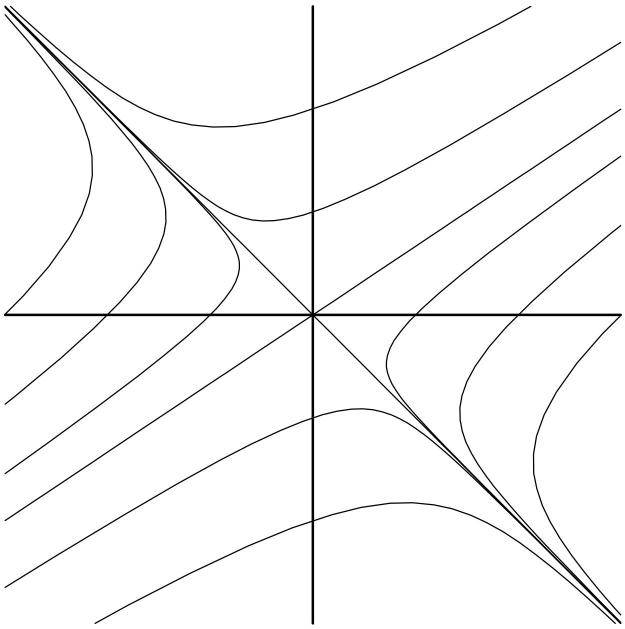
UNISA university of south africa

DEPARTMENT OF MATHEMATICAL SCIENCES



APPLIED MATHEMATICS, Mathematical Modelling (APM1514)

TUTORIAL LETTER 003, 2011 S1

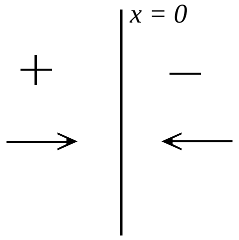
Solutions to Assignment 4

Question 1

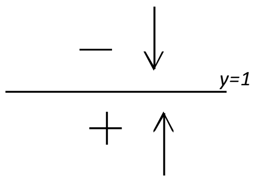
$$\frac{dx}{dt} = -x, \quad \frac{dy}{dt} = 1 - y$$

*x***–isoclines**: $dx/dt = 0$ when $-x = 0$ or equivalently when $x = 0$, that is, on the *y*–axis. Also, $dx/dt = -x$ is positive if $x < 0$ (so that is where motion is towards the right) and negative if $x > 0$ (motion towards the left). We summarise this information in the following diagram:

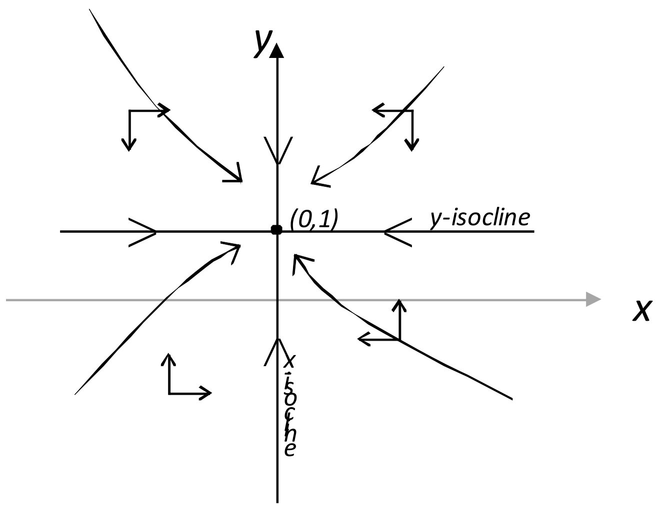
Sign of *dx/dt:*



*y***–isoclines**: $dy/dt = 0$ when $y = 1$. The sign of $dx/dt = y$ is positive (with motion upwards) when $y < 1$ and negative (with motion downwards) when $y < 1$.



There is one equilibrium point, $(0, 1)$, at the intersection of the *x* and *y* axes. The phase diagram is drawn below. (It is created by drawing in the same diagram the *x* and *y* isoclines, and the direction of motion arrows (up/down and left/right) copied to the appropriate areas from the two sketches above.)



