Tutorial letter 202/2/2018

DIFFERENTIAL EQUATIONS

APM2611

Semester 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains information about the October/November 2018 examination, Solutions to Assignment 2, and solutions to a previous year's examination paper

BARCODE

Define tomorrow.

UNISA university of south africa

Dear Student,

This tutorial letter intends to help you prepare for the examination. It contains the following information:

1 About the October/November 2018 examination.

2 Solutions to Assignment 2.

3 Solutions to a previous year's examination paper.

1 About the October/November 2018 examination

(i) You are allowed to use only a **non-programmable pocket calculator** in the examination, but I don't think you really need one!

(ii) It will be a closed book examination with **information sheets** containing some useful formula.

(iii) The examination is based on all parts of the study guide. In particular, *make sure that you know the following concepts or procedures*, and are able to apply them to solve problems:

- Solving a differential equations (DE) using one of the appropriate technique, remark or procedure (Separation of the variables, exact equation, integrating factor, Bernoulli type DE, etc.)
- Finding a general solution for a DE using the method of undetermined coefficients or the method of variation of parameters.
- Solving an initial-value problem by exploiting its auxiliary equation, using the power series method, using Laplace transforms or using the separation of variables,.
- Modeling: Expressing a real world problem into a DE and solve it.
- Calculating the Laplace transform of a function (sometimes from first principles or the table of Laplace transforms).
- Calculating the inverse Laplace transform a function.
- Computing the Fourier series for a function.

2 Solutions to Assignment 2

Question 1

Use the power series method to solve the initial value problem:

$$(1-x^2)y'' - 6xy' - 4y = 0; \qquad y(0) = 1, \quad y'(0) = 2.$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$. *Then* $y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, *and* $y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$. *Substitution into the DE yields*

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 6x \sum_{n=1}^{\infty} n a_n x^{n-1} - 4 \sum_{n=0}^{\infty} a_n x^n = 0.$$

Altering the range of summation in the first sum gives

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=2}^{\infty} n(n-1) a_n x^n - 6 \sum_{n=1}^{\infty} n a_n x^n - 4 \sum_{n=0}^{\infty} a_n x^n = 0,$$

or

$$2a_2 + 6a_3 x - 6a_1 x - 4a_0 - 4a_1 x + \sum_{n=2}^{\infty} \{(n+2)(n+1) a_{n+2} - n(n-1) a_n - 6n a_n - 4a_n\} x^n = 0.$$

Hence

$$\begin{aligned} 2a_2 - 4a_0 &= 0, \\ 6a_3 - 10a_1 &= 0, \\ (n+2)(n+1) a_{n+2} - (n^2 + 5n + 4) a_n &= 0 \quad \text{for} \quad n \geq 2. \end{aligned}$$

Since $n^2 + 5n + 4 = (n+1)(n+4)$ *we get*

$$a_{n+2} = \left(\frac{n+4}{n+2}\right) a_n \quad \text{for} \quad n \geq 2.$$

The initial values give $a_0 = 1$, $a_1 = 2$. *Hence*

$$\begin{aligned} a_2 &= 2a_0 = 2, \\ a_3 &= \frac{10a_1}{6} = \frac{10}{3}. \end{aligned}$$

Thus the first few terms of the power series solution are

$$y = 1 + 2x + 2x^2 + \frac{10}{3} x^3 + 3x^4 + \cdots .$$

Question 2

Calculate the Laplace transform of the following functions:

(2.1) $t^2 \sin t$.

$$\mathcal{L}(\sin t) = \frac{1}{s^2+1} \quad \text{from a Laplace Transform Table.}$$

Hence

$$\mathcal{L}\left\{t^2 \sin t\right\} = \frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{d}{ds}\left(\frac{-2s}{(s^2+1)^2}\right) = \frac{(s^2+1)^2(-2) - (-2s)(2s)2(s^2+1)}{(s^2+1)^4} = \frac{6s^2-2}{(s^2+1)^3}.$$

(2.2) $t^2\mathcal{U}(t-2)$.

Remember that the step function $\mathcal{U}(t-a)$ *is defined by*

$$\mathcal{U}(t-a) = \begin{cases} 0 & \text{if} \quad 0 \le t < a \\ 1 & \text{if} \quad t \ge a. \end{cases}$$

Then, from a table of transforms, we have

$$\mathcal{L}\{\mathcal{U}(t-2)\} = \frac{e^{-2s}}{s}.$$

Hence

$$\begin{aligned} \mathcal{L}\{t^2\mathcal{U}(t-2)\} &= \frac{d^2}{ds^2}\left(\frac{e^{-2s}}{s}\right) = \frac{d}{ds}\left(\frac{-2se^{-2s}-e^{-2s}}{s^2}\right) = \frac{d}{ds}\left[\frac{-e^{-2s}(2s+1)}{s^2}\right] \\ &= \frac{s^2[2e^{-2s}(2s+1)-2e^{-2s}]-[-e^{-2s}(2s+1)]2s}{s^4} \\ &= \frac{4s^3e^{-2s}+4s^2e^{-2s}+2se^{-2s}}{s^4} = \frac{2e^{-2s}(2s^2+2s+1)}{s^3}. \end{aligned}$$

