Tutorial Letter 202/1/2013

QUANTITATIVE MODELLING

Semester 1

Department of Decision Sciences

This tutorial letter contains the solutions for assignment 02.
Dear student

I hope that by this stage you have worked through about two-thirds of the work prescribed for this module and have completed your second assignment. As the assignments contain questions from old examination papers you are already, in a way, preparing for the examination. Practice makes perfect! Continue doing as many examples as possible, as the more examples you work through the more you will be able to recognise a problem and know how to solve it.

Remember, help is just a phone call or e-mail away. You are welcome to contact me if you need any help with the second assignment. My contact details and contact hours are as follows:

08:00 to 13:30 (Mondays to Fridays) (appointments and telephone)
13:30 to 16:00 (Mondays to Thursdays) (telephone only)

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I wish you everything of the best with your preparation for the third assignment.

Victoria Mabe-Madisa
ASSIGNMENT 2: SOLUTIONS

Question 1

We need to solve the following system of equations:

\[ x - 2y + 3z = -11 \] \hspace{1em} (1)
\[ 2x - z = 8 \] \hspace{1em} (2)
\[ 3y + z = 10 \] \hspace{1em} (3)

**Step 1:** Determine two equations with the **same** two unknowns (variables) by adding or subtracting two of the three equations at a time.

Now equations (2) and (3) are already equations in two variables. However, the variables in equation (2) are \( x \) and \( z \), and in equation (3) they are \( y \) and \( z \) and the equations have to have the same variables. Thus, to determine another equation with two variables we can subtract equation (2) from 2 times equation (1):

Now 2 times equation (1) is
\[ 2x - 4y + 6z = -22. \]

Two times equation (1) minus equation (2):

\[
\begin{align*}
2x - 4y + 6z &= -22 \\
-(2x - z = 8) &\quad \text{or} \quad 2x - 4y + 6z = -22 \\
-4y + 7z &= -30
\end{align*}
\]

Thus, equation (3): \( 3y + z = 10 \) and equation (4): \(-4y + 7z = -30\) are two equations with the same two variables, namely, \( y \) and \( z \).

**Step 2:** Next we solve two equations with the same two unknowns, using any of the two methods. Say we use the substitution method:

Make \( z \) the subject of equation (3) and substitute into equation (4) and solve for \( y \):

Now
\[ z = 10 - 3y. \]

Substitute the value of \( z \), namely, \( z = 10 - 3y \), into equation (4) and solve for \( y \):
Step 3: Substitute \( y = 4 \) into equation (3) and solve for \( z \):
\[
3y + z = 10
\]
\[
3(4) + z = 10
\]
\[
12 + z = 10
\]
\[
z = 10 - 12
\]
\[
z = -2
\]

Step 4: Substitute \( y = 4 \) and \( z = -2 \) into equation (1) or (2) and solve for \( x \). Say we use equation (2):
\[
2x - z = 8
\]
\[
2x - (-2) = 8
\]
\[
2x + 2 = 8
\]
\[
2x = 8 - 2
\]
\[
2x = 6
\]
\[
x = 6 / 2
\]
\[
x = 3
\]

Therefore, \( x = 3 \), \( y = 4 \) and \( z = -2 \).

Or alternatively,
\[
x - 2y + 3z = -11 \quad (1)
\]
\[
2x - z = 8 \quad (2)
\]
\[
3y + z = 10 \quad (3)
\]
Make $z$ the subject of equation (2) and $y$ the subject of equation (3):

$$z = 2x - 8 \quad (4)$$
$$y = \frac{10 - z}{3} \quad (5)$$

Substitute equation (4) into equation (5):

$$y = \frac{10 - (2x - 8)}{3} = \frac{18 - 2x}{3} \quad (6)$$

Substitute equation (2) and equation (6) into equation (1):

$$x - \frac{2}{3}(18 - 2x) + 3(2x - 8) = -11$$

$$x - 12 + \frac{4}{3}x + 6x - 24 = -11$$

$$\frac{3 + 4 + 18}{3}x = -11 + 12 + 24$$

$$\frac{25}{3}x = 25$$

$$x = 3$$

Substitute $x = 3$ into equation (4) and equation (6):

