the rolling condition, since the disc is known to roll without slipping. Here, the rolling condition is given as

$$\ddot{x} = -R\ddot{\theta}.$$

The sign (minus or plus) in the rolling condition depends on how you chose to measure $\ddot{x}$ and $\ddot{\theta}$. Here, we took $\ddot{\theta}$ to be positive counterclockwise, but when the disc rolls counterclockwise in the sketch above, the disc rolls down the slope while we decided to take $\ddot{x}$ positive up the slope – hence the minus sign in the rolling condition.

Also, since we know that the disc is rolling (rather than bouncing for instance) down the plane, we know that the centre of mass of the disc travels parallel to the plane and therefore $\ddot{y} = 0$.

If the disc does not move or rotate, we should have $\ddot{x} = 0$ and $\ddot{\theta} = 0$. If we substitute these to the equations of translation and rotation, we get the two equations

$$\begin{cases} T + f - Mg\sin\alpha = 0 \\ Rf - RT = 0 \end{cases}$$

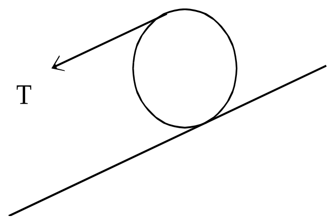
from which we can solve the value of T:

$$T = \frac{1}{2}Mg\sin\alpha.$$

Activity 13.1

Write down a complete solution to the following problem in your exercise book. Your solution must include a sketch, in which you should mark in all the forces acting on the disc. You must decide on and indicate in the sketch a coordinate system, and you must specify how you measure the linear and angular accelerations.

A disc with radius R and mass M is placed upright on a rough slope which forms the angle α with horizontal. A thin string is wound around the disc and pulled downwards parallel to the slope with tension T, as shown below. The string does not slip on the disc, and the disc rolls without slipping along the slope.



Find the angular acceleration of the disc as a function of M, T, α, R and g.

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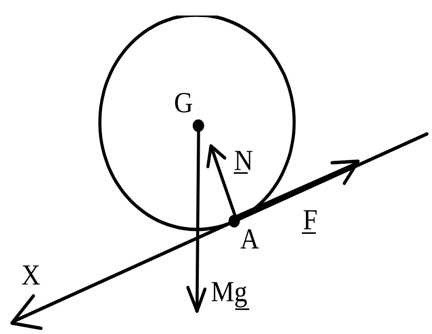
Feedback: You will need to find the rolling condition; make sure you are very clear in which direction the accelerations are measured, since that will determine the sign in the rolling condition! You should get $\ddot{\theta} = \frac{2}{3MR}\left(2T + Mg\sin\alpha\right)$ .The full solution is in your workbook.

13.4 More applications

Example 13.4
A ring with mass M and radius a rolls from rest, without slipping, down a plane inclined at an angle α to the horizontal. Show that the angular velocity of the ring is $\frac{\sqrt{gd\sin\alpha}}{a}$ when it has travelled the distance d down the plane.

Solution:
Let A be the point of contact between the ring and the plane, and let G denote the centre of the ring. The three forces acting on the ring are: $\underline{F}$, the force of friction and $\underline{N}$, the normal reaction exerted by the plane on the ring, both acting at point A; and $M\underline{g}$, the force of gravity, acting at point G. Let us take the X-axis to be down the plane (see the picture).



Firstly, we write down the equation for the translation of the centre of the ring. Since it moves in the X-direction, we only have to consider the components of all the forces along the X-axis. Newton's second law then gives

$$Mg\sin\alpha - F = M\ddot{x}$$

where $\ddot{x}$ is the acceleration of the centre of the ring down the plane.

Secondly, we write down the equation of rotation, by taking moments about G. Here, only $\underline{F}$ (friction) contributes because all the other forces act through point G. If a is the radius of the ring, we get

$$aF = I_G\ddot{\theta}$$

where $\ddot{\theta}$ is the angular acceleration of the ring, measured counterclockwise. But, $I_G = Ma^2$ for a ring. So,

$$\begin{aligned} aF &= Ma^2\ddot{\theta} \\ \therefore \quad F &= Ma\ddot{\theta}. \end{aligned}$$

Since there is no slipping, there is a link between $\ddot{x}$ and $\ddot{\theta}$, namely

$$\ddot{x} = a\ddot{\theta}.$$

Note that $\ddot{x}$ and $\ddot{\theta}$ have the same sign here, since counterclockwise rotation of the ring corresponds to motion in the direction of the positive X-axis. Substituting this into the first equation of motion, we see that the two equations describing the motion of the ring are

$$\begin{cases} Mg\sin\alpha - F = Ma\ddot{\theta} \\ F = Ma\ddot{\theta} \end{cases}$$

If we solve $\ddot{\theta}$ from these, we get

$$\ddot{\theta} = \frac{g\sin\alpha}{2a},$$

that is, the angular acceleration is constant. Finally, it remains to calculate the angular velocity when the ring has travelled the distance d down the plane. For this, we can apply the formula $\omega^2 = \omega_0^2 + 2\ddot{\theta}\theta$, which links (for rotation with constant angular acceleration, $\ddot{\theta}$) the values of the initial angular velocity ω_0, the final angular velocity ω and the total angle of rotation, θ. We have

$$\begin{aligned} \omega_0 &= 0 \text{ (initially, the ring is at rest);} \\ \omega &= \dot{\theta}, \quad \theta = \frac{1}{a}d. \end{aligned}$$

So,

$$\left(\dot{\theta}\right)^2 = 0 + 2\frac{g\sin\alpha}{2a}\frac{1}{a}d$$

so that

$$\dot{\theta} = \frac{\sqrt{dg\sin\alpha}}{a}.$$

Activity 13.2

A disc with radius r and mass M rolls down a plane inclined at an angle α to the horizontal. Find its angular and linear accelerations.

....

Feedback: you should get $\ddot{x} = \frac{2}{3}g\sin\alpha$ if $\ddot{x}$ is measured positive down the slope.

Activity 13.3

A uniform sphere with mass M and radius a rolls without slipping down a rough plane which makes an angle α with the horizontal. Find the acceleration of the centre of mass G and the force of friction.

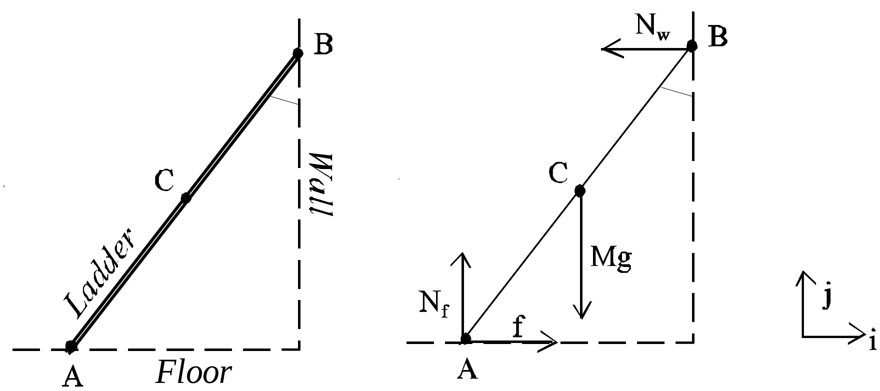
....

Feedback: $\ddot{x} = -\frac{5}{7} g \sin\alpha$ if $\ddot{x}$ is measured positive up the slope.

Example 13.5
A ladder of length ℓ and mass M leans against a smooth wall, supported on a rough floor. When the ladder is inclined at an angle $\theta = 30^\circ$ to the vertical, the ladder is about to slip. Find the coefficient of friction between the ladder and the floor. If $\theta = 25^\circ$, find how far up the ladder a 70 kg man can climb before the ladder slips.

Solution:
The motion of the ladder, whether it is sliding or stationary, is fully described by the equations of motion. The equation of motion for translation gives the acceleration of the centre of mass in terms of all the forces acting on the ladder, and the equation of motion for rotation gives the angular acceleration of the ladder for rotation about its centre of mass in terms of the moments of the forces acting on the ladder about that point.

If the ladder is just about to slip, then the forces acting on it, and their moments, must cancel out so that the angular and linear accelerations are zero. Therefore, to solve this exercise we shall find the forces acting on the ladder, write down the equations of rotation and translation, take the linear and angular acceleration to be zero, and solve for the requested information.



Let us assume that the ladder forms an angle of 30$^{\mathrm{o}}$ with the vertical. Let C denote the centre of the ladder (which is also its centre of mass), A the end of the ladder touching the floor, and B the end of the ladder touching the wall. Then the forces acting on the ladder are, as indicated in the sketch above, on the right:

- gravity $M\underline{g}$ acting at point C
- friction between the ladder and the floor, $\underline{f}$, acting at A
- the normal force between the floor and the ladder, $\underline{N}_f$, acting at A
- the normal force between the wall and the ladder, $\underline{N}_w$, acting at B

Note that there is no friction acting at B, since the wall is smooth.

Then, the equation of motion for translation is

$$M\underline{g} + \underline{f} + \underline{N}_f + \underline{N}_w = M\underline{\ddot{R}} \tag{13.6}$$

where $\underline{\ddot{R}}$ denotes the acceleration of the centre of mass.

The equation of motion for rotation about the centre of mass (point C) is

$$\left(\underline{CC} \times M\underline{g}\right) + \left(\underline{CA} \times \underline{f}\right) + \left(\underline{CA} \times \underline{N}_f\right) + \left(\underline{CB} \times \underline{N}_w\right) = I\ddot{\theta}\underline{k} \tag{13.7}$$

where I is the moment of inertia of the ladder when it rotates about point C, $\ddot{\theta}$ is the angular acceleration, and $\underline{CC}$, $\underline{CA}$ and $\underline{CB}$ are the position vectors from point C to points C, A and B respectively (where, of course, $\underline{CC} = \underline{0}$).

Next, we will introduce an XY coordinate system with $\underline{i}$ and $\underline{j}$ unit vectors, so that we can express (13.6) and (13.7) in a more concrete form. We will take the $\underline{i}$ and $\underline{j}$ unit vectors as shown in the sketch. Then we can express all the force vectors and position vectors in terms of the $\underline{i}$ and $\underline{j}$ unit vectors, as follows:

$$\begin{aligned} M\underline{g} &= -Mg\underline{j} & \underline{CC} &= \underline{0} \\ \underline{f} &= f\underline{i} & \underline{CA} &= -\frac{\ell}{2}\sin\left(30^{\mathrm{o}}\right)\underline{i} - \frac{\ell}{2}\cos\left(30^{\mathrm{o}}\right)\underline{j} \\ \underline{N}_f &= N_f\underline{j} & \underline{CB} &= \frac{\ell}{2}\sin\left(30^{\mathrm{o}}\right)\underline{i} + \frac{\ell}{2}\cos\left(30^{\mathrm{o}}\right)\underline{j} \\ \underline{N}_w &= -N_w i \end{aligned}$$

We can also write $\underline{\ddot{R}} = \ddot{x}\underline{i} + \ddot{y}\underline{j}$ to express the linear acceleration divided into its $\underline{i}$ and $\underline{j}$ components.

When we substitute the expressions above into (13.6) and separate the $\underline{i}$ and $\underline{j}$ components, we get the two equations

$$f - N_w = M\ddot{x} \tag{13.8}$$

$$-Mg + N_f = M\ddot{y} \tag{13.9}$$

When we substitute the expressions above into (13.7), and evaluate the vector products, we get

$$\begin{aligned} &\left(\underline{0} \times -Mg\underline{j}\right) + \left(-\frac{\ell}{2}\sin\left(30^{\mathrm{o}}\right)\underline{i} - \frac{\ell}{2}\cos\left(30^{\mathrm{o}}\right)\underline{j}\right) \times f\underline{i} \\ &\quad + \left(-\frac{\ell}{2}\sin\left(30^{\mathrm{o}}\right)\underline{i} - \frac{\ell}{2}\cos\left(30^{\mathrm{o}}\right)\underline{j}\right) \times N_f\underline{j} \\ &\quad + \left(\frac{\ell}{2}\sin\left(30^{\mathrm{o}}\right)\underline{i} + \frac{\ell}{2}\cos\left(30^{\mathrm{o}}\right)\underline{j}\right) \times \left(-N_w\underline{i}\right) = I\ddot{\theta}\underline{k} \end{aligned}$$

$$\therefore \frac{\ell}{2}\cos\left(30^{\mathrm{o}}\right) f - \frac{\ell}{2}\sin\left(30^{\mathrm{o}}\right) N_f + \frac{\ell}{2}\cos\left(30^{\mathrm{o}}\right) N_w = I\ddot{\theta}. \tag{13.10}$$

If μ denotes the coefficient of friction, then we will further have

$$f = \mu N_f.$$

We will substitute this into (13.8) and (13.10) and will thus get the three equations of motion

$$\begin{cases} \mu N_f - N_w = M\ddot{x} \\ -Mg + N_f = M\ddot{y} \\ \\ \frac{\ell}{2}\cos\left(30^{\mathrm{o}}\right)\mu N_f - \frac{\ell}{2}\sin\left(30^{\mathrm{o}}\right) N_f + \frac{\ell}{2}\cos\left(30^{\mathrm{o}}\right) N_w = I\ddot{\theta} \end{cases}$$

which fully describe the horizontal and vertical translation of the centre of mass, and the rotation about the centre of mass.

To find the value of μ for which the ladder is just about to slip, we shall have to take $\ddot{x} = 0$, $\ddot{y} = 0$, $\ddot{\theta} = 0$ and solve for μ. (Note that there are then three equations and three unknowns, μ, N_f and N_w).

$$\begin{cases} \mu N_f - N_w = 0 \\ -Mg + N_f = 0 \\ \\ \frac{\ell}{2}\cos\left(30^o\right)\mu N_f - \frac{\ell}{2}\sin\left(30^o\right)N_f + \frac{\ell}{2}\cos\left(30^o\right)N_w = 0 \end{cases}$$

We can, for instance, find μ as follows: substitute

$$N_w = \mu N_f$$

(from the first equation) and

$$N_f = Mg$$

(from the second equation) into the third equation, to get

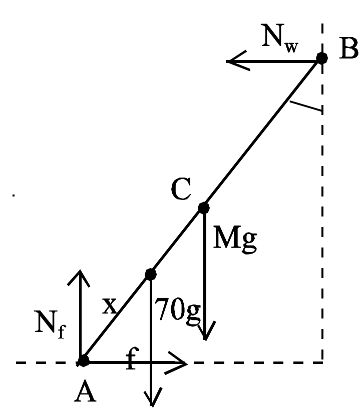
$$\frac{\ell}{2}\cos\left(30^o\right)\mu Mg - \frac{\ell}{2}\sin\left(30^o\right)Mg + \frac{\ell}{2}\cos\left(30^o\right)\mu Mg = 0$$

$$\therefore \quad \ell\cos\left(30^o\right)\mu Mg = \frac{\ell}{2}\sin\left(30^o\right)Mg$$

$$\therefore \quad \mu = \frac{1}{2}\tan\left(30^o\right) = \frac{1}{2}\sqrt{3}.$$

That is, if the coefficient of friction in $\mu = \frac{1}{2}\sqrt{3}$ then the ladder is just about to slip when it forms an angle of 30^o with the wall.

In the second part of the question, we consider the same ladder, now forming a smaller angle of 25^o with the wall. If the coefficient of friction is the same as before, then the ladder on its own will not slip. However, a man standing on the ladder will add a downwards force onto the ladder and, depending on his position, may cause the ladder to slip. To find how far up the ladder the man can climb, we shall assume that he stands at a point on the ladder which is a distance x from the end A of the ladder. We will write down the equations of motion for translation and rotation. To find out for which value of x the ladder is just about to slip, we will set $\ddot{\theta} = 0$, $\ddot{R} = 0$ in the equations of motion and solve for x.



The equations of motion for the translation of the centre of mass in the $\underline{i}$ and $\underline{j}$ directions, are

$$\begin{aligned} N_f - Mg - 70g &= M\ddot{y} \\ \mu N_f - N_w &= M\ddot{x}, \end{aligned}$$

so after setting $\ddot{x} = \ddot{y} = 0$, we get the two equations

$$N_f - Mg - 70g = 0 \tag{13.11}$$

$$\mu N_f - N_w = 0 \tag{13.12}$$

which must hold if the ladder is just about to slip.

In the equation of rotation, $\ddot{\theta} = 0$ means that the moments of all the forces about the centre of mass of the system (man + ladder) cancel out. But if that is true, then the angular acceleration about any other point of the ladder must also vanish, and accordingly the moments of all the forces taken about any other point of the ladder must also cancel out. We will calculate the moments of the forces about the end point A of the ladder, and set their sum to zero. We get

$$-x \sin\left(25^{\mathrm{o}}\right) 70g - \frac{\ell}{2}\sin\left(25^{\mathrm{o}}\right) Mg + \ell\cos\left(25^{\mathrm{o}}\right) N_w = 0. \tag{13.13}$$

We now have 3 equations, (13.11), (13.12) and (13.13), and 3 unknowns, N_f, N_w and x. We can find the value of x for instance as follows: from (13.11) we get

$$N_f = (M + 70)\, g$$

and therefore (13.12) gives

$$N_w = \mu N_f = \mu\,(M + 70)\, g.$$

When we substitute these into (13.13), it becomes

$$\begin{aligned} x \sin\left(25^{\mathrm{o}}\right) 70g &= \ell\cos\left(25^{\mathrm{o}}\right) N_w - \frac{\ell}{2}\sin\left(25^{\mathrm{o}}\right) Mg \\ &= \ell\cos\left(25^{\mathrm{o}}\right) \mu\,(M+70)\, g - \frac{\ell}{2}\sin\left(25^{\mathrm{o}}\right) Mg \\ \therefore x &= \frac{\ell\cos\left(25^{\mathrm{o}}\right)\mu\,(M+70)\,g - \frac{\ell}{2}\sin\left(25^{\mathrm{o}}\right) Mg}{\sin\left(25^{\mathrm{o}}\right) 70g}, \quad \mu = \frac{\tan\left(30^{\mathrm{o}}\right)}{2} \\ &= \frac{\ell}{2}\left(\frac{\tan\left(30^{\mathrm{o}}\right)}{\tan\left(25^{\mathrm{o}}\right)} - 1\right)\frac{M}{70} + \frac{\ell}{2}\frac{\tan\left(30^{\mathrm{o}}\right)}{\tan\left(25^{\mathrm{o}}\right)} \\ \therefore x &\approx 0.0017M\ell + 0.618\ell. \end{aligned}$$

Activity 13.4

A ladder of length 2ℓ and with mass m rests on a rough floor and against a smooth wall. The coefficient of friction between the ladder and the floor is μ. Find the smallest angle that the ladder could make with the floor.

....

Feedback: the answer will be $\tan^{-1}\left(\frac{1}{2\mu}\right)$.

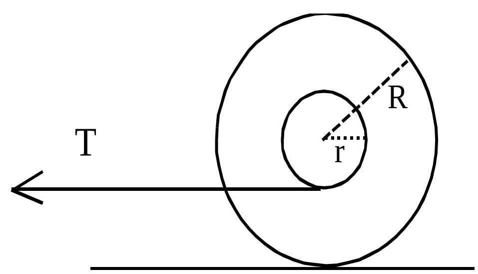
Activity 13.5

Prove that a ladder cannot rest on a smooth floor against a vertical wall, however rough the wall is.

