Tutorial Letter 202/1/2014

QUANTITATIVE MODELLING

Semester 1

Department of Decision Sciences

This tutorial letter contains the solutions
for Assignment 02.

Bar code

Dear Student

By now you should have worked through chapters 1 to 3 of the textbook and completed your first and second assignments. As the assignments contain questions from old examination papers, you are in fact already preparing for the examination. Do as many examples as possible – the more examples you work through, the better you will be able to recognise a problem and solve it.

Remember, help is just a phone call or an e-mail away. Please contact me if you need any help with the third assignment. I am available during the office hours indicated below and my contact details are as follows:

08:00 to 13:30 (Mondays to Fridays) (appointments and per telephone)

13:30 to 16:00 (Mondays to Thursdays) (only per telephone)

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I wish you everything of the best with your preparation for the third assignment.

Kind regards

Ms Victoria Mabe-Madisa

ASSIGNMENT 2: SOLUTIONS

Question 1

To determine the point of intersection of two lines, we need to determine a point $(x; y)$ so that the x and y values satisfy the equations of both lines. Thus we need to solve the two equations simultaneously. Different methods may be used to solve the set of equations.

(a) **Elimination method**:

Step 1: Eliminate one variable, say x, by adding or subtracting one equation or multiple of an equation from another equation:

Let: $x + 2y = 5$ (1)

and $2x - 3y = -4$ (2)

Now equation (2) minus 2 times equation (1) will eliminate x. But 2 × equation (1) is:

$2x + 4y = 10$ (3)

Now equation (2) minus 2 × equation (1) or equation (2) – equation (3) is:

$$\begin{array}{r} 2x - 3y = -4 \\ -(2x + 4y = 10) \\ \hline -7y = -14 \end{array} \quad \text{or} \quad \begin{array}{r} 2x - 3y = -4 \\ -2x - 4y = -10 \\ \hline -7y = -14 \end{array}$$

Now solve for y:

$$y = \frac{-14}{-7}$$
$$y = 2$$

Step 2: Solve for x. Substitute the value of y into **any one** of the equations and solve for x. Substitute the value of $y = 2$ into, say, equation (1):

$$\begin{aligned} x + 2(2) &= 5 \\ x + 4 &= 5 \\ x &= 5 - 4 \\ x &= 1 \end{aligned}$$

The two lines intersect at the point $(x; y) = (1; 2)$.

(b) **Substitution method:**

Step 1: Change one of the equations so that a variable is the subject of the equation, say x in equation (1):

Let: $x + 2y = 5$ (1)

then $x = -2y + 5$ (3)

Step 2: Substitute the value of x (equation (3)) into the unchanged equation (2) and solve for x. Substitute $x = -2y + 5$ into $2x - 3y = -4$:

$$\begin{aligned} 2x - 3y &= -4 \\ 2(-2y + 5) - 3y &= -4 \\ -4y + 10 - 3y &= -4 \\ -7y &= -4 - 10 \\ -7y &= -14 \\ y &= \frac{-14}{-7} \\ y &= 2 \end{aligned}$$

Step 3: Substitute the calculated value of the variable in step 2 into any equation and calculate the value of the other variable. Substitute $y = 2$ into equation (1) or equation (2). Let us choose equation (1):

$$\begin{aligned} x + 2(2) &= 5 \\ x + 4 &= 5 \\ x &= 5 - 4 \\ x &= 1 \end{aligned}$$

The two lines intersect in the point $(x; y) = (1; 2)$.

Question 2

We need to solve the following system of equations:

$$\begin{aligned} x + y + z &= 8 \qquad (1) \\ x - 3y &= 0 \qquad (2) \\ 5y - z &= 10 \qquad (3) \end{aligned}$$

Make x the subject of equation (2) and z the subject of equation (3):

$$x = 3y \qquad (4)$$
$$z = -10 + 5y \qquad (5)$$

Substitute equation (4) and equation (5) into equation (1):

$$x + y + z = 8$$
$$(3y) + y + (-10 + 5y) = 8$$
$$9y = 8 + 10$$
$$9y = 18$$
$$y = \frac{18}{9}$$
$$y = 2$$

Substitute $y = 2$ into equation (4) and equation (5):

$$x = 3y = 3 \times 2 = 6$$

and

$$z = –10 + 5y = –10 + 5(2) = –10 + 10 = 0.$$

Therefore $x = 6$; $y = 2$ and $z = 0$.

