Chapter 3

Interpolation and Polynomial Approximation

Chapter 3.1: Interpolation and the Lagrange Polynomial*

The important ideas in this section are:
- How can we want to find a polynomial that agrees with (interpolates) a given function at more than one point and remains as close to the given function as we want?
- How can we calculate a bound for the error involved in approximating a function by the interpolating polynomial?

The first thought might be to use a Taylor polynomial. However, although a Taylor polynomial agrees with the function near a specific point, it does not agree with the function closely over a specified interval. The Lagrange interpolating polynomial was developed specifically for this purpose.

Suppose \(x_0, x_1, \ldots, x_n\) are \(n+1\) distinct numbers and \(f\) is a function whose values are given at these numbers. Theorem 3.2 guarantees the existence of a unique polynomial \(P(x)\) of degree at most \(n\) with \(f(x_k) = P(x_k)\), for each \(k = 0, 1, \ldots, n\)

given by

\[
P(x) = f(x_0)L_{0, n}(x) + \ldots + f(x_n)L_{n, n}(x) = \sum_{k=0}^{n} f(x_k)L_{n, k}(x)
\]

where, for each \(k = 0, 1, \ldots, n\),

\[
L_{n, k}(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{k-1})(x - x_{k+1})\ldots(x - x_n)}{(x_k - x_0)(x_k - x_1)\ldots(x_k - x_{k-1})(x_k - x_{k+1})\ldots(x_k - x_n)} = \prod_{i=0, i \neq k}^{n} \frac{x - x_i}{x_k - x_i}
\]

Thus, a Lagrange interpolating polynomial of degree one that agrees with \(f\) at \((x_0, f(x_0)), (x_1, f(x_1))\) would be given by:

\[
P(x) = \frac{(x - x_1)}{(x_0 - x_1)}f(x_0) + \frac{(x - x_0)}{(x_1 - x_0)}f(x_1)
\]

and a Lagrange interpolating polynomial of degree two that agrees with \(f\) at \((x_0, f(x_0)), (x_1, f(x_1)), (x_2, f(x_2))\) would be given by:
\[
P(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}f(x_1) + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}f(x_2)
\]

Example 4 from the text shows how the error formula given by

\[
\frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)^{n+1}
\]

can be used to determine a bound on the error when using Lagrange interpolation.

Let's look at the following example where \(f(x) = \tan(x)\). The computer algebra system MAPLE was used to generate the polynomials of degree 1, 2, and the graphs.

```
restart:
with(plots):
f := x -> tan(x):
x_0 := 0:
x_1 := 0.6:
x_2 := 0.9:
L1 := \(\frac{(x-x_1)}{(x_0-x_1)}f(x_0) + \frac{(x-x_0)}{(x_1-x_0)}f(x_1)\):
P1 := x \rightarrow 1.140228014 x
P2 := x \rightarrow 0.866492613 x^2 + 0.620332446 x

plot([f, P1, P2], 0..0.9, axis = [gridlines = [colour = green, majorlines = 2]])
```
Notice that $P1$ agrees with the values of $f$ at $x_0$ and $x_1$ while $P2$ agrees with the values of $f$ at $x_0$, $x_1$, and $x_2$.

$f(0) = 0$
$P1(0) = 0.$
$P2(0) = 0.$

$f(0.6) = 0.6841368083$
$P1(0.6) = 0.6841368084$
$P2(0.6) = 0.6841368083$

$f(0.9) = 1.260158218$
$P1(0.9) = 1.026205213$
$P2(0.9) = 1.260158218$

The error bound for $P2(x)$ can be easily determined by maximizing $\left| \frac{f^3(\xi)}{(2 + 1)!} \right|$ on $[0,0.9]$. We see that the largest value obtained on $[0,0.9]$ is

$$M := \maximize \left( \frac{2 \sec^4(x) + 4 \sec(x) \cdot \tan(x)}{3!} \right), x = 0..0.9 \right) = 3.584078513.$$

Thus, the bound is
\[ M \cdot (x_2 - x_0) = 3.225670662 \]

\section*{Chapter 3.2: Data Approximation and Neville's Method*}

In the last section we found that the Lagrange polynomials gave us an explicit formula for the approximation of a function on a given interval. You will notice that occasionally the function we are trying to approximate is given in these sections, however, in reality we quite often only have a finite set of data available to us and do not know the function \( f \). In this situation we may not need to know the explicit formula for the approximating function, but we would like to be able to determine function values at certain \( x \) values of interest. Neville's method is used for this purpose.