(2.3) $\displaystyle\int_0^t \frac{1-e^{-u}}{u}\,du.$

We have that

$$\mathcal{L}\left\{\frac{1-e^{-t}}{t}\right\} = \mathcal{L}\left\{\frac{e^{0t}-e^{(-1)t}}{t}\right\} = \ln\left(\frac{s+1}{s}\right) \qquad \textit{from no. 39 of transform table,}$$

and $\mathcal{L}\{1\} = 1/s$. *Hence, from no. 56 of transform table*

$$\mathcal{L}\left\{\int_0^t \frac{1-e^{-u}}{u}\,du\right\} = \mathcal{L}\left\{\int_0^t \left(\frac{1-e^{-u}}{u}\right)(1)\,du\right\} = \frac{1}{s}\ln\left(\frac{s+1}{s}\right).$$

Question 3

Calculate the Laplace transform of the following function from first principles:

$$f(t) = \begin{cases} t & 0 \le t < 4 \\ 5 & t \ge 4 \end{cases}$$

$$\mathcal{L}\{f(t)\} = \int_0^4 te^{-st}dt + \int_4^\infty 5e^{-st}dt.$$

Then, integrating by parts, we get

$$\begin{aligned} \int_0^4 te^{-st}dt &= \left[\frac{-t}{s}e^{-st}\right]_{t=0}^{t=4} - \frac{-1}{s}\int_0^4 e^{-st}dt = \frac{-4}{s}e^{-4s} - \frac{1}{s^2}\left[e^{-st}\right]_{t=0}^{t=4} \\ &= -\frac{4}{s}e^{-4s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2} \qquad \textit{where } s > 0. \end{aligned}$$

Also

$$\int_4^\infty 5\,e^{-st}dt = -\frac{5}{s}\left[e^{-st}\right]_{t=4}^{t=\infty} = \frac{5}{s}e^{-4s} \qquad \textit{where } s > 0.$$

Hence

$$\mathcal{L}\{f(t)\} = -\frac{4}{s}e^{-4s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2} + \frac{5}{s}e^{-4s} = \frac{1}{s}e^{-4s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2} \qquad (s > 0).$$

Question 4

Calculate the following inverse Laplace transforms:

(4.1) $\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}$

Note that

$$\frac{6s-4}{s^2-4s+20} = \frac{6(s-2)+8}{(s-2)^2+16} = 6\left[\frac{s-2}{(s-2)^2+16}\right] + 2\left[\frac{4}{(s-2)^2+16}\right].$$

Hence, from the Laplace transform tables we get

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\} &= 6\mathcal{L}^{-1}\left\{\frac{s-2}{(s-2)^2+16}\right\} + 2\mathcal{L}^{-1}\left\{\frac{4}{(s-2)^2+16}\right\} \\ &= 6e^{2t}\cos 4t + 2e^{2t}\sin 4t.\end{aligned}$$

(4.2) $\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\}$

No. 40 *of the transform table gives*

$$\mathcal{L}^{-1}\left\{\ln\left(\frac{s^2+k^2}{s^2}\right)\right\} = \frac{2(1-\cos kt)}{t}.$$

Hence with $k = 1$ *we get*

$$\mathcal{L}^{-1}\left\{\ln\left(1+\frac{1}{s^2}\right)\right\} = \mathcal{L}^{-1}\left\{\ln\left(\frac{s^2+1}{s^2}\right)\right\} = \frac{2(1-\cos t)}{t}.$$

Question 5

Solve the following initial value problem by using Laplace transforms:

$$y''(t) + 9y = \cos 3t, \qquad y(0) = 2, \quad y'(0) = 5.$$

$$\begin{aligned}\mathscr{L}\{y''+9y\} &= \mathscr{L}\{\cos 3t\} \\ \mathscr{L}\{y''\} + 9\mathscr{L}\{y\} &= \frac{s}{s^2+9} \\ s^2\mathscr{L}\{y\} - sy(0) - y'(0) + 9\mathscr{L}\{y\} &= \frac{s}{s^2+9} \\ s^2\mathscr{L}\{y\} - 2s - 5 + 9\mathscr{L}\{y\} &= \frac{s}{s^2+9} \\ (s^2+9)\mathscr{L}\{y\} &= 2s + 5 + \frac{s}{s^2+9} \\ \mathscr{L}\{y\} &= 2\frac{s}{s^2+9} + \frac{5}{3}\frac{3}{s^2+9} + \frac{1}{6}\frac{6s}{(s^2+9)^2} \\ y &= 2\mathscr{L}^{-1}\left\{\frac{s}{s^2+9}\right\} + \frac{5}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+9}\right\} + \frac{1}{6}\mathscr{L}^{-1}\left\{\frac{6s}{(s^2+9)^2}\right\} \\ y &= 2\cos 3t + \frac{5}{3}\sin 3t + \frac{1}{6}t\sin 3t.\end{aligned}$$

Question 6

Compute the Fourier series for the function

$$f(x) = \begin{cases} 0 & -1 \le x \le 0 \\ x & 0 \le x \le 1 \end{cases}$$

on $[-1, 1]$.

