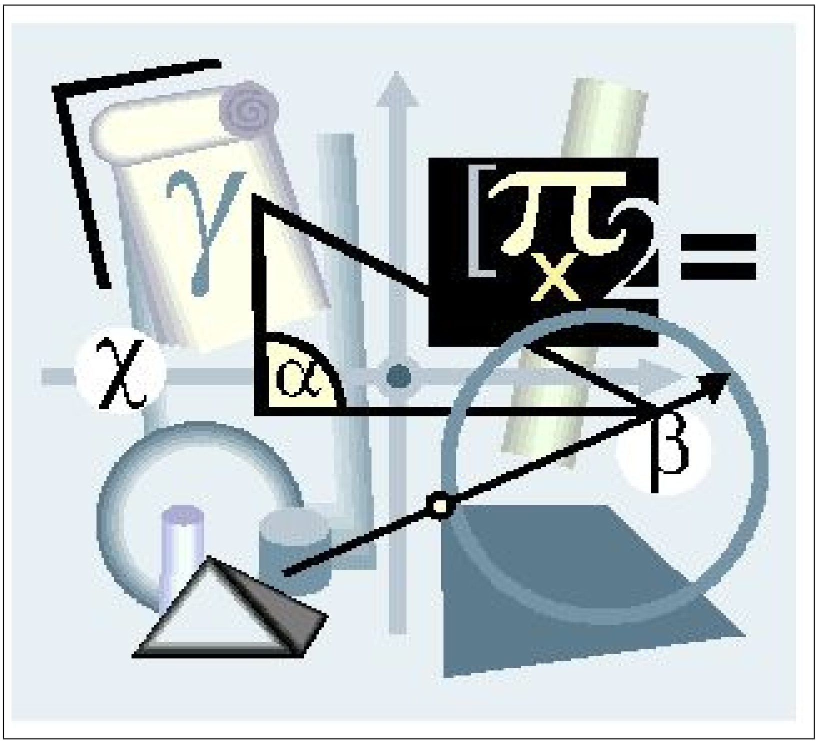
UNISA university of south africa

DEPARTMENT OF MATHEMATICAL SCIENCES



DIFFERENTIAL EQUATIONS (APM2611)

TUTORIAL LETTER 102/2011

EXAM PAPER AND MEMORANDUM

IMPORTANT INFORMATION: READ NOW

Dear Student,

This is the October/November 2009 examination paper (pp1-5). Also included in the Tutorial letter is the memorandum. Try to answer the questions without looking at the memo; then check to see if your answers are correct. Bear in mind the following:

1. **Always** give reasons for your statements - for example, it is not enough to simple write down the final answer - marks are awarded for your thinking process.
2. If you introduce a new variable that doesn't appear in the question, you must say clearly what it stands for.
3. State the final solution of the question clearly.

APM211-V
DIFFERENTIAL EQUATIONS

Time: 2 Hours **October/November 2009**
Examiners:
First: Mr. R.J. de Beer
Second: Dr. J.M. Manale **100 Marks**

Instructions to the candidate:

- The use of non-programmable calculators is allowed.
- This paper consists of 6 pages including formulas (pp. 5-6).
- Show **ALL** calculations.
- Try to answer **ALL** the questions.

QUESTION 1

(a) Solve the given differential equation by separation of variables.

$$dy - (y-1)^2\,dx = 0. \qquad (3)$$

(b) Determine the solution for the initial value problem

$$\frac{dx}{dt} = 4\left(x^2+1\right) \qquad x\left(\frac{\pi}{4}\right) = 1 \qquad (7)$$

[10]

QUESTION 2

(a) Show that $\left(x^3+y^3\right)dx + 3xy^2dy = 0$ is exact. (2)

(b) Solve the differential in part (a). (8)

[10]

QUESTION 3

(a) Find the differential operator that annihilates

$$1 + 6x - 2x^3. \tag{2}$$

(b) Use the D–Operator method to find the general solution of the differentiation equation

$$y'' - 2y' + y = x^3 + 4x. \tag{8}$$

[10]

NO MARK WILL BE AWARDED IF ANOTHER METHOD IS USED.

QUESTION 4

(a) Find the radius of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{x^{n-1}}{n \cdot 3^n}. \tag{5}$$

(b) Find the interval of convergence in part (a). (5)

[10]

QUESTION 5

Use the power series method to solve the initial value problem

$$(x+1)\, y'' - (2-x)\, y' + y = 0$$
$$y'(0) = 1, \qquad y(0) = 2,$$

where c_0 and c_1 are obtained from the initial conditions. **[10]**

NO MARK WILL BE AWARDED IF ANOTHER METHOD IS USED.