Feedback: there are many ways to approach this; the easiest way is to prove that there will be non-zero moments of force acting on the ladder, or that the total horizontal forces can never cancel out. But try to solve this using the equations of motion (rotation and translation). What do you get? Why can you conclude that the ladder will not be at rest?

Example 13.6

A wheel with mass M and radius R has an axle with radius r. The wheel rests upright on a horizontal table. A thin string is wrapped around the axle and is pulled with tension T in a horizontal direction. The moment of inertia of the wheel when it rotates about its centre is taken to be $I = \frac{1}{2}MR^2$



The coefficient of friction between the wheel and the table is denoted by μ.

(a) Write down the equation for the motion of the wheel for (i) linear translation, (ii) rotation.

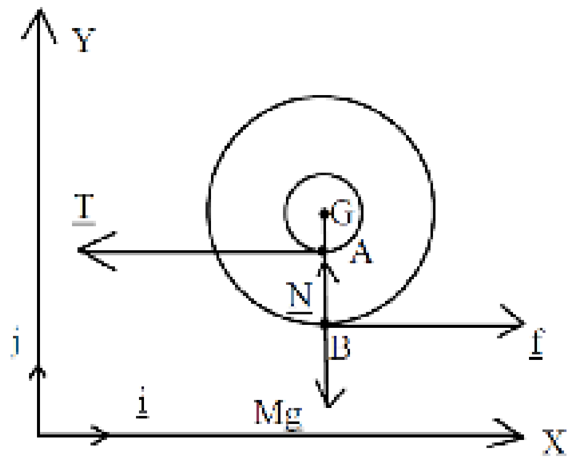
(b) Find the acceleration of the centre of the wheel in terms of M, R, r and T, if it rolls without slipping.

(c) What is the maximum coefficient of friction μ_0, such that the wheel slides without rolling?

(d) What is the direction of rotation in the sketch, if $\mu > \mu_0$?

Solution:

Let A denote the point where the string leaves the axle, B the point of contact between the wheel and the table, and G the centre of the wheel. The following forces act on the wheel: Tension $\underline{T}$ at point A, gravity $M\underline{g}$ at point G, the normal force $\underline{N}$ at B, and friction $\underline{f}$ at B.



If the coordinate system is chosen as indicated in the sketch, then we see that

$$\begin{aligned}
&\underline{GA} = -r\,\underline{j}, \quad \underline{GB} = -R\,\underline{j},\\
&\underline{N} = N\,j,\\
&M\underline{g} = -Mg\,\underline{i},\\
&\underline{f} = \mu\,N\,\underline{i} \ \ \text{(where } \mu \text{ is the coefficient of friction)},\\
&\underline{T} = -T\,\underline{i}\ .
\end{aligned}$$

(a) If $\ddot{R} = \left(\ddot{x}\,\underline{i} + \ddot{y}\,\underline{j}\right)$ is the acceleration of the centre of mass G, and $\ddot{\theta}$ is the angular acceleration of the disc, measured counterclockwise, then the equations of motion are as follows: (i) Linear translation (from (13.1)):

$$\begin{aligned}
M\ddot{R} &= \underline{T} + M\underline{g} + \underline{N} + \underline{f}\\
\therefore \quad M\left(\ddot{x}\,\underline{i} + \ddot{y}\,\underline{j}\right) &= \mu\,N\,\underline{i} - T\,\underline{i} + N\,\underline{j} - Mg\,\underline{j} \qquad (13.14)
\end{aligned}$$

(ii) Rotation (from (13.2)):

$$\begin{aligned}
&\left(\underline{GA}\right) \times T + \left(\underline{GB}\right) \times \underline{f} = I\,\ddot{\theta}\,\underline{k}\\
\therefore \quad &\left(-r\,\underline{j}\right) \times \left(-T\,\underline{i}\right) + \left(-R\,\underline{j}\right) \times \left(\mu\,N\,\underline{i}\right) = I\,\ddot{\theta}\,k\\
\therefore \quad &-rT + \mu\,R\,N = I\,\ddot{\theta} \qquad (13.15)
\end{aligned}$$

Dividing (13.14) into its $\underline{i}$ and $\underline{j}$ components, we get

$$\left\{\begin{array}{l} M\,\ddot{x} = \mu\,N - T\\ M\,\ddot{y} = N - Mg \end{array}\right. \qquad (13.16)$$

Obviously, $\ddot{y} = 0$ since the wheel moves along the table. Hence we see that we must have $N = Mg$. Therefore, the motion of the wheel is fully determined by the following two equations, obtained when we substitute $N = Mg$ into (13.14) and the first equation in (13.16) and use the fact that $I = \dfrac{1}{2}MR^2$ for the wheel:

$$\left\{\begin{array}{rl} M\,\ddot{x} = & \mu\,Mg - T\\ \\ \dfrac{1}{2}MR^2\,\ddot{\theta} = & \mu\,R\,Mg - r\,T \end{array}\right. \qquad (13.17)$$

(b) If the wheel rolls without slipping, then we must have $\ddot{x} = -R\ddot{\theta}$ (when the wheel rotates through an angle θ counterclockwise, the centre of the wheel has moved the distance $x = R\theta$ to the left, in the negative X–direction). Substituting $\theta = -\ddot{x}\,/\,R$ into the second equation in (13.17), we get the pair of equations

$$\begin{cases} M\ddot{x} = \mu Mg - T \\ \frac{1}{2} MR\ddot{x} = \mu R Mg - rT \end{cases}$$

which can be solved (e.g. by dividing the second equation by $-R$ and adding it to the first equation) to get

$$\ddot{x} = -\frac{2}{3}\frac{T}{M}\left(\frac{R-r}{R}\right).$$

Note that in rolling without slipping we therefore have $\ddot{x} < 0$ *and* $\ddot{\theta} = -\ddot{x}/R > 0$, *that is, the wheel rotates counterclockwise!*

(c) For the wheel to slide without rotating, we must have $\ddot{\theta} = 0$ in the second equation in (13.17). (Note that the rolling condition $\ddot{x} = -R\ddot{\theta}$ does **not** hold in this case!) So, we must have

$$0 = \mu R Mg - rT$$

which holds if

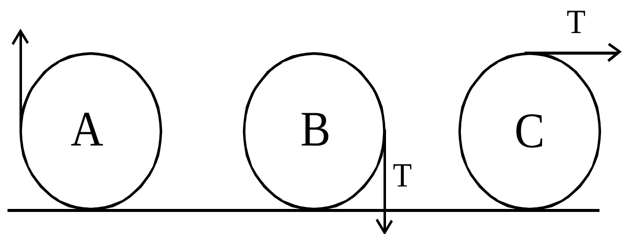
$$\mu = \mu_0 = \frac{rT}{R Mg}.$$

(d) If $\mu > \mu_0$ then, in the second equation in (13.17)

$$\begin{aligned} \frac{1}{2}MR^2\ddot{\theta} &= \mu R Mg - rT \\ &> \mu_0 R Mg - rT = 0 \end{aligned}$$

that is, $\ddot{\theta} > 0$, which means that rotation is counterclockwise.◀

Activity 13.6



Three identical discs A, B and C, each with radius R and mass M, are placed upright on a horizontal plane. A thin thread is wrapped clockwise around each disc. The end of the thread is then pulled with constant tension. The magnitude of the tension is T for each disc, but the direction in which the tread is pulled varies from disc to disc, as shown above. What are the directions of linear acceleration of the centre of each disc, in each of the following two cases?

(a) If there is no friction between the discs and the plane.

(b) If the discs roll without slipping on the plane.

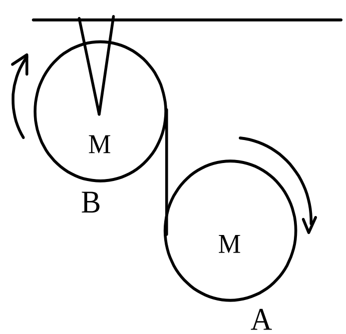
Make sure you can justify your answers, for instance by writing down the equations of motion!

....

Feedback: Case (a) A and B: no motion; C: to the right. Case (b): A, B and C all move towards the right.

Activity 13.7

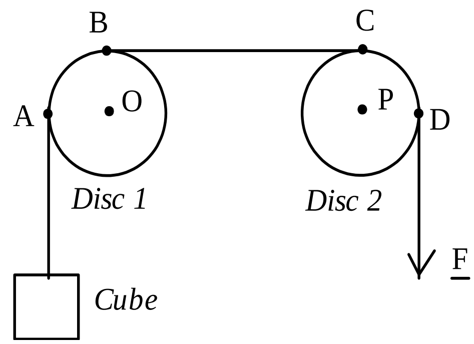
A solid cylinder A with mass M and radius R is suspended from a solid cylinder B, also with mass M and radius R, which is free to rotate about its axis (see the figure below). The suspension is in the form of a massless tape wound around the outside of each cylinder, and free to unwind, as shown. Both cylinders are initially at rest. Find the initial acceleration of cylinder A, assuming that it moves straight down.



....

Feedback: You must write down three equations of motions: one for rotation of B, one for rotation of B and one for translation of A! This means three variables, $\ddot{x}$, $\ddot{\theta}_A$ and $\ddot{\theta}_B$. Decide which way you want to measure each of these and then find the link between then — it will be one equation with all three variables in it! You will get $\ddot{x} = \frac{4}{5}g$.

Activity 13.8



The system in the sketch consists of two identical discs, both with radius R and mass M, fixed next to each other so that each is free to rotate about an axis through its centre. A massless string passes without slipping over both the discs, as shown. A cube with mass $3M$ is attached to one end of the string, and the other end is pulled downwards with a constant tension F. In the sketch, O and P denote the centres of the discs, and A, B, C, D the points where the string leaves the discs.

(a) Copy the sketch. List, and draw in your sketch, all the forces acting on the two discs and on the cube.

(b) Write down the equations of motion for the vertical motion of the cube, and the rotation of the two discs. Use $\ddot{y}$ to denote the vertical acceleration of the cube, with upwards motion being positive, and $\ddot{\theta}_1$, $\ddot{\theta}_2$, respectively, to denote the angular accelerations of the two discs, measured counterclockwise.

(c) What is the connection between $\ddot{y}$ and $\ddot{\theta}_1$, and $\ddot{\theta}_1$ and $\ddot{\theta}_2$?

(d) Solve the equations of motion to find the value of $\ddot{y}$ if $F = 2Mg$. In what direction will the cube move?

....

Feedback: In (d), you should get $\ddot{y} = -\frac{g}{4}$, so the cube moves down. See the workbook for a full worked out solution.

CONCLUSION

In this unit you have learned how to

- describe general motion as a combination of translation and rotation
- solve problems involving general motion
- apply the rolling condition

Remember to add the following tools to your toolbox:

- the concept of rolling without slipping
- the toolbox for solving problems involving rotation and translation

LEARNING UNIT 5

ENERGY METHODS

CONTENTS OF LEARNING UNIT 5

Study unit 14	Work and energy
Study unit 15	The energy conservation method

Introduction

Many problems to do with rigid bodies can be solved very easily by using energy methods, which are based on the principle of the conservation of energy. To derive the relevant formulas, we shall first have to introduce the concepts of **work** and **energy**. Once again, we start by first considering a single particle; later we shall see how these concepts apply to a **rigid body**.

The outcomes of Learning Unit 5

When you have worked through this Learning Unit of the study guide, you should be able to:

- explain what is meant by the work done by a force and how the concepts of kinetic and potential energy follow from this
- select a suitable zero energy level for the gravitational potential energy and use it to calculate the potential energy of particles, systems of particles and bodies
- calculate the kinetic energy for objects undergoing pure translation, pure rotation or a combination of translation and rotation
- explain what is meant by the energy conservation principle and in which cases it applies
- apply the energy conservation principle to solve problems

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Unit 14 WORK AND ENERGY

Key questions:

- *What is meant by work and by energy?*
- *What is the difference between potential and kinetic energy?*
- *How do we calculate the gravitational potential energy of a particle, a system, a rigid body?*
- *How do we calculate the kinetic energy in pure rotation, in pure translation, and in more general motion?*

In this unit, we will introduce the concepts of work and energy, and explain how the potential and kinetic energies are calculated. In the next unit, you will learn how to solve some problems quite easily, using the energy conservation principle which relies on potential and kinetic energy calculations!

Contents of this unit:

14.1 The one-particle case

14.2 Energy

14.3 The energy conservation principle

14.4 The energy of a system of particles

14.5 The kinetic and potential energy of a rigid body

What you are expected know before working through this unit:

This unit will refer you to the scalar product of vectors, discussed in unit 2, and to some line integrals, but you just need to be able to follow the reasoning here. The derivations of the translational and rotational kinetic energies refer to concepts of the centre of mass, and the moment of inertia and angular velocity, respectively.

14.1 The one-particle case

14.1.1 Work done by a constant force

We will start off with an explanation of what work and energy mean. These concepts are a bit complex, but please do read carefully through what follows — in the expectation that in the end, we will again come up with a very user-friendly and practical formula! The outcome of this unit is not really that you can calculate directly the work done in each situation; rather, we are building up to a formula for calculating kinetic and potential energies in various situations!

Consider a particle moving along a straight line, say along the X-axis. If a constant force F, also along the X-axis, acts on the particle, then the **work** done by the force on the particle as it moves some given distance $\triangle x$ is defined by

$$W = F \cdot \triangle x. \tag{14.1}$$

More generally, for a particle moving along a straight line in a three-dimensional space, which is acted upon by a general constant force, we must take into account the possibility that the force and the motion of the particle may be in different directions. Therefore, it will be convenient to use vector notation.

The motion of the particle is expressed in terms of a **displacement vector,** $\underline{\triangle r}$. If the motion of the particle is from a point P to a point Q, then the displacement $\underline{\triangle r}$ is the vector from P to Q, that is,

$$\underline{\triangle r} = \underline{PQ}.$$

Similarly, the constant force acting on the particle is now a vector, $\underline{F}$.

Now, only the component of the force which is parallel to the displacement does work. This can be formulated in terms of the scalar product of vectors, as follows:

Definition 14.1 (The work done by a constant force)
The work done by a constant force $\underline{F}$ acting on a particle, as it moves through a displacement $\underline{\triangle r}$ along a straight line, equals

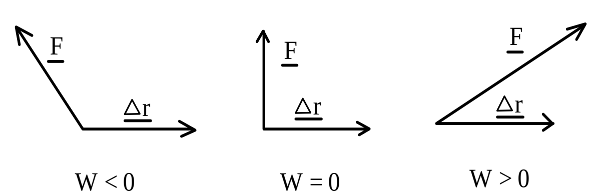
$$W = \underline{F} \bullet \underline{\triangle r}.$$

(Note that this equals $F_{\triangle r} \cdot \triangle r$ where $\triangle r$ is the length of the displacement and $F_{\triangle r}$ is the component of the force which is parallel to the displacement. This agrees with the definition in equation (14.1).)

Remember that the product in Definition 14.1 is a scalar product of two vectors, whose value of a real number (a scalar). We are using a big dot for denoting it in this section just to remind you of this fact! Please see Unit 2 for a reminder of how the scalar product works.

The following properties of work follow immediately from Definition 14.1.

1. The work done by a force $\underline{F}$ on a particle as it moves through a displacement $\underline{\triangle r}$ is **positive** if the force acts in the "same direction" as the displacement, that is, if the angle between them is smaller than $\pi/2 = 90^\circ$; and the work done is **negative** if the displacement and the force are in "opposite directions", that is, if the angle between them is larger than $\pi/2$.



2. If the displacement and the force are perpendicular to each other, then $\underline{F} \bullet \underline{\triangle r} = 0$, that is, the work done is zero.

3. If the displacement is zero: $\underline{\triangle r} = 0$, then no work is done, however large the force $\underline{F}$ is.

Example 14.1
An object with mass m is to be moved around. Calculate the force that has to be applied, and the work done by the force, in each of the following operations:

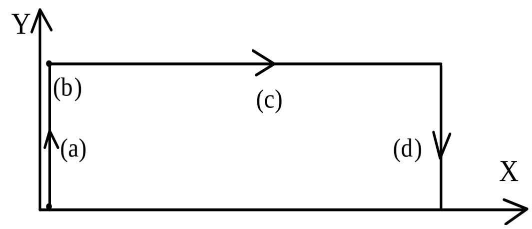
(a) The object is lifted up with a constant velocity from the floor to a height of 1 metre.

(b) The object is held in a fixed position 1 metre above the floor.

(c) The object is moved sideways 5 metres at a constant velocity at a height of one metre above the floor.

(d) The object is lowered back to the floor, at a constant velocity.

Solution:
If the origin of the coordinate system is at the initial position of the object, then the trajectory of the object looks like this:



The force needed in any of the operations can be determined by Newton's second law. In all four operations, the object moves with constant velocity (with zero velocity in (b)). So, the acceleration of the object is zero throughout. The only forces acting on the object are the unknown force $\underline{F}$ and gravity $m\underline{g} = -m g\underline{j}$, acting downwards; hence, according to Newton's second law,

$$\begin{aligned} \underline{0} &= -mg\underline{j} + \underline{F} \\ \therefore \quad \underline{F} &= mg\underline{j}. \end{aligned}$$

That is, in all four cases, we only have to apply enough force to counter the force of gravity.

(a) When the object is lifted up, the displacement vector is $\underline{\triangle r} = \underline{j}$. The work done by the force in this displacement is

$$W = \underline{\triangle r} \bullet \underline{F} = \underline{j} \bullet mg\underline{j} = mg.$$

(b) Here,

$$\begin{aligned} \underline{\triangle r} &= \underline{0} \text{ so} \\ W &= \underline{0} \bullet mg\underline{j} = 0. \end{aligned}$$

(c) Now, $\underline{\triangle r} = 5\underline{i}$; $\underline{\triangle r}$ and $\underline{F}$ are perpendicular to each other and thus

$$W = 5\underline{i} \bullet mg\underline{j} = 0.$$

(d) Here,

$$\underline{\triangle r} = -\underline{j},$$

so

$$W = \underline{\triangle r} \bullet \underline{F} = -\underline{j} \bullet mg\underline{j} = -mg.$$

Usually more than one force act on a particle, and then we can talk about the work done by each of the forces.

Example 14.2
Consider again Example 14.1. This time, let's find the work done by the force of gravity $-mg\underline{j}$ in the four operations.

(a) $W_{grav.} = \underline{j} \bullet \left(-mg\underline{j}\right) = -mg$

(b) $W_{grav.} = \underline{0} \bullet \left(-mg\underline{j}\right) = 0$

(c) $W_{grav.} = 5\underline{i} \bullet \left(-mg\underline{j}\right) = 0$

(d) $W_{grav.} = \left(-\underline{j}\right) \bullet \left(-mg\underline{j}\right) = mg.$

If n forces $\underline{F}_1, \underline{F}_2, \ldots, \underline{F}_n$ act on the particle and

$$\underline{F} = \sum_{i=1}^{n} \underline{F}_i$$

is the resultant force acting on the particle, then for W, the work done by the force $\underline{F}$ when the particle moves through some displacement, we have

$$W = \sum_{i=1}^{n} W_i$$

where W_i is the work done by force F_i. W is then the total work done on the particle in its displacement by all the forces acting on it. For instance, in Example 14.1 above, the resultant force is zero and thus the total work done on the object by all forces in all four operations equals zero.