$$z = 2 \times 3 - 8 = -2$$

and

$$y = \frac{18 - 2 \times 3}{3} = \frac{18 - 6}{3} = 4$$

Therefore, $x = 3$, $y = 4$ and $z = -2$. 
Question 2

We need to determine the difference between the quantity supplied and the quantity demanded if the price is equal to 38. First we determine what the quantity supplied is if the price is 38. Thus, we substitute the value \( P = 38 \) into the supply function and solve for the quantity supplied, \( S \). Therefore

\[
P = 14 + 1,5S \\
38 = 14 + 1,5S \\
-1,5S = 14 - 38 \\
-1,5S = -24 \\
S = \frac{-24}{-1,5} \\
S = 16
\]

Now we determine what the quantity demanded, \( D \), is if the price is 38. Thus, we substitute the value \( P = 38 \) into the demand function and solve for \( D \). Therefore

\[
P = 50 - 3D \\
38 = 50 - 3D \\
3D = 50 - 38 \\
3D = 12 \\
D = \frac{12}{3} \\
D = 4
\]

The supplier supplied 16 units and only 4 units were demanded. Thus, the difference between quantity supplied and quantity demanded is \( 16 - 4 = 12 \) units when the price is R38.

Question 3

When the market price per unit is \( P = 30 \), the consumer will purchase

\[
P = 48 - 0,2Q \\
30 = 48 - 0,2Q \\
30 - 48 = -0,2Q \\
-18 = -0,2Q
\]
\[ Q = \frac{-18}{-0.2} = 90 \text{ units} \]

- Draw a rough sketch of the graph.

Thus, the total amount that the consumer is willing to pay for the first 90 items = the area \( C \) (i.e. area \( A \) plus area \( B \)) under the demand function between \( P = 0 \) and \( P = 48 \).

Area \( C \) = area triangle \( A \) plus area square \( B \)

Area \( C \) = \( \left( \frac{1}{2} \times \text{base} \times \text{height} \right) + \left( \text{length} \times \text{width} \right) \).

\[
\text{Area } C = \left[ \frac{1}{2} \times 90 \times (48 - 30) \right] + 30 \times 90 \\
= \left[ \frac{1}{2} \times 90 \times 18 \right] + 2700 \\
= \frac{1620}{2} + 2700 \\
= 810 + 2700 \\
= 3510
\]

The total amount that the consumer is willing to pay for the first 90 units is, thus, R3510.

Now the amount which the consumer actually pays is the area of the square \( B \).
Area of the square (B) 
\[ = \text{(length} \times \text{breadth}) \]
\[ = 30 \times 90 \]
\[ = 2700 \]

The amount which the consumer actually pays is R 2 700.

Now the consumer surplus is defined as:
\[ \text{CS} = \text{amount willing to pay} \text{ – amount actual spent} \]
\[ = 3510 \text{ – 2700} \]
\[ = 810. \]

**Question 4**

The roots or solutions of a function can be found where the function, if drawn, intersects the x-axis. We therefore need to determine the value of x at the point(s) where the graph of the function intersects the x-axis, in other words where the function value is zero: \( y = 0 \)

Now, the function \( y + 6 = x^2 + x \) is given. Writing it in terms of \( y \) we get \( y = x^2 + x - 6 = 0. \)

We make use of the quadratic formula
\[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]
with \( a, b, \) and \( c \) being the values of the coefficients in the equation \( 0 = ax^2 + bx + c \) to determine the roots of a quadratic function.

Comparing the given equation \( 0 = x^2 + x - 6 \) with the general form \( 0 = ax^2 + bx + c \), we conclude that \( a = 1, \, b = 1 \) and \( c = -6. \) Substituting \( a, b \) and \( c \) into the quadratic formula gives

\[ x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-6)}}{2(1)} \]
\[ x = \frac{-1 \pm \sqrt{1 + 24}}{2} \]
\[ x = \frac{-1 \pm \sqrt{25}}{2} \]
\[ x = \frac{-1 \pm 5}{2} \]
\[ x = \frac{-1 + 5}{2} \quad \text{or} \quad \frac{-1 - 5}{2} \]
\[ x = \frac{4}{2} \quad \text{or} \quad \frac{-6}{2} \]
\[ x = 2 \quad \text{or} \quad -3. \]

The roots of the function \( y + 6 = x^2 + x \) or \( y = x^2 + x - 6 \) are 2 and -3.
Question 5

If two lines are parallel the slopes of the two lines are equal. This means that the value of the slope, namely \( m \), is the same for both lines.