QUESTION 6

Find $\mathcal{L}\{f(t)\}$ from first principles

$$f(t) = \begin{cases} t & 0 \le t < 1 \\ 1 & t \ge 1 \end{cases}.$$

[10]

QUESTION 7

Use Laplace transform to solve the initial value problem

$$y'' - 2y' + 1 = 1,$$
$$y(0) = 1, \qquad y'(0) = 2.$$

[10]

QUESTION 8

Find the Fourier series of the following function on the given interval:

$$f(x) = \begin{cases} 0 & -2 < x < 0 \\ x & 0 \le x < 1 \\ 1 & 1 \le x < 2 \end{cases}$$

[15]

QUESTION 9

Use the method of separation of variables to find the solution of the homogeneous heat conduction problem:

PDE: $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad (0 < x < L, \quad t > 0)$$

BCs: $$\frac{\partial u}{\partial x}(0,t) = 0, \qquad \frac{\partial u}{\partial x}(L,t) = 0$$

IC: $$u(x,0) = x$$

(a) Assume $u(x,t) = \phi(x)\,G(t)$ and derive the ODEs satisfied by $\phi(x)$ and $G(t)$, if $-\lambda$ is the separation constant. (2)

(b) If $\lambda > 0$, determine the general solution for $\phi(x)$ and hence use the boundary conditions to determine λ_n. (5)

(c) Write the eigenfunctions and corresponding eigenvalues of the above problem. (3)

(d) Write the product solution. (2)

(e) From the product solution show that

$$u(x,t) = \frac{L}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} e^{-\left(\frac{n\pi}{L}\right)^2 kt}. \qquad (3)$$

[15]

TOTAL: [100]

FORMULA SHEET

I. Properties of the inverse operator $\dfrac{1}{P(D)}$ applied to a few standard functions.

1.
$$\frac{1}{P(D)}[e^{ax}] = \frac{e^{ax}}{P(a)} \text{ if } P(a) \neq 0$$

2.
$$\frac{1}{P(D)}[e^{ax}y] = e^{ax}\frac{1}{P(D+a)}[y]$$

3.1
$$\frac{1}{P(D^2, D)}[\sin ax] = \frac{1}{P(-a^2, D)}[\sin ax],\ P(-a^2, D) \neq 0$$

3.2
$$\frac{1}{P(D^2, D)}[\cos ax] = \frac{1}{P(-a^2, D)}[\cos ax],\ P(-a^2, D) \neq 0$$

4.1
$$\frac{1}{D^2 + a}[\sin ax] = \frac{-x}{2a}\cos ax$$

4.2
$$\frac{1}{D^2 + a^2}[\cos ax] = \frac{x}{2a}\sin ax$$

5.1
$$\frac{1}{\lambda D + \mu}[\sin ax] = \frac{-1}{\lambda^2 a^2 + \mu^2}(\lambda a \cos ax - \mu \sin ax),\ \lambda, \mu \in \mathbb{R}$$

5.2
$$\frac{1}{\lambda D + \mu}[\cos ax] = \frac{1}{\lambda^2 a^2 + \mu^2}(\lambda a \sin ax + \mu \cos ax],\ \lambda, \mu \in \mathbb{R}$$

II.

Annihilator operator	functions annihilated
$(D-\alpha)^n$	$x^k e^{\alpha x}$ for each $k = 0, 1, \ldots, n-1$
D^n	x^k for each $k = 0, 1, \ldots, n-1$
$[D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$	$\begin{cases} x^k e^{\alpha x}\cos\beta x \\ x^k e^{\alpha x}\sin\beta x \end{cases}$ for each $k = 0, 1, \ldots, n-1$
$D^2 + \beta^2$	$\cos\beta x,\ \sin\beta x$
$(D^2 + \beta^2)^n$	$\begin{cases} x^k \cos\beta x \\ x^k \sin\beta x \end{cases}$ for each $k = 0, 1, \ldots, n-1$

III. LAPLACE–TRANSFORMS OF SOME BASIC FUNCTIONS

1.
$$\mathcal{L}\{1\} = \frac{1}{s}$$

2.
$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},\ n = 1, 2, 3, \ldots$$