14.1.2 Work done by a variable force

In practice, we cannot assume that all forces are constant. Also, we often wish to look at more general motion, rather than at motion along a straight line. A more general situation would look as follows: A particle moves along some path from point P_1 to P_2. A force, which depends on the position of the particle, acts on the particle – that is, if $\underline{r}$ is the position vector of the particle, then the force is a function of $\underline{r}$:

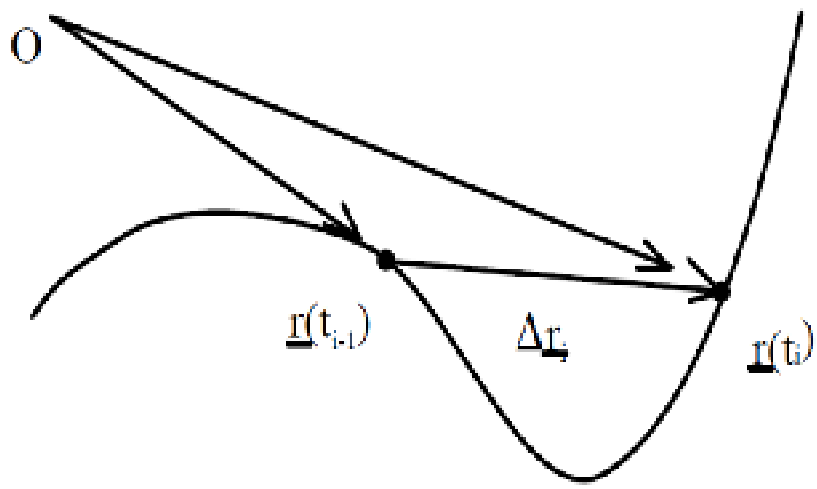
$$\underline{F} = \underline{F}\left(\underline{r}\right).$$

Now, how do we calculate the work done by the force as the particle moves from P_1 to P_2 along the given path? The obvious thing to do is to divide the path into short segments. Over a short enough segment, the force and the direction of motion are approximately constant, and the work done by the force over that little segment can be calculated approximately, as in the previous section. For the work done over the whole path, we simply add up the amounts of work done over all the subsegments. The technical details are as follows.

Let the path of a particle be described by the curve $\underline{r}(t)$, $0 \le t \le T$, where $\underline{r}(t)$ is the position of the particle at time t and in particular, $\underline{r}(0)$ and $\underline{r}(T)$ are the initial and final points of the path. We divide the path of the particle from $\underline{r}(0)$ to $\underline{r}(T)$ into n short path segments, by dividing the time interval $0 \le t \le T$ into small time intervals, $\left[t_{i-1}, t_i\right]$, where $0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = T$. Then over the ith segment of the path , from $\underline{r}(t_{i-1})$ to $\underline{r}(t_i)$, the force acting on the particle is approximately equal to $\underline{F}\left(\underline{r}(t_{i-1})\right)$ and the path of the particle over this segment is approximately described by the displacement vector

$$\triangle \underline{r}_i = \underline{r}(t_i) - \underline{r}(t_{i-1}).$$

Thus, the work done by the force on the particle as it travels along the ith path segment is approximately $\underline{F}\left(\underline{r}(t_{i-1})\right) \bullet \triangle \underline{r}_i$.



Adding up over all the path segments, the total work done by the force as the particle moves from $\underline{r}(0)$ to $\underline{r}(T)$ along the given path is approximately

$$W \approx \sum_{i=1}^{n} \underline{F}\left(\underline{r}(t_{i-1})\right) \bullet \triangle \underline{r}_i.$$

The approximation becomes exact when we take the limit as $n \to \infty$ and $\triangle t_i \to 0$; the sum becomes an integral, and we get the following definition:

Definition 14.2 (Work: the general case)
The work done by a force $\underline{F}$ on a particle as it moves from $\underline{r}(0)$ to $\underline{r}(T)$ along a given path is

$$W = \int_{\underline{r}(0)}^{\underline{r}(T)} \underline{F}\left(\underline{r}\right) \bullet d\underline{r}. \tag{14.2}$$

The integral (14.2) is called a **line integral**. Its value can be calculated by ordinary definite integrals, if we are able to write $\underline{F}\left(\underline{r}\right)$ and $d\underline{r}$ in their component forms and evaluate the scalar product. However, this may be difficult, because in general we would have to know the description of the path as an equation! In any case, you will not have to use the line integral directly in this module — we will just use it to derive certain results.

Note that the value of W could well depend on the path used to travel from $\underline{r}(0)$ to $\underline{r}(T)$. If the work done by a force is **independent** of the path taken, then we say that the force is **conservative**.

14.2 Energy

In this module, we are only going to look at two kinds of energy: kinetic energy and gravitational potential energy.

14.2.1 Kinetic energy

Kinetic energy is energy that an object possesses because of its motion. To see how this arises, let us investigate the behaviour of a particle which is acted upon by a force or forces. Let $\underline{F}$ be the resultant force acting on the particle. Let $\underline{r}(t)$ denote the position of the particle at time t. The force $\underline{F}$ could depend on the position of the particle, so we have in general $\underline{F} = \underline{F}\left(\underline{r}\right)$. According to Newton's second law, the motion of the particle is described by the equation

$$\underline{F} = m\ddot{\underline{r}} = m\frac{d\dot{\underline{r}}}{dt}. \tag{14.3}$$

What now is the work done by the force $\underline{F}$ on the particle, when it moves from position $\underline{r}(0)$ to position $\underline{r}(T)$ along some path? We know that the work done is given by the line integral

$$W = \int_{\underline{r}(0)}^{\underline{r}(\tau)} \underline{F} \bullet d\underline{r}.$$

From (14.3), we have

$$\underline{F} \bullet d\underline{r} = \left(m\frac{d\dot{\underline{r}}}{dt}\right) \bullet d\underline{r}$$

But this equals

$$= \frac{1}{2}m\frac{d}{dt}\left(\left|\dot{\underline{r}}\right|^2\right)dt,$$

since $\left|\dot{\underline{r}}\right|^2 = \dot{\underline{r}} \bullet \dot{\underline{r}}$ and thus, in terms of the product rule of differentiation,

$$\begin{aligned} \frac{d}{dt}\left|\dot{\underline{r}}\right|^2 &= \frac{d}{dt}\left(\dot{\underline{r}} \bullet \dot{\underline{r}}\right) \\ &= \left(\frac{d}{dt}\dot{\underline{r}} \bullet \dot{\underline{r}}\right) + \left(\dot{\underline{r}} \bullet \frac{d}{dt}\dot{\underline{r}}\right) = 2\left(\frac{d\dot{\underline{r}}}{dt} \bullet \dot{\underline{r}}\right) \end{aligned}$$

and $d\underline{r} = \dot{\underline{r}}dt$. We can therefore rewrite the line integral as a time integral:

$$\begin{aligned} W &= \int_0^\tau \frac{1}{2}m\frac{d}{dt}\left(\left|\dot{\underline{r}}(t)\right|^2\right)dt \\ &= \frac{1}{2}m\left|\dot{\underline{r}}(t)\right|^2\Big]_0^\tau = \frac{1}{2}m\left(\left|\dot{\underline{r}}(\tau)\right|^2 - \left|\dot{\underline{r}}(0)\right|^2\right) \\ &= \frac{1}{2}mv^2 - \frac{1}{2}mu^2 \end{aligned}$$

where $v = \left|\dot{\underline{r}}(\tau)\right|$ is the velocity at the final point and $u = \left|\dot{\underline{r}}(0)\right|$ is the velocity at the initial point of the path.

Definition 14.3 (The kinetic energy of a particle)
If $v = |\dot{\underline{r}}|$ is the speed of a particle of mass m, then the quantity

$$K = \frac{1}{2}mv^2$$

is called the kinetic energy of the particle.

If the velocity of the particle is zero ($v = 0$), then we say that the particle is at rest; for such a particle the kinetic energy obviously equals zero. Note that the kinetic energy of a particle does not depend on the direction of the velocity, but only on its magnitude!

Following from to the calculations above, we have the following result, known as the **Work–Energy Theorem**:

Result 14.4 (The Work–Energy Theorem)
The work done by the resultant force on a particle equals the change in the kinetic energy of the particle.

The resultant force acting on a particle causes acceleration, according to Newton's second law; it increases/decreases the velocity of the particle, changing its kinetic energy by an amount which equals the amount of work done by the force.

14.2.2 Potential energy

We say that a particle has **potential energy** if it has the capacity to do work by virtue of its **position,** in the sense that work can always be done simply by moving the particle to another position. Potential energy always corresponds to a conservative force (e.g. the force of a spring, or gravity). For a conservative force, the work done does not depend on the path taken. Thus it makes sense to talk about the work done on a particle when it moves from position P_1 to position P_2 when only conservative forces act on the particle. (If the non-conservative forces acted on the particle, this concept would not be well defined, as it might depend on the path taken!)

If we fix one point, let's call it O, as the "standard position", then we can define the potential energy at any other point P to be equal to the amount of work a force acting "against" the conservative force would need to do when moving the particle from position O to position P. That is, at the chosen standard position, the potential energy (denoted by V) of the particle is $V = 0$, and **the potential energy at any other position P equals the work done by the conservative force if the particle were to be moved from P back to O.** Note that the "standard" position, also called the zero potential energy position, can be arbitrarily chosen! Potential energy therefore has no absolute value, and only **differences** of potential energy at various points have any physical meaning.

The only conservative force and potential energy we are concerned with in this module is the **gravitational** force and the corresponding potential energy. The force of gravity acting on a particle of mass m is $m\underline{g}$. This is a constant force, so it is conservative. We will calculate the work done by the force of gravity on the particle, to obtain an expression for the gravitational potential energy.

Let us assume that the motion is in three dimensions, with the Z-axis identified with the up/down direction. That is, we let the negative Z-axis denote the downwards direction.

Then, the gravitational force acting on the particle is

$$\underline{F} = -mg\underline{k}.$$

Let the particle move from a point P_1 with position vector

$$\underline{r}_1 = x_1\underline{i} + y_1\underline{j} + z_1\underline{k}$$

to a point P_2 with position vector

$$\underline{r}_2 = x_2\underline{i} + y_2\underline{j} + z_2\underline{k}.$$

If the position vector of the particle is

$$\underline{r} = x\underline{i} + y\underline{j} + z\underline{k},$$

then

$$d\underline{r} = dx\underline{i} + dy\underline{j} + dz\underline{k}$$

and then the work done by the force of gravity on the particle as it moves from P_1 to P_2 is

$$\begin{aligned} W &= \int_{P_1}^{P_2} (-mg\underline{k}) \bullet \left(dx\underline{i} + dy\underline{j} + dz\underline{k}\right) \\ &= -mg\int_{P_1}^{P_2} \left((\underline{k} \bullet \underline{i})\, dx + \left(\underline{k} \bullet \underline{j}\right) dy + (\underline{k} \bullet \underline{k})\, dz\right) \\ &= -mg\int_{P_1}^{P_2} dz = -mg\,(z_2 - z_1)\,. \end{aligned}$$

That is, the work done by the force of gravity only depends on the difference between the Z-coordinates of the two positions P_1 and P_2. We arrive at the following definition:

Definition 14.5
If we fix an arbitrary level along the Z-axis, say $Z = z_0$, as the zero energy level of the gravitational energy, then the value of the potential energy of a particle with respect to this zero energy level is given by

$$V = mgh$$

where $h =$ the **height difference** (positive or negative) from the position of the particle to the zero energy level.

For any position above the chosen zero energy level, the potential energy V is positive; for any position below the zero energy level, V is negative.

14.3 The energy conservation principle

Let us assume that only a conservative force $\underline{F}$ acts on our particle, and let us investigate the potential and kinetic energies of the particle as it moves from a point P to a point Q. Let V_P and V_Q denote the potential energy of the particle at positions P and Q, respectively (with respect to an arbitrarily chosen zero energy position). Then, by the definition of potential energy, the work done by the conservative force on the particle as it moves from P to Q equals

$$W_{PQ} = V_P - V_Q.$$

On the other hand, we know that the work done by $\underline{F}$ on the particle equals the change in the kinetic energy of the particle, that is,

$$W_{PQ} = K_Q - K_P$$

when K_Q and K_P denote the kinetic energy of the particle at Q and P, respectively.

Thus, we have

$$V_P - V_Q = K_Q - K_P$$

so that

$$V_P + K_P = V_Q + K_Q \tag{14.4}$$

That is, the sum of the potential and kinetic energies are the same at point P and Q; and since P and Q are arbitrary points, we see that the sum of the potential and kinetic energies is always constant if only conservative forces act on a particle. The sum of the kinetic and potential energy is called the total mechanical energy of the particle, and is usually denoted by E.

14.4 The energy of a system of particles

In the case of a system of particles, the kinetic energy of the system is the sum of the kinetic energies of all the individual particles.

Similarly, the potential energy of a system of particles or a rigid body is the sum of the potential energies of the individual particles. However, the calculations of the potential energy of a system can be simplified by using the centre of mass. Let a system consist of n particles, each with a mass m_i and position vector $\underline{r}_i = x_i\underline{i} + y_i\underline{j} + z_i\underline{k}$. If we choose the XY-plane (with $Z = 0$) as the zero energy level for the gravitational potential energy, then the total potential energy of the system with respect to the chosen zero energy level is

$$V = \sum_{i=1}^{n} m_i g z_i.$$

We can re–write this in terms of the centre of mass of the system: If $M = \sum_{i=1}^{n} m_i$ is the total mass of the system then

$$V = Mg \ \frac{\sum_{i=1}^{n} m_i z_i}{M},$$

but in terms of the definition of the centre of mass, the last term here is the Z-coordinate of the centre of mass of the system.

Thus, to calculate the gravitational potential energy of a system of particles or a rigid body, we can replace the system with a particle of mass M, situated at the centre of mass of the system. Once the zero potential energy level has been chosen, the potential energy of the system is given by

$$PE = Mgh$$

where M is the mass of the system and h is the vertical distance from G, the centre of mass of the system, to the zero energy level; h is taken to be negative if G is below the zero energy level and positive if G is above the zero energy level.

In summary: The energy of a system of particles is just the sum of the energies of the individual particles:

Definition 14.6 The kinetic and potential energy of a system of particles
The kinetic energy of a system of particles is defined at the sum of the kinetic energies of all the particles.

The potential energy of a system of particles with respect to a fixed zero energy level is the sum of the potential energies of the particles, and can alternatively be found as

$$PE = Mgh$$

where M is the mass of the system and h is the vertical distance from G, the centre of mass of the system, to the zero energy level.

The principle of the conservation of mechanical energy extends to systems of particles, bodies, or even systems of bodies. In general terms we can express it as follows:

Result 14.7 (The law of the conservation of mechanical energy)
If only conservative forces act on a system of particles, then the sum of the kinetic and potential energies is constant:

$$P.E. + K.E. = E$$

where E (a constant) is called the total mechanical energy.

The law of the conservation of mechanical energy states that if only conservative forces act on a system of particles, then energy (potential or kinetic) cannot disappear from the system or be added to the system from outside. Note that an effort to add energy from outside, e.g. by pulling at one of the parts of the system, invariably involves a non-conservative force! Also, non-conservative frictional forces change energy from kinetic to heat energy; in those cases the total mechanical energy is not conserved.

However, it is possible for kinetic energy to change into potential energy, and the other way around. Indeed, a change in kinetic energy must be accompanied by a corresponding change in the potential energy. For instance, if the potential energy of the system decreases by a certain amount (which in the case of gravitational potential energy means that the centre of mass of the system moves lower down), then the kinetic energy must increase by the same amount.

The energy conservation principle will be very valuable to us in the next unit. We will restate it there as it applies to rigid bodies and systems.

14.5 The kinetic and potential energy of a rigid body

Once again, a rigid body can be considered to be just a special case of a system of particles. Reasoning as above, we can arrive at the following result for the potential energy of a rigid body.

Result 14.8
If we fix an arbitrary level along the Z-axis, say $Z = z_0$, as the zero energy level of the gravitational energy, then the value of the potential energy of a rigid body with respect to this zero energy level is given by

$$V = mgh$$

where $h =$ the **height difference** (positive or negative) from the centre of mass to the zero energy level.

Please remember that the zero energy level is not fixed, but can be chosen arbitrarily! Only differences in the potential energy have any real meaning.

Activity 14.1

A pendulum consists of a rod AB of length $2L$ and mass m with a thin disc of mass M and radius r attached rigidly at its centre to the rod's end point B. The pendulum rotates about an axis through point A of the rod, perpendicular to the plane of the disc. If the zero energy level of the (gravitational) potential energy is taken to go through point A, find the potential energy of the pendulum in the following cases:

(a) If the pendulum is horizontal;

(b) If the pendulum stands upside down vertically above point A;

(c) If the pendulum forms the angle 45° with a vertical line drawn directly downwards from A.

....

Feedback: You will need to locate the centre of mass of the object first, it is at the distance $x = L\left(1 + \frac{M}{m+M}\right)$ from point A. The potential energies are: (a) $PE = 0$; (b) $PE = gL\,(m + 2M)$; (c) $PE = -g\frac{1}{\sqrt{2}}L\,(m + 2M)$ (where the $1/\sqrt{2}$ comes from $\cos(45)$!)

We still need a concise expression for the total kinetic energy of a rigid body. One is tempted to guess that the kinetic energy would simply be

$$\text{“}K = \frac{1}{2}M\dot{R}^2\text{"},$$

analogous to similar results for the total angular momentum, linear momentum and potential energy, but this is wrong: what is missing here is the possibility of any energy involved in the motion of the various particles in relation to the centre of mass. We can use this definition for pure translation, but we will need another expression for the kinetic energy — both for pure rotation, and for combined rotation and translation.

Remember that pure translation is motion where all parts of the body always move at exactly the same velocity; pure rotation is motion where one point of the body always stays fixed and the other parts of the body rotate about it; and general motion can be expressed as a combination of the translation of the centre of mass and the rotation of the body about the centre of mass.

14.5.1 Kinetic energy in pure translation

We know that all the particles have the same velocity in pure translation. This velocity equals the velocity of the centre of mass. It follows that we can easily add up the kinetic energies of separate masses, and end up with the following definition.

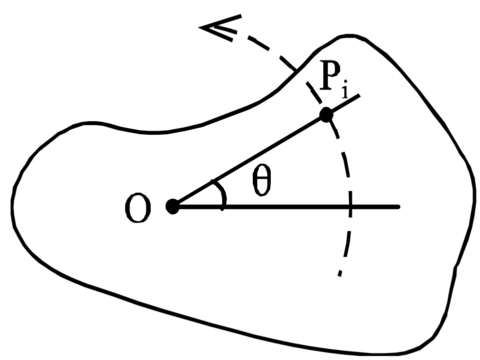
Result 14.9 (Kinetic energy in pure translation)
For a rigid body undergoing pure translation, the kinetic energy is equal to

$$KE = \frac{1}{2}M\dot{R}^2 \tag{14.5}$$

where M is the mass of the body and $\dot{R} = \left|\underline{\dot{R}}\right|$ is the velocity (speed) of the centre of mass.

14.5.2 Kinetic energy in pure rotation

Suppose now that the motion of the rigid body is pure rotation, and in particular that the rigid body rotates about a **fixed axis** through point O.