Now, given the equation of the line as \( y = -3x \). Comparing the given equation with the general format of a straight line, namely \( y = mx + c \), we can conclude that the slope of the given line is \( m = -3 \). Now, as stated, the new line must, because it is parallel to the line \( y = -3x \), also have a slope of \( m = -3 \). Thus, the new line has an equation of \( y = -3x + c \).

Next we need to determine the value of \( c \). It was given that the line passes through the point \((1;2)\). That means that the point \((1;2)\) is a realisation of the line, meaning that the \( y \)-value is equal to 2 if the \( x \)-value is equal to 1. Therefore

\[
y = -3x + c \\
2 = -3(1) + c \\
2 = -3 + c
\]

Solve \( c \) by adding 3 to both sides of the equation:

\[
2 + 3 = -3 + 3 + c \\
5 = c
\]

Thus, the equation of the line is \( y = -3x + 5 \) or \( y = 5 - 3x \).

The equation of the line that is parallel to the line \( y = -3x \) and that passes through the point \((1;2)\) is equal to \( y = 5 - 3x \).

Question 6

To determine the point of intersection of two lines we need to determine a point \((x ; y)\) so that the \( x \) and \( y \) values satisfy the equations of both lines. Thus, we need to solve the two equations with two unknowns or variables simultaneously. To do so, we can make use of the substitution method or the elimination method. I have chosen the substitution method as one of the variables is already the subject of one of the equations, namely, \( y \) in equation (1).

Substituting the value of \( y \) (equation (1)) into equation (2) and solving for \( x \), or substituting \( y = \frac{3}{4}x + 6 \) into \( 3x - 2y + 3 = 0 \):
Substitute the calculated value of the variable $x = 6$ into equation (1) or (2) and calculate the value of $y$. Substitute $x = 6$ into equation (1) or equation (2). Let’s say we choose equation (2):

\[
3x - 2y + 3 = 0
\]
\[
3(6) - 2y + 3 = 0
\]
\[
18 - 2y + 3 = 0
\]
\[
21 - 2y = 0
\]
\[
-2y = -21
\]
\[
y = \frac{-21}{-2} = \frac{21}{2}
\]

The two lines intersect at the point $(x ; y) = (6 ; \frac{21}{2})$.

**Question 7**

**Step 1:**

To graph a linear inequality we first change the inequality sign (≥ or ≤ or > or <) to an equal sign (=) and draw the graph of the line. But, we need two points to draw a line. Choose the two points where the lines cut the x-axis (x-axis intercept, thus $y = 0$) and y-axis (y-axis intercept, $x = 0$). Calculate $(0 ; y)$ and $(x ; 0)$ and draw a line through the two points. See the table below for the calculations.
Step 2: Determine the feasible region for each inequality by substituting a point on either side of this line into the equation of the inequality. The inequality region is the area where the selected point makes the inequality true. The calculations are given below:

Step 3: Determine the region where the inequality is true and shade the area.

\[ 25x + 40y \geq 2000 \]
\[ 10x + 4y \geq 400 \]
\[ y \geq 40 \]
\[ x, y \geq 0 \]

<table>
<thead>
<tr>
<th>Inequality</th>
<th>y-axis intercept</th>
<th>x-axis intercept</th>
<th>Inequality region</th>
</tr>
</thead>
<tbody>
<tr>
<td>25x + 40y \geq 2000 or 5x + 8y \geq 400</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y = \frac{400}{8} = 50 )</td>
<td>Point (0 ; 50)</td>
<td>( y = \frac{400}{5} = 80 )</td>
<td>Point: (80; 0)</td>
</tr>
<tr>
<td>10x + 4y \geq 400</td>
<td>10x + 4y \geq 400</td>
<td>10x + 4y \geq 400</td>
<td></td>
</tr>
<tr>
<td>( y = \frac{400}{4} = 100 )</td>
<td>Point (0 ; 100)</td>
<td>( x = \frac{400}{10} = 40 )</td>
<td>Point (40 ; 0)</td>
</tr>
<tr>
<td>y \geq 40</td>
<td>y \geq 40</td>
<td>No x-axis intercept</td>
<td></td>
</tr>
<tr>
<td>Point (0 ; 40)</td>
<td>0 \geq 40 – False</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( x, y \geq 0 )</td>
<td>Area above the x-axis and to the right of the y-axis</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Question 8