Rather than using the Lagrange formula to find the approximating polynomial as we did in Section 3.1, for \( f(x) = \tan(x) \), we use the formula for \( P(x) \) to calculate \( P(x^*) \) where \( x^* \) is some given value of \( x \). From our work in the previous section we know that \( f(x^*) \approx P(x^*) \). Thus, we can compute any approximation at a given value of \( x \) in this manner. For example, suppose we are given the following data: \( (0, 0), (0.6, 0.6841368083), \) and \( (0.9, 1.260158218) \) and we want to use the 2nd degree interpolating polynomial to approximate \( f \). The computations are in the following table:

\[ h := x \rightarrow \tan(x) \] The approximation \( h(.7) = 0.8422883805 \) is shown in the table below:

<table>
<thead>
<tr>
<th>( i )</th>
<th>( x_i )</th>
<th>( f(x_i) )</th>
<th>( P(x_i) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0.6</td>
<td>0.6841368083</td>
<td>0.7981596097</td>
</tr>
<tr>
<td>2</td>
<td>0.9</td>
<td>1.260158218</td>
<td>0.8761439450</td>
</tr>
<tr>
<td>3</td>
<td>0.95</td>
<td>1.9582589</td>
<td>0.8278915989</td>
</tr>
</tbody>
</table>

\[
\begin{align*}
\frac{1}{(0.6 - 0)} (0.7 - 0) & \quad \frac{1}{(0.9 - 0)} (0.7 - 0) \\
\cdot 0.6841368083 & \quad \cdot 0.707260734 \\
-0.6 & \quad -0.95 \\
= 0.7981596097 & \quad = 0.8761439450 \\
\end{align*}
\]

<table>
<thead>
<tr>
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<td>0.7981596097</td>
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</tr>
</tbody>
</table>

\[
\begin{align*}
\frac{1}{0.6} (0.7 - 0) & \quad \frac{1}{0.9} (0.7 - 0) \\
\cdot 0.6 & \quad \cdot 0.707260734 \\
= 0.7981596097 & \quad = 0.8761439450 \\
\end{align*}
\]

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= 0.7981596097 & \quad = 0.8761439450 \\
\end{align*}
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<th>( P(x_i) )</th>
</tr>
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<td>0.6</td>
<td>0.6841368083</td>
<td>0.7981596097</td>
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<td>1.260158218</td>
<td>0.8761439450</td>
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<td>0.95</td>
<td>1.9582589</td>
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\begin{align*}
\frac{1}{0.6} (0.7 - 0) & \quad \frac{1}{0.9} (0.7 - 0) \\
\cdot 0.6841368083 & \quad \cdot 0.707260734 \\
-0.6 & \quad -0.95 \\
= 0.7981596097 & \quad = 0.8761439450 \\
\end{align*}
\]
As illustrated in this example, iterated interpolation is used to generate successively higher degree polynomial approximations at a specific point, say $x^*$. 

Chapter 3.3: Divided Differences

The key question we want to address in this section is:
- What are other difference formulas for interpolation?

Newton's Divided Difference Method:
In the last section, iterated interpolation is used to generate successively higher degree polynomial approximations at a specific point, say $x^*$. In this section we will use divided differences to successively generate the polynomials.

Given a set of data, we will look at template that should act as a guide in this process.

<table>
<thead>
<tr>
<th>$x_i$</th>
<th>$f(x_i)$</th>
<th>First Divided Difference</th>
<th>Second Divided Difference (skip one $x$ between in denominator)</th>
<th>Third Divided Difference (skip two $x$s between in denominator)</th>
<th>Fourth Divided Difference (skip three $x$s between in denominator)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_0$</td>
<td>$f(x_0)$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>$f(x_1)$</td>
<td>$a_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$= \frac{f(x_1)}{x_1 - x_0}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_2$</td>
<td>$f(x_2)$</td>
<td>$a_2 = \frac{f(x_2)}{x_2 - x_1}$</td>
<td>$b_1 = \frac{a_2 - a_1}{x_2 - x_0}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_3$</td>
<td>$f(x_3)$</td>
<td>$a_3 = \frac{f(x_3)}{x_3 - x_2}$</td>
<td>$b_2 = \frac{a_3 - a_2}{x_3 - x_1}$</td>
<td>$c_1$</td>
<td>$= \frac{1}{(x_3 - x_1)}$</td>
</tr>
<tr>
<td>$x_4$</td>
<td>$f(x_4)$</td>
<td>$a_4 = \frac{f(x_4)}{x_4 - x_3}$</td>
<td>$b_3 = \frac{a_4 - a_3}{x_4 - x_2}$</td>
<td>$c_2$</td>
<td>$= \frac{1}{(x_4 - x_2)}$</td>
</tr>
</tbody>
</table>