Using Definition 11.2.1 (Fourier Series) in your textbook we get:

$$\begin{aligned}
a_0 &= \int_{-1}^{1} f(x)\,dx = \int_0^1 x dx = \frac{1}{2}. \\
a_n &= \int_{-1}^{1} f(x) \cos n\pi x dx = \int_0^1 x \cos n\pi x dx \\
&= \left[\frac{x \sin n\pi x}{n\pi} \right]_{x=0}^{x=1} - \frac{1}{n\pi} \int_0^1 \sin n\pi x dx \quad \textit{(integrating by parts)} \\
&= \frac{[\cos n\pi - 1]}{n^2\pi^2} = \frac{[(-1)^n - 1]}{n^2\pi^2}, \quad \textit{(for } n > 0). \\
b_n &= \int_{-1}^{1} f(x) \sin n\pi x dx = \int_0^1 x \sin n\pi x dx \\
&= -\left[\frac{x \cos n\pi x}{n\pi} \right]_{x=0}^{x=1} + \frac{1}{n\pi} \int_0^1 \cos n\pi x dx \qquad \textit{(integrating by parts)} \\
&= -\frac{\cos n\pi}{n\pi} + \frac{1}{n^2\pi^2} \sin n\pi = \frac{(-1)^{n+1}}{n\pi}, \quad (n \ge 1).
\end{aligned}$$

Hence $a_0 = 0$ *for* $n > 0$, n *even, and if* n *is odd, say* $n = 2r - 1$, *we get*

$$a_{2r-1} = \frac{-2}{(2r-1)^2 \pi^2}, \qquad r \ge 1.$$

Hence

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\pi x + b_n \sin n\pi x) \\
&= \frac{1}{4} - \frac{2}{\pi^2} \sum_{r=1}^{\infty} \frac{\cos[(2r-1)\pi x]}{(2r-1)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin n\pi x}{n\pi}.
\end{aligned}$$

Question 7

Using the method of separation of variables, solve the following boundary value problem:

$$\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}; \qquad u(0, y) = 8e^{-3y}.$$

Let $u = XY$, where X is a function of x only, and Y is a function of y only. Then

$$\frac{\partial u}{\partial x} = Y\frac{dX}{dx},$$
$$\frac{\partial u}{\partial y} = X\frac{dY}{dy}.$$

Substitution into the partial differential equation gives

$$\frac{1}{4X}\frac{dX}{dx} = \frac{1}{Y}\frac{dY}{dy}.$$

Since x and y are independent variables, and X depends only on x, Y depends only on y, each side of the equation (1) must be equal to the same constant c.
Hence

$$\frac{dX}{dx} - 4cX = 0 \quad \text{and} \quad \frac{dY}{dy} - cY = 0.$$

These 1st order linear equations are easy to solve. Check that you get

$$X = Ae^{4cx} \quad \text{and} \quad Y = Be^{cy},$$

where A, B are arbitrary constants. Hence

$$u(x, y) = XY = ABe^{c(4x+y)} = Ke^{c(4x+y)}.$$

Using the boundary condition we get

$$u(0, y) = 8e^{-3y} = Ke^{cy}.$$

Hence $K = 8$ and $c = -3$. Then

$$u(x, y) = 8e^{-3(4x+y)} = 8e^{-12x-3y}.$$

3 Questions and Solutions to a previous year's examination paper

QUESTIONS

Instructions to the candidate:

- The use of non-programmable calculators is allowed.
- This paper consists of 5 pages including formulas on page 12.
- Show **ALL** calculations.
- Try to answer **ALL** the questions.

This examination paper remains the property of the University of South Africa and may **NOT** be removed from the examination venue.

QUESTION 1

Consider the following first order ordinary differential equations and classify them as separable and / or homogeneous and / or linear and / or Bernoulli:

(a)

$$x\frac{dy}{dx} = xe^{-\frac{y}{x}} + y$$

(2)

(b)

$$\frac{dy}{dx} = y + y^2 + x^2y + x^2y^2$$

(2)

(c)

$$\left(x^2+1\right)\frac{dy}{dx} - xy = 2x$$

(3)

(d)

$$e^x\cos^2 y - (1+e^x)\frac{dy}{dx} = 0$$

(2)

[9]

QUESTION 2

Find the general solution of the following ordinary differential equations:

(a)

$$2e^x\cos^2 y - x\frac{dy}{dx} - e^x\frac{dy}{dx} + 2\cos^2 y = 0$$

(5)