Question 3

We need to represent $y \geq 3 + 3x$ graphically. To draw a linear inequality we first change the inequality sign (≥ or ≤ or > or <) to an equal sign (=) and draw the graph of the line, but we need two points to draw a line. Choose the two points where the line cuts through the x-axis (x-axis intercept, thus $y = 0$) and y-axis (y-axis intercept, $x = 0$). Calculate (0; y) and (x; 0) and draw a line through the two points.

Step 1: Change the ≥ sign to an equal sign (=). Choose the values of x and y randomly or use the points (0; y) and (x; 0) as below, sketch the two points and draw the line of the graph through the two points:

Let $x = 0$, then y is equal to:

$$y \geq 3 + 3x$$
$$y = 3 - 3(0)$$
$$y = 3$$

Let $y = 0$, then x is equal to:

$$\begin{aligned} y &\geq 3+3x \\ 0 &= 3+3x \\ -3 &= 3x \\ \frac{-3}{3} &= x \\ x &= -1 \end{aligned}$$

The two points calculated are (0; 3) and (−1; 0).

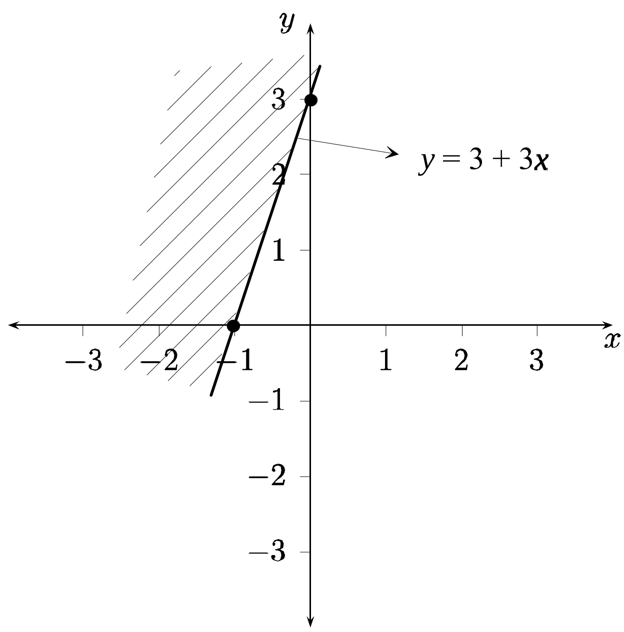
Step 2: Determine the feasible region for the inequality by substituting a point on either side of the drawn line into the equation of the inequality. The inequality region is the area where the selected point makes the inequality true. Shade the feasible area:

Select point (0; 0) to the right of the line.

Now the point (0; 0) on the right of the line makes the inequality

$$\begin{aligned} y &\geq 3+3x \\ 0 &\geq 3+3(0) \\ 0 &\geq 3 \end{aligned}$$

false. So the feasible region is to the left of the line.



Question 4

Demand function: $\text{Q} = 50 - 0,5P$
Supply function: $\text{Q} = 10 + 0,5P$

Equilibrium is the price and quantity where the demand and supply functions are equal, or the point where the lines of the demand and supply function intersect.

Therefore we need to determine the value of *P* and *Q* for which $P_d = P_s$ or $Q_d = Q_s$. Now given are $\text{Q} = 50 - 0,5P$ and $\text{Q} = 10 + 0,5P$. Thus:

$$\begin{aligned} Q_d &= Q_s \\ 50 - 0,5P &= 10 + 0,5P \\ -0,5P - 0,5P &= 10 - 50 \\ -1P &= -40 \\ \text{P} &= \frac{-40}{-1} \\ P &= 40 \end{aligned}$$

To calculate the quantity at equilibrium, we substitute the value of *P* into the demand or supply function and calculate *Q*. Say we use the demand function, then:

$$\begin{aligned} Q &= 50 - 0,5P \\ Q &= 50 - 0,5(40) \\ Q &= 50 - 20 \\ Q &= 30 \end{aligned}$$

The equilibrium price is equal to 40 and the quantity is 30.

