Study Unit 4 : Non-Linear functions

Chapter 4 : Sections 4.1– 4.4

Types of non-linear functions:

- **Polynomials**
  - Linear function : $y = mx + c$ – chapter 2
  - Quadratic function : $y = ax^2 + bx + c$
  - Cubic function : $y = ax^3 + bx^2 + cx + d$
  - *n*-th order polynomials : $y = ax^n + bx^{n-1} + cx^{n-2} +$ ……… + constant
- **Exponential functions**

  $y = a^x$: general format (a = constant)

  $y = 10^x$: scientific format

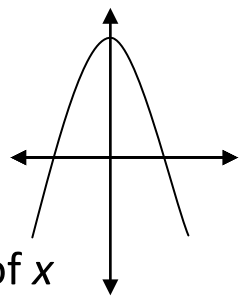
  $y = e^x$: natural format
- **Logarithm functions**

  $y = \log_a x$

  $y = \log_{10} x = \log x$

  $y = \log_e x = \ln x$
- **Hyperbolic functions** $y = \dfrac{a}{bx + c}$

1. Quadratic function



$$y = ax^2 + bx + c$$

- polynomial of **degree** 2 – highest power of *x*

- *c* : ***y*-axis intercept** : cuts *y*-axis
  - point where $x = 0$: (0 ; c)

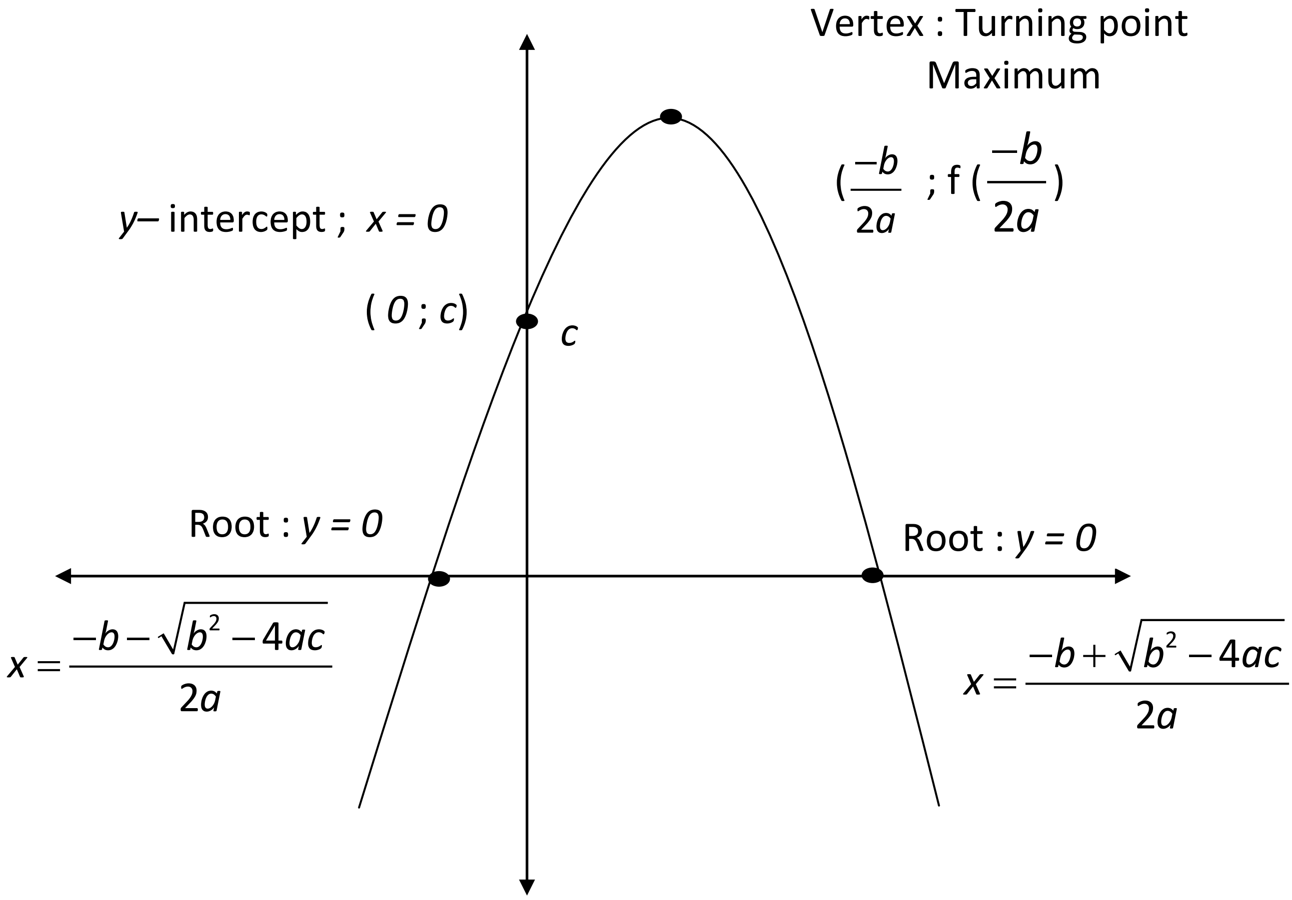
- *a* – **shape** of the function
  - $a > 0$: *a* positive : smiling face
  - $a < 0$: *a* negative : sad face