(b)

$$y' + \frac{1}{2}y\tan x = y^3$$

(6)

(c)

$$y'' - 2y' = e^x \sin x$$

(5)

(d)

$$\left(x^3 + y^3\right) dx + 3xy^2 dy = 0$$

(6)

[22]

QUESTION 3

(a) Separate the variables in the differential equation

$$(xy + y)\frac{dy}{dx} = xy^2 - x - y^2 + 1.$$

(2)

(b) Solve the differential equation in (a) with condition $y(-2) = 0$. (6)

[8]

QUESTION 4

A water tank has the shape obtained by revolving the parabola $x^2 = 4y$ around the y-axis. The water depth is 4 metres at 12 noon when a circular bottom plug is pulled. After one hour the depth of the water is 1 metre. Use the formula

$$A(y)\frac{dy}{dt} = -a\sqrt{2gy}.$$

where $A(y)$ denotes the area of the water at height y, a, the area of the outlet, and g the gravity, to determine the following:

(a) Find the depth $y(t)$ of the water remaining after t hours. (5)

(b) When will the tank be empty? (2)

[7]

QUESTION 5

Consider the initial value problem

$$y'' + 4y = 3e^{2x} + 7\cos x, \quad y(0) = 1, \quad y'(0) = 0.$$

5.1 Obtain y_c for the associated homogeneous equation. (2)

5.2 Given that $y_p = \frac{3}{8}e^{2x} + \frac{7}{3}\cos x$ is a particular solution of the differential equation. Find the solution of the initial value problem. (5)

[7]

QUESTION 6

Consider the differential equation

$$y''' - y = e^{2x}\sin 3x.$$

6.1 Find the solution for the homogeneous part of the differential equation, (3)

6.2 Use the operator method to find the particular solution, and (6)

6.3 Write down the general solution for the differential equation. (1)

[10]

QUESTION 7

Use $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ to find $\mathcal{L}^{-1}(G)$ if

$$G(s) = \frac{s-2e^{-s}}{s^2+6s+11}.$$

[4]

QUESTION 8

Find $\mathcal{L}\{f(t)\}$ from first principles if:

$$f(t) = \begin{cases} 0 & 0 < t < \frac{\pi}{2} \\ \cos t & t \geq \frac{\pi}{2} \end{cases}$$

[9]

QUESTION 9

Use the Laplace Transform to solve the initial value problem:

$$y'' + 6y' + 5y = 2\mathcal{U}(t-1)$$

subject to $y(0) = 3$ and $y'(0) = 0$. **[14]**

QUESTION 10

Consider the homogeneous heat conduction problem:

PDE: $$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \qquad (0 < x < 2, \quad t > 0)$$

BCs: $u(0,t) = 0, \qquad u(2,t) = 0$

IC: $u(x,0) = f(x)$

with

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

Use the method of separation of variables to find the following:

(a) Assume $u(x,t) = \phi(x)\,G(t)$, and derive the ODEs satisfied by $\phi(x)$ and $G(t)$, if $-\lambda$ is the separation constant. (2)

(b) If $\lambda > 0$, determine the general solution for $\phi(x)$ and hence use the boundary conditions to determine the eigenvalues λ_n. (3)

(c) Write down the eigenfunctions of the above problem. (2)

(d) By the superposition principle the solution is given by

$$u(x,t) = \sum_{n=1}^{n=\infty} A_n e^{-k(n^2\pi^2/L^2)t} \sin\frac{n\pi}{L}x.$$

Use the IC to calculate A_n if

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\;dx.$$

(2)

(e) Write down the solution $u(x,t)$. (1)

[10]

TOTAL: [100]

FORMULA SHEET

I LAPLACE–TRANSFORMS OF SOME BASIC FUNCTIONS

1.
$$\mathcal{L}\{1\} = \frac{1}{s}$$

2.
$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}},\ n = 1, 2, 3, \ldots$$

3.
$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

4.
$$\mathcal{L}\{\sin\ kt\} = \frac{k}{s^2+k^2}$$

5.
$$\mathcal{L}\{\cos\ kt\} = \frac{s}{s^2+k^2}$$

6.
$$\mathcal{L}\{\sinh\ kt\} = \frac{k}{s^2-k^2}$$

7.
$$\mathcal{L}\{\cosh\ kt\} = \frac{s}{s^2-k^2}$$

8.
$$\begin{aligned}\mathcal{L}\left\{f^{(n)}(t)\right\} &= s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) \\ &\quad - \ldots - f^{(n-1)}(0)\end{aligned}$$

where
$$F(s) = \mathcal{L}\{f(t)\}.$$

9.
$$\mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}.$$

10.
$$\mathcal{L}^{-1}\{f(s+1)\} = e^{-t}\mathcal{L}^{-1}\{f(s)\}.$$

SOLUTIONS

1. (a)

$$\begin{aligned} \frac{dx}{dy} &= e^{\frac{-y}{x}} + \frac{y}{x} \\ &= f(x,y) \\ &\Rightarrow \text{ homogeneous.} \end{aligned}$$