If P_i is a particle of mass m_i at a point with position vector $\underline{r}_i$ from O then, as the body rotates, P_i will move in a circle with radius $r_i = \left|\underline{r}_i\right|$ around the fixed axis. The velocity of P_i is tangential to the circle and of a magnitude $v_i = r_i\dot{\theta}$, where $\dot{\theta}$ is the angular velocity of the rotation. Thus the kinetic energy of P_i is $\frac{1}{2}v_i^2 m_i = \frac{1}{2}r_i^2\dot{\theta}^2 m_i$, and the total kinetic energy of the entire body is

$$\begin{aligned} KE &= \sum_{i=1}^{n} \frac{1}{2}r_i^2\dot{\theta}^2 m_i = \frac{1}{2}\left(\sum_{i=1}^{n} r_i^2 m_i\right)\dot{\theta}^2 \\ &= \frac{1}{2}I\dot{\theta}^2. \end{aligned}$$

Result 14.10 (Kinetic energy in pure rotation)
For a rigid body rotating about a fixed axis, the kinetic energy is equal to

$$KE = \frac{1}{2}I\dot{\theta}^2 \tag{14.6}$$

where I is the moment of inertia and $\dot{\theta}$ the angular velocity of the rotation.

If the angular velocity of the body rotating about a fixed axis is zero ($\dot{\theta} = 0$), then we say that the body is at rest, and then the (rotational) kinetic energy equals zero. (Note that this is only true for a body rotating about a fixed axis. In general motion, which we discuss in the next section, the motion could be a combination of rotation and translation, in which case the $\dot{\theta} = 0$ would not necessarily mean that the body is at rest — it would just mean that the body is not rotating.)

Example 14.3
Calculate the kinetic energy of

(a) a ring

(b) a disc

with mass M and radius a, rotating about an axis through its centre, with a constant angular velocity ω.

Solution:

(a) $I_G = Ma^2$ so that

$$KE = \frac{1}{2}\left(Ma^2\right)\omega^2 = \frac{1}{2}Ma^2\omega^2$$

(b) $I_G = \frac{1}{2}Ma^2$ so that

$$KE = \frac{1}{2}\left(\frac{1}{2}Ma^2\right)\omega^2 = \frac{1}{4}Ma^2\omega^2.$$

Thus, a rotating ring has twice as much kinetic energy as a disc with the same mass and radius, rotating at the same velocity. This follows directly from the fact that the ring has a greater moment of inertia, meaning that it is more difficult to set into rotating motion.◀

Activity 14.2

A thin rod AB has a length of 2ℓ and a mass $2M$. Particles with a mass m are attached at the end point B and at the centre of the rod. The combined object rotates about an axis perpendicular to the rod, through point A, with angular velocity ω. Find

(a) the moment of inertia of the object about the axis,

(b) the kinetic energy of rotation.

....

Feedback: $I_A = \left(5m + \frac{8}{3}M\right)\ell^2$, $K.E. = \frac{1}{2}\left(5m + \frac{8}{3}M\right)\ell^2\omega^2$.

Activity 14.3

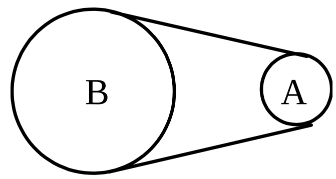
Calculate the kinetic energies for each of two uniform, solid cylinders, each rotating around its central axis. They have the same mass M and rotate with the same angular velocity ω, but the first has a radius r and the second has a radius $3r$.

....

Feedback: $\frac{1}{4}Mr^2\omega^2$ and $\frac{9}{4}Mr^2\omega^2$

Activity 14.4

Wheels A and B in the figure below are connected by a belt that does not slip. The radius of wheel B is three times the radius of wheel A. What would the ratio of the moments of inertia $I_A : I_B$ be if (a) both wheels had the same angular momenta and (b) both wheels had the same rotational kinetic energy?

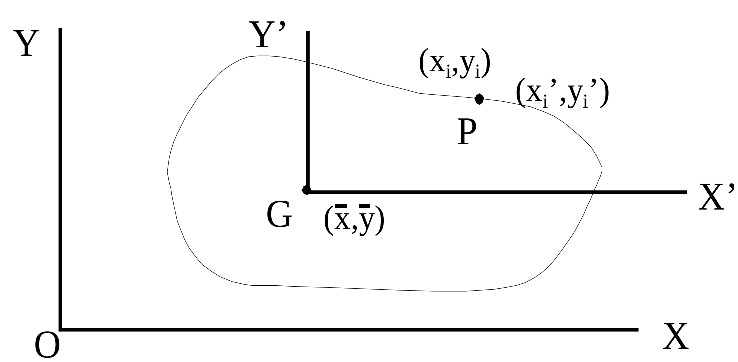


....

Feedback: (a) 1:3, (b): 1:9

14.5.3 The kinetic energy of a rigid body in arbitrary two-dimensional motion

In this section, we will develop a formula for the kinetic energy of the general motion of a rigid body, which can be a combination if translation and rotation. The kinetic energy can be written as a sum of rotational and translational kinetic energies, where the translation relates to translation of the centre of mass and the rotation involves rotation about an axis through the centre of mass.



Let X and Y be fixed coordinate axes with origin O, let G the centre of mass and let X' and Y' be coordinate axes which are parallel to X and Y respectively, but which have their origin at G and which move with the body. Let P have the coordinates (x_i, y_i) relative to the XY system and $\left(x_i', y_i'\right)$ relative to the $X'Y'$ system. If the coordinates of G are $(\overline{x}, \overline{y})$ relative to the XY system, we have

$$x_i = \overline{x} + x_i' \quad , \quad y_i = \overline{y} + y_i'.$$

Then the kinetic energy of the rigid body is given by

$$\begin{aligned} KE &= \sum_{i=1}^{n} \frac{1}{2} m_i \left(\dot{x}_i^2 + \dot{y}_i^2\right) \\ &= \sum_{i=1}^{n} \frac{1}{2} m_i \left[\left(\dot{\overline{x}} + \dot{x}_i'\right)^2 + \left(\dot{\overline{y}} + \dot{y}_i'\right)^2\right] \end{aligned}$$

$$\therefore \quad KE = \sum_{i=1}^{n} \frac{1}{2} m_i \left[\left(\dot{\bar{x}}^2 + \dot{\bar{y}}^2 \right) + \left(\dot{x}_i'^2 + \dot{y}_i'^2 \right) \right]$$

$$+ \sum_{i=1}^{n} m_i \dot{\bar{x}} \dot{x}_i' + \sum_{i=1}^{n} m_i \dot{\bar{y}} \dot{y}_i'$$

$$= \frac{1}{2} \left(\dot{\bar{x}}^2 + \dot{\bar{y}}^2 \right) \sum_{i=1}^{n} m_i + \frac{1}{2} \sum_{i=1}^{n} m_i \left(\dot{x}_i'^2 + \dot{y}_i'^2 \right),$$

since

$$\sum_{i=1}^{n} m_i \dot{\bar{x}} \dot{x}_i' = \dot{\bar{x}} \sum_{i=1}^{n} m_i \dot{x}_i' = 0$$

and

$$\sum_{i=1}^{n} m_i \dot{\bar{y}} \dot{y}_i' = \dot{\bar{y}} \sum_{i=1}^{n} m_i \dot{y}_i' = 0.$$

Now the first term can be expressed as $\frac{1}{2} M \dot{R}^2$, while in the second term $\dot{x}_i'^2 + \dot{y}_i'^2 = \dot{\theta} \left(x_i'^2 + y_i'^2 \right)$, where $\dot{\theta}$ refers to rotation about G. Hence we can write

$$KE = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} I_G \dot{\theta}^2.$$

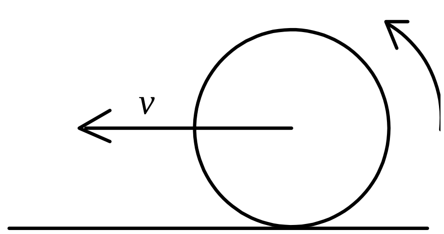
Result 4.11 (Kinetic energy in general two–dimensional motion)
For a rigid body moving on a plane, we have

$$KE = \frac{1}{2} M \dot{R}^2 + \frac{1}{2} I_G \dot{\theta}^2 \tag{14.7}$$

where $\dot{R} = \left| \underline{\dot{R}} \right|$ is the velocity of the centre of mass and I_G, $\dot{\theta}$ refer to the rotation about the centre of mass G.

Example 14.4
A disc with a radius r and mass M is rolling without slipping on a horizontal surface. It has a constant linear velocity υ. Calculate the total kinetic energy of the disc.

Solution:



The motion of the disc is a combination of rotation and translation. Since it rolls without slipping, it rotates with an angular velocity $\dot{\theta}$ where

$$\dot{\theta} = \frac{1}{r} \upsilon.$$

In terms of (14.7), the total kinetic energy of the disk is then

$$KE = \frac{1}{2} I_G \dot{\theta}^2 + \frac{1}{2} M v^2.$$

For the disc rotating about its centre, the moment of inertia is $I_G = \frac{1}{2} M r^2$. Thus, the total kinetic energy equals

$$\begin{aligned} KE &= \frac{1}{2} \left(\frac{1}{2} M r^2 \right) \frac{1}{r^2} v^2 + \frac{1}{2} M v^2 \\ &= \frac{3}{4} M v^2. \end{aligned}$$

Activity 14.5

A solid cylinder (mass M, radius r, length a) rolls without slipping along the top of a horizontal table, with constant angular velocity ω. What is the total kinetic energy of the cylinder? What proportion of the total kinetic energy is due to rotation, and what proportion is due to translation? (Hint: the rolling condition applies here!)

....

Feedback: The total kinetic energy is $KE = \frac{3}{4} M r^2 \omega^2$, of which 1/3 comes from rotation, and 2/3 from translation.

Activity 14.6

A ring and a disc, both with mass M and radius r, roll without slipping along a horizontal plane with constant angular velocity. Find the total kinetic energies for each object. Which has a larger kinetic energy?

....

Feedback: The ring has the larger kinetic energy.

Activity 14.7

A rod of length $2a$ with mass M is spinning horizontally on a smooth surface. It is spinning about its centre, with an angular velocity ω. The centre of the rod is moving along the surface at a speed u. Find the kinetic energy of the rod.

....

Feedback: You will get $KE = \frac{1}{6} M a^2 \omega^2 + \frac{1}{2} M u^2$. Note that this is not a case of rolling without slipping!

CONCLUSION

In this unit, your have learned

- what is meant by work, and by potential and kinetic energy
- how to specify the gravitational potential energies of particles, systems of particles and rigid bodies
- how to calculate the kinetic energy of rigid bodies for pure translation, pure rotation and general motion
- What is meant by the principle of energy conservation, and how it applies

Remember to add the following to your toolbox:

- the principle of the conservation of mechanical energy
- the definitions of potential and kinetic energy
- the definition of a conservative force
- the formulas for the kinetic energy of a rigid body in pure rotation, pure translation, and in general motion

Unit 15 THE ENERGY CONSERVATION METHOD

Key questions:

- *How does energy conservation apply to rigid bodies or systems of bodies?*
- *How can we solve problems using the concept of energy conservation?*
- *How do we know when to use energy methods, and when to use equations of motion?*

We will now proceed to solve problems using the energy conservation method. We will start with problems dealing with just one rigid body, but will later generalise the approach to analyse any systems of rigid bodies and/or particles. Finally, we will discuss problem solving and the problem of choosing between energy methods or the equations of rotation.

Contents of this unit:

15.1 Applications involving one rigid body

15.2 Applications involving systems of bodies

15.3 General problems: Which method to use?

What you are expected know before working through this unit:

The first two sections of this unit use the concept of kinetic and potential energies, as defined in Unit 14. In the final section, we bring together all the problem solving skills you have gained in the entire study guide!

15.1 Applications involving one rigid body

Let us first re-state the law of the conservation of mechanical energy for rigid bodies.

Result 15.1 (The law of the conservation of mechanical energy)
If only conservative forces act on a rigid body, then the sum of the kinetic and potential energies is constant:

$$PE + KE = E$$

where E (a constant) is called the total mechanical energy.

The energy conservation law can, for example, be used in any case where only gravity acts on the system. We can also ignore the following kinds of forces, since they **do not do work**, and thus do not affect the total energy of the system:

(i) the normal forces of reaction

(ii) internal forces

(iii) the forces of friction at the point of contact of bodies which roll without slipping

On the other hand, friction which resists motion along a surface (sliding friction) is not a conservative force and cannot be ignored.

In order to apply the energy conservation method, your will usually have to find the potential and kinetic energies of the rigid body at some initial position, and then at some final position. Equating the initial and final energies will give you an equation from which you can hopefully solve some of the unknown variables.

TOOLBOX FOR CALCULATING POTENTIAL AND KINETIC ENERGIES

Kinetic energy:

- Classify the motion of the object: is it pure rotation, pure translation or a combination of rotation and translation?
- Apply the appropriate formula to calculate the potential energy.
- In general motion, if the motion is rolling without slipping, then you can choose to express the kinetic energy in terms of either the angular velocity or the linear velocity of the centre of mass. Which one is better depends on the circumstances and the questions you are trying to answer.

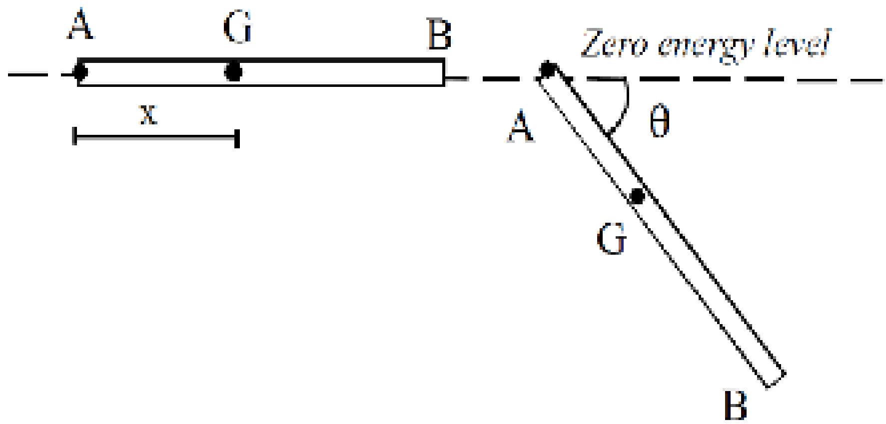
Potential energy:

- Decide on a zero energy level. Although the zero energy level can be chosen arbitrarily, a well-chosen one will make the calculations easier.

We shall start with examples which involve pure rotation. In the examples that follow, we have chosen the zero energy level to go through the axis of rotation, as it will make the calculations of the position of the centre of mass easier. Try to do these examples with some other choice of the zero energy level, and you will see how difficult things can get!

Example 15.1
A pendulum consists of a uniform rod AB with mass $2m$ and length 2ℓ, with a particle with a mass $3m$ attached to point B and a particle with a mass m attached to the centre of the rod. The object is free to turn on a vertical plane about a horizontal axis through A. The rod is first held in a horizontal position and then released. Find the angular velocity $\dot{\theta}$, and its maximum value.

Solution:
Since this is a conservative system, we can apply the energy conservation method. The axis of rotation goes through end A of the rod. Let us choose the zero energy level for the gravitational potential energy to go through point A. Then a sketch of the system might look as follows:



In the sketch we have drawn the rod in its initial position, when the rod is held horizontally at rest; and at an arbitrary position, when the rod forms the angle θ with the horizontal. We will calculate the potential energy (PE) and kinetic energy (KE) of the object, in both of these situations. To calculate the potential energy, we first have to find G, the centre of mass of the object. According to the rules for the centre of mass of a compound body, G is situated at a distance x from A on the rod, where

$$x = \frac{2m \cdot \ell + m \cdot \ell + 3m \cdot 2\ell}{2m + m + 3m} = \frac{3}{2}\ell.$$

Since the object rotates about a fixed axis, its kinetic energy is given by the formula

$$KE = \frac{1}{2} I_A \ \dot{\theta}^2.$$

A particle with a mass $3m$ is situated at B, a distance 2ℓ away from A, so its moment of inertia about A is $3m\,(2\ell)^2$; the particle at the centre of the rod with mass m has a moment of inertia $m\,(\ell)^2$ about A. For the rod of length 2ℓ and mass $2m$, the moment of inertia about A, according to the parallel axis theorem, is

$$\begin{aligned} I_A^{rod} &= I_{centre}^{rod} + (2m)\,\ell^2 \\ &= \frac{1}{3}(2m)\,\ell^2 + (2m)\,\ell^2. \end{aligned}$$

Thus the moment of inertia of the whole object about A is

$$\begin{aligned} I_A &= 3m\,(2\ell)^2 + m\ell^2 + \frac{1}{3}(2m)\,\ell^2 + (2m)\,\ell^2 \\ &= \frac{47}{3} m\ell^2. \end{aligned}$$

Initially,

$$PE = 0$$

(since the centre of mass is on the zero energy level); and

$$KE = 0$$

(since the rod is initially at rest).

Later, when the rod forms an angle θ with the horizontal,

$$PE = -(6m)\,gx\sin\theta$$

(since the centre of mass, G, lies a distance $x\sin\theta$ below the zero energy level and the total mass of the object is $6m$); and

$$KE = \frac{1}{2} I_A \ \dot{\theta}^2.$$

The energy conservation principle states that at all times the total mechanical energy of the system stays constant. That is,

$$PE_{initial} + KE_{initial} = PE_{later} + KE_{later}.$$

This gives

$$0 + 0 = -(6m)\, gx \sin\theta + \frac{1}{2} I_A \ \dot{\theta}^2.$$

Solving the angular velocity from this gives

$$\dot{\theta} = \sqrt{\frac{2\,(6m)\, gx \sin\theta}{I_A}}.$$

Substituting $I_A = \frac{47}{3} m\ell^2$, $x = \frac{3}{2}\ell$ into this, we get

$$\dot{\theta} = \sqrt{\frac{54}{47}\frac{g}{\ell}\sin\theta}.$$

The maximum value of this is

$$\dot{\theta} = \sqrt{\frac{54}{47}\frac{g}{\ell}},$$

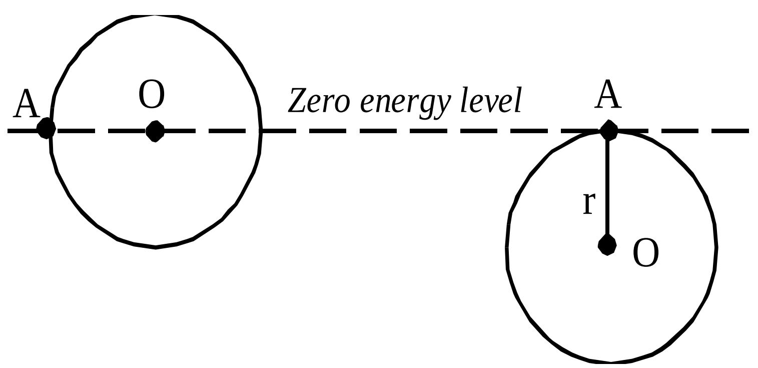
achieved when $\sin\theta = +1$, i.e. $\theta = \frac{\pi}{2}$; that is, when the rod is vertically below point A. ◀

Activity 15.1

A uniform rod of length 2ℓ and mass m is free to rotate on a vertical plane about an axis through one end of the rod. The rod is first held in a horizontal position and then released. Find the angular velocity, $\dot{\theta}$, as a function of the angle of rotation, and show that this has a *maximum value* of $\sqrt{3g/2\ell}$.