3.
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

4.
$$\mathcal{L}\{\sin\ kt\} = \frac{k}{s^2+k^2}$$

5.
$$\mathcal{L}\{\cos\ kt\} = \frac{s}{s^2+k^2}$$

6.
$$\mathcal{L}\{\sinh\ kt\} = \frac{k}{s^2-k^2}$$

7.
$$\mathcal{L}\{\cosh\ kt\} = \frac{s}{s^2-k^2}$$

8.
$$\begin{aligned}\mathcal{L}\left\{f^{(n)}(t)\right\} &= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \\ &\quad - \ldots - f^{(n-1)}(0)\end{aligned}$$

where
$$F(s) = \mathcal{L}\{f(t)\}.$$

9.
$$\mathcal{L}\{g(t)\,\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}.$$

10.
$$\mathcal{L}^{-1}\{f(s+1)\} = e^{-t}\mathcal{L}^{-1}\{f(s)\}.$$

Memorandum

1. (a)

$$\begin{aligned} \frac{dy}{(y-1)^2} &= dx \\ \ln|y-1| &= x+c \\ y-1 &= Ae^x \end{aligned}$$
$$y = 1 + Ae^x$$

(b) Separating variables,

$$\begin{aligned} \frac{dx}{1+x^2} &= 4dt \\ \arctan x &= 4t+c. \end{aligned}$$

Substituting $x(\pi/4) = 1$ we get

$$\begin{aligned} \frac{\pi}{4} &= \pi + c \\ c &= -\frac{3\pi}{4} \end{aligned}$$

so

$$x(t) = \tan\left(4t - \frac{3\pi}{4}\right).$$

2. (a)

$$\frac{\partial}{\partial y}\left(x^3+y^3\right) = 3y^2 = \frac{\partial}{\partial x}\left(3xy^2\right)$$

(b) There exists a function $f(x,y)$ such that $\dfrac{\partial f}{\partial x} = x^3 + y^3$, so

$$f = \frac{1}{4}x^4 + xy^3 + g(y).$$

As $\dfrac{\partial f}{\partial y} = 3xy^2$, we see that $g'(y) = 0$, so $f(x,y) = \frac{1}{4}x^4 + xy^3 + c$. A solution of the DE is given by setting $f(x,y) = 0$, so implicitly we see that

$$\begin{aligned} x^4 + xy^3 + c &= 0 \\ y &= (-cx^{-1} - x^3)^{1/3} \end{aligned}$$

3. (a)

$$D^4$$

.

(b) We write the DE as

$$\begin{aligned} (D^2 - 2D + 1)y &= x^3 - 4x \\ (D-1)^2 y &= x^3 - 4x \\ y &= \frac{1}{(D-1)^2}\left[x^3 - 4x\right] \\ y &= \left(\frac{1}{1-D}\right)^2\left[x^3 - 4x\right]. \end{aligned}$$

Now $\frac{1}{1-D} = 1 + D + D^2 + D^3 + \ldots$, so

$$\begin{aligned}\frac{1}{1-D}\big[x^3 - 4x\big] &= \big(1 + D + D^2 + D^3 + \ldots\big)\big[x^3 - 4x\big] \\ &= x^3 - 4x + 3x^2 - 4 + 6x + 6 \\ &= x^3 + 3x^2 + 2x + 2.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{1}{(1-D)^2}\big[x^3 - 4x\big] &= \big(1 + D + D^2 + D^3 + \ldots\big)\big[\frac{1}{1-D}\big[x^3 - 4x\big]\big] \\ &= \big(1 + D + D^2 + D^3 + \ldots\big)\big[x^3 + 3x^2 + 2x + 2\big] \\ &= x^3 + 3x^2 + 2x + 2 + 3x^2 + 2 + 6x + 6x + 6 + 6 \\ &= x^3 + 6x^2 + 14x + 16.\end{aligned}$$

Therefore $y = x^3 + 6x^2 + 14x + 16$.

4. (a) Set $a_n = \frac{x^{n-1}}{n3^n}$. By the ratio test, if the given series is to converge, we require that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. We calculate:

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left|\frac{x^n}{(n+1)3^{n+1}}\cdot\frac{n3^n}{x^{n-1}}\right| \\ &= \lim_{n\to\infty}\left|\frac{nx}{(n+1)3}\right| \\ &= \left|\frac{x}{3}\right| < 1\end{aligned}$$

if $|x| < 3$. So the radius of convergence is 3.

(b) From the answer above, we know that the series converges for all $x \in (-3, 3)$. to determine the interval of convergence, we must see what happens at $x = -3$ and $x = 3$.

First $x = 3$: the series becomes

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{x^{n-1}}{n3^n} &= \sum_{n=1}^{\infty}\frac{3^{n-1}}{n3^n} \\ &= \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n}\end{aligned}$$

which is divergent (the harmonic series, $\sum_{n=1}^{\infty}\frac{1}{n}$, is known to be divergent).

