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Cover: Eastern Transvaal, Lowveld (1928) J. H. Pierneef

J. H. Pierneef is one of South Africa's best known artists. Permission for the use of this work was kindly granted by the Schweickerdt family.

The tree structure is a recurring theme in various branches of the decision sciences.
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1. General

The prescribed textbook


forms the basis of the study material in this module. It is advisable to purchase the textbook as soon as possible. This study guide only contains study material that is supplemental to certain parts of the prescribed textbook. It contains additional explanations on certain topics of the textbook.

The textbook is your most important source and reference in this module. It is impossible to pass this module without the textbook.

For a detailed explanation of which parts of the textbook and study guide you need to study, read the sections “How to approach this module” and “Study plan” in Tutorial Letter 101, before you start with your studies.

1.1 Spreadsheets and the calculator

The textbook contains some examples that illustrate concepts using the Microsoft Excel spreadsheet software. These examples can be done in other spreadsheets, including the free OpenOffice Calc or Gnumeric. Students who do have access to a computer and spreadsheet software are encouraged to try these examples. However, the entire textbook can be used without a computer and you will, of course, not be allowed to take a computer into the examination hall. All students will need a scientific calculator in this module. The first chapter of the textbook contains some more detail in this regard as well as section 7 of Tutorial Letter 101.

1.2 South African conventions

We are using a UK textbook. You will notice that the currency symbol £ is used for the Pound, which is the international good practice. We need to draw your attention to the following important points regarding South African conventions in mathematics and finance.

1.2.1 The decimal comma

In South Africa, the decimal separator is (by law) the comma. Therefore, one half is 0,5 in South Africa and 1,000 is just one and not one thousand (as it is in the United States, for example). Other so-called “comma countries” include most of the rest of Africa, Germany and the rest of continental Europe, parts of Canada, Brazil, Indonesia and Russia. Countries using the point as decimal sign, include most English speaking countries, Mexico, China, Japan and Botswana. All authoritative business publications in South Africa (including the newspaper Business Day and the supplement Sake) use the comma and so should you. The use of anything other than a space as thousands separator can lead to confusion and should be avoided. In the United States one may therefore write 1000,5 and should avoid 1,000,5 and in South Africa we may write 1000,5 and rather not 1.000,5 for the number which is one half more than one
thousand. Iran and some Arabic speaking countries use a different sign altogether. Mature computer operating systems and software will let you set the convention you prefer (for your country).

1.2.2 The meaning of “billion”
The word billion is problematic because it can refer to two different numbers in English. In the US it has always meant 1,000,000,000 (a one followed by nine zeroes, or \(10^9\)) and this meaning has spread to other parts of the world. However in Britain and the Commonwealth it has traditionally meant 1,000,000,000,000 (a one followed by twelve zeroes, or \(10^{12}\)). The interpretation as \(10^{12}\) is still considered the standard in most other languages which have a similar word: French, Norwegian and Danish billion, German Billion, Dutch and Afrikaans biljoen, Spanish billón, Polish bilion and Portuguese (in Portugal, but not in Brazil) bilião. These languages use a variant of the now disused English term milliard for \(10^9\): French, Danish and Norwegian milliard, German Milliarde, Afrikaans and Dutch miljard, Hungarian milliárd and Spanish millardo. Slang of the London financial world has an unambiguous word for \(10^9\), a yard. Since very large numbers are not that often used, it is best to avoid the term “billion” altogether in English and use “thousand million” or “million million” instead. In languages where milliard is still known, it is unproblematic to use it to refer to \(10^9\).

1.3 Language
You are kindly reminded of the trilingual (English/Zulu/Afrikaans) glossary of the Department of Decision Sciences which you will find in Tutorial Letter 301. Please use it while working through the textbook and other study material and let us know if there are any terms that you would like to see included in it.

2. Study plan
Keep in mind that a semester is not longer than about 15 weeks. The study material has been subdivided into five units and you should therefore give yourself, on average, not more than three weeks to master each unit. The earlier units might contain some material that you are already familiar with and therefore you should try to master them more quickly. Refer to Tutorial Letter 101 for a detailed explanation on which parts of the book, and of the appendices, you need to study as well as lists of exercises to be completed in each unit. If you have worked up to page 45 (say) in the book, you should already be able to attempt the exercises up to that page.

3. Additional Material
The remaining pages contain Study material supplemental to parts of the prescribed textbook, namely:

- Appendix A: Numbers and working with numbers
- Appendix B: Functions and representations of functions
- Appendix C: Linear systems
Appendix A

Numbers and workings with numbers

On completion of this component you should have obtained basic numeracy skills.

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Subunit A.1  Priorities and laws of operations
Subunit A.2  Variables
Subunit A.3  Fractions
Subunit A.4  Powers and roots
Subunit A.5  Ratios, proportions and percentages
Subunit A.6  Signs, notations and counting rules

From material originally compiled for QMI101-X.
Subunit A.1   Priorities and laws of operations

Learning objectives: On completion of this Subunit you should be able to

• know and apply the priority rules in solving problems and
• know and apply the laws of operations in solving problems.

A.1.1   Priorities

What are the values of these expressions

$2 \times 3 + 4 \times 5$ and $8 \div 2 - 6 \div 3$?

I hope you will be able to answer 26 and 2 respectively, without using your calculator. How do you derive these results? In the first case $2 \times 3$ is equal to 6, and $4 \times 5$ is 20, and $6 + 20$ is 26. In the second case $8 \div 2$ is equal to 4; $6 \div 3$ is 2, and $4 - 2$ is 2. Elementary!

Yes. It is elementary but why?

It is essential that when we see an expression we all understand the same thing and obtain the same result or else all our numbers and operations will lead to nonsense. So, in fact, we all agree to perform the operations in a specific order.

Looking at the above two expressions and their accepted results, it is evident that multiplication takes precedence over addition, and division takes precedence over subtraction. And what about more complex expressions like

$2 \times 3 - 4 \times 5 + 6 \div 3$?

Here, too, multiplication and division take precedence over addition and subtraction, but whether you do the multiplication before the division and the subtraction before the addition, or vice versa, does not matter. Check for yourself.

The commonly accepted convention is thus: multiplication and division have a higher priority than addition and subtraction.

You may wonder whether there are operations that have a higher priority than multiplication and division. Yes, there are. Specifically, the operation "change sign" which changes a positive number to a negative number and vice versa, has the highest priority of all. Next comes exponentiation in its various forms. The accepted priority order is

Highest priority
   Brackets
   Change sign
Exponentiation in all its forms
Multiplication and division
Addition and subtraction
Lowest priority.
### Activity

Calculate each of the following expressions. In each case state the order in which the operations are performed, assuming that you operate from left to right when priorities are equal:

1. \[4 \times 3 + 2 + 5 \times 6 - 20 \div 4 \times 2;\]
2. \[3^2 \times 7 - \sqrt{9} \times 8 + 2^2 \div 4 \times 3.\]

### Answer

1. \[4 \times 3 + 2 + 5 \times 6 - 20 \div 4 \times 2 = 12 \div 2 + 30 - 5 \times 2 = 6 + 30 - 10 = 26;\]
2. \[3^2 \times 7 - \sqrt{9} \times 8 + 2^2 \div 4 \times 3 = 9 \times 7 - 3 \times 8 + 1 \times 3 = 63 - 24 + 3 = 42.\]

### A.1.2 Brackets

Say, for instance, that in the example

\[2 \times 3 + 4 \times 5\]

you actually want to add \(3 + 4\) first and then multiply by 2 and 5. How can we achieve this? The answer, as you are perhaps aware, is *brackets*. We write

\[2 \times (3 + 4) \times 5\]

and understand that the expression in brackets, namely \((3 + 4)\), must first be calculated and then the other operations. In this case we thus obtain, \(3 + 4\) is 7, \(2 \times 7\) is 14 and \(14 \times 5\) is 70.

Another example is the use of brackets is to indicate fractional exponents, for example \(8^{2/3}\) is \(8^{(2+3)}\).

Brackets are used to override the established priorities.

It is at this stage, I believe evident, that brackets are a very powerful means of casting expressions in a form that conveys the specific meaning that we want. The point is that any expression between brackets is regarded as a number which must be determined first before proceeding further with the calculation.

Any expression? Yes. But an expression itself may contain brackets. Does this mean that we can have brackets within brackets? Yes, indeed it does. That is, in fact, the beauty and the power of brackets. Let’s look at a more complex example.
Consider
\[ \sqrt{3 \times \sqrt{(4^2 + 3^2) + (4 + 3)^2}}. \]

This means that you take the square root of the expression between the outermost brackets. However, this cannot be done until we know the value of the expression contained by the innermost brackets. Thus the innermost brackets are calculated first. Note that within a set of brackets the usual priority rules apply, for example, exponentiation before addition in the case of the first inner bracket. Step by step, the calculation will take place as follows:

**Step**  **Reduced Expression**

1. \[ \sqrt{3 \times \sqrt{(4^2 + 3^2) + (4 + 3)^2}}; \]
2. \[ \sqrt{3 \times \sqrt{25 + 7^2}}; \]
3. \[ \sqrt{3 \times 5 + 49}; \]
4. \[ \sqrt{15 + 49}; \]
5. \[ \sqrt{64}; \]
6. \[ 8. \]

Whether we evaluate the first or second inner bracket first does not matter in this case.

---

**Activity**

Set out the steps which will be executed in the calculation of the following expression and check your result with your calculator:

\[ (4^2 - (6 \times 3 - 80 \div 5)^2) \div (9^2 - 77). \]

**Answer**

\[
\begin{align*}
(4^2 - (6 \times 3 - 80 \div 5)^2) \div (9^2 - 77) \\
= (4^2 - (18 - 16)^2) \div (9^2 - 77) \\
= (4^2 - 2^2) \div (9^2 - 77) \\
= (16 - 4) \div (81 - 77) \\
= 12 \div 4 \\
= 3.
\end{align*}
\]
This exercise illustrates, in some detail, the fact that any expression between brackets is simply a number which has to be determined before proceeding further.

For every left bracket in an expression there must be a corresponding right bracket and vice versa, otherwise the expression is not uniquely specified. Always work from the innermost bracket outwards.

Finally, a useful hint about brackets and priorities. If you are unsure about the relative priorities of different operations, use brackets to enforce the priority order you want. A set or two of extra brackets, even if they are redundant, often do much to enhance the readability of an expression.

A.1.3 Laws of Operations

There are specific laws that apply to combinations of the basic operations. The two most basic laws are illustrated in the following results. For addition:

\[
6 + 2 = 2 + 6 \quad \text{(and both are } 8) \\
3.1 + 9.3 = 9.3 + 3.1 \quad (= 12.4) \\
-7.6 + 2.3 = 2.3 + -7.6 \quad (= -5.3) \\
-11.1 + -4.4 = -4.4 + -11.1 \quad (= -15.5).
\]

For multiplication:

\[
5 \times 2 = 2 \times 5 \quad (= 10) \\
2.2 \times 3.5 = 3.5 \times 2.2 \quad (= 7.7) \\
-8.1 \times 9.4 = 9.4 \times -8.1 \quad (= -76.14) \\
-11 \times -13 = -13 \times -11 \quad (= 143).
\]

The two laws involved are the **commutative laws of addition and multiplication**.

**Commutative Law of Addition:**
The sum of two numbers is unique, that is it does not matter which number is placed first and which second, the result is the same.

**Commutative Law of Multiplication:**
The product of two numbers is unique. In other words, the multiplication may be done in any order.
What about subtraction and division? Does the commutative law apply to them too? A counter example shows us that it does not, for example

\[ 7 - 3 \neq 3 - 7 \quad \text{and} \quad 10 \div 2 \neq 2 \div 10. \]

The sign $\neq$ is read as "does not equal". In the first case, subtraction, commutation (i.e. changing the order) leads to the negative of the initial result (that is $-4$ instead of $4$); whereas in the second case, division, commutation leads to the reciprocal of the initial result (that is $1/5$ instead of $5$).

The conclusion is that, whereas for addition and multiplication we can change the order of the factors, for subtraction and division we cannot, or rather, if we do, we must do so with care.

The next two laws apply to addition and multiplication. Consider the following, where we use brackets to indicate the sequence in which the operations are to be performed.

For addition:

\[ 7 + (3 + 2) = (7 + 3) + 2 = 12 \]
\[ 6.1 + (-5.1 + 3.7) = (6.1 + -5.1) + 3.7 = 4.7 \]
\[ -9.3 + (2.2 + 4.5) = (-9.3 + 2.2) + 4.5 = -2.6. \]

For multiplication:

\[ 5 \times (4 \times 2) = (5 \times 4) \times 2 = 40 \]
\[ 6.1 \times (7.3 \times 5.5) = (6.1 \times 7.3) \times 5.5 = 244.915 \]
\[ -9 \times (5 \times -7) = (-9 \times 5) \times -7 = 315. \]

The two laws involved are the **associative laws of addition and multiplication**.

**The Associative Law of Addition:**
The sum of three numbers does not depend on which two are added first - the result is unique.

**The Associative Law of Multiplication:**
The product of three numbers does not depend on which two are multiplied first - the result is unique.

This implies, for example, that when adding long columns of numbers, it does not matter which numbers we add first and which last. A similar remark applies to multiplication, but when it comes to subtraction and division, we must be very careful, as the following examples show:

\[ 10 - (5 - 2) \neq (10 - 5) - 2 \quad \text{and} \quad 8 \div (4 \div 2) \neq (8 \div 4) \div 2. \]
This means that an expression like \( 8 \div 4 \div 2 \) is highly ambiguous because the result depends on the order in which the operations are performed. It is preferable that you indicate and enforce a specific order using brackets.

So far, we have stated laws for expressions containing one type of operation only - either addition or multiplication. Let us take a look at combinations of these two operations.

**Activity**

Confirm that the expressions to the left and the right of the equal sign yield the same results:

1. \( 6 \times (2 + 3) = 6 \times 2 + 6 \times 3 \);
2. \( 7.2 \times (2.2 + 3.3) = 7.2 \times 2.2 + 7.2 \times 3.3 \);
3. \( -5 \times (-4 + 6) = -5 \times -4 + -5 \times 6 \);
4. \( 12.3 \times (-3.4 + -6.6) = 12.3 \times -3.4 + 12.3 \times -6.6 \).

What we observe in this activity is an illustration of the following law.

**The Distributive Law of Multiplication over Addition:**

The product of a number with the sum of two other numbers is equal to the sum of the products of the first number with each of the other two numbers.

We can take this one step further. Since multiplication obeys the commutative law, the factors involved in the above statement may be reversed. Thus, for example

\[
6 \times (2 + 3) = 6 \times 2 + 6 \times 3 = 2 \times 6 + 3 \times 6 = (2 + 3) \times 6.
\]

In other words, it does not matter whether we multiply from the left or the right in expressions of this type. Similarly whether we write \( 2 + 3 \) or \( 3 + 2 \), the result is the same since the commutative law of addition applies.

Furthermore, although we cannot easily formulate similar laws for subtraction and division, we can frequently transform expressions containing subtraction and division to ones containing only addition and multiplication. To be specific, in the case of subtraction we change the sign of the number being subtracted and then add. For example,

\[
11 - 4 = 11 + -4 = -4 + 11
\]

where we have used the commutative law in the last step.

In the case of division we can replace the number being divided by, by its inverse (see subunit on roots) and multiply, for example

\[
9 \div 3 = 9 \times 3^{-1} = 3^{-1} \times 9
\]

again using the commutative law in the last step.
Or, consider an application of the distributive law:

\[(17 - 4) ÷ 5 = (17 + -4) \times 5^{-1}\]
\[= 17 \times 5^{-1} + -4 \times 5^{-1}.\]

Although each law is elementary almost to the extent of being self-evident, used in conjunction with each other, the three laws can be very powerful tools for rearranging expressions. We shall have ample opportunity to make good use of them later.