- **vertex** : turning point : function maximum or minimum
  - point (*x;y*) with $x = \frac{-b}{2a}$ and y the function value if $x = \frac{-b}{2a}$ (substitute the answer for x into the function and solve y)

- **Roots** or *x*-intercept : cuts *x*-axis
  - make $y = 0$ and solve *x* by
    - factorisation or quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Example :


Vertex : Turning point
Maximum
$\left(\frac{-b}{2a}\ ;\ f\left(\frac{-b}{2a}\right)\right)$
y– intercept ; *x = 0*
(*0* ; *c*)
c
Root : *y = 0*
Root : *y = 0*
$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$
$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$

Example:

Find the coordinates of the vertex of the graph $y = 4x^2 - x - 3$.

- Vertex = extreme point = turning point = max or min
- a > 0 thus graph = smiling face =>min exists
- x coordinate of minimum is $x = \frac{-b}{2a}$

Comparing $y = 4x^2 - x - 3$ with $y = ax^2 + bx + c$ => a =4, b = –1, c = –3. Thus

$$x = \frac{-b}{2a}$$

$$x = \frac{-(-1)}{2(4)}$$

$$x = \frac{1}{8} = 0{,}125$$

- y value if x = 0,125 is

$$y = 4x^2 - x - 3$$

$$y = 4(0{,}125)^2 - 0{,}125 - 3$$

$$y = -3{,}0625$$

Coordinates of the vertex of the graph $y = 4x^2 - x - 3$ are

(0,125; –3,0625)

Graph : $y = ax^2 + bx + c$

- Choose random x values and substitute into function to calculate y. Draw $(x ; y)$ coordinate points and graph function

  Or

- Calculate the
  - **Roots** or x-intercept
  - ***y*-intercept** : $(0;c)$
  - turning point or **vertex**

and draw coordinate points and graph of function.

Application:

- **supply and demand; break-even etc.**
- **Maximum or minimum**

Discussion class example 14

Question 14

The demand function for a commodity is $Q = 6\,000 - 30P$. Fixed costs are R72 000 and the variable costs are R60 per additional unit produced.

(a) Write down the equation of total revenue and total costs in terms of P.

(b) Determine the profit function in terms of P.

(c) Determine the price at which profit is a maximum, and hence calculate the maximum profit.

(d) What is the maximum quantity produced?

(e) What is the price and quantity at the break-even point(s)?