Example 15.2

A uniform, circular disc with mass m, radius r and centre O turns freely about a horizontal axis passing through a point A on the rim of the disc. The disc is released from rest in the position in which OA is horizontal and the disc is vertical. Find the angular velocity, $\dot{\theta}$, when AO first becomes vertical.



Solution:

In order to use the principle of energy conservation as expressed in Result 15.1, we first have to calculate the potential and kinetic energies at the initial time, say $t = 0$. We will take this to be the zero energy level, relative to which the potential energy at a later time can be calculated.

At $t = 0$ with AO horizontal:

$$PE = 0, \quad KE = 0.$$

When AO is vertical, we have:

$$KE = \frac{1}{2} I_A \dot{\theta}^2$$

where I_A represents the moment of inertia about the axis through A. The moment of inertia of the disc about O is given by $I_O = \frac{mr^2}{2}$. To calculate I_A, we shall use the parallel axes theorem:

$$I_A = I_O + mr^2 = \frac{3mr^2}{2}.$$

Hence

$$\begin{aligned} KE &= \frac{3mr^2}{4}\dot{\theta}^2 \\ PE &= (-1) \times (\text{distance of } O \text{ from the} \\ &\qquad \text{standard position}) \times mg \\ &= -mgr. \end{aligned}$$

Using the principle of the conservation of energy

$$\text{Initially} \quad : \quad KE + PE = 0 = E$$

$$\text{Finally} \quad : \quad KE + PE = \frac{3}{4} mr^2 \dot{\theta}^2 - mgr = E,$$

but E is a constant, so we have

$$\frac{3}{4} mr^2 \dot{\theta}^2 - mgr = 0.$$

Hence

$$\dot{\theta}^2 = \frac{4g}{3r} \quad \therefore \quad \dot{\theta} = \sqrt{\frac{4g}{3r}}.$$

Activity 15.2

A ring with radius r and mass M is free to rotate on a vertical plane about an axis through point A on its rim. Assume there is no friction. Initially the disc is held at rest so that the centre of the disc, G, is directly above A. The disc is then released. Find the size of the angular velocity when AG is horizontal.

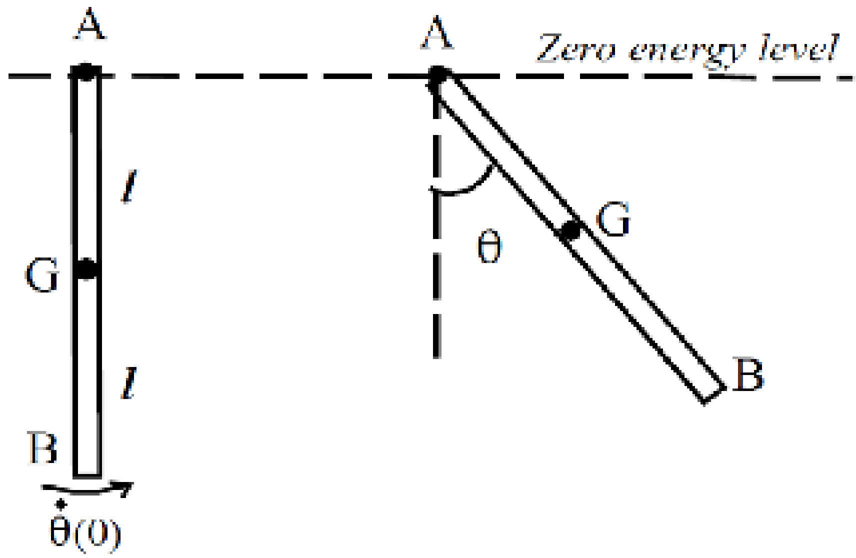
....

Feedback: $\dot{\theta} = \sqrt{\frac{g}{2r}}$.

Example 15.3

A uniform rod AB with a mass $3m$ and length 2ℓ is free to turn on a vertical plane about a horizontal axis through A. Initially the rod lies vertically below A and is given an angular velocity $\dot{\theta}(0) = \sqrt{3g/\ell}$. Find the values of $\theta(t)$ for which $\dot{\theta}(t) = 0$, where $t > 0$.

Solution



From the parallel axes theorem we have that

$$I_A = I_G + 3m\ell^2 = 4m\ell^2.$$

Let us take the horizontal line through A as the zero energy level. Then we get the following:

At time $t = 0$:

$$\begin{aligned} PE &= -3mg\ell \\ KE &= \frac{1}{2}I_A\dot{\theta}^2 = \frac{1}{2}\left(4m\ell^2\right)(3g/\ell) = 6gm\ell. \end{aligned}$$

At a time $t > 0$:
Assume that the rod makes an angle θ with the vertical at A.

$$\begin{aligned} PE &= -3mg\ell\cos\theta \\ KE &= \frac{1}{2}I_A\dot{\theta}^2 = 2m\ell^2\dot{\theta}^2. \end{aligned}$$

Using the principle of the conservation of energy,

$$\begin{aligned} -3mg\ell\cos\theta + 2m\ell^2\dot{\theta}^2 &= -3mg\ell + 6mg\ell \\ \dot{\theta}^2 &= \frac{3g}{2\ell}(1+\cos\theta). \end{aligned}$$

If $\dot{\theta} = 0$, then

$$\begin{aligned} \cos\theta &= -1 \\ \theta &= (2k-1)\pi \quad \text{for} \quad k \in \mathbb{Z}. \end{aligned}$$

That is, $\dot{\theta} = 0$ (meaning that the rotation of the rod stops momentarily) whenever the rod is vertically above A. ◄

Activity 15.3

A thin rod of length l and mass m is suspended freely from one end. It is pulled aside and allowed to swing like a pendulum, passing through its lowest position with an angular speed ω. (a) Calculate its kinetic energy as it passes through its lowest position. (b) How high does its centre of mass rise above its lowest position?

....

Feedback: (a) $ml^2\omega^2/6$, (b) $\frac{l^2\omega^2}{6g}$.

Activity 15.4

A uniform rod AB with a mass $3m$ and length 2ℓ rotates on a vertical plane about an axis through A. A particle of mass m is attached to the rod at B. Initially the rod hangs at rest vertically and is given an angular velocity of $\sqrt{2g/\ell}$. Use energy methods to find h, the height of B above the level of A when $\dot{\theta} = 0$.

....

Feedback: you will get $h = 6\ell/5$. (Note that you will need to find the angle at which this happens, and from that the value of h!)

The following examples deal with more general motion.

Example 15.4
A ring with radius R and mass M rolls without slipping along a slope inclined at angle α with horizontal. If it is given an initial linear velocity v up the slope, how long a distance will it roll up along the slope before it comes to an standstill and starts to roll down again?

Solution:
We can assume that the ring starts at the zero energy level — that is, the zero energy level goes through the centre of mass of the ring in its initial position. It is given an initial linear velocity v up the slope; it will roll up the slope, until finally it comes to a standstill, after which it will start rolling down the slope. According to the energy conservation principle, the ring will reach the level at which the entire initial kinetic energy has been fully converted into potential energy.

So, assume that, initially

$$\begin{aligned} \mathrm{PE}^{\mathrm{in}} &= 0, \\ \mathrm{KE}^{\mathrm{in}} &= \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \end{aligned}$$

where ω is the initial angular velocity, v is the initial linear velocity and

$$I = MR^2.$$

But in rolling without slipping, $\omega = \frac{1}{R}v$ must hold, so that

$$\mathrm{KE}^{\mathrm{in}} = \frac{1}{2}Mv^2 + \frac{1}{2}MR^2\left(\frac{v}{R}\right)^2 = Mv^2.$$

Finally, when the ring stops momentarily,

$$\begin{aligned} \mathrm{PE}^{\mathrm{final}} &= Mgh, \\ \mathrm{KE}^{\mathrm{final}} &= 0, \end{aligned}$$

where h_{disc} is the height that the ring reaches. By the energy conservation principle,

$$\mathrm{KE}^{\mathrm{in}} + \mathrm{PE}^{\mathrm{in}} = \mathrm{KE}^{\mathrm{final}} + \mathrm{PE}^{\mathrm{final}}$$

which gives the equation

$$Mv^2 = Mgh,$$

from which we can find

$$h = \frac{v^2}{g}.$$

This gives the height that the ring goes up; we were asked for the distance that it travels up along the slope, but this can easily be found as

$$d = \frac{h}{\sin(\alpha)} = \frac{v^2}{g\sin(\alpha)}.$$

You can also solve this by applying the equations of motion to the situation to find the acceleration and from that the distance covered, using the appropriate initial/final velocity, but this will be much more complicated!◀

Activity 15.5

A uniform disc with mass M and radius a is placed vertically on a plane inclined at an angle α to the horizontal. Assume that the disc rolls on the plane without slipping. If the disc is given an initial angular velocity of $\dot{\theta}(0) = \omega$ up the plane, how long a distance does the disc travel along the plane until it starts to roll downwards?

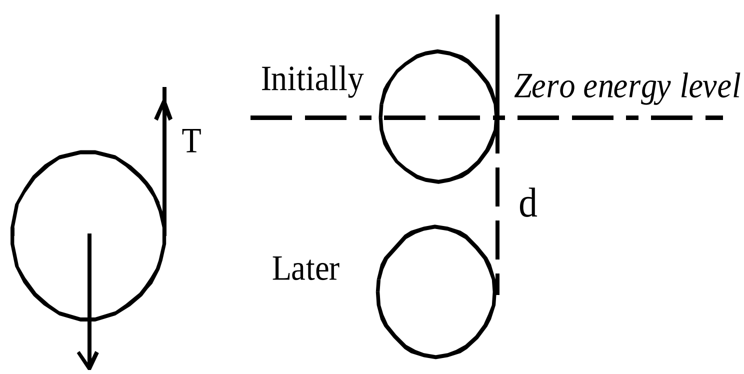
....

Feedback: $d = \frac{3a^2\omega^2}{4g\sin\alpha}$

Example 15.5

A light string is wound around the rim of a ring with a radius a and mass M. The end of the string is fixed and the ring is allowed to fall so that the string unwinds. What is the angular velocity of the ring when it has dropped the distance x vertically? How long does it take for the ring to drop the distance d?

Solution:



We can apply energy conservation methods, since only gravity works on the system. (The tension of the string, like friction in rolling without slipping, does not do work.) Let us take the initial level of the centre of the ring as the zero potential energy level for gravity; then initially,

$$\begin{aligned} P.E. &= 0 \\ K.E. &= 0 \end{aligned}$$

When the ring has dropped some distance x,

$$\begin{aligned} PE &= -Mgx \\ KE &= \frac{1}{2}I_G\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 \end{aligned}$$

where $\dot{\theta}$ is the angular velocity and $\dot{x}$ the linear velocity of the centre of mass. The principle of energy conservation then gives

$$PE_{initial} + KE_{initial} = PE_{later} + KE_{later},$$

so

$$0 \,+\, 0 = -Mgx \,+\, \frac{1}{2}I_G\dot{\theta}^2 \,+\, \frac{1}{2}M\dot{x}^2.$$

Since the string is unwinding without slipping, $\dot{\theta}$ and $\dot{x}$ are again related through $\dot{x} = a\dot{\theta}$. The moment of inertia of a ring rotating about its centre is $I_G = Ma^2$. Thus,

$$\begin{aligned} Mgx &= \frac{1}{2}\left(Ma^2\right)(\dot{x}/a)^2 + \frac{1}{2}M\dot{x}^2 \\ \therefore \quad (\dot{x})^2 &= gx. \end{aligned}$$

We can find the acceleration of the ring from this: differentiate both sides to get

$$\begin{aligned} 2\dot{x}\ddot{x} &= g\dot{x} \\ \therefore \quad \ddot{x} &= \frac{1}{2}g. \end{aligned}$$

Since the acceleration is constant, we can easily find the distance travelled, x, in terms of time t: We have $x = 0$, $\dot{x} = 0$ at $t = 0$, so

$$\begin{aligned} x &= \frac{1}{2}\ddot{x}t^2 = \frac{1}{4}gt^2 \\ \therefore \quad t &= \sqrt{\frac{4x}{g}} = 2\sqrt{\frac{x}{g}}. \end{aligned}$$

Thus, the time taken to travel the distance $x = d$ is $t = 2\sqrt{d/g}$. ◄

Activity 15.6

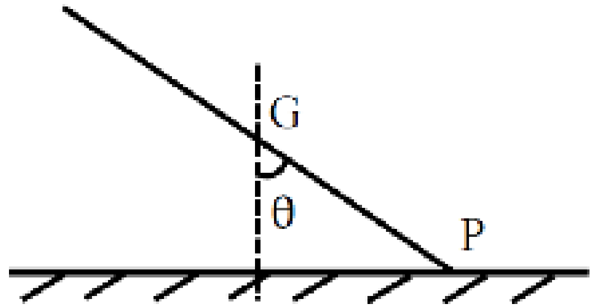
A ring with mass M and radius a rolls from rest, without slipping, down a plane inclined at an angle α to the horizontal. Use the energy conservation method to find the angular velocity of the ring when it has travelled the distance d down the plane.

....

Feedback: $\dot{\theta} = \frac{\sqrt{gd}\sin\alpha}{a}$.

Example 15.6

A uniform rod of length $2a$ with mass m is placed vertically, with the bottom end on a smooth (i.e. frictionless) table. The equilibrium of the rod is slightly disturbed so that it falls from rest. Let θ be the angle between the rod and the vertical at time t, as indicated in the sketch. Find the angular velocity $\dot{\theta}$ in terms of θ.



Solution:
The easiest way to calculate $\dot{\theta}$ is by applying the energy conservation method. We'll choose the table top as the zero energy level for potential energy. Then **initially** the rod is at rest and its centre of mass is the distance a above the table; thus

$$\begin{aligned} KE &= 0 \\ PE &= mga \end{aligned}$$

At time t, when the rod forms the angle θ with the vertical,

$$PE = mgy$$

$$KE = \frac{1}{2}m(\dot{R})^2 + \frac{1}{2}I_G(\dot{\theta})^2$$

where y denotes the height of the centre of mass G, and

$$I_G = \frac{1}{3}ma^2.$$

Let us take the X-axis to go parallel to the table, and the Y-axis to be perpendicular to it. Let $\underline{R}$ denote the position vector of point G :

$$\underline{R} = \underline{OG} = x\underline{i} + y\underline{j}.$$

First we note that all the forces acting on the rod are *vertical*, in other words, they act in the Y-direction. Therefore, G does not move horizontally, that is, $\dot{x} = 0$. So, the velocity of G is given by $\underline{\dot{R}} = \dot{y}\underline{j}$. But, since point P of the rod always touches the table,

$$\begin{aligned} y &= a\cos\theta \\ \dot{R} &= \dot{y} = -a\sin\theta\dot{\theta}. \end{aligned}$$

So,

$$KE = \frac{1}{2}ma^2(\sin^2\theta + \frac{1}{3})\dot{\theta}^2.$$

The principle of energy conservation,

$$PE + KE = \text{constant}$$

then gives

$$\begin{aligned} 0 + mga &= mga\cos\theta + \frac{1}{2}ma^2(\sin^2\theta + \frac{1}{3})\dot{\theta}^2 \\ \therefore \quad \dot{\theta} &= \sqrt{\frac{6g\,(1-\cos\theta)}{a\left(3\sin^2\theta + 1\right)}}. \end{aligned}$$

15.2 Applications involving systems of bodies

The following examples differ from the previous ones in that we shall now consider systems consisting of more than one body. The energy conservation principle still applies, but it is important to remember that we must now consider the kinetic and potential energies of the **whole system**. For the total kinetic energy we must add up the rotational and translational kinetic energies of all the parts of the system. For the total potential energy we similarly have to add up the potential energies of all the components.

Up to now we have chosen a zero energy level and calculated the potential energy of the body or system in relation to it. If the system is very complicated, this may be difficult. Sometimes it is easier to use the following reformulation of the energy conservation principle:

Result 15.2 (The law of the conservation of mechanical energy – reformulated)
If only conservative forces act on a system of particles, a rigid body or a system of bodies, then for any loss in the kinetic energy of the system there must be a corresponding increase in the potential energy, and the other way around:

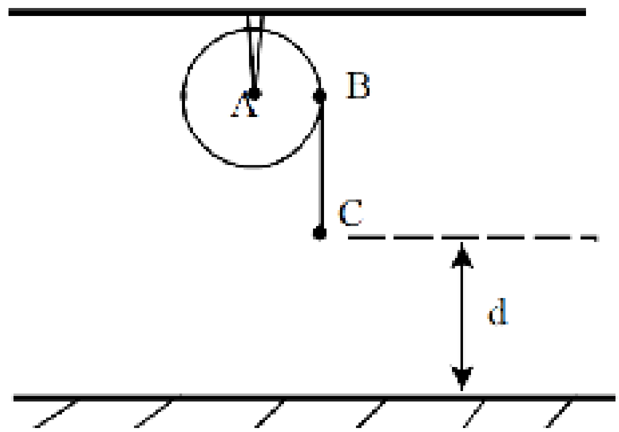
$$\Delta(PE) = -\Delta(KE).$$

If we use this formulation of the energy conservation principle, then we do not have to establish a zero energy level and calculate the values of the potential energies of all the components of the system in relation to the zero energy level. All we have to find is the change in potential energy for each component, which may sometimes be easier to establish. The following example illustrates these different approaches.

Example 15.7
A wheel with a radius r, a mass m and a moment of inertia $I = mr^2$ is free to rotate about a horizontal axis through the centre of the wheel, A. A massless string is wrapped around the rim of the wheel and a particle C with mass m is attached to the string, so that the wheel turns and the string unwinds as the particle drops. B denotes the point at which the string leaves the wheel. Initially the particle is held at rest at a distance d above the floor, and then it is released.

(a) Find the vertical velocity of C just before the particle hits the floor.

(b) Prove that however large the moment of inertia I of the wheel is, the particle will always move downwards when it is released.