**Exercise**

1. Apply the distributive law of multiplication over addition to expand the following expression to one containing the sum of four terms, each term being the product of three numbers. Don’t actually calculate the result. The expression is:

\[7 \times (6 \times (5 + 4) + 3 \times (2 + 1)).\]

2. Which of the commutative or associative laws are associated with the following expressions:

   (1) \[2 + (5 + 4) = (2 + 5) + 4;\]
   (2) \[(7 \times 5) \times 2 = 2 \times (7 \times 5);\]
   (3) \[(3 + 7) + 4 = (7 + 3) + 4;\]
   (4) \[2 \times (7 \times 4) = (4 \times 2) \times 7?\]

**Solutions**

1. \[7 \times (6 \times (5 + 4) + 3 \times (2 + 1))\]
   \[= 7 \times (6 \times 5 + 6 \times 4 + 3 \times 2 + 3 \times 1)\]
   \[= 7 \times 6 \times 5 + 7 \times 6 \times 4 + 7 \times 3 \times 2 + 7 \times 3 \times 1.\]

2. (1) \[2 + (5 + 4) = (2 + 5) + 4\] Associative Law of Addition;
(2) \[(7 \times 5) \times 2 = 2 \times (7 \times 5)\] Commutative Law of Multiplication;
(3) \[(3 + 7) + 4 = (7 + 3) + 4\] Commutative Law of Addition;
(4) \[2 \times (7 \times 4) = (4 \times 2) \times 7\] Associative Law of Multiplication.
Subunit A.2 Variables

Learning objectives: On completion of this Subunit you should be able to express relations between numbers in general terms, by making use of symbols or letters.

If you want to explain to someone how to find the area of a rectangle 4 cm by 3 cm, you can tell them to multiply 4 by 3. But if you do not have specific values for the rectangle, you can call the length of the rectangle \( L \) and its width \( W \), then you can say that the area is \( L \times W \) - this will be true for any value of \( L \) and \( W \). Consider the rule to express one quantity as a percentage of another: “put the first quantity on top of the second and multiply by 100” what a mouthful! But if the first number is denoted by \( x \) and the second by \( y \), then the rule boils down to \( \frac{x}{y} \times 100 \). This will be true for any values of \( x \) and \( y \). This is the great strength in using letters or symbols to represent numbers - we are then able to write down rules, expressions, and so on which are completely general, so that to find the answer in a particular case, all we need to do is substitute our particular values of \( x \) and \( y \) or whatever into the appropriated expression. (Note you could have used any letters or symbols not necessary \( x \) and \( y \))

The main purpose of using letters or symbols to represent numbers or quantities is that it enables us to express practical truths about the real world neatly, succinctly, and in more general terms, than we could if we insisted on sticking to definite numbers all the time.

This is illustrated with the laws discussed previously.

Consider the Commutative Law of Addition:

The sum of two numbers is unique, that is it does not matter which number is placed first and which second, the result is the same.

Suppose we use letters \( A \) and \( B \). Then this law can be written as \( A + B = B + A \).

By using the symbols or letters \( A \), \( B \) and \( C \), the various laws can be succinctly stated as follows.

Commutative Law of Addition:  
\[
A + B = B + A.
\]

Commutative Law of Multiplication:  
\[
A \times B = B \times A.
\]

Associative Law of Addition:
\[
(A + B) + C = A + (B + C).
\]

Associative Law of Multiplication:
\[
(A \times B) \times C = A \times (B \times C).
\]

The Distributive Law of Multiplication over Addition:
\[
A \times (B + C) = A \times B + A \times C.
\]
It would surprise me if you did not agree that in general these statements are more concise and clear than those of the previous subunit.

Four other fundamental rules which I am sure you have dealt with before are:

- the product of any number with zero is zero;
- the product of any number with 1 is the number;
- the sum of any number and its negative is zero;
- the product of any number and its inverse is 1.

In terms of letters or symbols these are very elegantly stated as:

- the product of any number with zero is zero
  \[ A \times 0 = 0; \]
- the product of any number with 1 is the number
  \[ A \times 1 = A; \]
- the sum of any number and its negative is zero
  \[ A + \neg A = 0; \]
- the product of any number and its inverse is 1
  \[ A \times A^{-1} = 1. \]

Again, we could have used any one of the symbols \( A \) to \( Z \) to formulate these two laws.

Everyday life abounds with examples of variables. A firm's gross monthly sales vary from month to month, usually in number and value. Your telephone account or water and lights account are unlikely to be the same each month. Nor are the rainfall or the average maximum temperature from day to day. Even seemingly constant items such as your flat or house rent (or your salary!) are adjusted periodically and are therefore variables.

A **variable** is something which can assume any one of several numeric values.

The symbols most often used to represent variables are letters of the alphabet, both upper case such as \( A, B, C, \ldots \) and lower case such as \( x, y, z \).

The **symbol** or letter which is used to represent a variable is simply a label. Its specific value has to be assigned directly or determined by calculation in each relevant instance. Always choose a symbol that is short and that makes sense.
Exercise

1. Determine the numerical value of the following:

   (a) \(12x + 17\) if \(x = 2\);   (e) \(\frac{x}{4} + 3\) if \(x = 5\);
   (b) \(x^2 - 3\) if \(x = 4\);   (f) \((x + 3)(x - 2)\) if \(x = 6\);
   (c) \(2x^2\) if \(x = 3\);   (g) \(10 - 3x\) if \(x = 1\);
   (d) \(4x - 1\) if \(x = 5\);   (h) \(\frac{x}{2} + \frac{x}{3}\) if \(x = 12\).

2. Substitute \(x\) by 3 and calculate:

   (a) \(5x + 7\);   (d) \(\frac{x + 4}{7}\);
   (b) \(x + 3x - 1\);   (e) \(2(x + 4)\);
   (c) \(5x^2 - 9\)   (f) \(7 - x + 2\).

3. If \(a = 2\), \(b = 1\) and \(c = 7\), then determine the value of the following:

   (a) \(2(a + b + -c) + c(b - a)\);   (c) \((a + b)(b - c)\);
   (b) \(a^2 + b^2 + c^2\);   (d) \(2b - \frac{c + 3}{a} + b^2\).

4. Write the following as a mathematical expression:

   (a) the sum of \(x\) and \(y\);
   (b) subtract the sum of \(a\) and \(b\) from 8;
   (c) three times \(x\) added to two times \(y\);
   (d) Robert’s age in seven years’ time if he is now \(y\) years old.
Solutions

1. (a) \(12 \times 2 + 17 = 24 + 17 = 41\); (c) \((5+7)/4 + 3 = 12/4 + 3 = 3 + 3 = 6\);
(b) \(4^2 - 3 = 16 - 3 = 13\); (f) \((6+3)(6-2) = 9 \times 4 = 36\);
(c) \(2 \times 3^2 = 2 \times 9 = 18\); (g) \(10 - 3 \times 1 = 10 - 3 = 7\);
(d) \(4 \times 5 - 1 = 20 - 1 = 19\); (h) \(12/2 + 12/3 = 6 + 4 = 10\).

2. (a) \(5 \times 3 + 7 = 15 + 7 = 22\); (d) \((3+4)/7 = 7/7 = 1\);
(b) \(3 + 3 \times 1 = 3 + 9 - 1 = 11\); (e) \(2(3 + 4) = 2 \times 7 = 14\);
(c) \(5 \times 3^2 - 9 = 5 \times 9 - 9 = 45 - 9 = 36\); (f) \(7 - 3 + 2 = 6\).

3. (a) \(2(2+1-7)+7(1-2) = 2(-4) + 7(-1) = 2 \times -4 + 7 \times -1 = -8 - 7 = -15\);
(b) \(2^2 + 1^2 + 7^2 = 4 + 1 + 49 = 54\);
(c) \((2 + 1)(1 - 7) = 3 \times -6 = -18\);
(d) \(2 \times 1 - (7+3)/2 + 1^2 = 2 - 10/2 + 1 = 2 - 5 + 1 = -2\).

4. (a) \(x + y\);
(b) \(8 - (a + b)\);
(c) \(3 \times x + 2 \times y = 3x + 2y\);
(d) \(y + 7\).
Subunit A.3  Fractions

Learning objectives: On completion of this Subunit you should be able to do all the basic mathematical operations on fractions.

The fraction 10/2 means that ten must be split into a number of groups of size two each. The question is, what is the number of groups?

The answer is $2 + 2 + 2 + 2 + 2 = 10$.

Five groups of size two each, added together, will give ten.

In short, $10/2 = 5$.

A.3.1 Division and multiplication of fractions

How do we divide 4 by 1/2?

In mathematical notation this is $4 \div \frac{1}{2}$ or $\frac{4 \div \frac{1}{2}}{}$.

The half poses a problem. How can one get rid of it? Maybe it can be changed to something that is easier to work with.

To make the $\frac{1}{2}$ more manageable, we multiply it by $\frac{2}{1}$, that is $\frac{1}{2} \times \frac{2}{1} = \frac{2}{2} = 1$.

To multiply fractions, we multiply the numerators with each other and the denominators with each other. The numerator is the top part of a fraction and the denominator is the bottom part of a fraction.

Take the rule that whatever is done on the left-hand side of an equation also has to be done on the right-hand side, then expand it to:

whatever is done in the top part of a fraction has to be done in the bottom part as well.

Thus:

$$\frac{4}{1} \times \frac{2}{1} = \frac{4 \times 2}{1} = 4 \times 2 = 8$$

(Note that \(\frac{2}{1} = 1\); that means we do not change our original problem, we multiply it with 1 in a manner that suits us).

Now that we have the idea, let us go for short cuts!

Take the bottom part of the fraction and turn it around, using division’s brother, the multiplication sign, as accomplice:
\[
\frac{4}{1/2} = 4 \div \frac{1}{2} = 4 \times \frac{2}{1} = 8
\]

which is exactly the same answer!

Let us try another one: \(15 \div \frac{1}{3}\)

\[
\frac{15}{\frac{1}{3}} = \frac{15}{\frac{1}{3}} \times \frac{3}{\frac{1}{3}} = \frac{45}{\frac{3}{3}} = 45
\]

or: \(\frac{15}{3} = 15 \times \frac{3}{1} = 45\) (quicker and easier!).

The 15 can be written as \(\frac{15}{1}\) and therefore

\[
\frac{15}{\frac{1}{3}} = \frac{15}{\frac{1}{3}} \times \frac{3}{\frac{1}{3}} = 15 \times \frac{3}{1}
\]

Let us take a closer look: the \(\frac{15}{1}\) stays as it is but the \(\frac{1}{3}\) changed to \(\frac{3}{1}\). If we take \(\frac{15}{\frac{1}{3}}\) and attach the following letters, we can write down the answer immediately.

The following fraction should not be so difficult then:

\[
\frac{3}{4} \div \frac{5}{6} = \frac{3 \times 6}{4 \times 5} = \frac{18}{20} \quad \text{or} \quad \frac{3}{4} \div \frac{5}{6} = \frac{3 \times 6}{4} \div \frac{5}{6}.
\]

**Activity**

Can you see a pattern in the answers of the following:

\[
\frac{10}{200}; \quad \frac{10}{20}; \quad \frac{10}{2}; \quad \frac{10}{1}; \quad \frac{10}{1/2}; \quad \frac{10}{1/20}.
\]
The smaller the denominator gets, the larger the answer becomes. Think about it - if you split 10 in groups of size two you get only five groups, but if you split 10 in groups of size a half each, you get for each unit two groups, therefore for 10 units you get 20 groups.

Since we are looking at division, what about dividing by zero? Look at the pattern:

\[
\begin{align*}
\frac{1}{1} &= 1; & \quad \frac{1}{10} &= 10; & \quad \frac{1}{100} &= 100; & \quad \frac{1}{1000} &= 1000; & \quad \frac{1}{10000} &= 10000.
\end{align*}
\]

We see that 1 is larger than \( \frac{1}{10} \), which is larger than \( \frac{1}{100} \), etcetera.

The smaller the number by which you divide, the larger the answer.

The closer the number by which you divide is to zero, the larger the answer. But you can never divide by zero, for that answer has not been mathematically defined. Let's look at it differently.

Question: how many two's do I have to add together to get six?
We know that \( 2 + 2 + 2 = 6 \). The answer is three two's.

Mathematically: \( \frac{6}{2} = 3 \).

**Question**: how many zeroes do I have to add together to get six?
**Answer**: this cannot be determined.
Mathematically: \( \frac{6}{0} \) is meaningless.

Activity

Calculate the following:

1. \( \frac{12}{3} \); 2. \( \frac{5 \times 4}{3} \).

Answer

1. \( 12 \times \frac{3}{1} = 36 \); 2. \( 5 \times 4 \times 5 = 100 \).

A.3.2 Adding and subtraction of fractions

How do we calculate \( \frac{5}{3} + \frac{2}{5} \)?

I have heard you! You have just said “with a calculator!”

Yes, that is the easy way, but do you really understand what you are doing?

Before one uses mechanical aids like calculators, one must make sense of what you are doing.

Let us start with something easy and try to find a method.

Take \( \frac{1}{2} + \frac{1}{2} = 1 \).

What we actually did was to write a common denominator for the two denominators, 2.

This is of course 2.

Then we fill in the numerator part: \( \frac{1 + 1}{2} = \frac{2}{2} = 1 \).

Similarly for \( \frac{3}{4} + \frac{1}{4} + \frac{6}{4} = \frac{3 + 1 + 6}{4} = \frac{10}{4} = \frac{5}{2} = 2 \frac{1}{2} \).

If we add fractions and they all have the same denominator, it is quite easy to do the sum.

But what about \( \frac{4}{13} + \frac{2}{5} \)?

A solution is to find the same denominator for both.
The easiest way is to multiply the two denominators:

\[ 13 \times 5 = 65. \]

But \( \frac{4}{13} \) is not \( \frac{4}{65} \)!

If we multiply the 13 by 5 we must also multiply the 4 by 5 before we add:

\[
\frac{4}{13} + \frac{2}{5} = \frac{20 + 26}{65} = \frac{46}{65}.
\]

Shortcut:

\[
\frac{4}{13} + \frac{2}{5} = \frac{4 \times 5 + 2 \times 13}{65} = \frac{20 + 26}{65} = \frac{46}{65}.
\]

Activity

Calculate the answers for the following:

1. \( \frac{3}{4} - \frac{1}{2} \);

2. \( \frac{17}{23} + 2 \);

3. \( \frac{8}{15} + \frac{11}{30} \);

4. \( 3\frac{1}{4} - \frac{9}{10} \).

Answer

1. \( \frac{3}{4} - \frac{1}{2} = \frac{3 - 1 \times 2}{4} = \frac{1}{4} \)

or \( \frac{3}{4} - \frac{2}{4} = \frac{3 - 2}{4} = \frac{1}{4} \)

or \( \frac{3}{4} - \frac{1 \times 2}{4} = \frac{1}{4} \);

2. \( \frac{17}{23} + 2 = \frac{17 + 2 \times 23}{23} = \frac{17 + 46}{23} = \frac{63}{23} \);

3. \( \frac{8}{15} + \frac{11}{30} = \frac{8 \times 2 + 11}{30} = \frac{16 + 11}{30} = \frac{27}{30} = \frac{9}{10} \);

4. \( 3\frac{1}{4} - \frac{9}{10} = \frac{13}{4} - \frac{9}{10} = \frac{13 \times 10 - 9 \times 4}{40} = \frac{130 - 36}{40} = \frac{94}{40} = \frac{47}{20} \).

_____________________________________________________________________

23
A.3.3 A decimal world

According to the new Collins Concise English dictionary, a decimal is:

1. also called decimal fraction. A fraction that has an unwritten denominator of a power of ten. It is indicated by a decimal point to the left of the numerator: $0,2 = \frac{2}{10}$;
2. any number used in the decimal system;
3. a) relating to or using powers of ten;
   b) of the base ten.

A decimate is (in ancient Roman Army) to kill every tenth man of a mutinous section.

Switch on your calculator and key in $1 \div 2 = , \text{ that is } \frac{1}{2} = .$

The answer is 0.5, that is $\frac{5}{10}$.

(The comma between the 0 and the 5 is the decimal comma. In most calculators this is given as a decimal point that is 0.5).

Every number that you key in, is interpreted according to the definition of a decimal as given above.

Without switching on the calculator, one realises that one is living in a number world dominated by “tens”.