Solution

(a) Given are the quantity demanded as $Q = 6000 - 30P$, the fixed costs of R72 000 and the variable costs per unit of R60. Now

$$\text{Total Revenue} = \text{Price} \times \text{Quantity}$$
$$TR = PQ$$
$$TR = P(6000 - 30P)$$
$$TR = 6000P - 30P^2$$

$$\text{Total cost} = \text{Fixed Cost} + \text{Variable Cost}$$
$$TC = 72000 + 60Q$$
$$TC = 72000 + 60(6000 - 30P)$$
$$TC = 72000 + 360\,000 - 1800P$$
$$TC = 432\,000 - 1800P$$

(b) Profit is total revenue minus total cost. Thus

$$\begin{aligned}\text{Profit} &= TR - TC \\ &= 6000P - 30P^2 - (432\,000 - 1800P) \\ &= -30P^2 + 7800P - 432\,000\end{aligned}$$

(c) The profit function derived in (b) is a quadratic function with

$$a = -30;\ b = 7800;\ \text{and } c = -432\,000.$$

As a < 0 the shape of the function looks like a "sad face" and the function thus has a maximum at the function's turning point or vertex (*P* ; *Q*).

The price *P* at the turning point or where the profit is a maximum is

$$P = -\frac{b}{2a} = -\frac{7800}{2\times -30} = \frac{-7800}{-60} = 130$$

and thus the maximum profit

$$\text{Profit} = -30(130)^2 + 7800(130) - 432\,000 = 75\,000\,.$$

(d) The maximum quantity produced at the maximum price of R130 calculated in (c) is

$$Q = 6000 - 30(130) = 2100\,.$$

(e) At break-even the profit is equal to zero. Thus

$$\text{Profit} = -30P^2 + 7800 - 432\ 000 = 0.$$

As the profit function is a quadratic function we use the quadratic formula with
a = –30 and b = 7800 and c = – 432 000 to solve P. Thus

$$P = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-7800 \pm \sqrt{(7800)^2 - 4(-30)(-432000)}}{2 \times -30}$$

$$= \frac{-7800 \pm \sqrt{9000000}}{-60}$$

$$= \frac{-7800 \pm 3000}{-60}$$

$$= \frac{-4800}{-60} \text{ or } \frac{-10800}{-60}$$

$$= 80 \text{ or } 180$$

Now if P = 80 then $Q = 6000 - 30(80) = 3600$ and if P = 180 then $Q = 6000 - 30(180) = 600$.

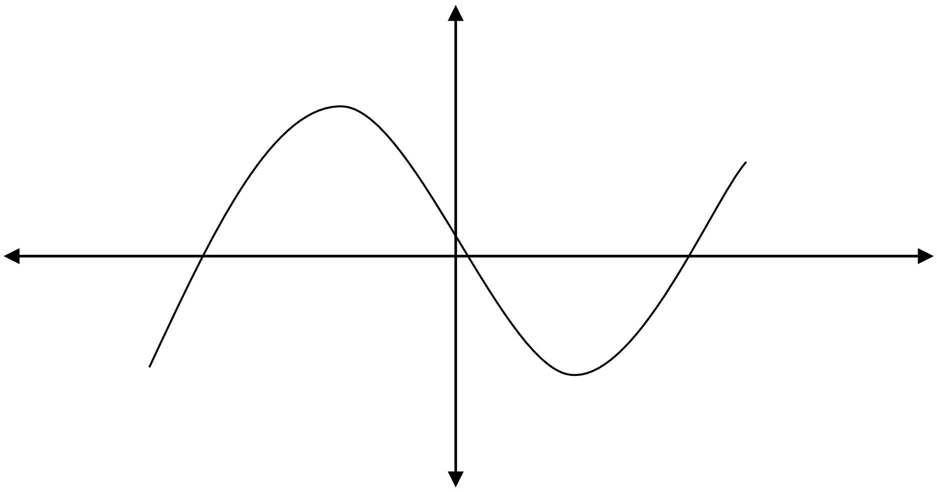
Thus the two break-even points are where the price is R80 and the quantity 3 600, and where the price is R180 and the quantity 600.