(2)

(b)

$$\begin{aligned} \frac{dy}{dx} &= y(1+x^2) + y^2(1+x^2) \\ &= (y+y^2)(1+x^2) \\ &\Rightarrow \text{ seperable, homogeneous and Bernouli.} \end{aligned}$$

(2)

(c)

$$\begin{aligned} (x^2+1)\frac{dy}{dx} - xy &= 2x \\ \frac{dy}{dx} &= \frac{2x+xy}{x^2+1} \\ &= \frac{x(2+y)}{x^2+1} \\ \frac{dy}{2+y} &= \frac{xdx}{x^2+1} \\ &\Rightarrow \text{ seperable or,} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} - \frac{x}{x^2+1}y &= \frac{2x}{x^2+1} \\ &\Rightarrow \text{ linear and homogeneous.} \end{aligned}$$

(3)

(d)

$$\begin{aligned} e^x\cos^2 y - (1+e^x)\frac{dx}{dy} &= 0 \\ \frac{dy}{dx} &= \frac{e^x\cos^2 y}{1+e^x} \\ \frac{dy}{\cos^2 y} &= \frac{e^x dx}{1+e^x} \\ &\Rightarrow \text{ seperable and homogeneous.} \end{aligned}$$

(2)

2.1

$$\begin{aligned}
2e^x\cos^2 y - x\frac{dy}{dx} - e^x\frac{dy}{dx} + 2\cos^2 y &= 0\\
(2+2e^x)\cos^2 y - (x+e^x)\frac{dy}{dx} &= 0\\
\left((2+2e^x)\cos^2 y\right)dx &= (x+e^x)dy\\
\int \frac{2+2e^x}{x+e^x}\,dx &= \int \frac{dy}{\cos^2 y}\\
2\ln|x+e^x| = -\cot y + A
\end{aligned}$$

(5)

2.2

$$y' + \frac{1}{2}y\tan x = y^3$$

Let $v = y^{-2}$ then $y = v^{\frac{-1}{2}}$, and $\frac{dy}{dx} = \frac{-1}{2}v^{\frac{-3}{2}}\frac{dv}{dx}$.
Then

$$\begin{aligned}
-\frac{1}{2}v^{\frac{-3}{2}}\frac{dv}{dx} + \frac{1}{2}v^{-\frac{1}{2}}\tan x &= v^{\frac{-3}{2}}\\
\frac{dv}{dx} - v\tan x &= -2
\end{aligned}$$

The intergration factor $p(x) = e^{\int -\tan x dx} = e^{\int \frac{-\sin x}{\cos x}dx} = e^{\ln\cos x} = \cos x$

We multiply through with the integration factor to obtain

$$\begin{aligned}
\int \frac{d}{dx}(v\cos x) &= \int -2\cos x dx\\
v\cos x &= -2\sin x + A\\
v &= -2\tan x + \frac{A}{\cos x}
\end{aligned}$$

Which gives you, after substituting the value of y back

$$y^{-2} = -2\tan x + \frac{A}{\cos x}.$$

(6)

2.3

$$y'' - 2y' = e^x\sin x$$

$$\begin{aligned}
r^2 - 2r &= 0\\
r(r-2) &= 0\\
r = 0, &\qquad r = 2
\end{aligned}$$

Therefore $y_c = c_1 + c_2e^{2x}$.

$$\begin{aligned}
y_p &= e^x(A\cos x + B\sin x)\\
y_p' &= e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)\\
y_p'' &= e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x)\\
&\quad + e^x(-A\cos x - B\sin x) + e^x(-A\sin x + B\cos x)\\
&= 2e^x(-A\sin x + B\sin x)
\end{aligned}$$

Substutute:

$$\begin{aligned} -2e^xA\sin x + 2e^xB\sin x - 2e^xA\cos x - 2e^xB\sin x + 2e^xA\sin x - 2e^xB\cos x &= e^x\sin x \\ -2Ae^x\cos x - 2Be^x\sin s &= e^x\sin x \end{aligned}$$

This gives us $A = 0$ and $B = \frac{-1}{2}$, and $y = c_1 + c_2e^{2x} - \frac{1}{2}e^x\sin x$. (5)

2.4

$$(x^3 + y^3)dx + 3xy^2dy = 0$$

$$\begin{aligned} \frac{d}{dy}(x^3 + y^3) &= 3y^2 \\ \frac{d}{dx}(3xy^2) &= 3y^2 \end{aligned}$$

The differential equation is exact. Find $f(x, y)$ such that

$$\begin{aligned} f_x &= x^3 + y^3 \\ f_y &= 3xy^2 \end{aligned}$$

Integrate the above equation for f_x on both sides with respect to x, to obtain

$$f = \frac{x^4}{4} + xy^3 + g(y).$$

Differentiate to y: $f_y = 3xy^2 + g'(y) = 3xy^2$. It is clear that $g'(y) = 0$, and therefore $g(y) = c$.