Let us write a few fractions as decimals:

$$\frac{1}{10} = 0,1 \quad \text{(10 has one zero therefore one number after decimal comma);}$$

$$\frac{1}{100} = 0,01 \quad \text{(100 has two zeroes therefore two numbers after decimal comma);}$$

$$\frac{1}{1000} = 0,001 \quad \text{(1 000 has three zeroes therefore three numbers after decimal comma);}$$

$$\frac{1}{10 000} = 0,0001 \quad \text{(10 000 has four zeroes therefore four numbers after decimal comma);}$$
\[
\frac{1}{100\ 000} = 0,00001
\]
(100 000 has five zeroes therefore five numbers after decimal comma).

So how about \( \frac{3}{10} \) as a decimal fraction?

Write down 3, which is actually 3,0 and move the decimal comma one place to the left, that is 0,3.

Similarly:

\[
\begin{align*}
\frac{3}{100} &= 0,03; \\
\frac{3}{1\ 000} &= 0,003; \\
\frac{3}{10\ 000} &= 0,0003; \\
\frac{3}{100\ 000} &= 0,00003.
\end{align*}
\]

Let us go back to the exercises on fractions:

\[
\frac{3}{4} - \frac{1}{2}.
\]

Switch on your calculator and key in \( \frac{3}{4} - \frac{1}{2} \). The answer is 0,25.

Then \( 0,25 = \frac{25}{100} = \frac{1}{4} \), which is what we have calculated.

___________________________________________________________

**Activity**

Go back to the previous activity. Use your calculator to calculate the answers.

_____________________________________________________________

**Answer**

1. 0,25;
2. 2,73913;
3. 0,90;
4. 2,35.
At this stage we have to talk about rounding. A decimal fraction like 0,43529 must sometimes be rounded to make sense.

If we want to express 0,43529 as a number with two decimal digits, we consider the first three decimal digits and then *round* it to two decimal digits.

Then 0,435 = 0,44 rounded to two decimal digits.

The rule we applied here is:

If the part of the number which is to be discarded begins with a digit greater than or equal to 5, then add 1 to the last digit retained; if not just drop the unwanted part.

Rounding is very important. If the interest rate is 16,25%, then you may not drop the 0,25%. There is a big difference if you calculate the interest on a loan using a 16,25% interest rate or a 16% interest rate. (This will be discussed in the component on the Mathematics of Finance.)

However, what does it mean if the price of an article is R1,75394?

When we are working with money, we will round to two decimal digits.

Do not round to integers unless it is either obvious, or asked for.

_____________________________________________________________________

**Activity**

1. Round the following number to two decimal digits:

   (a) 3,462;  (c) 8,998;  (e) 7,9747.
   (b) 10,495; (d) 12,034;

2. Round the following number to three decimal digits:

   (a) 0,0342; (c) 10,0004; (e) 4,57849.
   (b) 0,39887; (d) 9,8746;

---

**Answer**

1. (a) 3,46; (c) 9,00; (e) 7,97 (only look at the third decimal figure).
   (b) 10,50; (d) 12,03;

2. (a) 0,034; (c) 10,000; (e) 4,578 (only look at the fourth decimal figure).
   (b) 0,399; (d) 9,875;
Exercise

Calculate the answers to the following expressions:

1. \( \frac{\frac{8}{4}}{1} \); 2. \( \frac{\frac{12}{5}}{1} \);

3. \( \frac{\frac{8}{3}}{12} \); 4. \( \frac{\frac{3}{2}}{7} \);

5. \( \frac{\frac{8}{4}}{\frac{1}{2}} \); 6. \( \frac{\frac{17}{20} + \frac{1}{4} - \frac{3}{5}}{1} \);

7. \( \frac{\frac{2}{3} + 5 - \frac{6}{7}}{1} \); 8. \( \frac{\frac{5}{2} + \frac{3}{5} - \frac{6}{7}}{1} \);

9. \( \frac{\frac{3}{4}}{\frac{\left(\frac{5}{6} - \frac{1}{2}\right)}{1}} + \frac{3}{5} \).

Solutions

1. \( 8 \times \frac{4}{1} = 32 \); 2. \( 12 \times \frac{5}{1} = 60 \);

3. \( \frac{\frac{8}{13} \times \frac{25}{12}}{156} = \frac{200}{156} \); 4. \( \frac{\frac{7}{2}}{1} = \frac{7}{2} \times \frac{1}{7} = \frac{1}{2} \);

5. \( \frac{\frac{33}{4}}{\frac{1}{2}} = \frac{33}{4} \times \frac{2}{1} = \frac{66}{4} \);

6. \( \frac{\frac{17}{20} + \frac{1}{4} - \frac{3}{5}}{\frac{4}{5}} = \frac{17 \times 1 \times 5 - 3 \times 4}{20} = \frac{17 + 5 - 12}{20} = \frac{10}{20} = \frac{1}{2} \);

7. \( \frac{\frac{2}{3} + 5 - \frac{6}{7}}{\frac{21}{21}} = \frac{2 \times 7 + 5 \times 21 - 6 \times 3}{21} = \frac{14 + 105 - 18}{21} = \frac{101}{21} \).
8. 
\[
\frac{5}{2} + 3 \frac{2}{5} - \frac{6}{12} = \frac{11}{2} + \frac{17}{5} - \frac{79}{12} = \frac{11 \times 30 + 17 \times 12 - 79 \times 5}{60} = \frac{330 + 204 - 395}{60} = 139; 60
\]

(First multiply the two larger denominators with each other and check to see if the value is divisible by the other denominators).

9. 
\[
\frac{3}{4} \div \left( \frac{1}{5} - \frac{1}{2} \right) + \frac{3}{5} = \frac{3}{4} \div \left( \frac{11}{6} - \frac{1}{2} \right) + \frac{3}{5} = \frac{3}{4} \div \left( \frac{11 - 3}{6} \right) + \frac{3}{5} = \frac{3}{4} \div \frac{8}{6} + \frac{3}{5} = \frac{3}{4} \times \frac{6}{8} + \frac{3}{5} = \frac{18}{32} + \frac{3}{5} = \frac{9 \times 5 + 3 \times 16}{80} = \frac{45 + 48}{80} = \frac{93}{80} = 1 \frac{13}{80}.
\]

**NOTE:** Answers like \( \frac{200}{156} \) and \( \frac{66}{4} \) can be simplified into smaller fractions. Start by dividing the top and bottom part of the fraction by 2.

That is \( \frac{200}{156} = \frac{100}{78} = \frac{50}{39} \).

Also \( \frac{66}{4} = \frac{33}{2} \).
Subunit A.4 Powers and roots

**Learning objectives:** After completion of this Subunit you should be able to solve problems containing roots and/or powers.

Although roots and powers are not so much part of our everyday lives like a percentage, it is an extremely important part of mathematics and we must make friends with them. So, without much ado, let us start.

### A.4.1 Powers

Start with multiplying 10 by itself:

\[
10 \times 10 = 100 \\
10 \times 10 \times 10 = 1000 \\
10 \times 10 \times 10 \times 10 = 10000.
\]

It seems a very long way of doing it. Fortunately a shorter notation was developed:

\[
10 \times 10 = 10^2 = 100 \text{ (10 is multiplied 2 times by itself)} \\
10 \times 10 \times 10 = 10^3 = 1000 \text{ (10 is multiplied 3 times by itself)} \\
10 \times 10 \times 10 \times 10 = 10^4 = 10000 \text{ (10 is multiplied 4 times by itself)}.
\]

The number of times that 10 is multiplied by itself is the power of 10 involved, that is \(10^3\) is 10 to the power three.

Note that \(10 = 10^1\) or 10 to the power one.

And what about \(10^0\) or 10 to the power zero?

By definition, any number to the power zero is one.

\(10^0 = 1\), \(0^x = 0\) for \(x \neq 0\) and \(0^0 = 1\).

Consider \(10^3\) again, where 10 is called the base and 3 is called the exponent.

**Activity**

Which one is the base and which one is the exponent in the following:

\[
6^4; \quad 3^2; \quad 10^8?
\]
Answer

For $6^4$: 6 is the base and 4 is the exponent. It is read “six to the power four”.
For $3^2$: 3 is the base and 2 is the exponent. It is read “three to the power two”.
For $10^8$: 10 is the base and 8 is the exponent. It is read “ten to the power eight”.

Multiplication rule for exponents

The calculation $100 \times 100 = 10000$
can also be written as $10^2 \times 10^2 = 10^4$.

If the bases are the same, we can just add the exponents to get the answers. But beware! You cannot do this if the bases are different.

Activity

Using $2^3 \times 3^2$ as an example, decide whether the above statement is correct or not.

Answer

Is $2^3 \times 3^2 \neq 2^{3+2} \neq 2^5 \neq 32$ or $3^{3+2} \neq 3^5 \neq 243$?
We have that $2^3 = 2 \times 2 \times 2 = 8$
and $3^2 = 3 \times 3 = 9$.
Therefore $2^3 \times 3^2 = 72$.

It is clear that you can only add exponents if bases are the same.

What is the meaning of a negative power?

If $10^2 = 100$, what is $10^{-2}$?

To deal with this we introduce the concept of the inverse or reciprocal of a number. The inverse or reciprocal of a number is the result obtained when 1 is divided by the number.

The inverse of 10 is $\frac{1}{10}$.
If \( 10 = 10^1 \) then \( \frac{1}{10} = \frac{1}{10^1} = \frac{1 \times 10^{-1}}{10^1 \times 10^{-1}} = \frac{10^{-1}}{10^0} = \frac{10^{-1}}{1} = 10^{-1} \).

The superscript \(-1\) is used to indicate the inverse of a number.

Therefore \( \frac{1}{100} = \frac{1}{10^2} = \frac{1}{10^2} \times 10^{-2} = \frac{10^{-2}}{10^2 \times 10^{-2}} = \frac{10^{-2}}{10^0} \times \frac{10^{-2}}{1} = 10^{-2} \).

**Activity**

Derive a rule to find the inverse of any number written as a base number with an exponent.

**Answer**

If \( 10 = 10^{-1} \) then \( \frac{1}{10} = 10^{-1} \).

If \( 100 = 10^{-2} \) then \( \frac{1}{100} = 10^{-2} \).

Then for any number written as a base with an exponent, the inverse is found by using the same base, but changing the sign of the exponent.

**A.4.2 Roots**

We know that \( 10^2 = 10 \times 10 = 100 \) and we say “10 to the power 2 is 100”. What about going the other way, that is what number raised to the power 2 is 100? This is the problem of determining the root of a number.

We reverse the power story by using a root sign

\[
\sqrt{100} = 10, 
\]

and 10 is called the “square root” of 100.

Most calculators have a square root key. Convention has it that the little 2 is not written but just the root sign.

**Activity**

What is the square root of 9?
How will you describe the square root of 9 in words?

Answer

The answer is $\sqrt{9} = 3$.

In words we say 3 is the number when raised to the power two will give the answer of 9, that is

$3^2 = 9$.

NOTE! The $\sqrt{\phantom{0}}$ sign is taken away by squaring the 3. How does this work?

We see that $\sqrt{9}$ is actually $\frac{1}{2} \, 9^2$ (or 9 to the power half)

$\sqrt{9} = 3$

$9^{\frac{1}{2}} = 3$

$9^{\frac{1}{2} + \frac{2}{2}} = 3^2$

$9 = 3^2$.

What about something like

$\sqrt{25 - 16}$ and $\sqrt{16 + 9}$?

It is equal to

$\sqrt{25 - 16} = \sqrt{9}$ and $\sqrt{16 + 9} = \sqrt{25}$

$= 3$ and $= 5$.

Be careful:

$\sqrt{25} = 5$

$\sqrt{16} = 4$

and $5 - 4 = 1$ which is not the correct answer, 3, obtained above;

$\sqrt{16} = 4$

$\sqrt{9} = 3$

$4 + 3 = 7$ which is not the correct answer, 5, obtained above.
The moral of the story is:

*First finish all the calculations inside the sign before taking the sign away by squaring.*

**Exercise**

Determine the value of:

1. \(2^3\);  
2. \(4^2\);  
3. \(3^{-1}\);  
4. \(5^{-3}\);  
5. \(3^2 \times 3^0\);  
6. \(2^3 \times 2^2\);  
7. \(3^2 \times 4^2\);  
8. \(5^{-1} \times 5^2\).

Determine the following and describe each answer in words:

9. \(\sqrt{16}\);  
10. \(\sqrt{25}\);  
11. \(\sqrt[3]{16}\);  
12. \(\sqrt[3]{125}\).

13. Calculate:

(a) \(\sqrt{196} + \sqrt{144}\);  
(b) \((\sqrt{64})^2\);  
(c) \((\sqrt{5})^2\);  
(d) \((\sqrt{1})^2\).

14. Simplify:

(a) \(\sqrt{100} \div 4\);  
(b) \(2 \times \sqrt{144}\);  
(c) \(\sqrt[4]{\frac{36}{4}}\);  
(d) \(25 - \sqrt{4}\).

**Solution**

1. \(2^3 = 8\);  
2. \(4^2 = 16\);  
3. \(3^{-1} = \frac{1}{3}\);  
4. \(5^{-3} = \frac{1}{5^3} = \frac{1}{125}\);  
5. \(3^2 \times 3^0 = 3^2 \times 1 = 9\);  
6. \(2^3 \times 2^2 = 2^{3+2} = 2^5 = 32\);  
7. \(3^2 \times 4^2 = 9 \times 16 = 144\);  
8. \(5^{-1} \times 5^2 = 5^{-1+2} = 5\);  
9. \(\sqrt{16} = 4\)  
   \(4\) raised to the power two is 16;
10. \( \sqrt{25} = 5 \)  
    5 raised to the power 2 is 25;

11. \( \sqrt[4]{16} = 2 \)  
    2 raised to the power 4 is 16 
    \( 2^4 = 2 \times 2 \times 2 \times 2 = 16 \); 

12. \( \sqrt{125} = 5 \)  
    5 raised to the power three is 125 
    \( 5^3 = 5 \times 5 \times 5 = 25 \times 5 = 125 \).

13. (a) \( \sqrt{14} \times \sqrt{14} + \sqrt{12} \times \sqrt{12} = 14 + 12 = 26 \);  
    (b) \( (\sqrt{8} \times 8)^2 = (8)^2 = 8 \times 8 = 64 \);  
    (c) \( (\sqrt{5})^2 = \sqrt{5} \times \sqrt{5} = \sqrt{5 \times 5} = 5 \);  
    (d) \( (\sqrt{1})^2 = 1^2 = 1 \).

14. (a) \( \sqrt{25} = \sqrt{5 \times 5} = 5 \);  
    (b) \( 2 \times \sqrt{12} \times \sqrt{12} = 2 \times 12 = 24 \);  
    (c) \( \sqrt{\frac{6 \times 6}{2 \times 2}} = \frac{6}{2} = 3 \) or \( \sqrt{\frac{36}{4}} = \sqrt{9} = \sqrt{3 \times 3} = 3 \);  
    (d) \( 25 - \sqrt{2 \times 2} = 25 - 2 = 23 \).
Subunit A.5   Ratios, proportions and percentages

Learning objectives: On completion of this Subunit you should be able to relate numbers to each other using ratios, proportions and percentages.

In our day-to-day lives we are constantly confronted with situations in which we have to compare numbers: "One out of every three South Africans prefers sugar-free soft drinks", "Twenty percent of our national budget is spent on education and welfare", etcetera.

A.5.1   Ratios

A ratio is a way of comparing two (or more) numbers. For example we say “one out of three” and write 1 to 3 or 1:3 or 1/3. Similarly we speak of the ratio 2 to 1 or 2:1 or 2/1 in the case of “doubling”.

Activity

Write the ratios of each of the following pairs of numbers in each of the three ways: 7 and 16; 22 and 11; 9 and 12.

Answer

(1) 7 to 16 or 7:16 or 7/16;
(2) 22 to 11 or 22:11 or 22/11;
(3) 9 to 12 or 9:12 or 9/12.

Note that the last form the ratio is nothing more than a fraction or rational number. Furthermore it is often possible to reduce this ratio to a simpler form, as is the case in the last two examples above, namely

\[
\frac{22}{11} = \frac{2 \times 11}{11} = \frac{2}{1}
\]

and

\[
\frac{9}{12} = \frac{3 \times 3}{3 \times 4} = \frac{3}{4}.
\]

Thus the ratio 22 to 11 is the same as 2 to 1, and 9 to 12 is the same as 3 to 4.