Now for $x = -3$: the series becomes

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{x^{n-1}}{n3^n} &= \sum_{n=1}^{\infty}\frac{(-3)^{n-1}}{n3^n} \\ &= \frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\end{aligned}$$

which is convergent (the alternating harmonic series, $\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, is known to be convergent).

So the interval of convergence is $[-3, 3)$.

5. Let $y = \sum_{n=0}^{\infty} c_n x^n$. Then

$$\begin{aligned} y' &= \sum_{n=1}^{\infty} n c_n x^{n-1} \\ y'' &= \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} \end{aligned}$$

and substituting these into the DE yields

$$\begin{aligned} (x+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + (x-2)\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n &= 0 \\ \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-1} + \sum_{n=1}^{\infty} n c_n x^n - 2\sum_{n=1}^{\infty} n c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n &= 0. \end{aligned}$$

we make the following substitutions: in the first sum, $k = n-2$, in the second, $k = n-1$, in the third, $k = n$, in the fourth $k = n-1$ and in the fifth $k = n$. So we get

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} k(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} k c_k x^k - 2\sum_{k=0}^{\infty}(k+1)c_{k+1}x^k + \sum_{k=0}^{\infty} c_k x^k = 0.$$

Now taking out all terms where $k = 0$ (from the first, fourth and fifth sums), we obtain

$$2c_2 + c_0 - 2c_1 + \sum_{k=1}^{\infty}\big[(k+2)(k+1)c_{k+2} + k(k+1)c_{k+1} + kc_k + c_k\big]x^k = 0.$$

So we have the following equations to solve:

$$\begin{aligned} 2c_2 + c_0 - 2c_1 &= 0 \\ (k+2)(k+1)c_{k+2} + k(k+1)c_{k+1} + (k+1)c_k &= 0 \\ (k+2)c_{k+2} + kc_{k+1} + c_k &= 0 \\ c_{k+2} &= -\frac{1}{k+2}\big((k-2)c_{k+1} + c_k\big). \end{aligned}$$

From the Initial Conditions: $y(0) = c_0 = 2$ and $y'(0) = c_1 = 1$, so

$$c_2 = \frac{1}{2}(2c_1 - c_0) = 0.$$

Further terms are

$$\begin{aligned} c_3 &= -\frac{1}{3} \\ c_4 &= 0 \\ c_5 &= \frac{1}{15} \\ c_6 &= -\frac{1}{45}. \end{aligned}$$

Therefore the solution is $y = 2 + x - \frac{1}{3}x^3 + \frac{1}{15}x^5 - \frac{1}{45}x^6 + \ldots$.

6.

$$\begin{aligned}
\mathcal{L}(f(t)) &= \int_0^\infty f(t)e^{-st}dt \\
&= \int_0^1 te^{-st}dt + \int_1^\infty e^{-st}dt \\
&= -\frac{t}{s}e^{-st}\Big|_0^\infty + \frac{1}{s}\int_0^1 e^{-st}dt - \frac{1}{s}e^{-st}\Big|_1^\infty \\
&= -\frac{1}{s^2}e^{-st}\Big|_0^1 + \frac{1}{s}e^{-s} \\
&= -\frac{1}{s^2}e^{-s} + \frac{1}{s^2} + \frac{1}{s}e^{-s} \\
&= e^{-s}\left(\frac{1}{s} - \frac{1}{s^2}\right) + \frac{1}{s^2}
\end{aligned}$$

7. We compute the following:

$$\begin{aligned}
\mathcal{L}(y') &= sY - 1 \\
\mathcal{L}(y'') &= s^2Y - s - 2 \\
\mathcal{L}(y'' - 2y') &= 0 \\
s^2Y - s - 2 - 2(sY - 1) &= 0 \\
s^2Y - 2sY - s &= 0 \\
sY - 2Y - 1 &= 0 \\
Y(s-2) &= 1 \\
Y &= \frac{1}{s-2}
\end{aligned}$$

Therefore

$$y(t) = \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}.$$

8.