Thus $f(x, y) = \frac{1}{4}x^4 + xy^3 + c$ and the general solution of the differential equation is

$$\frac{1}{4}x^4 + xy^3 + c = 0$$

. (6)

[22]

3. (a)

$$\begin{aligned} (xy + y)\frac{dy}{dx} &= xy^2 - x - y^2 + 1 \\ xy\ dy + y\ dy &= xy^2\ dx - x\ dx - y^2\ dx + dx \\ y\ dy(x + 1) &= y^2(x - 1)\ dx - dx(x - 1) \\ y\ dy(x + 1) &= (y^2 - 1)(x - 1)\ dx \\ \frac{y}{y^2 - 1}\ dy &= \frac{x - 1}{x + 1}\ dx \end{aligned}$$

(2)

(b)

$$\begin{aligned} \int \frac{y}{y^2 - 1}\ dy &= \int \frac{x - 1}{x + 1}\ dx \\ \frac{1}{2}\ln|y^2 - 1| &= \int \frac{x}{x + 1}\ dx - \int \frac{1}{x + 1}\ dx \\ &= \int \frac{x + 1}{x + 1}\ dx - \int \frac{2}{x + 1}\ dx \\ &= x - 2\ln|x + 1| + c \end{aligned}$$

Initial conditions $y = 0$ and $x = -2$ produce

$$\frac{1}{2}\ln|-1| = -2 - 2\ln|-1| + c$$

We solve for c to obtain $c = 2$. The solution is

$$\ln|y^2 - 1| = 2x - 4\ln|x+1| + 4.$$

(6)
[8]

4. (a)

$$\begin{aligned} A(y)\frac{dy}{dt} &= -a\sqrt{2gy} \\ 4\pi y dy &= -a\sqrt{2gy}dt \\ \int 4\pi y^{\frac{1}{2}}\ dy &= -a\sqrt{2g}\int\ dt \\ \frac{8}{3}\pi y^{\frac{3}{2}} &= -a\sqrt{2g}\ t + c \end{aligned}$$

Initial conditions applied produces $\frac{8}{3}\pi(4)^{\frac{3}{2}} = c$, which leads to $c = \frac{64}{3}\pi$.

The condition $y(1) = 1$ leads to $-a\sqrt{2g} = \frac{56\pi}{3}$ and

$$y = (8 - 7t)^{\frac{2}{3}}$$

(5)

(b)

$$\begin{aligned} y = (8 - 7t)^{\frac{2}{3}} &= 0 \\ 8 - 7t &= 0 \\ t &= \frac{8}{7}\text{ hrs} \end{aligned}$$

(2)
[7]

5. 5.1

$$y'' + 4y = 3e^{2x} + 7\cos x, \quad \text{with} \quad y(0) = 1, \quad y'(0) = 0.$$

The characteristic equation for the homogeneous equation is

$$\begin{aligned} r^2 + 4 &= 0 \\ r &= \pm 2i \end{aligned}$$

So

$$y_c = c_1\cos 2x + c_2\sin 2x$$

(2)

5.2 The solution of the above DE is

$$y = c_1\cos 2x + c_2\sin 2x + \frac{3}{8}e^{2x} + \frac{7}{3}\cos x.$$

Use the initial conditions $y(0) = 1$ to find c_1:

$$\begin{aligned} 1 &= c_1 + \frac{3}{8} + \frac{7}{3} \\ c_1 &= \frac{24 - 9 - 56}{24} \\ &= \frac{-41}{24}. \end{aligned}$$

Now to find c_2, we use the initial condition $y'(0) = 0$:

$$\begin{aligned} y'(0) &= 2c_2 + \frac{3}{4} = 0 \\ c_2 &= \frac{-3}{8}. \end{aligned}$$

The solutions is therefore

$$y = \frac{-41}{24}\cos 2x - \frac{3}{8}\sin 2x + \frac{3}{8}e^{2x} + \frac{7}{3}\cos x.$$

(5)

[7]

6. 6.1

$$\begin{aligned} (t^3 - 1) &= (r-1)(r^2 + r + 1) \\ r = \pm 1, \quad r &= \frac{-1}{2} \pm \frac{\sqrt{3}}{2}i \end{aligned}$$

The solution of the homogeneous equation is

$$y_c + Ae^{-x} + Be^{-x}\sin\frac{\sqrt{3}}{2}x + Ce^{-x}\cos\frac{\sqrt{3}}{2}x.$$

(3)