Another way in which a ratio is reduced is by expressing the fraction in decimal form, in which case it is compared to one. To be specific, in the above three cases we have

\[
\frac{7}{16} = \frac{0.4375}{1} \quad \text{or} \quad 0.4375:1;
\]
Generally speaking this “comparison to one” formula is the one best suited to calculators and computers.

A room is 4 m wide and 6 m long. What is the ratio of the width to the length, and the ratio of the length to the width? Also express both ratios in the "comparison to one" form, working to three significant figures.

The ratio of the width to the length is 4 to 6 or \( \frac{4}{6} = \frac{2}{3} \) that is 0,667 to 1.

The ratio of the length to the width is 6 to 4 or \( \frac{6}{4} = \frac{3}{2} \) that is 1,50 to 1.

This example illustrates that we must be very careful to place the quantity we are comparing to, second. In other words, the ratio of width to length is not the same as that of length to width.

Activity

1. Last year the total sales for Water Walker Sailboards was R240 000 while the gross profit was R30 000. Determine the ratio of gross profit to total sales, reduce it, and express it as a comparison to one ratio.

2. Percy’s Programming Persons, a placement bureau for data processing personnel, has 3 female and 21 male job seekers on its books. What is the ratio of women to men? Reduce it and express it as a comparison to one ratio.

Answer

1. The ratio of gross profit to total sales is 30 000 to 240 000 or

\[
\frac{30\ 000}{240\ 000} = \frac{1}{8}
\]

that is 1 to 8 or 0,125 to 1.
2. The ratio of women to men is

\[ \frac{3}{21} = \frac{1}{7} \]

that is 1 to 7 or 0.143 to 1 (to three decimals).

Frequently it is necessary to divide a number into two parts that must satisfy a given ratio, as the next example illustrates.

Benjamin and Ernest start a transport service, BETS. Benjamin invests R25 000 and Ernest R20 000, and they agree to divide any profits in the same ratio as their capital investments. At the end of the first year the business shows a net profit of R9 000. What is each partner's share of the profit?

The ratio of their investments is 25 to 20. Now we must divide the profit into two parts which satisfy this ratio. This is done as follows.

The total number of parts in thousands of rand is 25 + 20 = 45.

Benjamin's share is then

\[ \frac{25}{45} \times 9 000 = R5 000. \]

Earnest's share is then

\[ \frac{20}{45} \times 9 000 = R4 000. \]

Why specifically '45 parts' in total? Would we get the same results if we worked with a reduced ratio? Yes, we would, but let's check:

\[ \frac{25}{20} = \frac{5}{4} = 1.25 \]

Consider \( \frac{5}{4} \) first. The total number of parts is then \( 5 + 4 = 9 \).

Benjamin's share is \( \frac{5}{9} \times 9 000 = R5 000 \) while Earnest's share is \( \frac{4}{9} \times 9 000 = R4 000 \) as before.

Or, working with 1.25 to 1, we have a total of \( 1.25 + 1 = 2.25 \) parts, and Benjamin's and Earnest's shares are respectively

\[ \frac{1.25}{2.25} \times 9 000 = R5 000 \] and \[ \frac{1}{2.25} \times 9 000 = R4 000. \]

In other words, it does not matter whether we work with the given ratio, or a reduced ratio, or a comparison to one ratio. We may choose whichever is most convenient.
Activity
Cleansweep, an office cleaning company, lands a big new contract and has to expand its staff from a total of 240 cleaners and 10 supervisors to a total of 400. If the same ratio of supervisors to cleaners is to be maintained, how many supervisors are required?

Answer
The ratio of supervisors to cleaners is 10 to 240, or 1 to 24 in reduced form (or 0.04167 to 1).

Thus we have a total of $1 + 24 = 25$ parts altogether.

The number of supervisors required is \[ \frac{1}{25} \times 400 \] which gives 16.

The number of cleaners is \[ \frac{24}{25} \times 400 = 384. \]

Note that we could have worked with the original ratio or the comparison to one ratio and obtained the same results, namely $10 + 240 = 250$ in total.

To check we calculate the ratio of supervisor to cleaners. It is 16 to 384 which is 0.04167 to 1.

Before moving on to the concept of proportion I must point out that ratios are not just used to compare two numbers. Three or more numbers can be compared using ratios, as the next example shows.

Benjamin and Ernest's Transport Service is doing well, so that when they recognise a need for a courier service for sensitive documents they decide to establish a new firm - Discreet Deliveries. They go into partnership with Salina. Their initial investments are as follows: Benjamin R7 500, Ernest R10 000 and Salina R2 500. If they agree to divide profits in the same ratio as their capital investments how will they share out the first year's profit of R8 000?

The ratio of their investments is $7500$ to $10000$ to $2500$ or, in reduced form, $3$ to $4$ to $1$ which we can also write $3 : 4 : 1$.

We add $3 + 4 + 1$ together to obtain 8 parts in total.

Thus Benjamin's share is $\frac{3}{8} \times R8 000 = R3 000$,

Ernest's share is $\frac{4}{8} \times R8 000 = R4 000$

and Salina's share is $\frac{1}{8} \times R8 000 = R1 000$.

The next activity requires the same approach.
Activity

Much to the chagrin of all his relatives Uncle Wilfred's will stipulated that his estate should be divided between his three pets, Percy the parrot, Bozo the bulldog and Sullivan the Siamese cat, in the ratio 7:5:4. The estate was worth R240 000. How much did each pet receive?

Answer

The total number of parts is $7 + 5 + 4 = 16$.

Thus Percy receives $\frac{7}{16} \times 240 000 = R105 000$,

Bozo receives $\frac{5}{16} \times 240 000 = R75 000$

and Sullivan receives $\frac{4}{16} \times 240 000 = R60 000$.

To check, note that $105 : 75 : 60$ reduces to $1,75 : 1,25 : 1$ (after division by 60), which is the comparison to one form of $7 : 5 : 4$.

A.5.2 Proportion

In ordinary language the words ratio and proportion are often used as synonyms and, indeed, for most purposes this slight confusion is acceptable and of little consequence. Mathematically speaking, however, there is a subtle difference.

A proportion is a statement of equality between ratios. Typically, statements of proportionality arise when ratios are reduced or when they are converted from one frame of units to another. For example:

$$\frac{15}{25} = \frac{3}{5} \text{ or } 15:25 = 3:5$$

which is read 15 is to 25 as 3 is to 5 and

$$24:16:8 = 3:2:1$$

which is read 24 is to 16 is to 8 as 3 is to 2 is to 1.

Statements of proportionality are used all the time to scale up, or down, the values for a known or accepted situation to new values for new situations.

If you are paid R1 580 for 5 working days, how much would you earn for 17?

You will probably do this calculation without thinking, but I would like you to take a few minutes as the basic step is the same for all proportion type problems.
The ratio of rands to days is:

\[
\frac{1 580}{5} \text{ which reduces to } 316:1 \text{ (rand per day or rands to days)}.\]

This is, of course, nothing more than the rate of pay. To obtain the wage for 17 days simply multiply this rate by 17 to obtain \(316 \times 17 = R5 \ 372\). Finally, check your answer by making sure that the two ratios are equal, that is

\[
\frac{5 \ 372}{17} = \frac{1 \ 580}{5}. \tag{2.1}
\]

**Activity**

Property tax is often assessed on a proportional basis. Suppose that a certain municipality charges R150 per year for every R10 000 assessed valuation. What would the annual tax be on a property valued at R125 000?

**Answer**

Again, we first determine the basic rate, which is, in fact, the comparison to one ratio. The ratio of tax to valuation is 150 to 10 000 or 0,015:1. Thus the annual tax is:

\[
0,015 \times 125 \ 000 = R1 \ 875. \tag{2.2}
\]

Check:

\[
\frac{1 \ 875}{125 \ 000} = \frac{150}{10 \ 000}. \tag{2.3}
\]

You will have noticed that the basic trick is to determine the comparison to one ratio, which we call the rate. "How do I know to which number I must compare? In other words, which number comes second in the ratio?" you may wonder. The answer is simple - the second number is always the one related to the multiplicand in the final step, for example days in the first example and valuation in the second example. And remember to always check your answer as shown above. With these hints as a guide try the next two activities.

**Activity**

1. You undertake a trip in your new car and find that for the first 300 km you use 25 litres of petrol. How many litres do you anticipate you will need for the next 360 km?
2. You obtain the licence for Gobbling Goblin, the latest video game, and make a profit of R5 200 on the first 650 cassettes sold. Assuming all prices remain the same, how much do you expect to make on the anticipated sales of 3 000 cassettes, for the next quarter?
**Answer**

1. The ratio required is 25 to 300 or 0,08333 to 1 (litre per km).

   You would therefore expect to use $0,08333 \times 360 = 30,0$ litres for the next 360 km.

   Check: $\frac{30}{360} = \frac{25}{300}$.

2. The ratio required is 5 200 to 650 or 8 to 1 (rands profit per game sold). Your anticipated profit for the next quarter is therefore

   $8 \times 3\,000 = \text{R}24\,000$.

   Check: $\frac{24\,000}{3\,000} = \frac{5\,200}{650}$.

---

### A.5.3. Percentage

There is probably not a day that goes by during which you will not hear some reference to the term **percent**. It is certainly the most commonly used way of indicating the relative size of two numbers and you are probably quite familiar with basic calculations using percentages. The recommended calculator has a percentage key. Please see Tutorial letter 101 how to use it.

The term percent comes from the Latin words “per” and “centum” and means “by the hundred”, or fractional part of one hundred or the ratio of a number to one hundred. It is denoted by the symbol %.

For example

\[
\begin{align*}
25\% & \quad \text{means} \quad \frac{25}{100} \\
67\% & \quad \text{means} \quad \frac{67}{100} \\
33\frac{1}{3}\% & \quad \text{means} \quad \frac{33\frac{1}{3}}{100} \\
150\% & \quad \text{means} \quad \frac{150}{100}.
\end{align*}
\]

You decide to buy a one-bedroom flat and apply to the building society for a loan. The purchase price is R180 000. The building society requires you to put down a deposit of 25%, and it is prepared to grant you a bond for the remaining 75%. How much must you deposit?
You will recognise that this is a proportion type calculation which we dealt with in the last paragraph. 25% deposit means that for every R100 of the purchase price you must deposit R25. We first calculate the ratio 25:100 which is 0,25:1.

Your total deposit is \(0,25 \times R180\,000 = R45\,000\).

As this example illustrates, in order to use percent in computations we need to express them as decimals (or fractions). Fortunately in the case of percent this is much simpler than for ratios in general. You will recall that division by 100 simply implies shifting the decimal point two places to the left. For example

\[
6\% = \frac{6}{100} = 0,06
\]

\[
10,5\% = \frac{10,5}{100} = 0,105
\]

\[
122,7\% = \frac{122,7}{100} = 1,227.
\]

Of course, if you are not sure in any specific instance you can always consult your calculator - or you can always enter a percent as a number followed by \(\div 100\), if you want to be on the safe side.

**Activity**

1. During a sale at Wendy’s Wardrobe goods are marked down 30%. How much will a dress that normally cost R188 be?

2. A salesman receives a commission that is 2½% of his sales. If his sales for the month total R120 000, what is his commission?

**Answer**

1. The percentage 30% is 0,30 so the amount off is \(188 \times 0,3 = R56,40\). The price to be paid is \(188 - 56,40 = R131,60\).

2. The percentage 2,5% is 0,025. Thus the commission is \(120\,000 \times 0,025 = R3\,000\).

Sometimes it is necessary to calculate the percentage rate applicable to a problem as the next example illustrates.

Frederick had R1 960 deducted from his gross monthly pay for income tax. If his gross pay was R8 800, what percentage of this is his income tax? Express your answer to 1 decimal place.
We reduce the rates 1 960:8 800 to 0,2227:1. However, percent means parts per hundred, so multiply by 100 to obtain 22,27%.

Thus 22,3% was paid as income tax.

**Activity**

1. A survey reveals that in a small town, with 11 275 families, 3 712 families own television sets. What percentage of families owns sets?

2. Sky High, a construction company, last year realised a net profit of R552 500 on a contract for R8 500 000. What percentage of the contract value was the net profit?

---

**Answer**

1. The ratio that owns TV sets is 3 712:11 275 which reduces to 0,329:1.

   Multiplying by a hundred gives 32,9%.

2. The ratio of net profit to contract value is 552 500:8 500 000 which reduces to 0,065:1.

   Multiplying by 100 gives 6,5%.

---

So far we have dealt with two types of calculations which involve percent. In the first case we were given a base number and a percentage and asked to calculate the corresponding amount, for example 10% of "so much" is …. In the second case we were asked to determine what percentage one amount is of another. Another form of calculation involving percent is when we know both the percentage and the final number but not the initial, or base number and want to work back to it. The following example illustrates this.

A video recorder is offered as a special for R3 600 and it states that this is 75% of the usual price. What was the price before discount?

The percentage 75% is 0,75. Divide 3 600 by 0,75 which gives R4 800 which is the normal price. This is because

\[
75\% \text{ of the price} = 3\,600 \\
0,75 \times \text{price} = 3\,600 \\
\text{price} = \frac{3\,600}{0,75} \\
= 4\,800.
\]

To check, calculate the ratio of 3 600:4 800 and see that it reduces to 75%.
**Activity**

1. Patrick’s Paintshop advertises “all goods at 80% of normal price”. For a quantity of paint and brushes you are charged R122,40. What would the normal price have been?

2. Benjamin and Ernest calculate that 55% of last year’s total expenses were for salaries. If the salary account was R121 000, what were the total expenses?

**Answer**

1. The percentage 80% is 0.8. Dividing R122,40 by 0.8 gives R153 which would have been the normal price.

   Check that 122,40 : 153 reduces to 80%,

   \[
   \begin{align*}
   80\% \text{ of the price} & = 22,40 \\
   0.8 \times \text{price} & = 22,40 \\
   \text{price} = \frac{122,40}{0.8} & = 153,0.:
   \end{align*}
   \]

   Normal price = R153,00.

2. The percentage 55% is 0.55. Dividing R121 000 by 0.55 gives R220 000 for the total expenses.

   Check that 121 000:220 000 reduces to 55%,

   \[
   \begin{align*}
   55\% \text{ of expenses} & = 121 000 \\
   0.55 \times \text{expenses} & = 121 000 \\
   \text{expenses} = \frac{121 000}{0.55} & = 220 000.
   \end{align*}
   \]

   Total expenses = R220 000.
Subunit A.6 Signs, notations and counting rules

Learning objectives: On completion of this Subunit you should know and be able to use

• all the basic signs and notations and
• the counting rules.

A.6.1 Signs and notation

The signs in mathematics, without which we cannot live are given below.

(i) \( = \) (equal)

This sign is used in equations and indicates that the left-hand side is the same as the right-hand side. An equation is sometimes like \( 3 \times 2 = 6 \).

(ii) \( < \) (less than)

Three is less than five, is written as \( 3 < 5 \) (3 is to the left-hand side of 5 on the numbers line).

The sign \(<\), indicates smaller than.

(iii) \( \leq \) (less than or equal)

This means that the left-hand side can be less than the right-hand side or equal to it.

Then \( a \leq 5 \) means that \( a \) can be any value smaller than 5, but that it can also be equal to 5.

(iv) \( > \) (greater than)

Three is greater than two, is written as \( 3 > 2 \) (3 is to the right-hand side of 2 on the numbers line).
(v) **(greater than or equal to)**

This means that the left-hand side of an expression can be more than the right-hand side of the expression, but can also be equal to it.

Then \( a \geq 5 \) means that \( a \) can be any value greater than 5, but that it can also be equal to 5.

(vi) **Play on words**

Many times the signs are not given, but only words like ... *at least* ..., ... *no more than* ... *at the most*.... How can we write this using only signs?

At least 2: this means *two or more, two included* and we write it as \( \geq 2 \).

![Diagram showing the range from 0 to 5]

No more than 3: this means *everything less than 3 and 3 included* and we write it as \( \leq 3 \).

![Diagram showing the range from -2 to 4]

At the most 3: this is the same as *not more than 3*.

Note that the words ‘*between* 2 and 5’ often are confusing. It must be stated clearly whether the endpoints, 2 and 5, are included.

(vii) Up until now we have worked with only positive integers to illustrate the symbols and signs. For negative integers, we need to keep some extra facts in mind.