We need to determine the maximum value of the function $P = 20x + 30y$ subject to the given constraints. To do this we determine all the corner points of the feasible region of the constraints and substitute them into the objective function (function you want to maximise or minimise) to determine the maximum or minimum value of the function. The method can be summarised as follows:

**Step 1:** Determine the coordinates of all corners of the feasible region by
- determining the point where the two lines intersect – solving two equations with two unknowns or
- read the coordinates of the intersection point from the graph.

**Step 2:** Substitute the corner points into the objective function and calculate the value of the objective function.

**Step 3:** Choose the corner point that results in the highest (maximisation) or the lowest (minimisation) objective function value.

**Step 1:**
The corner points of the feasible region in the graph below are the points A, B, C, D and the origin $(0; 0)$. 

![Graph with constraints and feasible region]

\[10x + 4y = 400\]
\[25x + 40y = 2000\]
\[y = 40\]
To determine the coordinates of the corners of the feasible region we can

- read the coordinates of the intersection points from the graph or
- determine the point where the lines intersect by solving two equations with two unknowns using the substitution or elimination methods.

**Point A:**

Point A is the point where line (2) cuts the $y$-axis. The coordinates of this point can be read from the graph as $(0 ; 70)$.

**Point B:**

Point B is the point where line (2) and (3) intersect. The coordinates of this point can be read from the graph as $(20 ; 60)$ or you can calculate them by solving two equations with two unknowns by, for example, using the substitution method:

We need to solve the following two equations simultaneously:

$$x + 2y = 140 \quad (2) \quad \text{and} \quad x + y = 80 \quad (3)$$

First we make $x$ the subject of equation (2) by subtracting $2y$ from both sides of equation (2):

$$x + 2y - 2y = 140 - 2y$$

$$x = 140 - 2y$$

Substitute the value of $x$, namely $x = 140 - 2y$, into equation (3) and solve for $y$: 
\[
\begin{align*}
\text{(3)} & \quad x + y = 80 \\
(140 - 2y) + y &= 80 \\
-\ y &= 80 - 140 \\
-\ y &= -60 \\
y &= 60
\end{align*}
\]

Substitute the value of \( y = 60 \) into equation (3) and solve for \( x \): \[
\begin{align*}
\text{(3)} & \quad x + y = 80 \\
x + 60 &= 80 \\
x &= 80 - 60 \\
x &= 20
\end{align*}
\]

The coordinates of Point B are \((20 ; 60)\).

**Point C:**

Point C is the point where line (1) and (3) intersect.

The coordinates of the point can be read from the graph as \((40 ; 40)\) or you can calculate them by solving two equations with two unknowns by using, for example, the substitution method:

We need to solve the following two equations simultaneously:

\[
\begin{align*}
2x + y &= 120 \quad (1) \\
x + y &= 80 \quad (3)
\end{align*}
\]

First we make \( y \) the subject of equation (1) by subtracting \( 2x \) from both sides of equation (1):

\[
\begin{align*}
2x + y - 2x &= 120 - 2x \\
y &= 120 - 2x \quad (4)
\end{align*}
\]

Substitute the value of \( y \) in equation (4) into equation (3) and solve for \( x \):

\[
\begin{align*}
\text{(3)} & \quad x + y = 80 \\
x + (120 - 2x) &= 80 \\
x + 120 - 2x &= 80 \\
-x &= 80 - 120 \\
-x &= -40 \\
x &= 40
\end{align*}
\]

Substitute the value of \( x = 40 \) into equation (3) and solve for \( y \):

\[
\begin{align*}
\text{(3)} & \quad x + y = 80 \\
40 + y &= 80 \\
y &= 80 - 40 \\
y &= 40
\end{align*}
\]

The coordinates of Point C are \((40 ; 40)\).
**Point D:**

Point D is the point where line (1) cuts the $x$-axis. The coordinates of this point can be read from the graph as $(60 ; 0)$.

