Tutorial Letter 203/1/2014

QUANTITATIVE MODELLING

Semester 1

Department of Decision Sciences

This tutorial letter contains the solutions
for Assignment 03.

Bar code

Dear Student

You have completed the assignments for the course and it is now time to start your revision for the examination. You will soon receive a tutorial letter containing a previous examination paper and some information about the examination. Work through all the assignments, the evaluation exercises and the previous examination paper in preparation for the examination. The questions in the May/June examination paper are similar to the problems in the previous paper. Do as many examples as possible – the more examples you work through, the better you will be able to recognise a problem and solve it.

Please contact me if you need any help. I am available during the office hours indicated below and my contact details are as follows:

08:00 to 13:30 (Mondays to Fridays) (appointments and per telephone)

13:30 to 16:00 (Mondays to Thursdays) (only per telephone)

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I wish you everything of the best with your preparation for the examination.

Kind regards

Ms Victoria Mabe-Madisa

ASSIGNMENT 3: SOLUTIONS

Question 1

$$
\begin{aligned}
&3(x^4)^2 + 4x^8y^0 + \frac{9x^{12}}{18x^4} \text{ if } y \neq 0 \\
&= 3x^8 + 4x^8 + \frac{1}{2}x^8 \\
&= 7\frac{1}{2}x^8 \\
&= 7{,}5x^8
\end{aligned}
$$

[Option 4]

Question 2

We need to determine the value of the expression:

$$\log_3\left(\frac{1073}{7}\right)$$

$= \dfrac{\ln\left(\frac{1073}{7}\right)}{\ln 3}$	$\log_a b = \dfrac{\ln b}{\ln a}$
$= \dfrac{\ln 1073 - \ln 7}{\ln 3}$	$\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$
$= 4{,}5806$	Using your calculator, rounded to 4 decimal places
$= 4{,}581$	Rounded to 3 decimal places

OR

$$\log_3\left(\frac{1073}{7}\right)$$

$= \dfrac{\ln 153{,}2857}{\ln 3}$	$\log_a b = \dfrac{\ln b}{\ln a}$
$= \dfrac{\ln 153{,}2857}{\ln 3}$	$\ln\left(\dfrac{a}{b}\right) = \ln a - \ln b$
$= 4{,}5806$	Using your calculator, rounded to 4 decimal places
$= 4{,}581$	Rounded to 3 decimal places

The value of the expression $\log_3\left(\frac{1073}{7}\right)$ is approximately equal to 4,581.

[Option 1]

Question 3

$$\begin{aligned}
40 &= 50 - 30e^{-0,05t} \\
\frac{40-50}{-30} &= e^{-0,05t} \\
\frac{-10}{-30} &= e^{-0,05t} \\
\ln\frac{1}{3} &= -0,05t\ln e \\
\frac{\ln\frac{1}{3}}{-0,05} &= t \\
t &= 21,97224577 \\
t &\approx 22
\end{aligned}$$

[Option 4]

Question 4

The roots or solutions of a function can be found where the function, if drawn, intersects the x-axis. We therefore need to determine the value of x at the point(s) where the graph of the function intersects the x-axis, in other words where the function value is zero:

$$y = 0$$

Now given is the function $y + 6 = x^2 + x$. Writing it in terms of y, we get $y = x^2 + x - 6 = 0$.

To determine the roots of a quadratic function we use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with a, b and c the values of the coefficients in the equation $0 = ax^2 + bx + c$.

Comparing the given equation $0 = x^2 + x - 6$ with the general form $0 = ax^2 + bx + c$, we conclude that $a = 1$, $b = 1$ and $c = -6$. Substituting a, b and c into the quadratic formula gives:

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-6)}}{2(1)}$$

$$x = \frac{-1 \pm \sqrt{1+24}}{2}$$

$$x = \frac{-1 \pm \sqrt{25}}{2}$$

$$x = \frac{-1 \pm \sqrt{5^2}}{2}$$

$$x = \frac{-1 \pm 5}{2}$$

$$x = \frac{-1+5}{2} \quad \text{or} \quad \frac{-1-5}{2}$$

$$x = \frac{4}{2} \quad \text{or} \quad \frac{-6}{2}$$

$$x = 2 \quad \text{or} \quad -3$$

The roots of the function $y + 6 = x^2 + x$ or $y = x^2 + x - 6$ are 2 and –3.