<table>
<thead>
<tr>
<th>The minus changes the sign next to it.</th>
</tr>
</thead>
<tbody>
<tr>
<td>A plus does <strong>not change</strong> the sign next to it:</td>
</tr>
<tr>
<td>that sign <strong>remains the same</strong> and the plus <strong>disappears</strong>.</td>
</tr>
</tbody>
</table>

What happens to pluses and minuses in multiplication and division?
Consider $3x - 2$. That is $+ 3x - 2$. Now arrange all the signs in front of the figures.

$$+ - 3 \times 2$$
$$= - 3 \cdot = - 6 \text{ (here the plus disappears but not the minus.)}$$

Likewise

$$- 2 \times - 2$$
$$= + 4 \text{ (the minus changes the sign next to it.)}$$

Let us look at what happens with

$$- 2 \times - 2 \times - 2 \text{ and } - 2 \times - 2 \times - 2 \times - 2$$

$$= - - 2 \times 2 \times 2$$
$$= + 1 \times - 2 \times - 2 \times - 2 \times - 2 \times - 2$$

We can now cancel the minus sign pairwise to obtain the final answer.

Answer $- 8$ Answer $+ 16$

Do you see what I see?

- When there is an **even** number of minuses in multiplication (ie 2, 4, 6, etc), the sign becomes a plus (+) and the answer is a **positive** number.

- When there is an **uneven** number of minuses (ie. 1, 3, 5, 7, etc) the sign becomes a minus (-) and the answer is a **negative** number.

**Summation**

The summation symbol, $\Sigma$, is an important symbol used when working with numbers. This is a very useful sign, because it is the mathematical shorthand sign for adding. The Greek capital letter for S is $\Sigma$ and is pronounced as sigma. Suppose we want the sum of all the integers between 1 and 10, 1 and 10 included. Instead of saying: “Add all the integers from 1 to 10 and write down the sum”, we write:

$$\sum_{i=1}^{10} i.$$

Below the $\Sigma$ sign we have $i = 1$. This indicates that you must add from the first position.

On top we have 10. This is the last value that $i$ can be.

Thus

$$\sum_{i=1}^{10} i = 1 + 2 + 3 + ... + 10.$$
If we have a set of observations we write \( \sum_{i=1}^{n} x_i \) where \( x_i \) indicates an observation.

The subscript \( i = 1 \) indicates that we start counting from the first observation and add all the observations up to the \( n \)-th observation.

Thus \( \sum_{i=1}^{10} x_i = x_1 + x_2 + \ldots + x_{10} \).

If we have a set of observations that must be added, we just write \( \Sigma x_i \). When no confusion is possible, it is not necessary to add the sub- and superscripts.

**Activity**

Calculate \( \sum_{i=1}^{5} 1 \).

**Answer**

The answer is \( \sum_{i=1}^{5} 1 = 1 + 1 + 1 + 1 + 1 = 5 \).

A.6.2 Counting rules

A.6.2.1 The multiplication rule

The registration number of cars in Gauteng consists of three letters and three numeric figures and GP (for Gauteng Province). An example is BCD 389 GP (with Gauteng’s coat of arms in front of the GP).

How many registrations are possible if no vowels (that is a, e, i, o and u) may be used?

There are 21 letters that may be used if no vowel is allowed.

The first position can be filled in 21 different ways.
The second position can be filled in 21 different ways.
The third position can be filled in 21 different ways.

For the numeric part we may use any of the 10 figures, 0, 1, 2, ..., 9. The first position can be filled in 10 ways; the second position can be filled in 10 ways and the third position can be filled in 10 ways.

The total number of registrations possible is

\[ 21 \times 21 \times 21 \times 10 \times 10 \times 10 = 9261000, \text{ which is quite a lot!} \]

The multiplication rule is defined as follows:
if an operation can be performed in \( n_1 \) ways, and then after it is performed in any one of these ways a second operation can be performed in \( n_2 \) ways, and after this second operation is performed in any one of these ways a third operation can be performed in \( n_3 \) ways, and so on for \( k \) operations, then the \( k \) operations can be performed in \( n_1 \times n_2 \times n_3 \times \ldots \times n_k \) ways.

**Activity**

If a parking garage has five entrances and three exits, in how many ways can a motorist enter and leave the garage?

**Answer**

The motorist can enter in 5 different ways.
The motorist can leave in 3 different ways.

The total number of ways in which he can enter and leave is \( 5 \times 3 = 15 \).

### A.6.2.2 Permutations

The exclamation mark, ! has a special role in mathematics.

Suppose we want to arrange or order five numbers. The first number can be placed in any of five places. This can be done in five ways. Then four places are left and the second number can take any of the four places. Similarly the third number can take any of three places. The total number of ways is

\[
5 \times 4 \times 3 \times 2 \times 1 = 120.
\]

A shorthand way of writing this is \( 5! \) And we say: “5 factorial”.

**Therefore:** \( n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1 \). **Note:** \( 0! = 1 \).

Then

\( 5! \) Can also be written as \( 5 \times 4! \) Or \( 5 \times 4 \times 3! \) Or \( 5 \times 4 \times 3 \times 2! \) Or \( 5 \times 4 \times 3 \times 2 \times 1! \)

**Activity**

Seven horses run in a race. What is the total number of ways in which they can complete the race?

**Answer**

The first horse can take any of seven places. The next horse can take any of six places.
The total number of ways is \( 7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5 \, 040 \).
How do we handle the following situation?

A list of 10 investment possibilities are presented to the directors of a company. Each director must order the five projects he considers as the best, in order of importance. How many different arrangements are possible?

It is clear that order of placement is of importance and that only five must be chosen out of the ten possibilities. When order is of importance, we use permutations.

We want to determine the number of permutations of five out of ten objects. The formula is

\[ P_{10}^5 = \frac{10!}{(10 - 5)!} = \frac{10!}{5!} = \frac{10 \times 9 \times 8 \times 7 \times 6 \times 5!}{5!} = 30,240 \]

or, in general, the number of permutations of \( x \) objects out of \( m \) objects is:

\[ P_m^x = \frac{m!}{(m - x)!} \]

The notations \( P_{10}^5 \) or \( P(10, 5) \) are also used.

**Activity**

How many arrangements are possible for the first three places in a race with eight horses?

**Answer**

The answer is

\[ P_8^3 = \frac{8!}{(8 - 3)!} = \frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5!}{5!} = 8 \times 7 \times 6 = 336 \]

**A.6.2.3 Combinations**

When order of placement is not important, we use combinations instead of permutations.

Suppose we have four workers of equal competence. In how many different ways can we select two workers?

Suppose the workers are A, B, C and D.

The possible choices of two out of four is

A B  B C  C D  A C  A D  B D

The formula used here is:
4C_2 = \frac{4!}{2!2!} = \frac{4 \times 3 \times 2!}{2! \times 2!} = \frac{4 \times 3}{2} = 6.

Generally: the combination of \(x\) objects out of \(m\) possible objects is

\[ mC_x = \frac{m!}{(m - x)! \, x!} \quad \text{(or} \quad ^mC_x \quad \text{or} \quad C(m, x) \quad \text{or} \quad \binom{m}{x} \text{)} \]

(If order of placement were important, then we would have:


---

**Activity**

In how many ways can a police captain choose any three of his seven detectives for a special assignment?

**Answer**

Order of placement is not important. The possible number of combinations is:

\[ 7C_3 = \frac{7!}{(7 - 3)! \, 3!} = \frac{5040}{24 \times 6} = 35. \]

---

**Exercise**

1. Use the symbols \(<, >\) or = to make the following true:
   
   (a) \(-5\) \(\square\) \(-2\); \hspace{1cm} (d) \(-6\) \(\square\) \(-12\);
   
   (b) \(9\) \(\square\) \(-2\); \hspace{1cm} (e) \(2\) \(\square\) \(0\);
   
   (c) \(-100\) \(\square\) \(7\); \hspace{1cm} (f) \(+3\) \(\square\) \(3\).

2. Write down the following and complete the missing parts:
   
   (a) \(x < 5\) and \(x \geq 0\) can also be written as \(0\) \(\square\) \(x\) \(\square\) \(5\);
   
   (b) \(x \geq -3\) and \(x < 3\) can also be written as \(-3\) \(\square\) \(x\) \(\square\) \(3\);
   
   (c) \(-6 < x \leq 5\) can also be written as \(\ldots\ldots\ldots\ldots\ldots\ldots\) and \(\ldots\ldots\ldots\ldots\ldots\ldots\);
   
   (d) \(0 \leq x < 6\) can also be written as \(\ldots\ldots\ldots\ldots\ldots\ldots\) and \(\ldots\ldots\ldots\ldots\ldots\ldots\).
3. Graph the following inequalities on a number line (x is integer):
   
   (a) $x \geq -3$;    (c) $x < 5$;
   
   (b) $-3 < x \leq 7$;    (d) $x \geq -4$ and $x < 5$.

4. (a) How many four letter words (including those not making sense) are possible if a character may appear more than once in the same word?
   
   (b) How many meals are possible if there is a choice of four starters, ten main courses and six desserts?
   
   (c) Any three people out of twelve can be chosen for a committee. How many possible arrangements are there?
   
   (d) How many four letter words are possible if a letter may not occur more than once in the same word (including words not making sense)?

Solutions

1. (a) $-5 < -2$;    (b) $9 > -2$;    (c) $-100 < 7$;
   
   (d) $-6 > -12$;    (e) $2 > 0$;    (f) $+3 = 3$.

2. (a) $0 \leq x < 5$;    (c) $x \leq 5$ and $x > -6$;
   
   (b) $-3 \leq x < 3$;    (d) $x < 6$ and $x \geq 0$.

3. (a) ![Graph of $x \geq -3$]

(b) ![Graph of $-3 < x \leq 7$]

(c) ![Graph of $x \geq -4$ and $x < 5$]
4. (a) There can be $26 \times 26 \times 26 \times 26 = 456,976$ words. 
**Remember!** A letter may appear more than once.

(b) The number of different possible meals are $4 \times 10 \times 6 = 240$.

(c) Because the order of placement is not important, the answer is a combination:

$$12C_3 = \frac{12!}{9!3!} = \frac{12 \times 11 \times 10 \times 9!}{9! \times 3!} = \frac{12 \times 11 \times 10}{3!} = 220.$$ 

(d) In this case the order of placement is important. For example:

ABCD, BCAD and DBCA all form different words. In the question stated **words** it would have been a combination.

Then ABCD, BCAD and DBCA would have been the same, because the order of placement is not important. The answer is

$$26P_4 = \frac{26!}{(26-4)!} = \frac{26!}{22!} = \frac{26\times25\times24\times23\times22!}{22!} = 26 \times 25 \times 24 \times 23 = 358,800.$$
Appendix B
Functions and representations of functions

On completion of this component you should be able to

- explain the concept of a function;
- represent linear functions graphically.

CONTENTS

Subunit B.1  What is a function?
Subunit B.2  Linear functions

From material originally compiled for QMI101-X.
Subunit B.1 What is a function?

Learning objectives: On completion of this subunit you should be able to explain the concepts of a formula and a function.

B.1.1 Formulae

Expressions like \( A \times B \), \( A \times B + A \times C \), \( A^2 + B^2 \), \( A^M \times N \) which do not contain numbers only but also variables, are referred to as algebraic expressions in contrast to the arithmetic expressions which contain only numbers. In general, we will not, however, make this subtle distinction and simply refer to expressions.

A formula is a specific algebraic expression which results when symbols or letters are used to represent in a concise and clear way the relationship between different variables. It is in fact a recipe for calculating the value of some desired variable, called the dependent variable, from the values of the relevant independent variables.

Actually, you are well acquainted with the use of formulae although, possibly, you may not generally resort to the use of symbols but use words instead. An example of a “word” formula is the following.

Bank balance at month’s end = bank balance at month’s beginning + sum of all transactions during the month.

Using symbols or letters we could write

\[ B_e = B_b + T. \]

In this example \( B_e \) is the dependent variable and \( B_b \) and \( T \) are the independent variables. This is so because the value of \( B_e \) is determined by the values of \( B_b \) and \( T \).

There is, please note, no binding reason to use these specific symbols and we could in fact have written

\[ A = B + C. \]

The calculation of a worker’s wage is:

Wage (in rands) = Hours worked \( \times \) Rate (in rands per hour).

Using obvious symbols we write

\[ W = H \times R. \]

Here \( W \) is the dependent variable, and \( H \) and \( R \) are the independent variables.

From these two examples alone, it is obvious that a major advantage of formulae is their brevity. Following from this is the fact that they may be easily manipulated.
B.1.2 The concept of a function

When the value of one variable is dependent on the value of several others, we make the statement that the dependent variable is a function of the independent variable.

The way in which the value of the dependent variable is determined from the values of the independent variables must be clearly stated and unambiguous, and only one single value for the dependent variable must result.

The worker’s wage was written as \( W = H \times R \). \( W \) is a function of \( H \) and \( R \) and is dependent on the values of \( H \) and \( R \).

---

**Activity**

Consider the formula \( P = \frac{I}{RT} \)

which is obtained from the basic simple interest formula by rearrangement. Identify the dependent and independent variables in this formula.

**Answer**

The dependent variable is \( P \) and \( I, R \) and \( T \) are the independent variables.

The term "function of" occurs so often in mathematics that there is a special notation to denote it, namely \( f (...) \). The letter \( f \) obviously stands for function while all the independent variables, are listed in the brackets, separated by commas or semicolons. In this notation the function \( S = P \times (1+RT) \) is written as

\[
S = f(P, R, T)
\]

which is read: \( S \) is a function of \( P, R \) and \( T \).

Take note, however, that with this notation no information with regard to the specific form (that is rule applicable) is conveyed. It is merely stated that \( S \) is the dependent variable which is a function of \( P, R \) and \( T \). In this sense the notation does not discriminate between the above function and any other function of \( P, R \) and \( T \).

The function is only completely specified once the *functional form* or *relationship*, that is the rule for determining the value of the dependent variable from the values of the independent variables, is given.

Thus a complete specification of the function \( S = P \times (1+RT) \) is

\[
S = f(P, R, T) = P \times (1 + RT).
\]

Why then do we use this notation if it does not convey complete information about the function? The answer is that the notation provides us with a concise and clear way of
indicating that in a particular function, which has been previously defined, particular numerical values are to be substituted for the independent variables.

Suppose we consider the function

\[ S = f(P, R, T) = P \times (1 + RT), \]

\(f(1; 2; 3)\) means that 1 is substituted for \(P\), 2 for \(R\) and 3 for \(T\), and its value is

\[ f(1; 2; 3) = 1 \times (1 + 2 \times 3) = 7. \]

Similarly:

\[ f(1; 1; 1) = 1 \times (1 + 1 \times 1) = 2, \]
\[ f(2000; 0,1; 3) = 2000 \times (1 + 0,1 \times 3) = 2600, \]
\[ f(350; 0,08; 2,5) = 350 \times (1 + 0,08 \times 2,5) = 420 \text{ and so on.} \]

**Note:** where confusion may arise with the decimal, we use a semicolon ( ; ) to separate the variables.

**Activity**

Calculate the following values of the function \(S = f(P, R, T) = P \times (1 + RT)\):

(1) \(f(10\,000; 0,1; 10)\);
(2) \(f(1\,500; 0,075; 4)\).

**Answer**

(1) The function is \(f(10\,000; 0,1; 10) = 10\,000 \times (1 + 0,1 \times 10) = 20\,000; \)
(2) The function is \(f(1\,500; 0,075; 4) = 1\,500 \times (1 + 0,075 \times 4) = 1\,950. \)
Subunit B.2    Linear functions

Study objectives: On completion of this subunit you should be able to make a graphical representation of a linear function.

B.2.1    The set of axes

We want to represent the linear function \( y = f(x) \) graphically. Then \( y = f(x) \) means that \( y \) is a function of \( x \). The dependent variable is \( y \) and the independent variable is \( x \).

To represent a linear function graphically, we use a set of axes.