Solution:
Assuming that the wheel turns with no friction, only the conservative force of gravity works on the system consisting of the wheel and particle. (The string can be ignored, since it is massless.) Therefore, we can apply energy conservation methods. Let us choose the zero potential energy so that in the initial position, the total potential energy of the system is zero.
Then, **initially**, the potential and kinetic energies of the system are

$$PE_{\text{initial}} = 0,$$

$$KE_{\text{initial}} = 0.$$

(Initially the system is at rest.)
Later, at the moment when the particle hits the floor, both the potential and kinetic energies have changed. Firstly, while the wheel is of course still at its initial level, the particle has moved the distance d downwards. Therefore, the system has lost the amount $-mgd$ of potential energy, and thus the potential energy is now

$$PE_{\text{final}} = -mgd.$$

Secondly, both the particle and the wheel are now in motion, rather than at rest. The particle is moving downwards at the unknown vertical velocity v, and the wheel is rotating about its fixed axis with some clockwise angular velocity $\dot{\theta}$. The kinetic energy of the system is the sum of the kinetic energies of these two motions (the translation of the particle and the pure rotation of the wheel):

$$\begin{aligned} KE_{\text{final}} &= \frac{1}{2}mv^2 + \frac{1}{2}I\dot{\theta}^2 \\ &= \frac{1}{2}mv^2 + \frac{1}{2}mr^2\dot{\theta}^2. \end{aligned}$$

Now, according to the principle of energy conservation, the total energy (the sum of the kinetic and potential energies) of the system is constant, so that

$$PE_{\text{initial}} + KE_{\text{initial}} = PE_{\text{final}} + KE_{\text{final}}.$$

Therefore, we must have

$$0 + 0 = -mgd + \frac{1}{2}mv^2 + \frac{1}{2}mr^2\dot{\theta}^2. \tag{15.1}$$

(a) We wish to find the value of the unknown velocity, v, from these equations. The angular velocity of the wheel, $\dot{\theta}$, is also unknown, but since the string unwinds without slipping or stretching from the wheel, the values v and $\dot{\theta}$ are linked by the equation

$$v = \dot{\theta} r \qquad \therefore \dot{\theta} = \frac{v}{r}.$$

(When the wheel turns clockwise through an angle θ, the particle drops a distance θr.) Substituting this into (15.1), we get

$$\begin{aligned} 0 &= -mgd + \frac{1}{2}mv^2 + \frac{1}{2}mv^2 \\ \therefore mv^2 &= mgd \\ \therefore v &= \sqrt{gd}. \end{aligned}$$

Alternatively, we could have taken the zero energy level of gravity to be at the level of the floor. Then the potential energy of the system is the sum of the potential energies of the wheel and of the particle. (The string can be ignored, since it is massless.) In particular, in the initial position, the total potential energy is

$$PE_{\text{initial}} = mgh + mgd$$

if h denotes the distance from the floor to the centre of the wheel, A. In the final position, the potential energy of the wheel has not changed, but the particle's potential energy is now zero. Therefore, in the final state the total potential energy is

$$PE_{\text{final}} = mgh + 0.$$

We don't know the value of h, but that does not matter, since we are only interested in the amount of change in the potential energy. The mgh terms will cancel out in the energy conservation equation!

Alternatively, we could use the fact that the lost potential energy of the particle is translated into kinetic energy of the wheel and the particle. As the particle drops the distance d, but the wheel stays at the same level, the particle loses the potential energy mgd, but the potential energy of the wheel does not change. Therefore, the system consisting of the particle and the wheel has lost the potential energy mgd. On the other hand, the system has gained the kinetic energy $\frac{1}{2}mv^2 + \frac{1}{2}mr^2\dot{\theta}^2$. The loss of potential energy must equal the gain in kinetic energy, and therefore the equation

$$mgd = \frac{1}{2}mv^2 + \frac{1}{2}mr^2\dot{\theta}^2$$

must hold.

(b) Let us repeat the energy conservation argument of (a) above, this time with an arbitrary value I for the moment of inertia of the wheel. We get

$$mgd = \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}mv^2$$

and, applying the rolling condition,

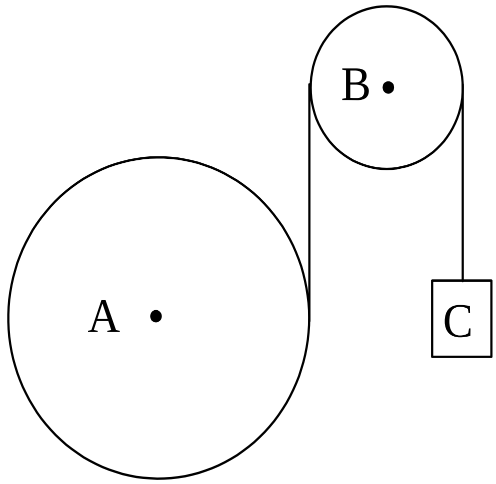
$$mgd = \frac{1}{2}I\left(\frac{v}{r}\right)^2 + \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{I}{r^2} + m\right)v^2.$$

This gives

$$\begin{aligned} v^2 &= \frac{mgd}{\frac{1}{2}\left(\frac{I}{r^2} + m\right)} \\ \therefore v &= \sqrt{\frac{mgd}{\frac{1}{2}\left(\frac{I}{r^2} + m\right)}}. \end{aligned}$$

But this is always strictly positive (never zero) if we assume that m, d, r and I are strictly positive, rather than zero. (However, v can be made to be arbitrarily small by taking I large enough: $v \to 0$ as $I \to 0$, when all the other values (m, d, r) are kept constant.)◀

Activity 15.7



Two discs A and B are fixed on a wall in the positions shown, such that each disc rotates freely about a horizontal axis through its centre. The vertical distance between the centres of A and B is h. Disc A has mass $4M$ and radius $2R$, and disc B has mass M and radius R. A tape is wound around disc A, then passes around disc B, and at its other end is attached to a weight C of mass m. Initially the centre of mass of weight C is at level with the centre of disc A, and C is then released. Use the energy method to find the vertical velocity of C when it has dropped the distance d.

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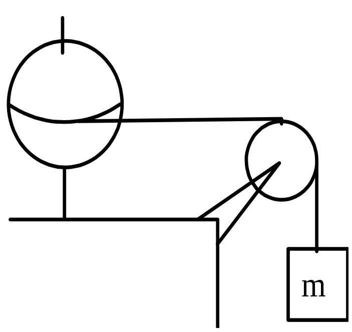
Feedback: $v = \sqrt{\frac{mgd}{\frac{5}{4}M+\frac{1}{2}m}}$. Remember to include the two rotational energies of the discs, and the translational energy of the weight! You will also need to use the link between the two angular velocities and the linear velocity.

Activity 15.8

A uniform, hollow sphere with mass M and radius R rotates about a vertical axis on frictionless bearings. A massless cord passes around the equator of the sphere, over a pulley with a moment of inertia I and radius r, and is attached to a small object of mass m that is otherwise free to fall under the influence of gravity. There is no friction on the pulley's axle; the cord does not slip on the pulley. What is the speed v of the object after it has fallen a distance h from rest?



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Feedback: You will get $v = \sqrt{\dfrac{2gh}{1+\dfrac{I}{mr^2}+\dfrac{2M}{3m}}}$.

15.3 General problems: Which method to use?

So far, in this study guide, we have come across two general methods for solving problems: either, by writing down the equations of motion and rotation; or, by using the principle of energy conservation. How do we decide which of these methods to apply?

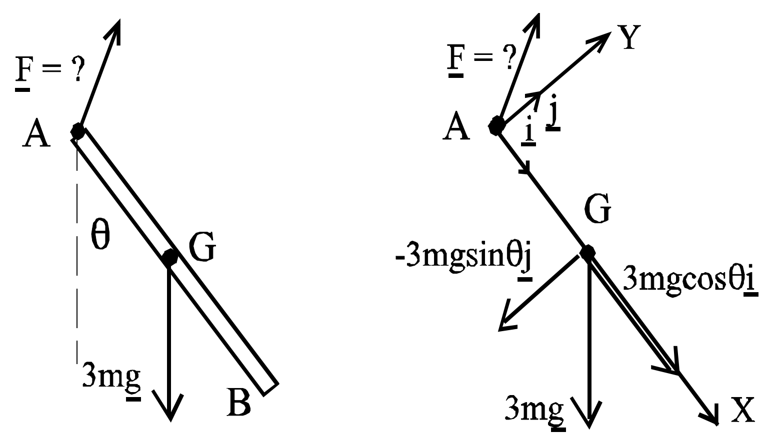
In this last section, we shall first give a toolbox with some instructions on how to decide on a method, and then we shall give some examples.

TOOLBOX FOR DECIDING BETWEEN EQUATIONS OF MOTION AND THE ENERGY CONSERVATION METHOD

To decide which to apply, consider the following questions:

- Is the system conservative? If not, the energy conservation principle cannot be applied.
- Do we know enough to apply all the necessary equations of motion?
- What type of information is asked for? Remember that
  - the energy conservation method deals with velocities
  - equations of motion deal with accelerations and forces
- Is the problem stated in terms of an initial and a final state?

Example 15.8
Find an expression for the magnitude of the force of reaction $\underline{F}$ acting on the rod at the axis at A, in Example 15.3.



Solution:
Although in Example 15.3 we applied the conservation energy method, we are now asked to find the force of reaction. Accordingly, we have no choice but to use the equations of motion. Choose a set of perpendicular axes X and Y at A as shown, along which lie the unit vectors $\underline{i}$ and $\underline{j}$ respectively. Since the rod only rotates on the XY-plane, there are no forces acting perpendicularly to this plane. Let the components of $\underline{F}$ along the X- and Y-axes be given by

$$\underline{F} = F_x \underline{i} + F_y \underline{j}.$$

The centre of mass of the rod lies at G, so that its position vector is $\underline{R} = \ell\underline{i}$. In order to calculate $\ddot{\underline{R}}$, we note that G travels along a circle with centre A and radius ℓ. Then, from the equations describing circular motion (which you should have come across in previous physics modules), we have

$$\begin{aligned} \text{tangential acceleration} &= \ell\ddot{\theta}\,\underline{j} \\ \text{normal acceleration} &= -\frac{\ell^2\dot{\theta}^2}{\ell}\underline{i} = -\ell\dot{\theta}^2\underline{i}. \end{aligned}$$

The gravitational force $3m\underline{g}$ has components

$$3m\underline{g} = 3mg\left(\cos\theta\,\underline{i} - \sin\theta\,\underline{j}\right).$$

Applying the equation for the translation of the centre of mass now gives us:

$$\begin{aligned} \underline{F} + 3m\underline{g} &= m\ddot{\underline{R}} \\ \therefore\quad (F_x + 3mg\cos\theta)\,\underline{i} + \left(F_y - 3mg\sin\theta\right)\underline{j} &= 3m\left(-\ell\dot{\theta}^2\underline{i} + \ell\ddot{\theta}\,\underline{j}\right). \end{aligned}$$

If we equate the coefficients of $\underline{i}$ and $\underline{j}$ on both sides of the equation respectively, we get

$$\begin{aligned} (F_x + 3mg\cos\theta)\,\underline{i} &= -3m\ell\dot{\theta}^2\underline{i} \\ \left(F_y - 3mg\sin\theta\right)\underline{j} &= 3m\ell\ddot{\theta}\,\underline{j} \end{aligned}$$

Hence

$$F_x = -3m\left(\ell\dot{\theta}^2 + g\cos\theta\right) \tag{15.2}$$

$$F_y = 3m\left(\ell\ddot{\theta} + g\sin\theta\right). \tag{15.3}$$

From Example 15.3 we have that

$$\dot{\theta}^2 = \frac{3g}{2\ell}(1+\cos\theta)$$

and if we differentiate this equation again with respect to t we get

$$\frac{d}{dt}\left(\dot{\theta}^2\right) = 2\dot{\theta}\ddot{\theta} = \frac{3g}{2\ell}\sin\theta\,\dot{\theta}$$

so that

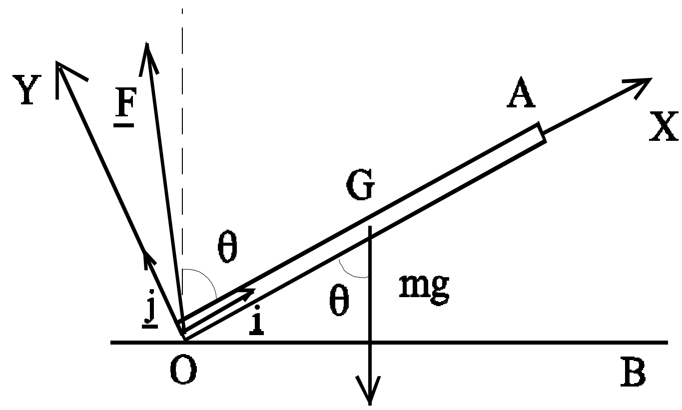
$$\ddot{\theta} = -\frac{3g}{4\ell}\sin\theta.$$

Substituting into (15.2) and (15.3) we get

$$\begin{aligned} F_x &= -\frac{3mg}{2}(3+5\cos\theta) \\ F_y &= \frac{3mg}{4}\sin\theta. \end{aligned}$$

Example 15.9

A uniform rod of mass m rotates about a fixed end point O. The length of the rod is $2a$. Initially the other end point A is held at rest vertically above O, and then the rod is released. Find the angular velocity $\dot{\theta}$, as well as the force of reaction at O when the rod forms the angle θ with the vertical.



Solution:
This time we are asked for both an angular velocity and the force of reaction. Since the energy conservation method is easier to use, we shall first use that one to find an expression for the angular velocity, and then we hope to be able to use it to find the angular acceleration, and hence the force.

Choose axes X and Y as shown. Then the translation of the centre of mass of the rod is described by

$$\underline{F} + m\underline{g} = m\underline{\ddot{R}}$$

which we write as

$$\left(F_x\underline{i} + F_y\underline{j}\right) + mg\left(-\cos\theta\underline{i} - \sin\theta\underline{j}\right) = m\underline{\ddot{R}}$$

where

$$\underline{\ddot{R}} = a\ddot{\theta}\left(-\underline{j}\right) - a\dot{\theta}^2\underline{i}$$

so that we get

$$F_x - mg\cos\theta = -ma\dot{\theta}^2 \tag{15.4}$$
$$F_y - mg\sin\theta = -ma\ddot{\theta} \tag{15.5}$$

On the other hand, we can apply the energy conservation principle:
At $t = 0$

$$KE = 0$$

and if we choose OB as the zero energy level, then

$$PE = mga.$$

At $t > 0$

$$\begin{aligned} KE &= \frac{1}{2}I_O\dot{\theta}^2 = \frac{1}{2}\left(I_G + ma^2\right)\dot{\theta}^2 \\ &= \frac{1}{2}\left(\frac{4ma^2}{3}\right)\dot{\theta}^2 = \frac{2ma^2\dot{\theta}^2}{3}, \\ PE &= mga\cos\theta. \end{aligned}$$

From the principle of energy conservation we get

$$mga = \frac{2}{3}ma^2\dot{\theta}^2 + mga\cos\theta$$

so that

$$\dot{\theta}^2 = \frac{3}{2a}g\left(1 - \cos\theta\right). \tag{15.6}$$

Differentiating (15.6) on both sides with respect to t gives us:

$$2\dot{\theta}\ddot{\theta} = \frac{3g}{2a}\sin\theta\,\dot{\theta}$$

so that

$$\ddot{\theta} = \frac{3g}{4a}\sin\theta. \tag{15.7}$$

Substituting (15.6) and (15.7) into (15.4) and (15.5) respectively gives us

$$\begin{aligned} F_x &= mg\cos\theta - ma\frac{3g}{2a}\left(1 - \cos\theta\right) = \frac{mg}{2}\left(5\cos\theta - 3\right), \\ F_y &= mg\sin\theta - ma\frac{3g}{4a}\sin\theta = \frac{mg\sin\theta}{4}. \end{aligned}$$

Example 15.10

A uniform ring, disc and sphere, all with the same radius R and mass M, are released simultaneously from rest at the top of a ramp whose length is L and which forms the angle α with horizontal.

(a) Which object reaches the bottom first?

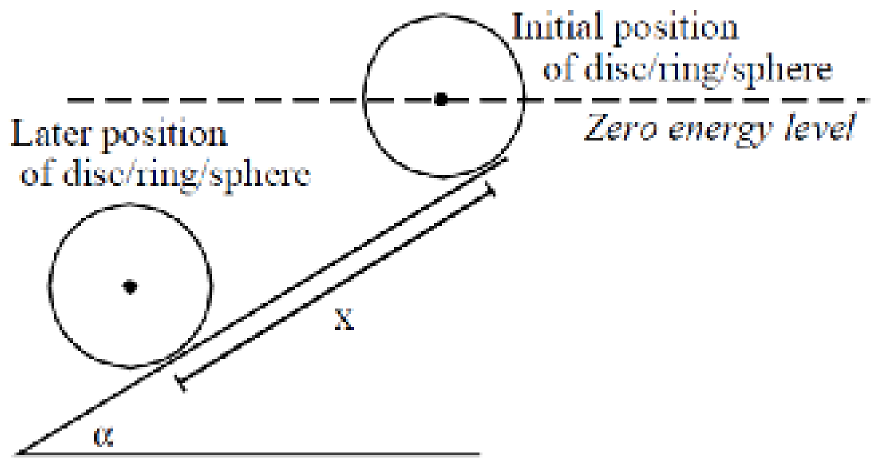
(b) How fast is each of the objects moving at the bottom of the ramp?

Solution:

This is clearly a situation where objects have a known initial potential energy, and we are asked to investigate a later time (when each of the objects reaches the bottom of the ramp). Also, in (b), we are asked to find the velocity of the objects at the bottom of the ramp. This suggests that energy methods can be used. As for (a), it is not initially obvious that such a question can also be answered in terms of velocities. Certainly, if one of the objects has a greater velocity than the others at every point of the ramp then that object will reach the bottom first. But what if one of the objects starts off slower than the others, but then speeds up and overtakes the others? Accelerations may be needed to answer that question. However, since the energy method usually takes less effort, we might as well try that first. We'll use the energy conservation methods to find the velocity of each object at any point on the ramp. This will enable us to answer both of the questions.

Let us choose the zero potential energy level as shown in the sketch below, so that at the initial position at the top of the ramp, all the objects have zero potential energy and zero kinetic energy (since they are initially at rest). That is, the total energy of each object at the initial position equals zero. But then, according to the energy conservation principle, the total energy must also equal zero at any subsequent time when the objects are rolling down the ramp.

The total energy at any time is the sum of the potential and kinetic energies. At any position on the ramp, the potential energy of an object is easy to calculate, and subtracting the potential energy from the total energy will give us the kinetic energy, from which we can in turn find the angular and linear velocities of the objects at any given position as it rolls down the slope.



Some of the following calculations are the same for all three objects (the ring, disc and sphere), as they have the same mass and radius. We will use PE_*, KE_*, $\dot{x}_*$, I_* and so on to denote the potential energy, kinetic energy, linear velocity, moment of inertia etc. of "object *" where * can be the ring, the disc or the sphere. (Note that since we want the velocities of each object separately, we cannot just apply the energy method to a system consisting of the three objects!)

At the initial position, each of the objects is at rest, with its centre of mass at the zero energy level. Therefore, we have

$$\begin{aligned} PE_*^{\text{initial}} &= 0, \\ KE_*^{\text{initial}} &= 0, \end{aligned}$$

and thus the total initial energy is

$$E_*^{\text{initial}} = PE_*^{\text{initial}} + KE_*^{\text{initial}} = 0. \tag{15.8}$$

At a later time, when object * has travelled the distance x down the slope, it will have lost some of its potential energy since it is now lower down, but on the other hand it will have gained some kinetic energy since it is no longer at rest. The potential energy is now

$$PE_*^{\text{later}} = -Mgx\sin\alpha,$$

since the distance x **along** the slope corresponds to a **vertical drop** of $x\sin\alpha$.

The kinetic energy is given by the sum of the rotational and translational energies:

$$KE_*^{\text{later}} = \frac{1}{2}I_*\dot{\theta}_*^2 + \frac{1}{2}M\dot{x}_*^2$$

where I_* is the moment of inertia of object *, $\dot{\theta}_*$ its angular velocity and $\dot{x}_*$ the linear velocity of its centre of mass. If we assume that the objects are rolling without slipping down the slope, then for each object the angular and linear velocities are linked by the rolling condition:

$$\dot{\theta}_* = \frac{\dot{x}_*}{R}$$

Hence, the kinetic energy is

$$KE_*^{\text{later}} = \frac{1}{2}\left(\frac{I_*}{R^2} + M\right)\dot{x}_*^2,$$

and the total energy at this later time is

$$\begin{aligned} E_*^{\text{later}} &= PE_*^{\text{later}} + KE_*^{\text{later}} \\ \therefore E_*^{\text{later}} &= -Mgx\sin\alpha + \frac{1}{2}\left(\frac{I_*}{R^2} + M\right)\dot{x}_*^2. \end{aligned} \tag{15.9}$$

In terms of the energy conservation principle,

$$E_*^{\text{initial}} = E_*^{\text{later}},$$

so using the values of E_*^{initial} and E_*^{later} calculated in (15.8) and (15.9), we see that we must have

$$0 = -Mgx\sin\alpha + \frac{1}{2}\left(\frac{I_*}{R^2} + M\right)\dot{x}_*^2$$

$$\therefore \quad \dot{x}_*^2 = \frac{2Mg\sin\alpha}{\left(\frac{I_*}{R^2} + M\right)}x. \tag{15.10}$$

This equation gives the velocity of any of the objects when it has travelled the distance x down the slope.