2. Cubic function

$$y = ax^3 + bx^2 + cx + d$$

- polynomial of **degree** 3 – highest power of x
- 1 or 3 roots
- 0 or 2 turning points

Example:



Graph : Choose random *x* values and substitute into function to calculate *y*. Draw (*x* ; *y*) coordinate points and graph function

Application:

- supply and demand; break-even etc.
- Maximum or minimum - Differentiation

3. Exponential functions

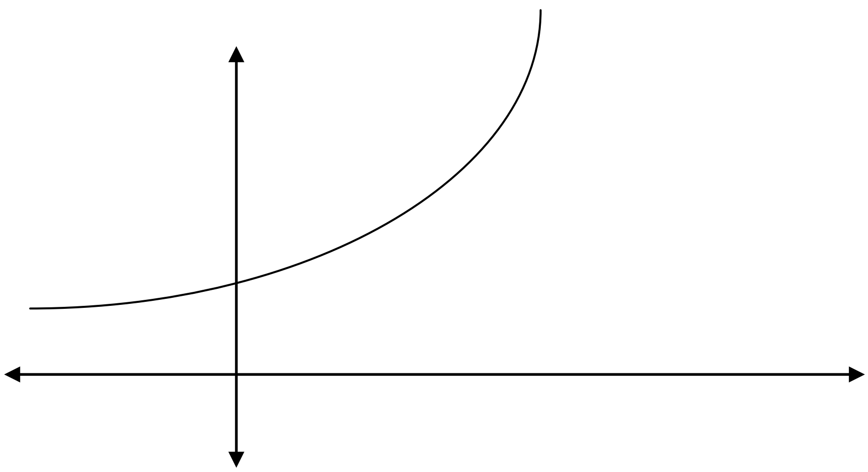
$y = a^x$: general format (a = constant)

$y = 10^x$: scientific format

$y = e^x$: natural format

- a = base = constant
- x = index or power = variable
- e = unending number = 2,7182818….

Example



Properties:

- Continuously pass through point (0 ; 1)
- If power > 0 and a > 0, curve increases : growth curve
- If power < 0 and a > 0, curve decrease ; decay curve
- If a > 0 curve above x-axis
- If a < 0 curve below x-axis

Graph : Choose random *x* values and substitute into function to calculate *y*. Draw (*x* ; *y*) coordinate points and graph function

Rules: Let A and B be any two bases and *x* and *y* any two powers then

1. $A^x \times A^y = A^{x+y}$ $2^2 \times 2^3 = 2^{2+3} = 2^5$
2. $A^x \div A^y = A^{x-y}$ $2^4 \div 2^3 = 2^{4-3} = 2^1$
3. $(A \times B)^x = A^x \times B^x$ $(2 \times 3)^3 = 2^3\, 3^3$
4. $(A/B)^x = A^x / B^x$ $(2 / 3)^3 = 2^3 / 3^3$
5. $(A^x)^y = A^{x \times y}$ $(2^3)^3 = 2^{3\times 3} = 2^9$

Note: If a is any number

1. $(a)^0 = 1$ but $0^0 = 0$ $4^0 = 1$
2. $(a)^1 = a$
3. $\frac{1}{a^n} = a^{-n}$ $2/x^4 = 2x^{-4}$
4. $\sqrt[n]{a} = a^{\frac{1}{n}}$ $\sqrt{24} = 24^{\frac{1}{2}}$
5. $a^{2x+4} = a^{15}$ (base the same) **then 2*x*** + 4 = 15

Discussion class example 15

Application:

Discussion class example 16a

Question 15

Simplify the following expression

$$\left(\frac{4L^2}{L^{-2}}\right)^2$$

Solution

$$\left(\frac{4L^2}{L^{-2}}\right)^2 = (4L^2 \times L^2)^2 \qquad \text{since } \frac{1}{a^b} = a^{-b}$$

$$= (4L^{2+2})^2 \qquad \text{since a}^b \times a^c = a^{b+c}$$

$$= (4L^4)^2$$

$$= 4^2 L^{4\times 2} \qquad \text{since } (a^a)^b = a^{a\times b}$$

$$= 16L^8$$

Question 16

An investment in a bank is said to grow according to the following formula:

$$P(t) = \frac{6\,000}{1 + 29e^{-0,4t}}$$

where t is time in years and P is the amount (principle plus interest).