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-2}^{2} f(x)dx = \frac{1}{2}\big[\int_0^1 xdx + \int_1^2 dx\big] \\
&= \frac{1}{2}[\frac{1}{2}+1] = \frac{3}{4} \\
a_n &= \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\big[\int_0^1 x\cos\frac{n\pi x}{2}dx + \int_1^2 \cos\frac{n\pi x}{2}dx\big] \\
&= \frac{1}{2}\big[\frac{2}{n\pi}x\sin\frac{n\pi x}{2}\big|_0^1 - \frac{2}{n\pi}\int_0^1 \sin\frac{n\pi x}{2}dx + \frac{2}{n\pi}\sin\frac{n\pi x}{2}\big|_1^2\big] \\
&= \frac{1}{2}\big[\frac{2}{n\pi}\sin\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\big|_0^1 - \frac{2}{n\pi}\sin\frac{n\pi x}{2}\big|_1^2\big] \\
&= \frac{1}{2}\big[\frac{4}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{4}{n^2\pi^2}\big] \\
&= \frac{2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{2}{n^2\pi^2} \\
b_n &= \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = \frac{1}{2}\big[\int_0^1 x\sin\frac{n\pi x}{2}dx + \int_1^2 \sin\frac{n\pi x}{2}dx\big] \\
&= \frac{1}{2}\big[-\frac{2}{n\pi}x\cos\frac{n\pi x}{2}\big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos\frac{n\pi x}{2}dx - \frac{2}{n\pi}\cos\frac{n\pi x}{2}\big|_1^2\big] \\
&= \frac{1}{2}\big[-\frac{2}{n\pi}\cos\frac{n\pi}{2} + \frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\big|_0^1 - \frac{2}{n\pi}\cos n\pi + \frac{2}{n\pi}\cos\frac{n\pi}{2}\big] \\
&= \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{1}{n\pi}(-1)^n.
\end{aligned}$$

Therefore the Fourier series of f is:

$$f(x) = \frac{3}{8} + \sum_{n=1}^{\infty}\big[\frac{2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{2}{n^2\pi^2}\big]\cos\frac{n\pi x}{2} + \big[\frac{2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{1}{n\pi}(-1)^n\big]\sin\frac{n\pi x}{2}.$$

9. (a)

$$\begin{aligned}
u_t &= \phi(x).G'(t) \\
u_{xx} &= \phi''(x).G(t).
\end{aligned}$$

Substituting into the PDE, we get

$$\phi G' = k\phi'' G$$

$$\frac{\phi''}{\phi} = \frac{1}{k}\frac{G'}{G} = -\lambda.$$

Therefore the two ODEs are

$$\begin{aligned}
\phi'' + \lambda\phi &= 0 \\
G' + k\lambda G &= 0.
\end{aligned}$$

(b) The characteristic equation of $\phi + \lambda\phi'' = 0$ is $r^2 + \lambda = 0$, so $r = \pm i\sqrt{\lambda}$. The solution of the ODE is

$$\phi(x) + A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x.$$

Now

$$\phi'(x) = \sqrt{\lambda}\big(A\cos\sqrt{\lambda}x - B\sin\sqrt{\lambda}x\big)$$

$$\phi'(0) = \sqrt{\lambda}A = 0$$

Therefore $A = 0$ and $\phi(x) = B\cos\sqrt{\lambda}x$. Hence

$$\begin{aligned} \phi'(L) &= -B\sin\sqrt{\lambda}L = 0 \\ \sin\sqrt{\lambda}L &= 0 \\ \sqrt{\lambda}L &= n\pi \text{ for } n = 0, 1, 2, \ldots \\ \lambda_n &= \frac{n^2\pi^2}{L^2}. \end{aligned}$$

(c) These $\lambda_n = (n^2\pi^2)/L^2$ are the eigenvalues; the corresponding eigenfunctions $u_n(x,t)$ are given by

$$\begin{aligned} u_n(x,t) &= \phi_n(x)G_n(t) \\ &= B_n \cos\frac{n\pi}{L}x \; e^{\frac{-n^2\pi^2kt}{L^2}}. \end{aligned}$$

(d) By the principle of superposition, we see that

$$u(x,t) = \sum_{n=0}^{\infty} B_n \cos\frac{n\pi}{L}x \; e^{\frac{-n^2\pi^2kt}{L^2}}.$$

(e) When $t = 0$, we use the boundary condition $u(x,0) = x$ to see that

$$\sum_{n=0}^{\infty} B_n \cos\frac{n\pi}{L}x = x.$$

In other words, we have a Fourier sine series expansion of the function x. From the general theory of Fourier series, we know that the constant term is given by

$$\begin{aligned} \frac{1}{2}\left[\frac{2}{L}\int_0^L x \; dx\right] &= \frac{1}{2L}x^2\Big|_0^L \\ &= \frac{L}{2}. \end{aligned}$$

Thus

$$u(x,t) = \frac{L}{2} + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi}{L}x \; e^{\frac{-n^2\pi^2kt}{L^2}}.$$