For the ring, $I_{\text{ring}} = MR^2$ and therefore

$$\dot{x}_{\text{ring}}^2 = g\sin\alpha x.$$

For the disc, $I_{\text{disc}} = \frac{1}{2}MR^2$ and therefore

$$\dot{x}_{\text{disc}}^2 = \frac{4}{3}g\sin\alpha x.$$

For the sphere, $I_{\text{sphere}} = \frac{2}{5}MR^2$ and therefore

$$\dot{x}_{\text{sphere}}^2 = \frac{10}{7}g\sin\alpha x.$$

(a) Which object reaches the bottom first? We can answer this question based on the calculations above. Since

$$1 < \frac{4}{3} < \frac{10}{7},$$

we see that at **any** point of the ramp the velocity of the **sphere** is the greatest. It then follows that it must reach the bottom first. Alternatively, we could use (15.10) to find the acceleration of each object. Equation (15.10) links together the velocity, $\dot{x}_*^2$, and the distance covered, $x = x_*$, of each object at a certain time t :

$$\dot{x}_*^2 = \frac{2Mg\sin\alpha}{\left(\frac{I_*}{R^2} + M\right)}x_*.$$

To find the acceleration, we can differentiate each side of this equation with respect to time, to get

$$\frac{d}{dt}(\dot{x}_*^2) = \frac{d}{dt}\left(\frac{2Mg\sin\alpha}{\left(\frac{I_*}{R^2}+M\right)}x_*\right)$$

$$\therefore \quad 2\dot{x}_*\ddot{x}_* = \frac{2Mg\sin\alpha}{\left(\frac{I_*}{R^2}+M\right)}\dot{x}_*$$

$$\therefore \quad \ddot{x}_* = \frac{Mg\sin\alpha}{\left(\frac{I_*}{R^2}+M\right)}.$$

Applying again the moments of inertia of each object, we see that the accelerations are

$$\begin{aligned}\ddot{x}_{\text{ring}} &= \frac{1}{2}g\sin\alpha,\\ \ddot{x}_{\text{disc}} &= \frac{2}{3}g\sin\alpha,\\ \ddot{x}_{\text{sphere}} &= \frac{5}{7}g\sin\alpha.\end{aligned}$$

Thus, each object has a constant acceleration, and the sphere has the largest acceleration, so that it reaches the bottom of the ramp first.

The exact time taken by each object to reach the bottom of the ramp is also easy to calculate. If an object has constant acceleration α and initial velocity $v = 0$, then in time t it has travelled the distance

$$d = \frac{1}{2}\alpha t^2.$$

It follows that it takes the time

$$t = \sqrt{\frac{2d}{\alpha}}$$

to travel the distance d. If we apply this formula to the three objects travelling the length of the ramp L, we find that the times taken by each object to reach the bottom are, respectively

$$\begin{aligned}t_{\text{ring}} &= \sqrt{\frac{4L}{g\sin\alpha}},\\ t_{\text{disc}} &= \sqrt{\frac{3L}{g\sin\alpha}},\\ t_{\text{sphere}} &= \sqrt{\frac{14L}{5g\sin\alpha}}.\end{aligned}$$

(c) How fast is each of the objects moving at the bottom of the ramp? We have already calculated the values of $\dot{x}^2_{\text{ring}}$, $\dot{x}^2_{\text{disc}}$ and $\dot{x}^2_{\text{sphere}}$ when the objects have travelled the distance x down the slope. Applying these, we find that at the bottom of the ramp (the case $x = L$), the velocities of the objects are

$$\begin{aligned} \dot{x}_{\text{ring}} &= \sqrt{g \sin \alpha L} \\ \dot{x}_{\text{disc}} &= \sqrt{\frac{4}{3}}\sqrt{g \sin \alpha L} \\ \dot{x}_{\text{sphere}} &= \sqrt{\frac{10}{7}}\sqrt{g \sin \alpha L}. \end{aligned}$$

Activity 15.9

A uniform, solid sphere with radius R rolls from rest, without slipping, on a plane inclined at the angle α to the horizontal. Find the velocity of the centre of mass of the sphere:

(a) after time t

(b) after moving a distance d down the plane.

Note that you might find it easier to use a different methods in the two cases!

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Feedback: (a) $\dot{x} = \frac{5}{7}(\sin \alpha)\, gt$, (b) $\dot{x} = \sqrt{\frac{10}{7}(\sin \alpha)\, gd}$.

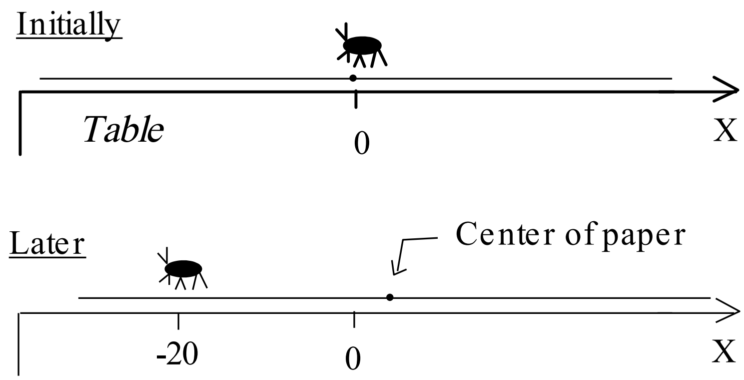
Example 15.11

A sheet of paper, weighing 10 g, lies on top of a smooth (frictionless) table top. An insect, which weights 2 g, stands on top of it. If the insect walks 20 cm to the left, how does the paper move?

Solution:

Here, we are looking at a situation stated in terms of an initial and a final state. However, the system is not conservative, since the insect is applying force to the paper when it walks along the paper. Therefore, the energy conservation method will not work here. But then it does not seem as if we know enough to apply the equations of motion either, since we do not know how much force the insect exerts on the paper (that depends on the friction between the insect and the paper, and presumably on the strength of the insect as well...) But, since we do know that there is no friction between the paper and the table, we can just consider the system formed by the paper and the insect. Then, the force needed for the insect to move is an internal force and of no concern to us. Since there is no friction between the table and any parts of the system, no external forces parallel to the table act on the system, and the vertical forces (gravity and the normal force) cancel each other out. According to the law of the motion of the centre of mass of the system, this means that $\underline{\ddot{R}} = 0$, that is, the acceleration of the centre of mass of the system is zero. Since the system is at rest initially, this means that the position of the centre of mass does not change at any time.

Let us choose the X-axis along the table top. We can assume that the insect and the centre of the paper are at the origin initially, and that the insect walks in the direction of the negative X-axis.



Initially the centre of mass has the X-coordinate

$$\overline{x}_{\text{in}} = \frac{0 \cdot 2 + 0 \cdot 10}{2 + 10} = 0.$$

Later, the insect has moved 20 cm to the left, to position (-20) on the X-axis. Let x_p denote the new X-coordinate of the centre of the paper. The centre of mass of the system is now

$$\overline{x}_{\text{later}} = \frac{-20 \cdot 2 + x_p \cdot 10}{2 + 10} = \frac{1}{12}\,(10x_p - 40).$$

But $\overline{x}_{\text{in}} = \overline{x}_{\text{later}}$, so we must have

$$\begin{aligned} \frac{1}{2}(10x_p - 40) &= 0 \\ \therefore x_p &= +4 \end{aligned}$$

Hence the paper has moved 4 cm to the right.◀

CONCLUSION

In this unit you have learned how to apply the energy conservation method to solve problems, and how to decide between using the energy method and the approach involving forces and equations of motion.

Remember to add the following to your toolbox:

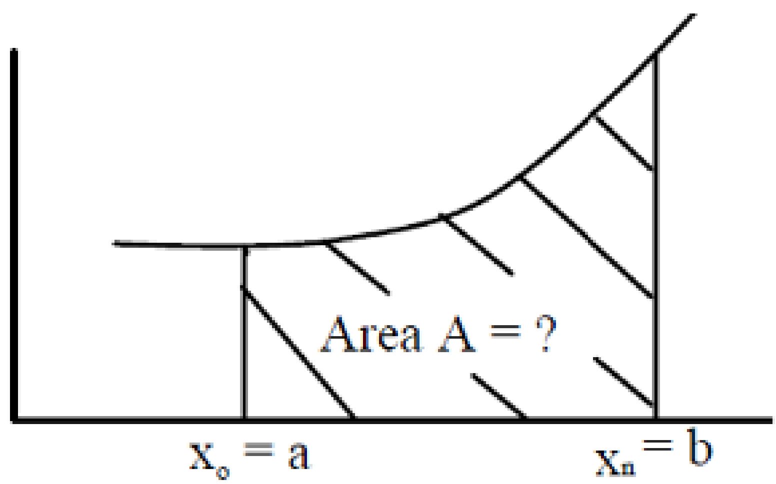
- the toolbox for calculating kinetic and potential energies
- the toolbox for deciding between equations of motion and the energy method

Appendix A DEFINITE INTEGRALS

The following is a brief non–rigorous account on the subject of **definite integrals.**

DEFINITE INTEGRALS AS AREAS

The concept of a definite integral arises naturally if we consider a method for finding areas. Suppose that we wish to find the area enclosed by the X- and Y-axes and under the curve $y = f(x)$ in the interval $a \leq x \leq b$. Let us denote this area by A.

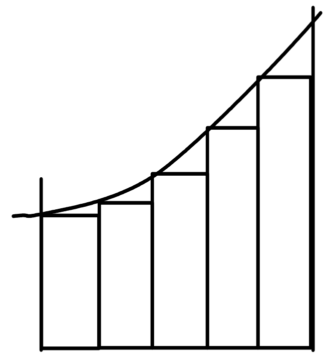


One way of finding this area is as follows: We will divide the interval $a \leq x \leq b$ into n subintervals of lengths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$, by introducing points

$$x_0 = a \leq x_1 \leq x_2 \leq \ldots \leq x_{n-1} \leq x_n = b$$

such that $\Delta x_i = x_i - x_{i-1}$.

Now let's approximate the original function $f(x)$ by a "step function", which has a constant value over each of our intervals. We can do this, for instance, by taking the value of the step function between x_{i-1} and x_i to be equal to $f(x_{i-1})$.



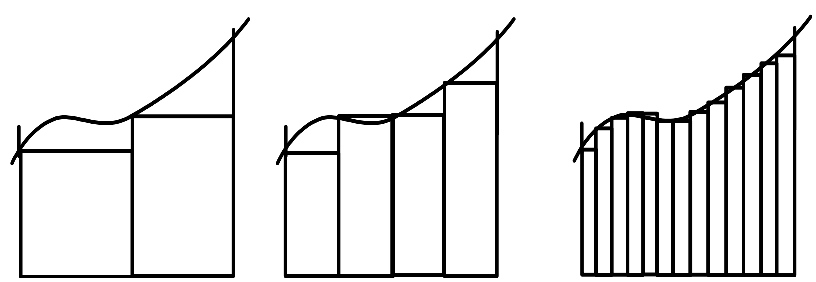
The area under the step function is easy to calculate: The area consists of n rectangles, with widths $\Delta x_1, \Delta x_2, \ldots, \Delta x_n$ and heights $f(x_0) = f(a)$, $f(x_1), f(x_2), \ldots, f(x_{n-1})$, respectively. The area of rectangle number i is equal to $f(x_{i-1})\,\Delta x_i$, and so the total area under the step function (summing up over all the rectangles) is

$$\sum_{i=1}^{n} f(x_{i-1})\,\Delta x_i .$$

Now, it is clear that we can use the area under the **step function** as an approximation for A, the area under the original function $f(x)$:

$$A \approx \sum_{i-1}^{n} f(x_{i-1})\,\Delta x_i .$$

Moreover, it is clear that by increasing the number of subdivision points, and by decreasing the "step lengths" $\triangle x_i$, we will get more and more accurate approximations for the area A, since the step function will more closely approximate the function $f(x)$.



At limit, when we let n increase so that Δx_i tend towards zero, we get the exact area:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x_i .$$

The limit on the right has a value (a number) which does not depend on the specific choice of all the division points x_i. The limit of the sums on the right is denoted by a "definite integral":

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i-1})\,\Delta x_i .$$

This is the definition of a definite integral.

$$\int_a^b f(x)\,dx$$

is called the definite integral of $f(x)$ between a and b. (Note that there are various restrictions on the function $f(x)$ that apply to this definition, but for our purposes we can neglect them.)

So, an integral is a limit of sums. Please note the resemblance between the original sums

$$\sum_{i=1}^{n} f\left(x_{i-1}\right) \Delta x_i$$

and the integral

$$\int_a^b f(x)\, dx$$

which is the limit of these sums as $\triangle x_i \to 0$. Here, x is called the variable of integration, the interval $\lfloor a, b \rceil$ is called the range of integration, and b is the upper limit and a the lower limit of integration. We also say that $f(x)$ is integrated over $[a, b]$, or from $x = a$ to $x = b$

The following properties can easily be proved for the integral:

$$\begin{aligned} \int_a^b \left(f(x) + g(x)\right) dx &= \int_a^b f(x)\, dx + \int_a^b g(x)\, dx \\ \int_a^b A f(x)\, dx &= A \int_a^b f(x)\, dx. \end{aligned}$$

(where A is any constant.)

THE EVALUATION OF INTEGRALS

The next step is to establish a procedure for the calculation of definite integrals. This is achieved by applying the fact that, in a way, integration is the inverse operation of differentiation. This is made clear in the concept of antiderivatives (indefinite integrals) and the fundamental theorem of calculus.

Let $f(x)$ be given. Then any function $F(x)$ such that

$$\frac{dF(x)}{dx} = f(x)$$

is called the antiderivative of $f(x)$. Alternatively, we say that $F(x)$ is the indefinite integral of $f(x)$. Clearly, if $F(x)$ is an indefinite integral of $f(x)$, then so is $F(x) + c$, where c is an arbitrary constant. We use the notation

$$\int f(x)\, dx$$

to denote any indefinite integral of $f(x)$. If one antiderivative $F(x)$ of $f(x)$ is known, then we have

$$\int f(x)\, dx = F(x) + c;$$

this gives all the antiderivatives / indefinite integrals of $f(x)$.

So far we have a definite integral, defined as a limit of sums, and an indefinite integral, defined as the inverse operation of differentiation. How are these operations linked? The connection is stated in the Fundamental Theorem of Calculus.

The Fundamental Theorem of Calculus

Let $f(x)$ be a given function. If $F(x)$ is an indefinite integral of $f(x)$ (that is, an antiderivative of $f(x)$), then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

This theorem enables us to calculate definite integrals directly, without having to use the definition of the definite integral as a limit of sums, whenever any antiderivative of the function to be integrated is known. To calculate the value of $\int_a^b f(x)\,dx$, we proceed as follows:

1) Find $F(x)$, an antiderivative / indefinite integral of $f(x)$.
2) The value of the definite integral is then equal to $F(b) - F(a)$.

We use the notation

$$F(x)]_a^b = F(b) - F(a)\,;$$

thus we have

$$\int_a^b f(x)\,dx = F(x)]_a^b$$

when $F(x)$ is any antiderivative of $f(x)$.

Example A.1
Find the value of $\int_1^2 x^2 dx$.
Since $\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$, an antiderivative of x^2 is $\frac{1}{3}x^3$, that is, $\int x^2 dx = \frac{1}{3}x^3 + c$. Thus,

$$\begin{aligned}\int_1^2 x^2 dx &= \left.\frac{1}{3}x^3\right]_1^2 = \frac{1}{3}(2)^3 - \frac{1}{3}(1)^3 \\ &= \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.\end{aligned}$$

Antiderivatives of some usual functions are given in the following:

$$\int u^n du = \tfrac{1}{n+1}u^{n+1}, \quad n \neq -1$$

$$\int \tfrac{1}{u} du = \ell n\,|u|$$

$$\int \sin u du = -\cos u$$

$$\int \cos u du = \sin u$$

$$\int e^{ax} dx = \tfrac{1}{a}e^{ax}$$

(Note that we should in fact add an arbitrary constant c to the right hand-side of all these formulas, to obtain all the possible antiderivatives. However, in evaluating indefinite integrals that constant will cancel out anyway, so we can drop it.)

Example A.2
Evaluate

$$\int_1^2 \left(x^2 + 2x - 1\right).$$

We have

$$\int x^2 dx = \frac{1}{3}x^3, \quad \int 2x dx = x^2, \quad \int 1 dx = x.$$

Thus,

$$\begin{aligned} & \int_1^2 \left(x^2 + 2x - 1\right) \\ = \quad & \left(\frac{1}{3}x^3 + x^2 - x\right)\Big]_1^2 \\ = \quad & \left(\frac{1}{3}2^3 + 2^2 - 2\right) - \left(\frac{1}{3}1^3 + 1^2 - 1\right) \\ = \quad & \left(\frac{8}{3} + 4 - 2\right) - \left(\frac{1}{3} + 1 - 1\right) = \frac{13}{3} \end{aligned}$$

Example A.3

$$\begin{aligned} \int_{-1}^0 2e^{-x} dx &= 2\left(-e^{-x}\right)\Big]_{-1}^0 \\ &= \left(-2e^{-0}\right) - \left(-2e^{-(-1)}\right) \\ &= 2e - 2. \end{aligned}$$

Appendix B STRATEGIES

In these last pages of the study guide, we will reproduce all the list of strategies and tool-boxes that we mentioned in the main Learning Units of the study guide. You may wish to cut them out!

GENERAL TOOLBOX

(1) UNDERSTANDING THE PROBLEM

Here, you must understand what the object/system/situation is like, and what you are asked to do. To make sure that you have understood the problem, answer the following questions:

- What is given and what is wanted? What conditions apply?
- Can you describe the situation in your own words?

You could make use of the following tools:

- Knowledge of the language of mechanics problems, and using keywords for clues about positions, objects and their properties, types of motion, etc.
- Sketches and diagrams
- Real-life examples and experiments
- Listing in standard mathematical notation the known and unknown quantities

2. PLANNING A SOLUTION

Most of the time solving a problem in mechanics involves deciding on the correct principles or results of physics to apply in a given situation. Hence, one important class of tools consists of your knowledge of these:

- The principles, definitions and results of mechanics – add the tools here
- Knowledge about when the principles and results apply and when not – add the tools here
- Sub-toolboxes you may already have designed for other tasks – add here

To decide which of these you should apply to a particular problem, you may wish to use the following strategic questions as tools:

- Can you find similar, already solved examples and problems? Can you use their method, or their results? (Similarity could mean dealing with a similar situation, or dealing with the same type of unknown.)
- Which mechanics principles could be applied in this situation?
- Which definitions, principles, results deal with the given type of unknown?
- Do we have all the information necessary to apply the definitions, principles or results we have decided on? If not, can we determine the information from the given? Alternatively, can we introduce the information as another unknown? Which definitions, principles, results deal with the new unknown?