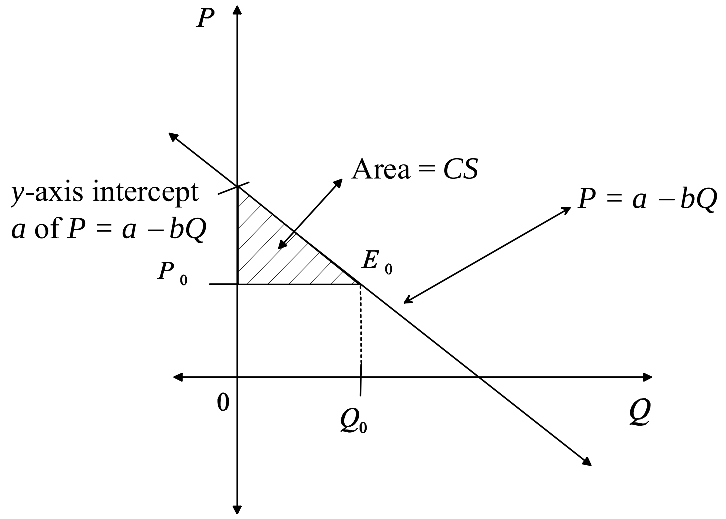
Question 5

If you need to determine the demand surplus for a demand function of $P = a - bQ$, then the consumer surplus can be calculated by calculating an area of the triangle $P_0 E_0 a$, which is equal to

$$\tfrac{1}{2} \times \text{height} \times \text{base} = \tfrac{1}{2} \times (a - P_0) \times (Q_0 - 0) = \tfrac{1}{2} \times (a - P_0) \times (Q_0)$$

with

- P_0 the value given as the market price,
- Q_0 the value of the demand function if *P* equals the given market price (substitute P_0 into the demand function and calculate Q_0), and
- *a* the *y*-intercept of the demand function $P = a - bQ$, also known as the value of *P* if $Q = 0$ or the point where the demand function intercepts the *y*-axis.



The steps to determine the consumer surplus can be summarised as follows:

Method:

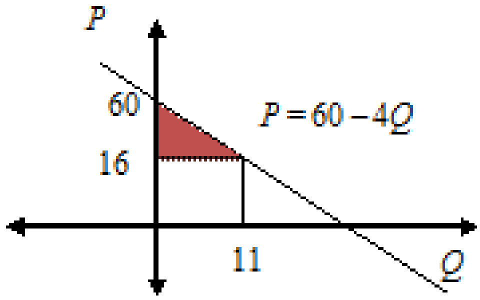
1. Calculate Q_0 if P_0 is given.
2. Draw a rough graph of the demand function.
3. Read the value of a from the demand function – the y-intercept of the demand function.
4. Calculate the area of CS = ½ × $(a - P_0) \times (Q_0)$.

First we calculate Q if $P = 16$. Therefore:

$$\begin{aligned} P &= 60 - 4Q \\ 16 &= 60 - 4Q \\ 4Q &= 60 - 16 \\ 4Q &= 44 \\ Q &= 11 \end{aligned}$$

The consumer surplus is the area of the shaded triangle on the right:

$$\begin{aligned} CS &= \frac{1}{2} \times \text{base} \times \text{height} \\ &= \frac{1}{2} \times 11 \times (60 - 16) \\ &= \frac{1}{2} \times 11 \times 44 \\ &= \frac{482}{2} \\ &= 242 \end{aligned}$$



Question 6

Profit is equal to revenue (sales) minus total cost. The cost to produce x number of sports hats is given as: $c = 200 + 25x$, and the profit as R3 000. Thus:

$$\text{Profit} = \text{Sales} - \text{cost, or } 3\,000 = 45x - (200 + 25x)$$

Now we have to solve x:

$$\begin{aligned}
3\,000 &= 45x - (200 + 25x)\\
3\,000 &= 45x - 200 - 25x\\
3\,000 &= 20x - 200\\
3\,000 + 200 &= 20x\\
3\,200 &= 20x\\
\frac{3\,200}{20} &= x\\
160 &= x
\end{aligned}$$

160 hats were sold to make a profit of R3 000.

Question 7

First we define the variables. Let x be the number of radios manufactured and y the number of television sets manufactured. To help us with the formulation we summarise the information given in a table with the headings: resources (items with restrictions), the variables (x and y) and capacity (amount or number of the resources available).

Resources	**Radio**	**Television**	**Capacity**
Manufacturing time (min)	90	150	95×60 = 5 700 minutes
Testing time (min)	5	15	9×60 = 540 minutes
Production cost	175	850	13 500

Using the table, we can formulate our linear program as follows:

Let: x = number of radios

y = number of television sets

Then:

$$\begin{array}{ll} 90x + 150y \leq 5\ 700 & \text{manufacturing time} \\ 5x + 15y \geq 540 & \text{testing time} \\ 175x + 850y \leq 13\ 500 & \text{production cost} \\ x, y \geq 0 & \text{non-negative} \end{array}$$

Question 8

First we define the variables.

Let x and y be the number of products A and B manufactured respectively.