Note: The table entries for $c_1$ and $c_2$ are not fully filled as they depend on the values of $a_1, a_2, a_3,$ and $a_4$. The entries for $b_1, b_2,$ and $b_3$ are derived from the divided differences, and $c_1$ and $c_2$ are used in the final polynomial approximation.
The Newton's divided difference polynomial then becomes

\[ P_4(x) = f(x_0) + a_1(x - x_0) + b_1(x - x_0)(x - x_1) + c_1(x - x_0)(x - x_1)(x - x_2) + d_1(x - x_0)(x - x_1)(x - x_2)(x - x_3) \]

**Newton's Forward, Backward & Centered Difference Formulas:**

The following table shows the difference formulas rooted in the Aitkin's A^2 method:

<table>
<thead>
<tr>
<th>( x_i )</th>
<th>( f(x_i) )</th>
<th>First Divided Difference (same as first column in previous table)</th>
<th>Second Divided Difference (same as second column in previous table &amp; also divide by 2)</th>
<th>Third Divided Difference (same as third column in previous table &amp; also divide by 3)</th>
<th>Fourth Divided Difference (same as fourth column in previous table &amp; also divide by 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_0 )</td>
<td>( f(x_0) )</td>
<td>( = (f(x_1) - f(x_0)) )</td>
<td>( = (a_2 )</td>
<td>( = (b_2 )</td>
<td>( = (c_2 )</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>( f(x_1) )</td>
<td>( a_1 )</td>
<td>( = (f(x_2) - f(x_1)) )</td>
<td>( = (a_2 )</td>
<td>( = (b_2 )</td>
</tr>
<tr>
<td>( x_2 )</td>
<td>( f(x_2) )</td>
<td>( a_2 )</td>
<td>( = (b_2 )</td>
<td>( = (c_2 )</td>
<td>( = (d_1 )</td>
</tr>
<tr>
<td>( x_3 )</td>
<td>( f(x_3) )</td>
<td>( a_3 )</td>
<td>( = (c_2 )</td>
<td>( = (d_1 )</td>
<td></td>
</tr>
<tr>
<td>( x_4 )</td>
<td>( f(x_4) )</td>
<td>( a_4 )</td>
<td>( = (d_1 )</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

When approximating a value when \( x \) is close to the end of the tabulated values, we make use of the
earliest data points closest to x.

The forward difference polynomial is defined to be:

\[ P_4(x) = f(x_0) + s_1 \cdot h \cdot a_1 + s_1 \cdot s_2 \cdot h^2 \cdot b_1 + s_1 \cdot s_2 \cdot s_3 \cdot h^3 \cdot c_1 + s_1 \cdot s_2 \cdot s_3 \cdot s_4 \cdot h^4 \cdot d_1 \]

where \( h = \frac{(x_4 - x_0)}{4} \) and \( s_i = \frac{(x - x_i)}{h} \) for \( i = 0, \ldots, 4 \).

When approximating a value when x is close to the end of the tabulated values, we make use of the earliest data points closest to x.

The backward difference polynomial is defined to be:

\[ P_4(x) = f(x_4) + s_4 \cdot h \cdot a_4 + s_4 \cdot s_3 \cdot h^2 \cdot b_3 + s_4 \cdot s_3 \cdot s_2 \cdot h^3 \cdot c_2 + s_4 \cdot s_3 \cdot s_2 \cdot s_1 \cdot h^4 \cdot d_1 \]

where \( h = \frac{(x_4 - x_0)}{4} \) and \( s_i = \frac{(x - x_i)}{h} \) for \( i = 4, \ldots, 0 \) for this example.