[Option 1]

Question 5

The given quadratic profit function is $y = -2x^2 + 10x - 8$. We need to determine the number of products that would maximise the given profit function. The point (x; y), where a function has a minimum or maximum, is also called the extreme value, the turning point or the vertex of a function. As y is given as a quadratic profit function and x as number of products, we need to determine the value of x that would maximise the profit or y. There are different ways to determine the x-coordinate of the maximum point of a quadratic function.

Method 1:

The x-coordinate of the turning point, vertex or extreme point can be calculated using the formula

$$x = \frac{-b}{2a}$$

with a, b and c the coefficients in the standard quadratic function $y = ax^2 + bx + c$, which also describes the axis of symmetry of the parabola.

The function $y = -2x^2 + 10x - 8$ is given. Comparing it with the standard quadratic form of the quadratic function $y = ax^2 + bx + c$, we can conclude that $a = -2$, $b = 10$ and $c = -8$ for the given function. As the a-value is negative, the graph of the function looks like a sad face, thus a maximum extreme point exists for the function.

Therefore the x-value of the extreme point is:

$$\begin{aligned} x &= \frac{-b}{2a} \\ &= \frac{-(10)}{2\times -2} \\ &= \frac{-10}{-4} \\ &= 2,5 \\ &\approx 3 \end{aligned}$$

Therefore 3 units must be manufactured to maximise the profit.

Method 2:

Determine the vertex or minimum or maximum value of a quadratic function by using the symmetry of the graph of the quadratic function. Because the graph of the quadratic function is symmetrical, the vertex (x; y) occurs halfway between the two roots of the quadratic function. Therefore we determine the x-value of the vertex by calculating the roots; x is halfway between the two roots namely (root 1 + root 2)/2. This method is actually a roundabout version of method 1 above.

To determine the roots we use the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

with a, b and c the coefficients in the function $y = ax^2 + bx + c$.

For the given function $y = -2x^2 + 10x - 8$, we note that $a = -2$, $b = 10$ and $c = -8$.

Substituting the values of a, b and c into the formula, we calculate the roots of the function:

$$
\begin{aligned}
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\
&= \frac{-(10) \pm \sqrt{(10)^2 - 4(-2)(-8)}}{2(-2)} \\
&= \frac{-10 \pm \sqrt{100 - 64}}{-4} \\
&= \frac{-10 \pm \sqrt{36}}{-4} \\
&= \frac{-10 \pm 6}{-4} \\
&= \frac{-10 + 6}{-4} \qquad \text{or} \qquad \frac{-10 - 6}{-4} \\
&= \frac{-4}{-4} \qquad \text{or} \qquad \frac{-16}{-4} \\
&= 1 \qquad \text{or} \qquad 4
\end{aligned}
$$

Now the x-value of the vertex lies halfway between the two roots. Thus the x-value of the vertex is:

$$
\begin{aligned}
x &= \frac{root1 + root2}{2} \\
&= \frac{1+4}{2} \\
&= \frac{5}{2} \\
&= 2\frac{1}{2} \\
&= 2{,}50 \\
&\approx 3
\end{aligned}
$$

Therefore 3 units must be manufactured to maximise the profit.

[Option 3]

Question 6

The total cost function *TC* is given. We need to determine the marginal cost if $Q = 10$. First we need to determine the marginal cost function. The marginal cost function *MC* is the differentiated total cost function *TC*.

$$\begin{aligned}
MC &= \frac{\mathrm{d}TC}{\mathrm{d}Q} \\
MC &= \frac{\mathrm{d}}{\mathrm{d}Q}(2Q^3 - Q^2 + 80Q + 150) \\
MC &= (2)(3)Q^{3-1} - (2)(1)Q^{2-1} + 80Q^{1-1} + 0 \\
MC &= 6Q^2 - 2Q + 80
\end{aligned}$$

To determine the marginal cost if $Q = 10$, we calculate $MC(10)$:

$$\begin{aligned}
MC(10) &= 6 \times (10)^2 - 2 \times (10) + 80 \\
&= 600 - 20 + 80 \\
&= 660
\end{aligned}$$

The marginal cost is equal to 660 when Q equals 10.