Draw two lines at right angles to each other as shown in Figure B.1. The horizontal line is generally used for the independent variable and the vertical for the dependent variable. The point where they cross is the common origin. A convenient scale, which need not be the same for both lines, is indicated on each. Just as it is customary to associate points to the right of the origin with positive values of the independent variable, and points to the left with negative values, so it is customary to associate points above the origin with positive values and points below it with negative values of the dependent variable. Furthermore, it is customary to refer to the horizontal line as the \( x\)-axis and the vertical line as the \( y\)-axis. A scaled set of axes introduced in this way is referred to as a rectangular co-ordinate system. As indicated in the figure, it divides the plane (that is the sheet of paper) into four sections which are known as quadrants and which are numbered as shown.

![Figure B.1](image)
If \( x \) is the independent variable and \( y \) the dependent variable then \( x \) and \( y \) are both positive in the first quadrant. In the second quadrant \( x \) is negative and \( y \) is positive, in the third quadrant \( x \) and \( y \) are both negative, and in the fourth quadrant \( x \) is positive and \( y \) is negative. Since most business problems deal with positive quantities, we shall be concerned mainly with points in the first quadrant, but if we regard losses as negative profits, deductions as negative additions, deficits as negative income, etcetera, we shall also have the occasion to work with points in the other three quadrants. Whenever "reading" a graph, you carefully establish the variables represented on each axis and the relevant scales. **Always label the axes clearly.** Remember the variables you want to draw on the axes need not be \( x \) and \( y \), but any variables, for example \( A \) and \( B \) or \( x_1 \) and \( x_2 \).

Now according to the definition of a function, a function assigns one value of \( y \) to each value of \( x \) within its domain. Thus a set of ordered pairs of data which we can write as \((x; y)\) are established. Each of these pairs corresponds to a point in the plane, and if we plotted all these points we would obtain what is called a graph of a given function.

In other words, the graph of the function \( y = f(x) \) consists of all ordered pairs \((x; y)\) which satisfy \( y = f(x) \). We speak of the **coordinates** \((x; y)\) of each point \( P \), and call \( x \) the **abscissa** and \( y \) the **ordinate** of \( P \).

Of course, it is not only points on the graph of \( y = f(x) \) which may be referred to in this way. Any point in the plane is located by the specification of an ordered pair of numbers \((x; y)\). That is in fact why we refer to a rectangular **co-ordinate system**.

We start by considering the **straight line** or **linear function**, as it is known, which is the most elementary, and perhaps the most important, of all functions.

The general functional expression of a straight line is

\[
y = ax + b \quad \text{or} \quad y = mx + c \quad \text{where} \quad a \text{ and } b \quad \text{or} \quad m \text{ and } c \quad \text{are constants.}
\]

### B.2.2 The intercept of a straight line

Let us look at the general properties of a straight line.

The point where the **line cuts the y-axis**, is called the **y-intercept**. This is where \( x = 0 \). At this point the value of \( y \) is

\[
y = a \times 0 + b \\
= b.
\]

In words we say that the **intercept on the y-axis** is equal to the constant term \( b \) in the functional expression for the straight line.

The point where the **line cuts the x-axis**, is called the **x-intercept or root**. The value of the **intercept on the x-axis** is where \( y = 0 \), this is \( ax + b = 0 \).
We can easily solve this to obtain the value of $x$.

Add $-b$ to both sides which gives:

$$ax + b - b = -b$$

this is $ax = -b$ (since $+b - b = 0$).

Next, assuming that $a$ is not 0, we divide by $a$:

$$\frac{ax}{a} = -\frac{b}{a} \quad \text{this is} \quad x = -\frac{b}{a}.$$

See B.2.6, Special cases for the discussion when $a = 0$.

This is the value of the intercept on the $x$-axis.

Thus we can conclude that the line cuts the axes at the points $(0 ; b)$ and $(-\frac{b}{a} ; 0)$ as shown in Figure B.2.

![Figure B.2](image.png)

Note that in this figure $a$ and $b$ are assumed positive.
The four specific cases which can occur are illustrated in Figure B.3.

Figure B.3
Activity

Determine the intercepts on the x- and y-axes of the following straight lines:

(1) \( y = 1 + x; \)  
(2) \( y = 2 - 4x; \)  
(3) \( y = -6 + 9x; \)  
(4) \( y = -25 - 5x. \)

Answer

The intercepts on the axes are as follows.

(1) Determine the y-intercept: if \( x = 0 \) then \( y = 1 + 0 \), thus \( y = 1 \).  
    Determine the x-intercept: if \( y = 0 \) then \( 0 = 1 + x \), thus \( x = -1 \).

(2) Determine the y-intercept: if \( x = 0 \) then \( y = 2 - 4 \times 0 \), thus \( y = 2 \).  
    Determine the x-intercept: if \( y = 0 \) then \( 0 = 2 - 4x \),  
    Thus \( -2 = -4x \)  
    \( \frac{-2}{-4} = x \)  
    \( \frac{1}{2} = x \)  
    \( x = \frac{1}{2} \).

(3) Determine the y-intercept: if \( x = 0 \) then \( y = -6 + 9 \times 0 \), thus \( y = -6 \).  
    Determine the x-intercept: if \( y = 0 \) then  
    \( 0 = -6 + 9x \)  
    \( 6 = 9x \)  
    \( \frac{6}{9} = x \)  
    \( x = \frac{2}{3} \).
(4) Determine the y-intercept: if \( x = 0 \) then \( y = -25 - 5 \times 0 \), thus \( y = -25 \).

Determine the x-intercept: if \( y = 0 \) then

\[
0 = -25 - 5x \\
25 = -5x \\
-\frac{25}{5} = x \\
x = -5.
\]

---

B.2.3 The slope of the straight line

The steepness with which straight lines ascend or descend is called the slope.

The slope, \( a \), is the ratio of the change in \( y \) values to a given change in \( x \) values.

In terms of two arbitrary points, \( P_1 \) with co-ordinates \((x_1; y_1)\) and \( P_2 \) with co-ordinates \((x_2; y_2)\) on the straight line, we can write:

\[
slope = m = \frac{y_2 - y_1}{x_2 - x_1}.
\]

This is depicted in Figure B.4.

![Figure B.4](image)

Now it is clear from Figure B.4 that \( a \) is a measure of the steepness of a straight line. The greater the change in \( y \) for a given change in \( x \), the steeper the line. Furthermore, as we saw in Figure B.3, if \( a > 0 \), then the line is ascending (from left to right); if \( a < 0 \), then the line is descending.
B.2.4 Using two points to determine the equation of a straight line

The expressions of the previous paragraphs can be used to determine the specific functional expression for the straight line passing through two given points, as the next example shows.

Determine the expression for the straight line passing through the points (1; 3) and (3; 7).

The general expression is \( y = mx + c \) or \( y = ax + b \).

But \[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

for any two points. Taking \((x_1; y_1) = (1; 3)\) and \((x_2; y_2) = (3; 7)\), we find

\[ m = \frac{7 - 3}{3 - 1} = \frac{4}{2} = 2. \]

The general expression thus reduces to \( y = 2x + c \).

How do we find the value of \( c \)? Since the line must pass through both given points, either point can be used. Substitute the \( x \) and \( y \) values of the first point in \( y = 2x + c \). This gives:

\[ 3 = 2 \times 1 + c = 2 + c \]

Subtracting 2 from both sides gives \( c = 1 \).

The expression for the line passing through the two points (1; 3) and (3; 7) is therefore

\[ y = 2x + 1. \]

Note

(1) We could just as well have used the point (3; 7) to find the value of \( c \), namely

\[ 7 = 2 \times 3 + c \]
\[ 7 = 6 + c. \]

Subtracting 6 from both sides gives \( c = 1 \).

(2) It does not matter which point we call \( P_1 \) and which \( P_2 \). Had we numbered them the other way around above we would have found:

\[ m = \frac{3 - 7}{1 - 3} = \frac{-4}{-2} = 2 \text{ as before}. \]
Activity

Determine the expression for the straight line passing through the points (–2 ; 8) and (4 ; 1).

Answer

The general expression is \( y = mx + c \).

The slope, \( m = \frac{y_2 - y_1}{x_2 - x_1} \) for any two points.

Taking \((x_1; y_1) = (–2 ; 8)\) and \((x_2; y_2) = (4 ; 1)\), we find

\[
m = \frac{-8 - 8}{4 - (-2)} = -\frac{7}{6}.
\]

The general expression thus reduces to

\[
y = -\frac{7}{6}x + c.
\]

How do we find the value of \( c \)?

Since the line must pass through both given points, either point can be used. Substitute the \(x\)- and \(y\)-values of the first point in \( y = -\frac{7}{6}x + c \). This gives:

\[
8 = -\frac{7}{6} \times (-2) + c
\]

\[
8 = \frac{7}{3} + c.
\]

Subtracting \( \frac{7}{3} \) from both sides gives \( c = \frac{17}{3} \).

The expression for the line passing through the two points (–2 ; 8) and (4 ; 1) is therefore

\[
y = -\frac{7}{6}x + \frac{17}{3}.
\]
Sometimes the two points are not given to you, but you must unravel them from the information given, as illustrated in the following example.

Miriam bakes vetkoek. If she bakes 20 vetkoek, her cost is R8,00, and if she bakes 40 vetkoek, her cost is R13,00. Determine the linear cost function, if it is assumed that a linear relationship exists between the cost and number of vetkoek baked.

Let \( x \) = the number of vetkoek baked.
Let \( y \) = the cost to bake the vetkoek.

The following data for \( x \) and \( y \) are given:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>8</td>
</tr>
<tr>
<td>40</td>
<td>13.</td>
</tr>
</tbody>
</table>

Thus two data points that satisfy the linear relationship are:

\((20 \; ; \; 8) \; \text{and} \; \; (40 \; ; \; 13)\).

The general expression is \( y = mx + c \).

The slope is \( m = \frac{y_2 - y_1}{x_2 - x_1} \) for any two points.

Taking \((x_1 ; y_1) = (20 \; ; \; 8) \; \text{and} \; (x_2 ; y_2) = (40 \; ; \; 13)\), we find

\[
m = \frac{13 - 8}{40 - 20} = \frac{5}{20} = \frac{1}{4} = 0.25.
\]

The general expression thus reduces to

\( y = 0.25x + c \)

How do we find the value of \( c \)?
Since the line must pass through both given points, either point can be used.
Substitute the \( x \)- and \( y \)-values of the first point in \( y = 0.25x + c \). This gives:

\[
8 = 0.25 \times 20 + c \\
8 = 5 + c.
\]

Subtracting 5 from both sides gives \( c = 3 \).

The expression for the line passing through the two points \((20 \; ; \; 8) \; \text{and} \; (40 \; ; \; 13)\) is therefore

\( y = 0.25x + 3 \).
Activity
Mr BR Wash sells BB (Brighter and Better) washing powder. If he charges R19 per box, he has a weekly demand of 26 000 boxes and if he charges R21 per box the weekly demand is 16 000. If \( p \) is the price per box and \( d \) is the weekly demand, derive an expression for the linear weekly demand.

Answer

Let \( d \) = weekly demand and
\[ p = \text{price per box.} \]

Thus the weekly demand \((d)\) can be written as a linear function in terms of the price \((p)\).

The general expression is \( d = mp + c \).

But \( m = \frac{d_2 - d_1}{p_2 - p_1} \) for any two points.

Taking \((p_1; d_1) = (19; 26 000)\) and \((p_2; d_2) = (21; 16 000)\), we find
\[ m = \frac{16000 - 26000}{21 - 19} = \frac{-10000}{2} = -5000. \]

The general expression thus reduces to
\[ d = -5000p + c. \]

How do we find the value of \( c \)? Since the line must pass through both given points, either point can be used. Substitute the \( x \)- and \( y \)-values of the first point in \( d = -5000p + c \)
\[ 26 000 = -5000 \times 19 + c \]
\[ 26 000 = -95 000 + c \]

Add \(-95 000\) to both sides:
\[ c = 121 000. \]

The expression for the line passing through the two points \((19; 26 000)\) and \((21; 16 000)\) is therefore
\[ d = -5000p + 121 000. \]
B.2.5 Representing a function on a set of axes

Only two points are needed to determine an equation of a straight line. If you are not convinced of this, just mark two points on a piece of paper and try to put more than one straight line through them. To draw the graph of a straight line we make use of the general method:

1. draw your axes and label them;
2. choose the scale of the axes;
3. plot the two given points;
4. draw a line through the two plotted points.

Activity

Graph the straight line that passes through the points (1 ; 4) and (4 ; 2).

Answer

And if we do not have two points but the expression of the line \( y = mx + c \)?

In order to draw a straight line we only need two points. These two points must, however, consist of a \( x \)-value and a corresponding \( y \)-value. To obtain these points we substitute any two values of the one variable into the function and calculate the other. Two useful points to use, and which are easy to calculate are the \( y \)-intercept (where \( x = 0 \)) and the \( x \)-intercept (where \( y = 0 \)). Thus select \( x = 0 \) and calculate \( y \), and then select \( y = 0 \) and calculate \( x \).

I should point out that there is no reason why we must choose the points where the line crosses the axes. Any two points which satisfy the expression \( y = mx + c \) will do. In fact, we are often only interested in the first quadrant that is points for which both \( x \) and \( y \) are greater than or equal to zero. In many actual problems, such as the production
process referred to above, the variables can only assume positive values. In such cases we can select any two points in the first quadrant to draw the straight line.

Draw the line \( y = 2 + x \).

To draw the line we need to determine two points through which the line passes.

If \( x = 0 \) then \( y = 2 + 0 = 2 \). The point is \((0; 2)\).

If \( y = 0 \) then \( 0 = 2 + x \) thus \( x = -2 \). The point is \((-2; 0)\).

Plot the two points and draw a line through the two points.

---

**Activity**

Draw the following straight lines:

1) \( y = x + 1 \);
2) \( y = 2 - 4x \);
3) \( y = 9x - 6 \);
4) \( y = -25 - 5x \).
Answer

(1) If \( x = 0 \) then \( y = 1 + 0 = 1 \).
   If \( y = 0 \) then \( 0 = 1 + x \), thus \( x = -1 \).

Thus 2 points on the line are (0; 1) and (-1; 0).

(2) If \( x = 0 \) then \( y = 2 - 4(0) = 2 \).
   If \( y = 0 \) then \( 0 = 2 - 4x \), thus \( x = 1/2 \).

The two points on the line are (0; 2) and (1/2; 0).

(3) If \( x = 0 \) then \( y = -6 + 9(0) = -6 \).
   If \( y = 0 \) then \( 0 = -6 + 9x \), thus \( x = 2/3 \).

The two points on the line are (0; -6) and (2/3; 0).
(4) If \( x = 0 \) then \( y = -25 - 5(0) = -25 \).
If \( y = 0 \) then \( 0 = -25 - 5x \), thus \( x = -5 \).

The two points on the line are \((0, -25)\) and \((-5, 0)\).

B.2.6 Special cases

- The case for which the constant term is zero, this is \( c = 0 \).

This means that the intercepts on both axes are zero. That is, the line goes through the origin as shown in Figure B.5.

![Figure B.5](image-url)
• The case of a zero valued slope, this is \( m = 0 \).

What does this mean? Looking at our expression for \( m \), that is \( m = \frac{{y_2 - y_1}}{{x_2 - x_1}} \), we see that this can only be the case if \( y_2 = y_1 \), that is if the function values are the same. \( y \) is not dependent on \( x \) - it is in fact a constant. This is represented by a straight line parallel to the \( x \)-axis (a horizontal line) as depicted in Figure B.6.

\[
\begin{align*}
\text{Figure B.6} \\
\end{align*}
\]

Notice that in this case there is no intercept on the \( x \)-axis.

• A straight line parallel to the \( y \)-axis.

In this case we would have \( x_2 = x_1 \) and

\[
m = \frac{{y_2 - y_1}}{{0}}.
\]

Division by zero is not defined. We say that the slope becomes infinite in this case. The line is vertical, that is, as shown in Figure B.7.

\[
\begin{align*}
\text{Figure B.7} \\
\end{align*}
\]

As indicated, the expression for this line is \( x = c \) where \( c \) is the intercept on the \( x \)-axis. There is no intercept on the \( y \)-axis.
• Two straight lines \( y \) and \( \overline{y} \) which have the same slope but different \( y \)-intercepts.