<table>
<thead>
<tr>
<th>x</th>
<th>f(x)</th>
<th>First Divided Difference (same as first column in previous table)</th>
<th>Second Divided Difference (same as second column in previous table &amp; also divide by 2)</th>
<th>Third Divided Difference (same as third column in previous table &amp; also divide by 3)</th>
<th>Fourth Divided Difference (same as fourth column in previous table &amp; also divide by 4)</th>
</tr>
</thead>
<tbody>
<tr>
<td>x_0</td>
<td>f(x_0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x_1</td>
<td>f(x_1)</td>
<td>( a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x_2</td>
<td>f(x_2)</td>
<td>( a_2 = \frac{f(x_2) - f(x_1)}{x_2 - x_1} ) ( b_1 = \frac{a_2}{2(x_2 - x_1)} )</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>x_3</td>
<td>f(x_3)</td>
<td>( a_3 = \frac{f(x_3) - f(x_2)}{x_3 - x_2} ) ( b_2 = \frac{a_3}{2(x_3 - x_2)} )</td>
<td></td>
<td>( c_1 = \frac{b_2}{3(x_3 - x_1)} )</td>
<td></td>
</tr>
<tr>
<td>x_4</td>
<td>f(x_4)</td>
<td>( a_4 = \frac{f(x_4) - f(x_3)}{x_4 - x_3} ) ( b_3 = \frac{a_4}{2(x_4 - x_3)} )</td>
<td></td>
<td>( c_2 = \frac{b_3}{3(x_4 - x_1)} )</td>
<td>( d_1 = \frac{c_2}{4(x_4 - x_0)} )</td>
</tr>
</tbody>
</table>
When approximating a value when \(x\) is close to the middle of the tabulated values, say near \(x_3\), we make use of the earliest data points closest to \(x\).

**Stirling's centered difference** polynomial is defined to be:

\[
P_4(x) = f(x_3) + s_4 \cdot h \cdot a_4 + s_4 \cdot s_3 \cdot h^2 \cdot b_3 + s_4 \cdot s_3 \cdot s_2 \cdot h^3 \cdot c_2 + s_4 \cdot s_3 \cdot s_2 \cdot s_1 \cdot h^4 \cdot d_1
\]

where \(h = \frac{(x_4 - x_0)}{4}\) and \(s = \frac{(x - x_3)}{h}\) for this example.

### Chapter 3.4: Hermite Interpolation*

In this section we look at osculating polynomials that agree with values of \(f\). The text provides a detailed example of the Hermite interpolating polynomial, so we provide a visual illustration of the graph of \(f\) and the Hermite polynomial for the example in the text.

```
restart:

\[
ff := x \rightarrow 1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 - \frac{1}{2304} x^6:
\]

```

```
ff1 = 0.6200860 \cdot (10 \cdot x - 12) \cdot \left( \frac{50}{9} x^2 - \frac{175}{9} x + \frac{152}{9} \right)^2 + 0.4554022 \cdot \left( -\frac{100}{9} x^2 + \frac{320}{9} x - \frac{247}{9} \right)^2 + 0.2818186 \cdot 10 \cdot (2 - x) \cdot \left( \frac{50}{9} x^2 - \frac{145}{9} x + \frac{104}{9} \right)^2 - 0.5220232 \cdot (x - 1.3) \cdot \left( \frac{50}{9} x^2 - \frac{175}{9} x + \frac{152}{9} \right)^2 - 0.5698959 \cdot (x - 1.6) \cdot \left( -\frac{100}{9} x^2 + \frac{320}{9} x - \frac{247}{9} \right)^2 - 0.5811571 \cdot (x - 1.9) \cdot \left( \frac{50}{9} x^2 - \frac{145}{9} x + \frac{104}{9} \right)^2:
```

```
HI := x \rightarrow 1.001944063 - 0.0082292208 \cdot x - 0.2352161732 \cdot x^2 - 0.01455607812 \cdot x^3 + 0.02403178946 \cdot x^4 - 0.002774691277 \cdot x^5:
```

```
plot([ff, HI], 1.2..4.0, axis = [gridlines = [colour = green, majorlines = 2]])
```
Chapter 3.5: Cubic Spline Interpolation

Chapter 3.6: Parametric Curves