Secondly, to help us with the formulation, we summarise the information in a table with the headings:

- Resources (items with restrictions)
- The variables (x and y)
- Capacity (amount or number of the resources available)

	x	y	
Resource	**A**	**B**	**Capacity**
Processing	30 minutes	12 minutes	4 hours
Assembly	18 minutes	72 minutes	6 hours
Packaging	24 minutes	48 minutes	4,8 hours
Number of products			Never negative

The system of inequalities that best describes this situation in minutes:

$$\begin{aligned} 30x + 12y &\leq 240 \\ 18x + 72y &\leq 360 \\ 24x + 48y &\leq 288 \\ x,\ y &\geq 0 \end{aligned}$$

OR

The system of inequalities that best describes this situation in hours:

$$\begin{aligned} \frac{1}{2}x + \frac{1}{5}y &\leq 4 \\ \frac{3}{10}x + \frac{6}{5}y &\leq 6 \\ \frac{2}{5}x + \frac{4}{5}y &\leq 4{,}8 \\ x, y &\geq 0 \end{aligned}$$

Question 9

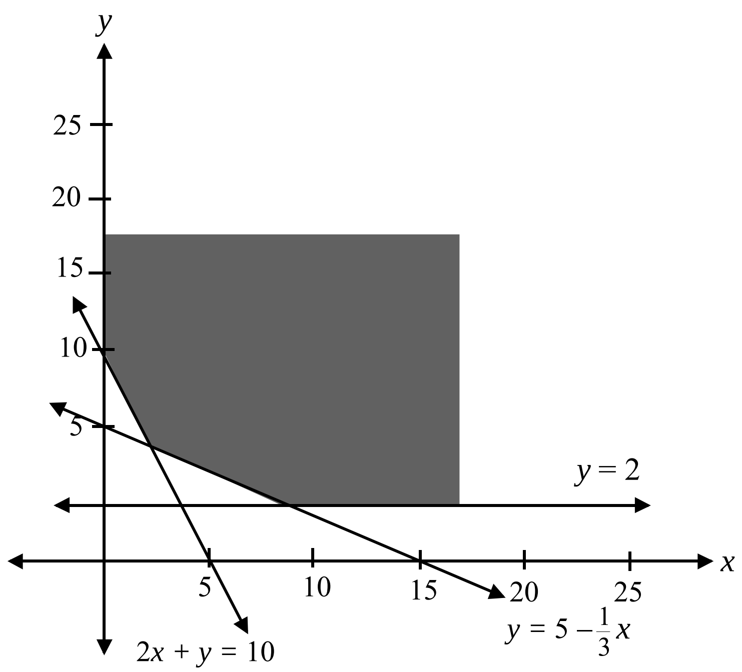
Step 1: To graph a linear inequality we first change the inequality sign (≥ or ≤ or > or <) to an equal sign (=) and draw the graph of the line, but we need two points to draw a line. Choose the two points where the lines cut the *x*-axis (*x*-axis intercept, thus $y = 0$) and *y*-axis (*y*-axis intercept, thus $x = 0$). Calculate (0; *y*) and (*x*; 0), and draw a line through the two points. See the table below for the calculations.

Step 2: Determine the feasible region for each inequality by substituting a point on either side of this line into the equation of the inequality. The inequality region is the area where the selected point makes the inequality true. The calculations are given below:

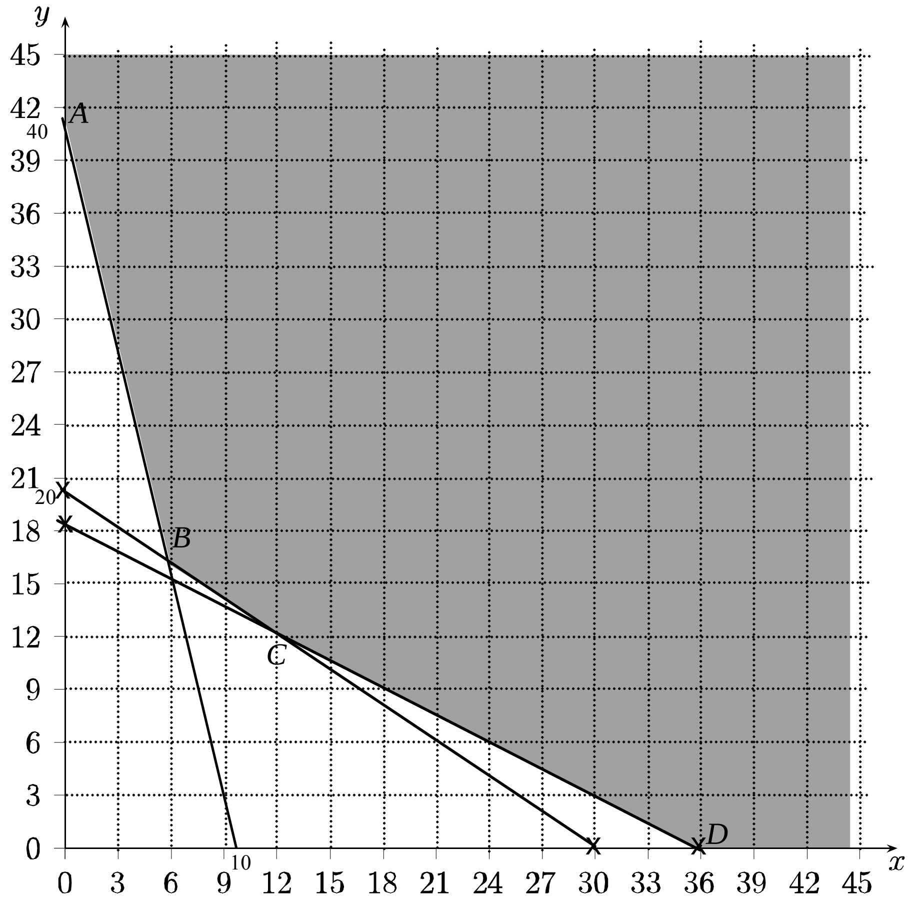
	Step 1		**Step 2**
Inequality	***y*-axis intercept** **point (*x*; *y*) if** **$x = 0$**	***x*-axis intercept** **point (*x*; *y*) if** **$y = 0$**	**Inequality region**
$x+3y \geq 15$ $y \geq 5 - \frac{1}{3}x$ (1)	$y \geq 5 - \frac{1}{3}x$ $y = 5 - \frac{1}{3}(0)$ $y = 5$ Point: (0; 5)	$y \geq 5 - \frac{1}{3}x$ $0 = 5 - \frac{1}{3}x$ $\frac{1}{3}x = 5$ $x = 5 \times 3$ $x = 15$ Point: (15; 0)	Select the point (0; 0) below the line $y \geq 5 - \frac{1}{3}x$. Thus $0 \geq 5 - \frac{1}{3}(0)$ $0 \geq 5$ is false. As the point (0; 0) lies below the line $y \geq 5 - \frac{1}{3}x$, the area below the line $y \geq 5 - \frac{1}{3}x$ is false. Thus the area that makes the inequality true lies above the

			line $y \geq 5 - \frac{1}{3}x$.
$2x + y \geq 10$ $y \geq 10 - 2x$ (2)	$y \geq 10 - 2x$ $y = 10 - 2(0)$ $y = 10$ Point: (0; 10)	$y \geq 10 - 2x$ $0 = 10 - 2x$ $-10 = -2x$ $x = \frac{-10}{-2}$ $x = 5$ Point: (5; 0)	Select the point (0; 0) below the line $y \geq 10 - 2x$. Thus $0 \geq 10 - 2(0)$ $0 \geq 10$ is false. As the point (0; 0) lies below the line $y \geq 10 - 2x$ the area below the line $y \geq 10 - 2x$ is false. Thus the area that makes the inequality true lies above the line $y \geq 10 - 2x$.
$y \geq 2$			$y \geq 2$ $0 \geq 2$ is false. The area above the line $y = 2$ makes the inequality true.
$x \geq 0$			Area above the x-axis.
$y \geq 0$			Area to the right of the y-axis.

Step 3: The feasible region (area shaded grey) is the one where all the inequalities are true simultaneously.



Question 10



A = (0; 40)

$$20x + 30y = 600 \text{ and } 4x + y = 40$$

$$20 - \frac{2}{3}x = \quad 40 - 4x$$

$$\left(4 - \frac{2}{3}\right)x = 20$$

$$x = 6$$

$$y = 40 - 4(6) = 16$$

B = (6; 16)

$$20x+30y=600 \text{ and } 10x+20y=360$$
$$20-\frac{2}{3}x = 18-\frac{1}{2}x$$
$$2=\left(\frac{2}{3}-\frac{1}{2}\right)x$$
$$x=2\times 6=12$$
$$y=18-\frac{1}{2}(12)=12$$

C = (12; 12)

D = (36; 0)

Cornerpoints	Cost
A(0;40)	$C=5\times 0+4\times 40=160$
B(6;16)	$C=5\times 6+4\times 16=94$
C(12;12)	$C=5\times 12+4\times 12=108$
D(36;0)	$C=5\times 36+4\times 0=180$

$C=5x+4y$ **;** Minimum occurs at B where x = 6 and y = 16 and C = 94.