6.2

$$\begin{aligned} (D^3 - 1)y &= e^{2x}\sin 3x \\ y_p &= \frac{e^{2x}\sin 3x}{D^3 - 1} \\ &= e^{2x}\frac{1}{(D+2)^3 - 1}\sin 3x \\ &= e^{2x}\frac{1}{(D^2 + 4D + 2)(D+2) - 1}\sin 3x \\ &= e^{2x}\frac{1}{D^3 + 4D^2 + 2D + 2D^2 + 8D + 4}\sin 3x \\ &= e^{2x}\frac{1}{D^3 + 6D^2 + 10D + 4}\sin 3x \\ &= e^{2x}\frac{1}{-9D + 6(-9) + 10D + 4}\sin 3x \\ &= e^{2x}\frac{-1}{50^2 + 9}(-50\sin 3x + 3\cos 3x) \\ &= e^{2x}\frac{-1}{2509}(-50\sin 3x + 3\cos 3x) \end{aligned}$$

(6)

6.3

$$y = \frac{-41}{24}\cos 2x - \frac{3}{8}\sin 2x + \frac{3}{8}e^{2x} + \frac{7}{3}\cos x + e^{2x}\frac{-1}{2509}(-50\sin 3x + 3\cos 3x). \quad (1)$$

[10]

7.

$$\begin{aligned} G(s) &= \frac{s}{(s+3)^2+2} - \sqrt{2}(\frac{\sqrt{2}e^{-s}}{(s+3)^2+2}) \\ &= \frac{s+3}{(s+3)^2+2} - \frac{3}{(s+3)^2+2} - \sqrt{2}(\frac{\sqrt{2}e^{-s}}{(s+3)^2+2}). \end{aligned}$$

Therefore

$$\mathcal{L}^{-1}\{G(s)\} = g(t) = e^{-3t}(\cos\sqrt{2}\ t - \frac{3}{\sqrt{2}}\sin\sqrt{2}\ t) - \sqrt{2}e^{-3(t-1)}\sin\sqrt{2}\ \mathcal{U}(t-1) \quad t \geq 1$$

[4]

8.

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^\infty e^{-st}f(t)\ dt \\ &= \int_{\frac{\pi}{2}}^\infty e^{-st}\cos t\ dt \end{aligned}$$

Integrate by parts:

$$\begin{aligned} u = e^{-st}, &\qquad du = -se^{-st} \\ dv = \cos t, &\qquad v = \sin t \end{aligned}$$

Now we have

$$\begin{aligned} \int_{\frac{\pi}{2}}^\infty e^{-st}\cos t\ dt &= e^{-st}\sin t|_{\frac{\pi}{2}}^\infty + s\int_{\frac{\pi}{2}}^\infty e^{-st}\sin t\ dt \\ &= (0 - s^{\frac{-s\pi}{2}}) + \frac{1}{s}(-\cos te^{-st}|_{\frac{\pi}{2}}^\infty) + s^2\int_{\frac{\pi}{2}}^\infty e^{-st}(-\cos t)\ dt \end{aligned}$$

$$\begin{aligned} \int_{\frac{\pi}{2}}^\infty e^{-st}\cos t\ dt + s^2\int_{\frac{\pi}{2}}^\infty e^{-st}(\cos t)\ dt &= -e^{\frac{-s\pi}{2}} \\ \int_{\frac{\pi}{2}}^\infty e^{-st}(\cos t)\ dt(1+s^2) &= -e^{\frac{-s\pi}{2}} \\ \int_{\frac{\pi}{2}}^\infty e^{-st}(\cos t)\ dt &= \frac{-e^{-\frac{s\pi}{2}}}{(1+s^2)} \end{aligned}$$

[10]

9. We take the Laplace Transform of the DE to obtain:

$$\begin{aligned}
\mathcal{L}\{y''\}+6\mathcal{L}\{y'\}+5\mathcal{L}\{y\} &= 2\mathcal{L}\{\mathcal{U}(t-1)\}\\
s^2Y(s)-sY(0)-Y'(0)+6(sY(s)-Y(0))+5Y(s) &= \frac{2e^{-s}}{s}\\
s^2Y(s)-3s-0+6(sY(s)-Y(0))+5Y(s) &= \frac{2e^{-s}}{s}\\
(s^2+6s+5)Y(s)-3s-18 &= \frac{2e^{-s}}{s}\\
(s^2+6s+5)Y(s) &= 3s+18+\frac{2e^{-s}}{s}
\end{aligned}$$

Therefore

$$Y(s)=\frac{3s+18}{s^2+6s+5}+\frac{2e^{-s}}{s(s^2+6s+5)}$$

Partial fractions are done on the first term of the above:

$$\begin{aligned}
\frac{3s+18}{s^2+6s+5} &= \frac{A}{s+5}+\frac{B}{s+1}\\
3s+18 &= A(s+1)+B(s+5)\\
3 &= A+B\\
18 &= A+5B\\
&= 3-B+5B\\
15 &= 4B\\
B &= \frac{15}{4}\\
A &= 3-\frac{15}{4}\\
&= -\frac{3}{4}
\end{aligned}$$