For example:

\[
y = mx + c \quad \text{and} \quad \overline{y} = mx + \overline{c}
\]

have the same slope \( a \), but different intercepts on the \( y \)-axis, namely \( c \) and \( \overline{c} \). (Note the use of the bar on \( y \) and \( c \) to distinguish the two cases.) If we subtract the one expression from the other we obtain:

\[
y - \overline{y} = c - \overline{c}.
\]

This indicates that the distance between the two lines does not depend on the value of \( x \), it is constant. In other words, the lines are parallel. This is depicted in Figure B.8.

![Figure B.8](image)

**Activity**

Draw the following lines on the same set of axes:

1. \( y = 4 \);
2. \( x = 6 \);
3. \( y = 2x \);
4. \( y = 2x + 4 \);
5. \( y = 2x - 1 \).

Can you notice anything in regard to \( (4) \) and \( (5) \)?

**Answer**

1. Horizontal line at \( y = 4 \).
2. Vertical line at \( x = 6 \).
(3) If $x = 0$ then $y = 2(0) = 0$. The point is $(0 ; 0)$.
If $y = 0$ then $0 = 2x$, thus $x = 0$. The point is $(0 ; 0)$. Thus we still have only one point.

Take any other $y$-value and determine a $x$-value.
If $y = 1$ then $1 = 2x$, thus $x = 1/2$.

Thus two points on the line are $(0 ; 0)$ and $(1/2 ; 1)$.

(4) If $x = 0$ then $y = 2(0) + 4 = 4$.
If $y = 0$ then $0 = 2x + 4$, thus $x = -2$.

Thus two points on the line are $(0 ; 4)$ and $(-2 ; 0)$.

(5) If $x = 0$ then $y = 2(0) - 1 = -1$.
If $y = 0$ then $0 = 2x - 1$, thus $x = 1/2$.

Thus two points on the line are $(0 ;-1)$ and $(1/2 ; 0)$.

Plot the data points for the different graphs and draw the necessary lines through them. The following graph is obtained.

Lines (4) and (5) are parallel. The two lines have the same slope.

---

**Summary**

The general expression for a straight line or linear function is $y = mx + c$
where $c$ is the intercept on the $y$-axis, and $m$ is the slope of the line.

The formula for the slope $m$ in terms of the co-ordinates $(x_1; y_1)$ and $(x_2; y_2)$ of two points on the line is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$
Exercises

1. (a) Determine the equation of the straight line through the points \((1 ; 2)\) and \((3 ; 3)\).
   (b) Find the intercepts on the \(x\)- and \(y\)-axes of the line in (a).
   (c) Is the line in (a) parallel to the line \(y = 2 + x\)? Why or why not?
   (d) Draw the lines of (a) and (c) on one graph.

2. Consider the lines \(y = 5 + 2x\) and \(y = 2 + x\). What are their intercepts on the axes? Are they parallel or not? What is the vertical distance between the lines at \(x = 3,5\)?

3. Draw the following lines on one graph:
   (a) \(x = 2\);
   (b) \(y = 4x\);
   (c) \(y = -2x - 3\).

4. A bus agency has room for 60 people on a bus tour. If they charge R600 per person, they will be able to fill the bus. They know from experience that if they increase the price of the tour by R50 they will lose three customers. Determine the price function if the price \(p\) (in rand) is a linear function of the demand (number of customers).

Solutions

1.(a) Straight line : \(y = mx + c\).
   (i) Determine \(m\): using \(m = \frac{y_2 - y_1}{x_2 - x_1}\).
   - Given two points \((1 ; 2)\) and \((3 ; 3)\), select any one of the two points to be \((x_1 ; y_1)\) and the other one to be \((x_2 ; y_2)\).
   - Let \((1 ; 2) = (x_1 ; y_1)\) and \((3 ; 3) = (x_2 ; y_2)\), then \(m = \frac{3 - 2}{3 - 1} = \frac{1}{2} = 0,5\).
   Thus \(y = 0,5x + c\).

   (ii) Determine \(c\).
   Take any one of the two points and substitute these values for \(x\) and \(y\) into the equation \(y = 0,5x + c\). Say we choose the point \((1 ; 2)\):
\[ y = 0.5x + c \\
2 = 0.5 \times 1 + c \\
2 = 0.5 + c \\
2 - 0.5 = c \\
c = 1.5. \]

Thus the equation of the line that cuts through the points \((1 ; 2)\) and \((3 ; 3)\) is 
\[ y = 0.5x + 1.5. \]

(b) Determine the intercepts on the \(x\)- and \(y\)-axes for \(y = 0.5x + 1.5\).

The intercept on the \(x\)-axis is where the line cuts through the \(x\)-axis; meaning the \(x\) value where the \(y\)-value is 0:

\[ 0 = 0.5x + 1.5 \\
0 - 1.5 = 0.5x \\
0.5x = -1.5 \\
x = -3. \]

The \(y\)-axis intercept is where the line cuts through the \(y\)-axis; where \(x = 0\):

\[ y = 0.5 \times 0 + 1.5 \\
y = 1.5. \]

(c) Two lines are parallel if the slopes of the two lines are the same.

Line in (a): \(y = 0.5x + 1.5\) \hspace{1em} slope = 0.5.
Line in (c): \(y = 2 + x\) \hspace{1em} slope = 1.

The two lines are not parallel.

(d) Line in (a): plot the two points \((1 ; 2)\) and \((3 ; 3)\) and draw a line through them.
Line in (c): calculate two points on the line \(y = 2 + x\).
For example: if \(x = 0\) then \(y = 2 + 0 = 2\). The data point is \((0 ; 2)\).
If \(y = 0\) then \(0 = 2 + x\), thus \(x = -2\). The data point is \((-2 ; 0)\).
Plot the two points and draw a line through the points.
2. For \( y = 5 + 2x \) the intercepts are:

- on the \( y \)-axis (where \( x = 0 \)): 5;
- and on the \( x \)-axis (where \( y = 0 \)): \(-\frac{5}{2}\).

For \( y = 2 + x \) the intercepts are:
- on the \( y \)-axis: 2;
- and on the \( x \)-axis: \(-\frac{2}{1} = -2\).

The lines are not parallel since the slopes are different, namely 2 and 1.

At \( x = 3.5 \) the \( y \) values for the two lines are respectively
\(- y_1 = 5 + 2 \times 3.5 = 12 \) and \(- y_2 = 2 + 3.5 = 5.5\).

The vertical distance between the two, at \( x = 3.5 \), is \( 12 - 5.5 = 6.5 \).

3. 

4. Let \( p = \text{price} \) and \( x = \text{number of customers} \).

The point \((x; p)\), that is \((60; 600)\), is given.

The general expression is \( y = mx + c \), written in terms of our variables,
\[ p = mx + b. \]

The slope \( m \), gives you the rate of change in the \( p \)-value for a unit change in \( x \)-value. It is given that if the price increases by R50 the number of customers decreases by 3.

Thus \( m = \frac{50}{-3} \).

The general expression reduces to \( p = \frac{50}{-3} x + c \).

How do we find the value of \( c \)? Since the line must pass through the given point \((60; 600)\), substitute the \( x \)- and \( p \)-values into the last expression. This gives:

\[
600 = \frac{50}{-3} \times 60 + c
\]

\[
600 = -1000 + c.
\]

Adding 1000 to both sides gives: \( c = 1600 \).

The expression for the line is therefore

\[
p = \frac{50}{-3}x + 1600 \text{ or } p = -16.67x + 1600.
\]
APPENDIX C

Linear systems

CONTENTS

Study unit C.1 Linear equations in one variable
Study unit C.2 Simultaneous linear equations in two variables
Study unit C.3 Linear inequalities in one variable
Study unit C.4 Systems of linear inequalities in two variables

From material originally compiled for QMI101X.
Study unit C.1 Linear equations in one variable

**Learning objectives:** On completion of this study unit you should be able to solve an equation in one variable algebraically.

C.1.1 What is an equation in one variable?

An equation in one variable is a statement containing an = sign with algebraic expressions to the left and right of the sign, using only one variable. The values of the variable which make the statement true are called solutions or roots of the equation.

A few examples should make this definition clear. In each case the values on the right (the solutions) make the statements on the left (the equations) true.

<table>
<thead>
<tr>
<th>Equation</th>
<th>Solution(s) or root(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x + 3 = x + 4$</td>
<td>$x = 1$</td>
</tr>
<tr>
<td>$A + 1 = 3$</td>
<td>$A = 2$</td>
</tr>
<tr>
<td>$2^x = 4$</td>
<td>$x = 2$</td>
</tr>
<tr>
<td>$y^2 + 3y + 1 = y + 4$</td>
<td>$y = 1$ and $y = -3$</td>
</tr>
</tbody>
</table>

The process of determining which values of the variables make the statement true is known as solving the equation.

C.1.2 Solving linear equations in one variable algebraically.

Linear equations are equations in which the unknown variable only appears in linear form:

$3x + 1 = 4x + 3$

$5A - 40 = 0$

$-5s + 2 = s + 8$. 
Solving linear equations is simply a matter of juggling and manipulating the equation until the variable is alone on the left-hand side. The golden rule is that we can perform the same operation to the expressions on both sides of the equals sign without altering the solution. More specifically we may:

1. add (or subtract) the same number or expression to (or from) both sides of the equation;
2. multiply (or divide) both sides of the equation by the same non-zero number or expression. (See Component I for a complete discussion of this.)

The new equation obtained by any one of these operations is equivalent to the original equation.

Let us do a few examples.

(1) Solve $3x + 5 = 2x - 3$.

Subtract $2x$ from both sides: $3x - 2x + 5 = 2x - 2x - 3$

$x + 5 = -3$.

Subtract 5 from both sides: $x + 5 - 5 = -3 - 5$

$x = -8$.

Note that the associative and commutative laws of addition allow us to enter the term that we are adding at any position on each side of the equation, for example whether we write

$3x - 2x + 5$ or $-2x + 3x + 5$ or $3x + 5 - 2x$,

the result is the same.

(2) Solve $4A - 25 = 0$.

Add 25 to both sides: $4A - 25 + 25 = 25$

$4A = 25$.

Divide both sides by 4: $1/4 \times 4A = 1/4 \times 25$

$A = 25/4 = 6.25$.

(3) Solve $5s - 6 = 10 - (s/6)$.

Add $s/6$ to both sides: $5s + s/6 - 6 = 10 - s/6 + s/6$

$(5 + 1/6)s - 6 = 10$

where we have used the distributive law to add the two terms containing the $s$.

Then we have $31/6 s - 6 = 10$ because $5 + 1/6 = 30/6 + 1/6 = 31/6$. 

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Add 6 to both sides: \[ \frac{31}{6} s - 6 + 6 = 10 + 6 \]
\[
\frac{31}{6} s = 16.
\]
Multiply both sides by \(\frac{6}{31}\):
\[
\frac{6}{31} \times \frac{31}{6} s = 16 \times \frac{6}{31}
\]
\[
s = \frac{(16 \times 6)}{31} = \frac{96}{31}.
\]
Note that if you are not adept in adding fractions together, such as \(5 + \frac{1}{6}\), you may use your calculator to obtain \(5,166...\) (and finally \(s = 3,096774194\) for the solution). (Try it!)

---

**Activity**

1. Solve \(25 + 3x = 50 - 7x\).

2. Solve \(y - 3 = 2y + 4\)

---

**Answer**

1. The equation is: \(25 + 3x = 50 - 7x\).

   Add +7x to both sides: \(25 + 3x + 7x = 50 - 7x + 7x\)
   \[
   25 + 10x = 50.
   \]

   Subtract 25 from both sides: \(-25 + 25 + 10x = 50 - 25\)
   \[
   10x = 25.
   \]

   Divide both sides by 10: \(x = \frac{25}{10} = 2.5\).

2. The equation is: \(y - 3 = 2y + 4\).

   Subtract 2y from both sides: \(-2y + y - 3 = -2y + 2y + 4\)
   \[
   -y - 3 = 4.
   \]

   Add 3 to both sides: \(-y - 3 + 3 = 4 + 3\)
   \[
   -y = 7.
   \]

   Multiply both sides by -1: \(-1 \times -y = -1 \times 7\)
   \[
   y = -7.
   \]
Although we have been working step-by-step here, there is no reason why you cannot use short-cuts and add several terms at once, as the next example illustrates.

Solve $4x + 30 = 16x - 54$.

Add $-16x - 30$ to both sides: $-16x + 4x + 30 - 30 = -16x + 16x - 54 - 30$

$$-12x = -84.$$  

Multiply both sides by $-1/12$: $-1/12 \times -12x = -1/12 \times -84$

$$x = 84/12 = 7.$$  

However, if you are unsure at all, rather be on the safe side and work step by step.

Any linear equation can be solved by using the following method:

1. manipulate the linear equation by operations of the above type until it is in the form: $ax + b = 0$ or $mx+c$,

   the left-hand side is the general expression for a linear function, hence the name linear equation;

3. this equation is solved by adding $-b$ to both sides to get $ax + b - b = -b$ or $ax = -b$;

3. divide by $a$, assuming $a \neq 0$, to obtain the root or solution $x = \frac{-b}{a}$.

Do you recognise the last expression? You should. It is, in fact, the expression for the intercept on the $x$-axis of the linear function $y = ax + b$. Since this intercept occurs at $y = 0$ this result should not surprise you.

In other words we can always interpret the root of a linear equation in one variable as the $x$-axis intercept of the corresponding linear function as shown below.

![Figure C.1](image-url)
To summarise:

We can solve any linear equation quite easily by rearranging it into the form $ax + b = 0$, whereafter the solution is simply $x = -b/a$.

### Activity

1. Solve $5x + 6 = -3x - 10$ by writing it in the form $ax + b = 0$.
2. Solve $1.1x + 3.4 = 2.5x + 1.3$.

### Answer

1. The equation is: $5x + 6 = -3x - 10$.
   Add $3x + 10$ to both sides: $5x + 3x + 6 + 10 = 0$
   $8x + 16 = 0$
   $x = -16/8 = -2$.

2. The equation is: $1.1x + 3.4 = 2.5x + 1.3$.
   Add $-(2.5x + 1.3)$ to both sides: $(1.1 - 2.5)x + 3.4 - 1.3 = 0$
   $-1.4x + 2.1 = 0$
   $x = -2.1/-1.4 = 1.5$. 

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Study unit C.2   Simultaneous linear equations in two variables

**Learning objectives:** On completion of this study unit you should be able to solve algebraically simultaneous linear equations in two variables.

Consider the following two functions:

\[ y = 3 - 2x \quad \text{and} \quad y = 2 + x. \]

If we consider them separately, then there exists a whole range of \( y \)-values for the \( x \)-values.

The point where the two graphs intersect is common to both functions. In this case it is the point \( \left( \frac{1}{3}, \frac{2}{3} \right) \).

But how do we solve it algebraically?

Set \( y = 3 - 2x \) equal to \( y = 2 + x \):

\[
\begin{align*}
3 - 2x &= 2 + x \\
3 - 2 &= 2x + x \\
1 &= 3x \\
x &= \frac{1}{3}.
\end{align*}
\]
To determine the value of \( y \) we substitute \( x = \frac{1}{3} \) in any one of the two equations, that is
\[
y = 3 - 2 \times \frac{1}{3} = 2 \frac{1}{3}.
\]

The solution for the system of simultaneous equations
\[
y = 3 - 2x \quad \text{and} \quad y = 2 + x
\]
is the point \( \left( \frac{1}{3}; 2 \frac{1}{3} \right) \).

**Activity**

Suppose we have the system of simultaneous equations \( 2x + 5y = 13 \) and \( 3x + 4y = 9 \). Rewrite the equations to make \( y \) the subject of the expression and solve.