Note that neither the *x*–isoclines nor the *y*–isoclines can be crossed; the motion on the *y*–isoclines is horizontal and the motion on the *x*–isoclines is vertical. The direction of motion along the axes is in each case the same as in the neighbouring regions.

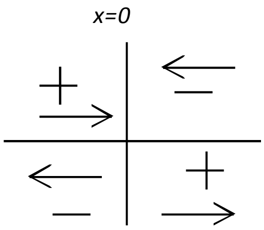
Solutions starting with $x = 0$, $y = 1$ (the equilibrium point) stay there; solutions starting elsewhere converge towards the point $(0, 1)$. The equilibrium point $(0, 0)$ is stable.

Question 2

$$\frac{dx}{dt} = -xy, \quad \frac{dy}{dt} = x - 1$$

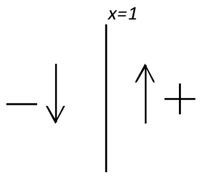
x–isoclines and sign of dx/dt

Here $dx/dt = -xy$ is zero when either x or y is zero, that is, on the y and x axes. The sign of dx/dt will be negative on the first and third quadrants (where x and y have the same sign) and positive on the second and fourth quadrants (where their signs are opposite).

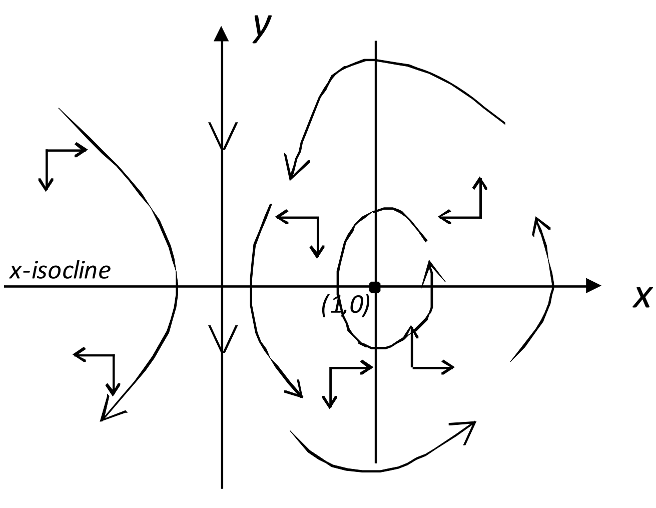


y–isoclines and sign of dy/dt

$dx/dt = 0$ when $x - 1 = 0$, that is, where $x = 1$. The sign of $(x - 1)$ is positive if $x > 1$, negative otherwise.



There is one equilibrium point at $(1, 0)$. The phase diagram looks like this:



Motion on x–isoclines must be horizontal, which here means that the x–isocline $y = 0$ must be crossed vertically and on the y–axis motion is downwards. Likewise, the y–isocline $x = 1$ must be crossed horizontally.

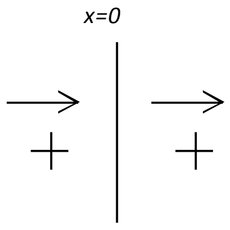
Solutions starting with $y_0 < 0$ will eventually move towards $(-\infty, -\infty)$, and solutions starting on the y–axis will stay on the y axis with y decreasing towards $-\infty$. Solutions starting with $y_0 > 0$ seem to be rotating counter-clockwise around the equilibrium point $(1, 0)$. That equilibrium point seems to be stable, although we cannot be sure!

Question 3

$$\frac{dx}{dt} = x^2, \quad \frac{dy}{dt} = -y$$

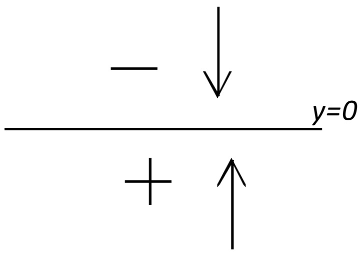
x–isoclines and sign of dx/dt

$dx/dt = 0$ holds when $x = 0$, that is, on the y–axis. dx/dt is always non-negative, so motion is always towards the right.



