c) The initial boundary-value problem can thus be constructed:

$$\textcircled{1}: \quad \frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \quad ; \quad 0 < r < a, \quad t > 0.$$

$$\textcircled{2}: \quad \frac{\partial u}{\partial r} = -hu(a,t)$$

$$\textcircled{3}: \quad u(r,0) = c \quad , \quad |u(0,t)| < \infty \quad \forall\ t.$$

To find the solution, $u(r,t)$, we first need to seperate the variables. Therefore,

$$u(r,t) = R(r)T(t) \quad \ldots \quad \textcircled{A}$$

Substituting Ⓐ into ① yields:

$$\frac{\partial u}{\partial t} = \frac{\partial(R(r)T(t))}{\partial t} = R(r)\frac{\partial T(t)}{\partial t} = RT'$$

$$\frac{\partial u}{\partial r} = \frac{\partial(R(r)T(t))}{\partial r} = T(t)\frac{\partial R(r)}{\partial r} = TR'$$

$$\frac{\partial^2 u}{\partial r^2} = \frac{\partial(T(t)R'(r))}{\partial r} = T(t)\frac{\partial^2 R(r)}{\partial r^2} = TR''.$$

Now, ① becomes:

$$RT' = k\left(TR'' + \frac{1}{r}TR'\right)$$

$$\Rightarrow \frac{1}{kT}T' = \frac{1}{R}\left(R'' + \frac{1}{r}R'\right)$$

Now, we set $RH = LH = -\lambda^2$

$$\Rightarrow R'' + \frac{1}{r}R' + R\lambda^2 = 0$$

and $$T' + kT\lambda^2 = 0$$

$$\Rightarrow rR'' + R' + rR\lambda^2 = 0 \quad \ldots \quad \textcircled{4}$$

and $$T' + kT\lambda^2 = 0 \quad \ldots \quad \textcircled{5}$$

The general solution of (4) is given by:

$$y(x) = A_1 J_p(x) + A_2 Y_p(x).$$

$y(x)$ is the general solution for the Bessel equation.
$J_p(x)$ is the Bessel function of the first kind and
$Y_p(x)$ is the Bessel function of the second kind.
$J_p(x)$ is bounded at 0; $Y_p(x)$ is unbounded at 0.

Proceeding with the general solution of (4), we know it is of the form:

$$R(r,\lambda) = A_1 J_0(\lambda r) + A_2 J_1(\lambda r). \quad \ldots\ldots (B)$$

The general solution for (5) is simply

$$T = A_3 e^{-k\lambda^2 t}.$$

Since the problem is bounded at $r=0$ (from (3)), A_2 in (B) $= 0$.

$$\Rightarrow \text{Solution} \quad u(r,t) = R(r)T(t)$$
$$= A_1 J_0(\lambda r) A_3 e^{-k\lambda^2 t}$$
$$= A J_0(\lambda r) e^{-k\lambda^2 t}.$$

(2) now becomes: $\lambda J_0'(\lambda a) + hAJ_0(\lambda a) e^{-k\lambda^2 t} = 0$

When we apply superposition of the two solutions, we find

$$u(r,t) = \sum_{n=1}^{\infty} A_n e^{-k\lambda^2 t} J_0(\lambda_n r).$$

Since by (3), $u(r,0) = C = \sum_{n=1}^{\infty} A_n J_0(\lambda_n r)$