**Step 2:**

Substitute the corner points of the feasible region into the objective function and determine the value of the objective function for each corner point:

<table>
<thead>
<tr>
<th>Corner points of feasible region</th>
<th>Value of $P = 20x + 30y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A: $x = 0; y = 70$</td>
<td>$P = 20(0) + 30(70) = 2100$</td>
</tr>
<tr>
<td>B: $x = 20; y = 60$</td>
<td>$P = 20(20) + 30(60) = 2200 \rightarrow \text{Maximum}$</td>
</tr>
<tr>
<td>C: $x = 40; y = 40$</td>
<td>$P = 20(40) + 30(40) = 2000$</td>
</tr>
<tr>
<td>D: $x = 60; y = 0$</td>
<td>$P = 20(60) + 30(0) = 1200$</td>
</tr>
<tr>
<td>Origin: $x = 0; y = 0$</td>
<td>$P = 20(0) + 30(0) = 0$</td>
</tr>
</tbody>
</table>

**Step 3:**

Choose the corner point which results in the highest (maximisation) objective function value.

The maximum of $P$ is at point B, where $x = 20, y = 60$ and $P = 2200$.

**Question 9**

First we define the decision variables. Let $x$ be the number of Product A manufactured and $y$ the number of Product B manufactured, respectively.

To help us with the formulation, we summarise the information given in a table with the headings: resources (items with restrictions and that are used in the manufacturing process), the variables ($x$ and $y$) and capacity (amount or number of the resources available).

<table>
<thead>
<tr>
<th>Resources</th>
<th>$x$ Number of Product A</th>
<th>$y$ Number of Product B</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Processing</td>
<td>30 minutes 0,5 hours</td>
<td>12 minutes 0,2 hours</td>
<td>4 hours 240 minutes</td>
</tr>
<tr>
<td>Assembly</td>
<td>18 minutes 0,3 hours</td>
<td>72 minutes 1,2 hours</td>
<td>6 hours 360 minutes</td>
</tr>
<tr>
<td>Packaging</td>
<td>24 minutes 0,4 hours</td>
<td>48 minutes 0,8 hours</td>
<td>4,8 hours 288 minutes</td>
</tr>
</tbody>
</table>

Number of Product A ($x$) and Product B ($y$) Never negative
Now the units of the resources and capacity must be the same. Using the table, the following constraints can be defined if we use minutes as the units:

\[
\begin{align*}
30x + 12y &\leq 240 \\
18x + 72y &\leq 360 \\
24x + 48y &\leq 288 \\
x, y &\geq 0
\end{align*}
\]

Using the table, the following constraints can be defined if we use hours as the units:

\[
\begin{align*}
0,5x + 0,2y &\leq 4 \\
0,3x + 1,2y &\leq 6 \\
0,4x + 0,8y &\leq 4,8 \\
x, y &\geq 0
\end{align*}
\]

**Question 10**

Firstly, we define the variables.

Let \(x\) and \(y\) be the number of dining room and lounge chairs manufactured respectively.

Secondly, to help us with the formulation, we summarise the information in a table with the headings:

- resources (items with restrictions),
- the variables (\(x\) and \(y\)) and
- capacity (amount or number of the resources available).

<table>
<thead>
<tr>
<th>Resource</th>
<th>dining room chairs</th>
<th>lounge chairs</th>
<th>Capacity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sanding</td>
<td>2 hours</td>
<td>2 hours</td>
<td>78 hours</td>
</tr>
<tr>
<td>Staining</td>
<td>4 hours</td>
<td>3 hours</td>
<td>96 hours</td>
</tr>
<tr>
<td>Number of products</td>
<td></td>
<td></td>
<td>Never negative</td>
</tr>
</tbody>
</table>

A system of linear inequalities that describes these constraints is:

\[
\begin{align*}
2x + 2y &\leq 78 \\
4x + 3y &\leq 96 \\
x, y &\geq 0
\end{align*}
\]