3. EXECUTING THE PLAN

To complete this step, you will probably have to apply the following tools:

- Sketches and diagrams
- Mathematical notation, symbols for variables, coordinate systems
- Equations and formulas
- Mathematical tools (integration, solving equations, etc.)

Turn over!

4. ANALYSING THE SOLUTION

To check the correctness of the solution, you can

- see whether the solution makes sense
- try to think of other ways to solve the same problem
- compare the end result with other known, similar results
- compare the result with experiments and guesses based on real-life objects
- work in a group and compare your results with those of others

If your solution seems to be wrong, you should

- find out where you went wrong, by checking the argument and the calculations
- go back to step 1 or step 2

To reflect and learn from the solution, you can

- try to invent similar problems
- compare this problem with other examples and problems that you have come across, and ask yourself what the differences and similarities are

TOOLBOX: IDENTIFYING ALL THE FORCES ACTING ON A SYSTEM

- Draw a picture of the system as a whole, and, if necessary, separate diagrams for all the components of the system, with the forces acting on them. Draw each force as a vector (an arrow) which starts at the point of action of that force.
- For all forces, consider the corresponding reaction force. Does the reaction force act on one of the objects we are interested in? If so, remember to include it!
- Does the collection of forces make sense? Remember that if the object is motionless, then the forces acting on it must balance out – that is, their sum must be zero. If the object is supposed to be still, but all the forces act in the same direction, then something is wrong – you may have let out some forces.

TOOLBOX FOR SELECTING A COORDINATE SYSTEM

Before we can even start the task of finding the centre of mass of a system of particles, we need to have a coordinate system in place! If one is already given, fine; if not, then we must decide on a suitable one. The reason why we need a coordinate system is because the formulas (3.1) or (3.2), (3.3) and (3.4) help us to find the position of the centre of mass from the position of the particles. However, we cannot talk about the position of a particle without having a way to refer to it! The formulas (3.1) to (3.4) refer to the position vectors or coordinates of the particles of the system. But position vectors are meaningless unless we have a reference point (position vectors from *where*?) and, similarly, the coordinates of a point do not mean anything unless we have specified our coordinate axes.
There are many possible coordinate systems, any one of which would do; but some are more suitable than others, because they make calculations easier. Here are some guidelines on how you can go about to select a good coordinate system.

- Draw a sketch of the system. The sketch will make sure that you understand the situation, and will make it easier to select a suitable coordinate system! You might want to label the particles in the sketch. You may have to assume values for distances, masses and positions when they are not fully specified!
- Determine the dimension of the system. If all particles are along one straight line, or along one plane, then the system is in fact one- or two-dimensional, respectively. In that case you do not have to introduce a complete three-dimensional coordinate system.
- Now, you can proceed to select the direction of the coordinate axes and the position of the origin. Things to take into account here are symmetry (more about that later), and the need to be able to find the position vectors or coordinates of all particles as easily as possible!

TOOLBOX FOR SIMPLIFYING THE TASK OF FINDING CENTRES OF MASS OF RIGID SYSTEMS OR BODIES

(1) Draw a sketch of the system (or body). Remember that you can turn a rigid system around any way you like, and therefore there are many possible ways to draw the system! A well-chosen orientation in you sketch may make it easier for you to decide on a coordinate system later on, but you can always re-do your sketch if necessary.

(2) Are there axes or planes of symmetry?

(3) What is the dimension of the system?

(4) Can the system be expressed as a composite body, where the centres of mass of the components are easier to find?

(5) Select a final coordinate system, based on all the considerations above. (This may involve re-drawing the system in another orientation.)

TOOLBOX FOR SLICING AND INTEGRATING TO FIND CENTRES OF MASS

(1) Draw a sketch of the body.

(2) Check whether you can apply any of the simplifying tricks:
   - symmetry; dimension of the problem
   - interpreting the object as a composite body — in which case you should proceed to slice and integrate the components first
   - suitable selection of coordinates; re-drawing the system if necessary

(3) Decide what would be the best way to slice the object.

(4) Select the coordinate system, taking into account the considerations above.

(5) Identify your integration variable. Find the centre of mass and the position of the centre of mass of each slice in terms of density and the variable of integration. Identify the upper and lower limits of integration.

(6) Evaluate the integrals to find the centre of mass with respect to the chosen coordinate system.

(7) Express the position of the centre of mass in relation to the object.

(8) Check the solution.

TOOLBOX FOR FINDING CENTRES OF MASS

1. UNDERSTANDING THE PROBLEM

- What is the system like? What are the shapes, sizes, masses, compositions, positions of the parts? Where are the parts in relation to each other?
- Can you describe the system in your own words?

You could make use of the following tools:

- Knowledge of the language of mechanics problems, and using keywords for clues about the positions and properties of objects
- Sketches and diagrams
- Real-life examples
- Using symbols for referring to parts of the system, positions, distances etc.

2. PLANNING A SOLUTION

We have the following principles, definitions, results and sub-toolboxes available to us for finding centres of mass:

- The original definition, using a sum, for a system of particles
- Slicing and integrating, applicable to bodies with continuous structure
- The equation for the centre of mass of a composite system, put together from any kinds of components
- We have sub-toolboxes for
  - selecting coordinate systems
  - simplifying the task of finding centres of mass
  - slicing and integrating
- Also, we have a special trick for dealing with objects with parts removed.
- If we need to integrate, we may use polar coordinates.
- Finally, we have ways of finding the positions of the centres of mass of various objects, including those of solids of revolution and laminas bounded by functions.

To decide on which of these you should apply to a particular system, ask yourself:

- Can you find similar, already solved examples and problems?
- Are all the components particles? If not, we shall have to find the centres of mass of the components with continuous structure, and then the following list of questions applies:
  - Do we already know where the centre of mass of the object is, based on an already solved example?
  - Can the toolbox for simplifying be applied? Is this an object with parts removed?
  - If all else fails, we can use slicing and integrating. If this is necessary, is the case similar to one we have already done?
- Can the toolbox for simplifying be applied to the entire system?

Turn over!

3. EXECUTING THE PLAN

To complete the calculation of the centre of mass, you will have to

- introduce mathematical notation
- find the centres of mass and the masses of the components , if necessary
- introduce a suitable coordinate system, draw a sketch of the entire system with the coordinates, and express the centres of mass of the components in terms of this coordinate system
- apply the relevant formula to find the centre of mass of the entire system in terms of the coordinates
- express the centre of mass in relation to the system itself.

4. ANALYSING THE SOLUTION

To check the correctness of the solution you can

- see whether the solution makes sense. Compare the end result to the centre of mass of other similar objects
- try to think of alternative ways to find the centre of mass
- re-do the calculations with a different coordinate system
- compare your solution with experiments and guesses based on real-life objects
- work in a group and compare your results with those of others

If your solution seems to be wrong, you should

- find out where you went wrong, by checking the argument and the calculations
- go back to step 1 or step 2

To reflect and learn from the solution, you can

- think of other systems where a similar approach would work; compare this problem with other systems that you have come across: what are the differences and similarities?

TOOLBOX FOR APPLYING THE EQUATION FOR THE MOTION OF THE CENTRE OF MASS OF A SYSTEM

The equation for the motion of the centre of mass of a system, given by $\underline{F} = M\ddot{\underline{R}}$ links

- the acceleration of the centre of mass,
- the external forces acting on the system, and
- the total mass of the system.

Thus, given two of these we can find the third. Usually, we wish to find the acceleration of the centre of mass from the mass and forces acting on the system, so we shall design our toolbox around that problem and leave it for you to modify the toolbox for the other two cases!

To find the acceleration of the centre of mass of a system, we shall have to identify the mass and the resultant external force acting on the system. The following checklist will help you do that.

(1) **UNDERSTANDING THE SYSTEM**

Here, you must understand what the system is like:

- What are the components? (size, shape, consistency: uniform, massless,..)
- How are the components related to each other? (relative positions, do they touch each other, linked by a string, is there friction between them...)

You must also make sure that the system is what you think it is. Change your definition of the system if necessary.

You might make use of the following tools:

- knowledge of the language of mechanics problems, and using keywords for clues about positions, objects and their properties, types of motion etc.
- sketches and diagrams
- real-life examples and experiments
- mathematical notation for known and unknown quantities

2. **PLANNING A SOLUTION**

To be able to apply the equation of motion, we need to find the mass of the system and the external forces acting on it. Do we have the information necessary for doing that?

MASSES:

- Are we given the masses of all the components? If not, can we calculate them? Or do we know the relative sizes of the masses?

FORCES:

- For each component which forms a part of the system, identify all the forces acting on it. (List them, and also draw them in your sketch.) Categorise the forces acting on the component into internal ones (due to another component which forms part of the system) and external ones.
- Check your categorisation: All internal forces should appear in action-reaction pairs.
- Now, ignore all the internal forces, but list all the external forces acting on the various components. These all form the external forces acting on the system.
- Are some of the external forces unknown? If so, then we may need further information linking the motions of the components of the system.

If you cannot identify the masses and/or the forces, you may have to check that you have chosen your system correctly.

Turn over!

3. **EXECUTING THE PLAN**

To apply the equation of motion, you will have to

- introduce mathematical notation and symbols for the masses and forces.
- calculate the masses of the components, if necessary
- introduce a suitable coordinate system; draw a sketch of the entire system and the external forces with the coordinate system; express the vectors of the external forces in terms of this coordinate system; introduce notation for the acceleration of the centre of mass based on this system
- write down the equation for the motion of the centre of mass of the entire system in terms of the coordinate system
- add equations describing extra information about the motion of the system if some of the forces are not known
- solve the equation(s) for the acceleration of the centre of mass
- express this acceleration in relation to the system itself, if required

4. **ANALYSING THE SOLUTION**

- Does the solution make sense? Compare it with experiments and guesses based on real-life objects.
- Try to think of alternative ways to solve the problem.

If your solution seems to be wrong, you should

- find out where you went wrong, by checking the argument and the calculations
- make sure that your system is as you intended

To reflect and learn from the solution, you can

- think of other systems where a similar approach would work; try to generalise the result; compare this problem with other systems that you have come across: what are the differences and similarities?

If some of the forces are unknown, and if you need information relating to the relative motion of the components, then you may also have to write down the equations of motion for the individual components!

HOW TO FIND THE MOMENT OF A FORCE

- Firstly, we must make sure that the question is well defined — remember the moment of a force is only defined with respect to a point of reference O. We'll have to identify the point O (about which we plan to take the moment), the point P (at which the force acts), and the force $\underline{F}$ (its magnitude and direction).
- If a coordinate system is already given, fine; otherwise we will have to introduce one.
- Next, we shall express the vectors $\underline{r} = \underline{OP}$ and $\underline{F}$ in terms of the unit vectors $\underline{i}$, $\underline{j}$ and $\underline{k}$.
- Finally, we shall calculate the value of the cross product $\underline{M} = \underline{r} \times \underline{F}$.

TOOLBOX FOR FINDING MOMENTS OF INERTIA BY SLICING AND INTEGRATING

(1) Understanding the problem: Make sure you know what the body is like, and where the axis of rotation lies! Draw a sketch of the body and the axis.

(2) Check whether you can apply any of the simplifying tricks:
- symmetry, identical axes
- interpreting the object as a composite body – in which case you should proceed to slice and integrate the components first

(3) Decide on the best way to slice the object. Remember that you wish the slices to be such that you can find their moments of inertia dI easily! Also make sure that you know what kind of an object you get when slicing!

(4) Select a coordinate system. The moment of inertia is an absolute quantity, which does not need to be referred to in terms of a specified coordinate system. However, if we are going to apply integration to find it, we shall have to use an integration variable, which means that we will need at least one coordinate axis! If the slicing is done perpendicularly to the axis of rotation, then sometimes we can take that axis as one of our coordinate axes, for example the X-axis, and integration will then be over the variables x which denote the position on the X-axis of each slice. In general, the choice of the coordinate system is closely linked to the decision of how you will "slice" the object! Add the coordinate system to your sketch.

(5) Identify your integration variable. Find the value of the moment of inertia dI for all the small mass elements, in terms of the integration variable. This will usually involve the mass of the small mass element, which may be found by applying the concept of density and the volume, area or length of the small element!. Identify the upper and lower limits of integration.

(6) Evaluate the integral. The end result may be in terms of the density ρ, in which case we also have to apply the link between the total mass M, the density ρ and the dimensions of the body, to express the result in terms of M instead.

(7) Check the solution.

TOOLBOX FOR THE TASK OF FINDING THE MOMENT OF INERTIA OF A RIGID OBJECT

(1) UNDERSTAND THE PROBLEM

Make sure that you understand, firstly, what the object is like; and secondly, where the axis of rotation lies in relation to the body. Some of the following tactics may help you to make sure you achieve this!

- Draw a sketch of the object.
- Think of a real-life example of the situation.

2. PLANNING THE SOLUTION

We have the following ways of finding moments of inertia:

- For systems of particles, $I = \sum m_i r_i^2$.
- For objects with a continuous structure, slicing and integrating: $I = \int dI$.

We have also introduced several simplifying tools:

- symmetries, identical axes of rotation
- the parallel and perpendicular axes theorems
- the rule for compound bodies

Finally, you usually have at your disposal a set of basic or previously calculated moments of inertia for certain objects: rods, rings, discs, etc...

You will need to decide which of these tools apply for the particular object or its components, and in which order you should apply them.

3. EXECUTING THE PLAN

You will now have to do the calculations you have decided on. The following points should help you here:

- Introduce notation for the axes, objects etc. involved.
- If you have to integrate, you will also need to decide on the variable of integration.
- The link between density and mass will help you express the end result in terms of the mass of the object, where necessary.

4. ANALYSING THE SOLUTION

- Do basic checks for correctness: The moment of inertia should be positiveness, increase when mass increases, and so on.
- Re-calculate, using another method.
- Compare with other results for the same object with different axes, or different objects with the same axis.

TOOLBOX FOR SOLVING PROBLEMS INVOLVING PURE ROTATION

(1) UNDERSTANDING THE PROBLEM

Here, you must understand what the rotating object and any other components of the system are like, and what are you asked to do. To make sure that you achieve this, you might make use of the following tools:

- Look for keywords for hints about
  - the position of the rotating object: horizontal, vertical, tangential etc.
  - the position of the axis of rotation
  - the shape and composition of the rotating object (disc, rod; uniform, composite etc.)
  - any other objects which form part of the system, their way of motion, their links with the rotating object: pulleys, ropes, etc.
- Use sketches and diagrams of the whole system and its components.
- Use real-life examples and experiments.
- Try to rephrase the problem in your own words.
- Use standard mathematical notation for the known and unknown quantities.

2. PLANNING WHAT TO DO

Review the available principles, results and definitions:

- The equation of pure rotation, the moment of inertia, angular position and acceleration
- From previous Learning Units: Newton's equations for the motion of particles, the centre of mass, the equation for the motion of the centre of mass, and all the tools listed in Learning Unit 1

To decide on which principles you should apply to the system and/or its various components, you may wish to try the following tools:

- Know when the principles apply and when not.
- Find similar problems and examples.
- Look for principles dealing with the types of variables which are given and wanted.

To make sure that a plan will work, also check the following:

- Do you have all the information necessary to apply the definitions, principles and results decided on? If not, can you find the information from what is given? Alternatively, can one introduce the information as another unknown? Which definitions, principles or results deal with the new unknown?
- Have you used all the given facts and all the conditions in the problem statement?

3. EXECUTING THE PLAN

This is where you will have to set up the equations and solve them. You may need:

- mathematical notation, symbols for variables, coordinate systems
- mathematical tools (integration, solving equations etc.)
- sketches and diagrams
- already calculated results, tables of moments of inertia

4. ANALYSING THE SOLUTION

To check the correctness of the solution, you can do the following:

- See whether the solution makes sense. Compare the end result to other known, similar results.
- Try to think of other ways to solve the same problem.
- Compare it with experiments and guesses based on real-life objects; work in a group and compare your results with those of others.

To reflect on and learn from the solution, you can think of other systems where a similar approach would work. Try to generalise the result. Compare this problem with other systems that you have come across: what are the differences and similarities?

TOOLBOX FOR SOLVING PROBLEMS BY MEANS OF EQUATIONS OF TRANSLATION AND ROTATION

(1) UNDERSTANDING THE PROBLEM

Here, you must understand what the object/system/situation is like, and what you are asked to do. To make sure that you have understood the problem, make sure that you can answer the following questions:

- What is given and what is wanted? What conditions hold?
- Can you describe the situation in your own words?

You might make use of the following tools:

- your knowledge of the language of mechanics problems, and using keywords for clues about objects and their properties, about positions, types of motion, links between the different components etc.
- sketches and diagrams of the whole system and its components
- real-life examples and experiments
- listing in standard mathematical notation the known and unknown quantities

2. PLANNING THE SOLUTION

Analyse the motion of the different components: what type of motion does each undergo? What are the connections between the different components and their motions?

The three different types of motion each have their related principles, results and definitions:

- the equation of motion for pure translation
- the equation of motion for pure rotation
- the equation of motion for a combination of rotation and translation

Check the following:

- Do we have all the information necessary to apply the equations of motion decided on? If not, can we find/calculate the information from what is given? Alternatively, can we introduce the information as another unknown? Which definitions, principles, results deal with the new unknown?
- Is the number of equations equal to the number of unknowns?

If something seems to be missing,

- have you used all the given facts and all the conditions in the problem statement?

3. EXECUTING THE PLAN

This is where we shall set up the equations and solve them. You may need

- mathematical notation, symbols for variables, coordinate systems
- mathematical tools (integration, solving equations etc.)
- sketches and diagrams
- already calculated results, tables of moments of inertia

4. ANALYSING THE SOLUTION

To check the correctness of the solution you can do the following:

- See whether the solution makes sense. Compare the end result to other known, similar results.
- Try to think of other ways to solve the same problem.
- Compare the solution with experiments and guesses based on real-life objects.
- Work in a group and compare your results with those of others.

To reflect on and learn from the solution, you can do the following:

- Think of other systems where a similar approach would work. Try to generalise the result. Compare this problem with other systems that you have come across: what are the differences and similarities?

TOOLBOX FOR CALCULATING POTENTIAL AND KINETIC ENERGIES

Kinetic energy:

- Classify the motion of the object: is it pure rotation, pure translation or a combination of rotation and translation?
- Apply the appropriate formula to calculate the potential energy.
- In general motion, if the motion is rolling without slipping, then you can choose to express the kinetic energy in terms of either the angular velocity or the linear velocity of the centre of mass. Which one is better depends on the circumstances and the questions you are trying to answer.

Potential energy:

- Decide on a zero energy level. Although the zero energy level can be chosen arbitrarily, a well-chosen one will make the calculations easier.

TOOLBOX FOR DECIDING BETWEEN EQUATIONS OF MOTION AND THE ENERGY CONSERVATION METHOD

To decide which to apply, consider the following questions:

- Is the system conservative? If not, the energy conservation principle cannot be applied.
- Do we know enough to apply all the necessary equations of motion?
- What type of information is asked for? Remember that
  - the energy conservation method deals with velocities
  - equations of motion deal with accelerations and forces
- Is the problem stated in terms of an initial and a final state?