[Option 3]

Question 7

The basic rule of differentiation states that $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$ when $n \neq 0$. To use this rule we first need to simplify the expression. We can write $\sqrt{Q}$ as $Q^{\frac{1}{2}}$ when changing it from square root form to exponential form. Therefore:

$$\begin{aligned}
\frac{4\sqrt{Q}}{Q^2} &= \frac{4Q^{\frac{1}{2}}}{Q^2} && \text{because } \sqrt{a} = a^{\frac{1}{2}} \\
&= 4Q^{\frac{1}{2}}Q^{-2} && \text{because } \frac{1}{a^c} = a^{-c} \\
&= 4Q^{\frac{1}{2}-2} && \text{because } a^b \times a^c = a^{b+c} \\
&= 4Q^{-1\frac{1}{2}} \text{ or } 4Q^{-\frac{3}{2}}
\end{aligned}$$

Now we differentiate the simplified expression, using the basic rule of differentiation namely $\frac{\mathrm{d}}{\mathrm{d}x}x^n = nx^{n-1}$ when $n \neq 0$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left[4Q^{-\frac{3}{2}}\right] &= 4(-\frac{3}{2})Q^{-\frac{3}{2}-1}\\ &= -\frac{12}{2}Q^{-\frac{5}{2}}\\ &= -6Q^{-\frac{5}{2}}\end{aligned}$$

[Option 1]

Question 8

$$\begin{aligned}f(x) &= x^2 + 5x + \sqrt{x^{\frac{3}{2}}}\\ &= x^2 + 5x + (x^3)^{\frac{1}{2}}\\ &= x^2 + 5x + x^{\frac{3}{2}}\\ f'(x) &= 2x + 5 + \frac{3}{2}x^{\frac{1}{2}}\\ &= 2x + 5 + \frac{3}{2}\sqrt{x}\end{aligned}$$

[Option 4]

Question 9

To integrate the function, we use the basic rule of integration, namely $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ when $n \neq -1$,

but we need to simplify the expression $x^3(1+\frac{1}{x^2})$ before we can use the rule. Therefore:

$$\begin{aligned}\int x^3(1+\frac{1}{x^2})\ dx &= \int (x^3 + \frac{x^3}{x^2})\ dx\\ &= \int (x^3 + x^{3-2})\ dx\\ &= \int (x^3 + x^1)\ dx\\ &= \frac{x^{3+1}}{3+1} + \frac{x^{1+1}}{1+1} + c\\ &= \frac{x^4}{4} + \frac{x^2}{2} + c\end{aligned}$$

[Option 3]

Question 10

To determine the definite integral, we first use the basic rule, namely $\int x^n dx = \frac{x^{n+1}}{n+1} + c$ when $n \neq -1$, to integrate the function:

$$\int_1^5 (2x + x^3)\ \mathrm{d}x = \left(\frac{2x^{1+1}}{1+1} + \frac{x^{3+1}}{3+1} \right) \Bigg|_1^5$$

$$= \left(\frac{2x^2}{2} + \frac{x^4}{4} \right) \Bigg|_1^5$$

$$= \left(x^2 + \frac{x^4}{4} \right) \Bigg|_1^5$$

Secondly we substitute the given values between which we must integrate (1 and 5 in this case) into the integrated function and subtract the two values $F(x)\big|_{x=b}^{x=a} = F(a) - F(b)$:

$$= \left(x^2 + \frac{x^4}{4} \right) \Bigg|_1^5$$

$$= \left((5)^2 + \frac{(5)^4}{4} \right) - \left((1)^2 + \frac{(1)^4}{4} \right)$$

$$= \left(25 + \frac{625}{4} \right) - \left(1 + \frac{1}{4} \right)$$

$$= 181{,}25 - 1{,}25$$

$$= 180$$

[Option 1]