And for the second term:

$$\begin{aligned}
\frac{C}{s}+\frac{D}{s+5}+\frac{E}{s+1} &= \frac{1}{s(s+5)(s+1)}\\
C(s+5)(s+1)+Ds(s+1)+Es(s+5) &= 1\\
5C &= 1 \;\Rightarrow\; C=\frac{1}{5}
\end{aligned}$$

and

$$\begin{aligned}
5(6C+D+5E) &= 0\\
\frac{6}{5}+D+5E &= 0
\end{aligned}$$

and

$$\begin{aligned}
s^2(C+D+E) &= 0\\
\frac{1}{5}+D+E &= 0\\
1+4E &= 0\\
E &= -\frac{1}{4}
\end{aligned}$$

This gives

$$D = \frac{1}{20}$$

Now

$$\begin{aligned}
Y(s) &= \frac{3s+18}{s^2+6s+5} + \frac{2e^{-s}}{s(s^2+6s+5)} \\
&= \frac{15}{4}\frac{1}{(s+1)} - \frac{3}{4}\frac{1}{(s+5)} + \frac{2e^{-s}}{5s} + \frac{e^{-s}}{10(s+5)} - \frac{1}{2}\frac{e^{-s}}{(s+1)} \\
y(t) &= \mathcal{L}^{-1}\{\frac{15}{4}\frac{1}{(s+1)} - \frac{3}{4}\frac{1}{(s+5)} + \frac{2e^{-s}}{5s} + \frac{e^{-s}}{10(s+5)} - \frac{1}{2}\frac{e^{-s}}{(s+1)} \\
&= \frac{15}{4}e^{-t} - \frac{3}{4}e^{-5t} + \frac{2}{5}\mathcal{U}(t-1) + \frac{1}{10}e^{-5(t-1)}\mathcal{U}(t-1) - \frac{1}{2}e^{-(t-1)}\mathcal{U}(t-1)
\end{aligned}$$

[14]

10. (a)

$$\begin{aligned}
u_t &= \phi(x).G'(t) \\
u_{xx} &= \phi''(x).G(t).
\end{aligned}$$

Substituting into the PDE, we get

$$\begin{aligned}
\phi G' &= k\phi'' G \\
\frac{\phi''}{\phi} &= \frac{1}{k}\frac{G'}{G} = -\lambda.
\end{aligned}$$

Therefore the two ODEs are

$$\begin{aligned}
\phi'' + \lambda\phi &= 0 \\
G' + k\lambda G &= 0.
\end{aligned}$$

(2)

(b) The characteristic equation of $\phi + \lambda\phi'' = 0$ is $r^2 + \lambda = 0$, so $r = \pm i\sqrt{\lambda}$. The solution of the ODE is

$$\phi(x) + A\sin\sqrt{\lambda}x + B\cos\sqrt{\lambda}x.$$

Now

$$\begin{aligned}
\phi'(x) &= \sqrt{\lambda}\big(A\cos\sqrt{\lambda}x - B\sin\sqrt{\lambda}x\big) \\
\\
\phi'(0) &= \sqrt{\lambda}A = 0
\end{aligned}$$

Therefore $A = 0$ and $\phi(x) = B\cos\sqrt{\lambda}x$. Hence

$$\begin{aligned}
\phi'(L) &= -B\sin\sqrt{\lambda}L = 0 \\
\sin\sqrt{\lambda}L &= 0 \\
\sqrt{\lambda}L &= n\pi \text{ for } n = 0, 1, 2, \ldots \\
\lambda_n &= \frac{n^2\pi^2}{L^2}.
\end{aligned}$$

(3)

(c) These $\lambda_n = (n^2\pi^2)/L^2$ are the eigenvalues; the corresponding eigenfunctions $u_n(x,t)$ are given by

$$\begin{aligned} u_n(x,t) &= \phi_n(x)G_n(t) \\ &= B_n \cos\frac{n\pi}{L}x \; e^{\frac{-n^2\pi^2 kt}{L^2}}. \end{aligned}$$

(2)

(d) By the principle of superposition, we see that

$$u(x,t) = \sum_{n=0}^{\infty} B_n \cos\frac{n\pi}{L}x \; e^{\frac{-n^2\pi^2 kt}{L^2}}.$$

(2)

(e) When $t = 0$, we use the boundary condition $u(x,0) = x$ to see that

$$\sum_{n=0}^{\infty} B_n \cos\frac{n\pi}{L}x = x.$$

In other words, we have a Fourier sine series expansion of the function x. From the general theory of Fourier series, we know that the constant term is given by

$$\begin{aligned} \frac{1}{2}\Big[\frac{2}{L}\int_0^L x \; dx\Big] &= \frac{1}{2L}x^2\Big|_0^L \\ &= \frac{L}{2}. \end{aligned}$$

Thus

$$u(x,t) = \frac{L}{2} + \sum_{n=1}^{\infty} B_n \cos\frac{n\pi}{L}x \; e^{\frac{-n^2\pi^2 kt}{L^2}}.$$

(1)

[10]

Total [100]