**Answer**

If \( 2x + 5y = 13 \), then \( y = 13 - 2x \)
\[
y = \frac{13}{5} - \frac{2}{5}x.
\]

If \( 3x + 4y = 9 \), then \( 4y = 9 - 3x \)
\[
y = \frac{9}{4} - \frac{3}{4}x.
\]

Now we have
\[
\frac{13}{5} - \frac{2}{5}x = \frac{9}{4} - \frac{3}{4}x
\]
\[
\frac{13}{5} - \frac{9}{4} = \frac{2}{5}x - \frac{3}{4}x
\]
\[
\frac{52 - 45}{20} = \frac{8 - 15}{20}x
\]
\[
\frac{7}{20} = -\frac{7}{20}x
\]
\[
x = -1
\]

Substitute into
\[
y = \frac{13}{5} - \frac{2}{5}x = \frac{13}{5} - \frac{2}{5} \times (-1) = \frac{13}{5} + \frac{2}{5} = \frac{15}{5} = 3
\]

The solution for the system of simultaneous equations is \((-1; 3)\).
An alternative method is the following: consider the system of simultaneous equations

\[
\begin{align*}
2x + 5y &= 13 \\
3x + 4y &= 9,
\end{align*}
\]

From the first equation we have

\[
x = \frac{13}{2} - \frac{5}{2}y.
\]

Substitute into the second equation:

\[
3 \left( \frac{13}{2} - \frac{5}{2}y \right) + 4y = 9
\]

\[
\frac{39}{2} - \frac{15}{2}y + 4y = 9
\]

\[
\frac{39}{2} - \frac{7}{2}y = 9
\]

\[-\frac{7}{2}y = -\frac{21}{2}
\]

\[-7y = -21
\]

\[y = 3.
\]

Substitute \(y = 3\) into \(x = \frac{13}{2} - \frac{5}{2}y\) gives

\[
x = \frac{13}{2} - \frac{15}{2} = -\frac{2}{2} = -1
\]

and we have the same solution obtained previously!
Exercises

4. Solve the system of equations:

\[ 7x + 5y = -4 \]
\[ 3x + 4y = 2 \]

5. Solve the system of equations:

\[ 2x + 2y = 3 \]
\[ 5x + (1/2)y = -6. \]

3. Solve the system of equations:

\[ x + 4y = 49 \]
\[ -2x + y = 1. \]

Solutions

1. The equations are

\[ 7x + 5y = -4 \] \hspace{1cm} (1)
\[ 3x + 4y = 2 \] \hspace{1cm} (2)

From (1):

\[ 7x = -4 - 5y \]
\[ x = -\frac{4}{7} - \frac{5}{7}y \] \hspace{1cm} (3)

Substitute into (2):

\[ 3\left(-\frac{4}{7} - \frac{5}{7}y\right) + 4y = 2 \]

\[ -\frac{12}{7} - \frac{15}{7}y + 4y = 2 \]

\[ -\frac{15}{7}y + \frac{28}{7}y = 2 + \frac{12}{7} \]

\[ 13y = 26 \]
\[ y = 2. \]

Substitute for \( y \) into (3):

\[ x = -\frac{4}{7} - \frac{5}{7} \times 2 = -\frac{14}{7} = -2. \]

The solution is the point \((-2; 2)\).
2. The equations are
\[ 2x + 2y = 3 \quad (1) \]
\[ 5x + \frac{1}{2}y = -6 \quad (2) \]

From (1):
\[ 2x = 3 - 2y \]
\[ x = \frac{3}{2} - \frac{2y}{2} = \frac{3}{2} - y \quad (3) \]

Substitute into (2):
\[ 5 \left( \frac{3}{2} - y \right) + \frac{y}{2} = -6 \]
\[ \frac{15}{2} - 5y + \frac{y}{2} = -6 \]
\[ 10y + y = -15 \]
\[ 9y = -27 \]
\[ y = 3 \]

Substitute \( y = 3 \) into
\[ x = \frac{3}{2} - y \]
\[ x = \frac{3}{2} - 3 = -\frac{3}{2} \]

The solution is \((-\frac{3}{2}, 3)\).

3. The equations are
\[ x + 4y = 49 \quad (1) \]
and
\[ -2x + y = 1 \quad (2) \]

From (1):
\[ x = 49 - 4y. \]

Substitute into (2):
\[ -2(49 - 4y) + y = 1 \]
\[ -98 + 8y + y = 1 \]
\[ 9y = 99 \]
\[ y = 11. \]

Substitute \( y = 11 \) into \( x = 49 - 4y = 49 - 4 \times 11 = 49 - 44 = 5 \).

The solution is \((5, 11)\).
Study unit C.3    Linear inequalities in one variable

Learning objectives: On completion of this study unit you should be able to solve a linear inequality in one variable.

You are already familiar with the concepts "greater than", which is denoted by the symbol \( > \), and "less than" which is denoted by the symbol \(<\).

We say that \( b \) is greater than \( a \) if and only if the difference \( b - a \) is positive.

Thus \( b > a \) means \( b - a > 0 \).

We say that \( b \) is less than \( a \) if and only if the difference \( b - a \) is negative.

Thus \( b < a \) means \( b - a < 0 \).

If \( b = a \) or \( a = b \) then \( a \) and \( b \) are the same point.

Sometimes the symbol = is combined with \( > \) or \( < \) as follows:

\[ b \geq a \text{ means } b - a \geq 0; \text{ and } b \leq a \text{ means } b - a \leq 0. \]

If we replace the = sign in any equation, or system of equations, with one of \( >, <, \geq \) or \( \leq \), then we obtain an inequality or system of inequalities. The inequality, or system of inequalities, is linear or non-linear as the functions involved are linear or non-linear.

To solve an inequality means to find all values of the variable for which the inequality is true.

For example:

solve the inequality \( 5x - 15 < 0 \) means determine the values of \( x \) for which this statement is true.

How is this done?

First we note that \( b > a \) implies \( a < b \)

and vice versa. That is, if \( b \) is greater than \( a \), then \( a \) is less than \( b \). (For example \( 4 > 1 \) implies \( 1 < 4 \)). Although you might say that this is obvious, it is nevertheless very useful at times when we want to interchange the left and right hand sides.

Secondly, if \( b > a \), then \( b + c > a + c \) for any number \( c \).

That is, we may add the same number to both sides of an inequality.
For example, since $4 > 1$ then $4 + 3 > 1 + 3$ that is $7 > 4$
and $4 - 5 > 1 - 5$ that is $-1 > -4$.

**Thirdly**, if $b > a$ and $c > 0$

Then $b \times c > a \times c$ and $b/c > a/c$.

That is, we may multiply or divide both sides of an inequality by the same positive number.

For example, since $4 > 1$ thus $4 \times 8 > 1 \times 8$, that is $32 > 8$
and $4 ÷ 20 > 1 ÷ 20$, that is $0,2 > 0,05$.

**Fourthly**, if $b > a$ but $c < 0$

then $b \times c < a \times c$
and $b/c < a/c$.

That is, we may multiply or divide both sides of an inequality by the same negative number, but **then we** must reverse the sense of the inequality that is, change $>$ to $<$, or change $<$ to $>$.

For example since $4 > 1$ then $4 \times -5 < 1 \times -5$, that is $-20 < -5$
and $4 ÷ -8 < 1 ÷ -8$, that is $-0,5 < -0,125$.

These are the four rules that we need to solve linear inequalities. Summarised they are,

if $b > a$ then:

1. $a < b$;
2. $b + c > a + c$ for any $c$;
3. $b \times c > a \times c$ and $b/c > a/c$ for any $c > 0$;
4. $b \times c < a \times c$ and $b/c < a/c$ for any $c < 0$. 

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Study unit C.4  Systems of linear inequalities in two variables

Learning objectives: On completion of this study unit you should be able to solve a system of linear inequalities in two variables.

C.4.1  Linear inequalities in two variables

Although the rules in study unit C.3 may be applied to linear inequalities in two, or more variables, they are not of much use when it comes to solving a system of linear inequalities in two or more variables. The trouble is that the solution is not generally a single point but usually an infinite sequence of points. In fact, in the two variable cases it is usually even more than that - it is a whole area, or region as it is known, in the $x$-$y$ plane. This means that the most successful approach to solving linear inequalities in two variables is by means of graphs.

Suppose we have to graph the set of points which obey the inequality

$$-x + 2y - 2 \geq 0.$$

Now if we ignore the $>$ sign for a moment and just consider the $=$ sign we can easily draw a graph of

$$-x + 2y - 2 = 0.$$  

This is the straight line $y = \frac{1}{2} x + 1$ depicted below.

![Figure C.3](image)
Now any point on the line satisfies the original inequality (or rather the = part of it). Thus, all points on the line are in the solution space. What about points not on the line? Well, the simple approach is to simply test a few and see whether they satisfy the inequality or not.

Consider \((1; 1)\). It does not satisfy the inequality since \(-1 + 2 \times 1 - 2 = -1 > 0\) (read this as "is not greater than 0"). Also note that it lies below the straight line in the graph.

So, too, do the points \((4; 1)\) and \((3 ; 2)\). Neither do they satisfy the inequality. On the other hand the points \((3; 3)\), \((-1 ; 2)\) and \((-5 ; 0)\) all satisfy the inequality and all lie above the straight line, for example for \((3 ; 3)\) we find \(-3 + 2 \times 3 - 2 = 1 > 0\).

If we carry on in this fashion we will find that all points above the line satisfy the inequality whereas all points below the line do not. Thus the line divides the \(x-y\) plane into two regions - those that satisfy the inequality and those that do not. In this case the points on the line also satisfy the inequality (since the = sign is included in the statement). We can use this result as the basis of a prescription for graphing inequalities.

**Rules for Graphing Linear Inequalities:**

1. Graph the line which results when the inequality is changed to an equality;
2. Select any point not on the line;
3. If the coordinates of the point satisfy the inequality then all points on the same side of the line satisfy the inequality;
4. If the coordinates of the point do not satisfy the inequality then all points on the opposite side of the line satisfy the inequality;
6. If the = sign is part of the inequality then the points on the line also satisfy the inequality, otherwise they do not.

This means that all you have to do in order to solve a single inequality in two variables is to graph the corresponding straight line and to examine a single point in the plane.

**Exercises**

1. Graph the linear inequality \(3x + y - 3 > 0\).
2. Graph the linear inequality \(2x + 4y + 1 \leq x + y - 2\).
Solutions

1. The inequality is $3x + y - 3 > 0$.

The corresponding straight line is $3x + y - 3 = 0$ or $y = -3x + 3$.

This is depicted below.

Now consider the point $(2;2)$, which is to the right of and above the line. Substituted in the left hand side of the inequality it gives:

$$3 \times 2 + 2 - 3 = 5 > 0.$$

Thus all the points to the right and above the line satisfy the inequality. Since the inequality contains no $=$ sign, the points on the line do not satisfy it. We indicate this by drawing a dashed line.

2. The inequality is $2x + 4y + 1 \leq x + y - 2$.

Add $-x - y + 2$ to both sides: $2x - x + 4y - y + 1 + 2 \leq 0$

$$x + 3y + 3 \leq 0.$$

The corresponding straight line is $x + 3y + 3 = 0$ or $y = \frac{-1}{3}x - 1$. 
This is shown in the graph.

Consider the point \((0 ; 0)\). Substituted in the inequality this gives:

\[
0 + 3 \times 0 + 3 \leq 0.
\]

That is, it does not satisfy the inequality. Thus all the points below and to the left of the line as well as those on it (why?) satisfy the inequality.

### C.4.2 Systems of linear inequalities in two variables

Just as systems of linear equations can be formulated, so can systems of linear inequalities. When solving a system we must determine all points that simultaneously satisfy all linear inequalities in the system. **A system of inequalities means that there is more than one line involved. The lines must be drawn on ONE set of axes and therefore there is ONLY ONE solution space.** Once again the solution is generally a region in the \(x-y\) plane. This is demonstrated by an example below.

Consider the system of inequalities:

\[
-x + y - 1 \leq 0 \quad \text{and} \quad 2x + y - 4 < 0.
\]

If we examine each inequality separately we can graph its solution along the lines discussed in the previous study unit. The solution of the first inequality is the region including all points on the line \(-x + y - 1 = 0\) and all those below and to the right of this line. The solution of the second inequality is all points below and to the left of the line \(2x + y - 4 = 0\), but not the points on the line.
The two solutions are depicted graphically in Figure C.4, the first with horizontal lines and the second with vertical lines.

![Graph of solutions](image)

Figure C.4

Now the region in the figure where the horizontal and vertical lines cross, the so-called cross-hatched or grey space, is the region in which both inequalities are satisfied. It is, in other words, the solution of the system of inequalities. Note that the first line is included in the solution while the second is not, as is indicated by dashing the second line. An important point is the corner of the region where the two lines intersect at (1; 2). It is known as an extreme point of the region of solution.

To determine the solution of a system of inequalities, solve each inequality separately and determine the region which is common to all solutions. This region is known as the solution space of the system of inequalities.

**Activity**

Solve the system of inequalities graphically:

\[
\begin{align*}
x + \ y + 2 &\geq 0 \\
-x + 2y + 2 &< 0.
\end{align*}
\]

**Answer**

The inequalities are \( x + y + 2 \geq 0 \) and \( -x + 2y + 2 < 0 \).

The solution of the first inequality is the region including all the points on, above and to the right of the line \( x + y + 2 = 0 \). This is the region indicated by the horizontal lines in the sketch below. The solution of the second inequality is the region including all points below and to the right of the line \( -x + 2y + 2 = 0 \), but not the points on the line. This region is indicated by the vertical lines in the sketch below. The solution of the system of two inequalities is the cross-hatched region.
So far we have only considered systems of two inequalities in two unknowns. However, it is quite feasible, and very often necessary in practice, to consider systems with a greater number of inequalities than the number of unknowns. This is illustrated below, where we consider a system of five inequalities in two unknowns. The procedure for solving the system is exactly as before, namely solve each inequality separately and then determine which region is common to all solutions.

**Activity**

Solve the following system of inequalities graphically:

\[-x + y - 3 \leq 0
\]
\[x + y - 5 \leq 0
\]
\[x - 3 \leq 0
\]
\[x \geq 0
\]
\[y \geq 0.
\]

**Answer**

We look at the last two of these inequalities first. The inequality \(x \geq 0\) simply means the region to the right of and including the \(y\)-axis whereas \(y \geq 0\) means the region above and including the \(x\)-axis. Taken together, these two inequalities imply the first quadrant of the \(x\)-\(y\) plane so we can restrict our considerations to this region.

Now the first inequality is satisfied by all points on, below and to the right of the line

\[-x + y - 3 = 0,
\]
that is the region shaded with horizontal lines. The second is satisfied by all points on, below and to the left of the line \( x + y - 5 = 0 \), that is the region shaded with vertical lines. Finally, the inequality \( x - 3 \leq 0 \) or \( x \leq 3 \) is satisfied by all points on and to the left of the vertical line \( x = 3 \). The solution is the diagonally cross-hatched area shown in Figure C.5 below.

\[ x = 3 \]
\[ -x + y - 3 = 0 \]
\[ x + y - 5 = 0 \]

Figure C.5

Exercise

1. Solve the following system of inequalities graphically:

\[
\begin{align*}
2x + y - 5 &\leq 0 \\
x - 2 &\leq 0 \\
y - 4 &\leq 0 \\
x, y &\geq 0
\end{align*}
\]

2. Solve \( 3x - 7 \leq 5x + 2 \) and indicate your solution on the real line.

3. Solve \( 5x + y + 1 < -x - y - 1 \) graphically.
Solutions

1. The system of inequalities is:

\[ 2x + y - 5 \leq 0 \]
\[ x - 2 \leq 0 \]
\[ y - 4 \leq 0 \]
\[ x \geq 0 \]
\[ y \geq 0. \]

The solution of the first inequality is the region on, below and to the left of the line \( 2x + y - 5 = 0 \).

The second implies all \( x \) values on or to the left of the line \( x = 2 \) while the third implies all \( y \) values on or below the line \( y = 4 \). The last two inequalities imply the first quadrant, including the axes. The solution is graphed below and indicated by the grey area.

2. The inequality is \( 3x - 7 \leq 5x + 2 \).

Add +7:
\[ 3x \leq 5x + 9. \]
Subtract 5x:
\[ -2x \leq 9. \]
Divide by -2:
\[ x \geq -9/2. \]

Thus the solution is all values of \( x \) greater or equal to \(-9/2\) (that is \(-4,5\)), as shown.
3. The inequality is $5x + y + 1 < -x - y - 1$.

Add $x + y + 1$: $6x + 2y + 2 < 0$.

The corresponding straight line is $6x + 2y + 2 = 0$ or $y = -3x - 1$.

Graphically it is represented in the following graph.

The point $(0; 0)$ does not satisfy the inequality since $6 \times 0 + 2 \times 0 + 2 = 2$

Thus all the points to the left and below the line satisfy the inequality, but the points on the line are excluded. (Why?)