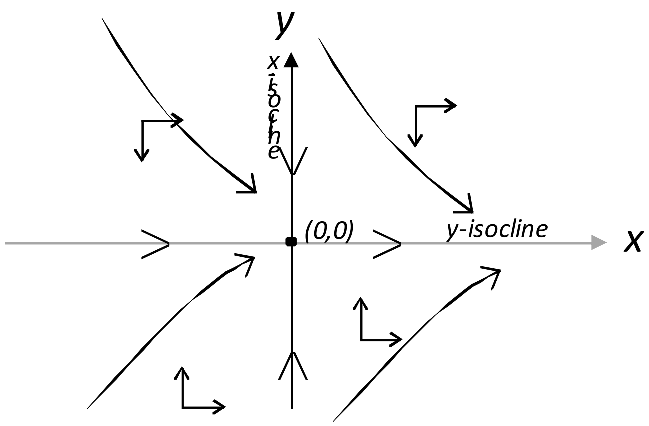
y–isoclines and and sign of dy/dt

$dy/dt = 0$ when $y = 0$, that is, on the x–axis. We find that $dy/dt > 0$ below this line and $dy/dt < 0$ above it.



There is one equilibrium point, $(x, y) = (0, 0)$, at the intersection of the x– and y–isoclines.

The phase diagram looks like this:



Solutions starting with $y_0 \leq 0$ will converge towards the point $(0, 0)$ while solutions starting with $y_0 > 0$ will approach the positive x–axis asymptotically. The equilibrium point $(0, 0)$ is unstable.

Question 4

The model is given by

$$\frac{dx}{dt} = xy - 3x \quad (1)$$

$$\frac{dy}{dt} = -xy - y. \quad (2)$$

(a) If y is not present, then the difference equation (1) ruling the behaviour of the x–species will just be

$$\frac{dx}{dt} = -3x$$

which is a Malthusian population with negative growth constant, that is, a population which is dying out. If x is not present, then the difference equation (2) for the y–population becomes

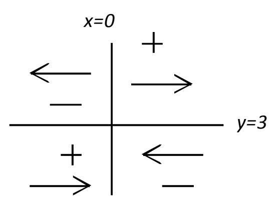
$$\frac{dy}{dt} = -y$$

is also a Malthusian population which is dying out (but slower than the x–population). Both populations will die out when the other population is present.

(b) Since the coefficient of the interaction term xy is positive in equation (1), species x benefits from each interaction; on the other hand, in (2) the interaction is negative, so species y suffers from each interaction. So, it could be true that x is a predator and y is a prey species.

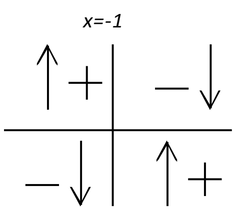
(c) x**–isoclines and sign of** dx/dt :

Here $dx/dt = xy - 3x = x\,(y-3)$ is zero when either $x = 0$ or $y = 3$. The signs of dx/dt and the corresponding horizontal directions of motion are shown below.

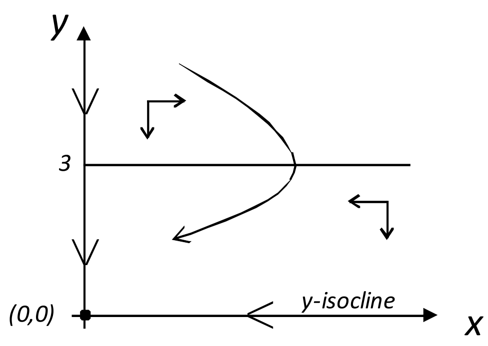


y**–isoclines and sign of** dy/dt

$dx/dt = 0$ when $y = 0$ or $x = -1$. The signs of dy/dt, and the corresponding vertical directions, in the different regions are shown below.



Combining this information we get the following phase diagram.



Only the first quadrant ($x, y \geq 0$) is shown. There is one equilibrium point, $(0, 0)$. (Note that (0,3) is not an equilibrium point!) We see that all solutions starting in this quadrant converge towards $(0, 0)$.

(d) The solution curve with $(x_0, y_0) = (1, 2)$ is shown below (dotted line).