(a) What is the initial amount invested?

(b) Determine algebraically the time in years when the amount will be R4 000.

Solution

(a) Initial means $t = 0$

$$P = \frac{6000}{1 + 29e^{-0,4\times 0}}$$

Using your calculator's e^x key

$$P = \frac{6000}{30} = 200.$$

(b) If *P* = 4 000 then

$$4000 = \frac{6000}{1+29e^{-0,4t}}$$

$$1+29e^{-0,4t} = \frac{6000}{4000}$$

$$1+29e^{-0,4t} = \frac{3}{2}$$

Divide nominator and denominator by 2000

$$29e^{-0,4t} = \frac{3}{2} - 1$$

$$29e^{-0,4t} = \frac{1}{2}$$

$$e^{-0,4t} = \frac{1}{2} \div \frac{29}{1}$$

$$e^{-0,4t} = \frac{1}{2} \times \frac{1}{29}$$

$$e^{-0,4t} = \frac{1}{58}$$

$$\ln(e^{-0,4t}) = \ln(\frac{1}{58})$$ Take ln on both sides

$$-0,4t\ln e = \ln(\frac{1}{58})$$ $\ln a^b = b\ln a$

$$t = \frac{\ln(\frac{1}{58})}{-0,4}$$ Since $\ln e = 1$

$t = 10,15110753$ Using your calculator, rounded to 8 decimal places

$t = 10,2$ years Rounded to one decimal place

4. Logarithmic functions

$$y = \log_a x$$

$$y = \log_{10} x = \log x$$

$$y = \log_e x = \ln x$$

- Logs is the power of a number : log 10 = 1; log 100 = 2
- $\log_{base}$ number = power same as Number = base ^{power}

$$\log_2 8 = 3 \quad => \quad 8 = 2^3$$

Rules:

1. $\log (u \text{ x } v) = \log u + \log v$
2. $\log (u \div v) = \log u - \log v$
3. $\log u^j = j \log u$
4. $\log_a x = \log x / \log a = \ln x / \ln a$

Discussion class example 17 + 18

Application:

Discussion class example 16(b)

Question 17

Evaluate $\dfrac{\log_3 12,34}{\ln\sqrt{12,34}}$

Solution

Since $\log_a b = \dfrac{\ln b}{\ln a}$ we can write

$$\frac{\log_3 12,34}{\ln\sqrt{12,34}} = \frac{\ln 12,34}{\ln 3} \times \frac{1}{\ln\sqrt{12,34}}$$

Using your calculator, rounded to 3 decimal places

$= 1,820$

Question 18

Solve for Q if $\log Q - \log\left(\frac{Q}{Q+1}\right) = 0,8$

Solution

$$\log(Q) - \log\left(\frac{Q}{Q+1}\right) = 0,8$$

$$\log\left(\frac{Q}{\frac{Q}{Q+1}}\right) = 0,8 \qquad \log\frac{a}{b} = \log a - \log b$$

$$\log\left(Q \times \frac{Q+1}{Q}\right) = 0,8$$

$$\log(Q+1) = 0,8$$

$$Q+1 = 10^{0,8} \qquad \log_a b = c \text{ can be written as } a^c = b$$

$$Q = 10^{0,8} - 1$$

Using your calculator, rounded to 9 decimal places

$$Q = 5,309573445$$

$$Q = 5,31 \qquad \text{Rounded to 2 decimal places}$$