Lecture 03-04

Program Efficiency & Complexity Analysis

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Algorithm Definition

A finite set of statements that guarantees an optimal solution in finite interval of time

GOOD ALGORITHMS?

- Run in less time

- Consume less memory

But computational resources (time complexity) is usually more important

Measuring Efficiency

- The efficiency of an algorithm is a measure of the amount of resources consumed in solving a problem of size n.
  - The resource we are most interested in is time
  - We can use the same techniques to analyze the consumption of other resources, such as memory space.
- It would seem that the most obvious way to measure the efficiency of an algorithm is to run it and measure how much processor time is needed
- Is it correct ?

Factors

- Hardware
- Operating System
- Compiler
- Size of input
- Nature of Input
- Algorithm

Which should be improved?

RUNNING TIME OF AN ALGORITHM

- Depends upon
  - Input Size
  - Nature of Input
- Generally time grows with size of input, so running time of an algorithm is usually measured as function of input size.
- Running time is measured in terms of number of steps/primitive operations performed
- Independent from machine, OS

FINDING RUNNING TIME OF AN ALGORITHM / ANALYZING AN ALGORITHM

- Running time is measured by number of steps/primitive operations performed

- Steps means elementary operation like
  - ,+, *,<, =, A[i] etc

- We will measure number of steps taken in term of size of input

Simple Example (1)

```
// Input: int A[N], array of N integers
// Output: Sum of all numbers in array A

int Sum(int A[], int N)
{
  int s=0;
  for (int i=0; i< N; i++)
    s = s + A[i];
  return s;
}
```

How should we analyse this?

SIMPLE EXAMPLE (2)

```
// Input: int A[N], array of N integers
// Output: Sum of all numbers in array A

int Sum(int A[], int N){
    int s=0;                          ①

    for (int i=0; i< N; i++)
     ②             ③      ④
        s = s + A[i];
     ⑤          ⑥    ⑦
    return s;
}                 ⑧
```

1,2,8: Once

3,4,5,6,7: Once per each iteration of for loop, N iteration

Total: 5N + 3

The *complexity function* of the algorithm is : $f(N) = 5N + 3$

Simple Example (3) Growth of 5n+3

Estimated running time for different values of N:

N = 10	=> 53 steps
N = 100	=> 503 steps
N = 1,000	=> 5003 steps
N = 1,000,000	=> 5,000,003 steps

As N grows, the number of steps grow in *linear* proportion to N for this function *"Sum"*

What Dominates in Previous Example?

What about the +3 and 5 in 5N+3?

- As N gets large, the +3 becomes insignificant
- 5 is inaccurate, as different operations require varying amounts of time and also does not have any significant importance

What is fundamental is that the time is *linear* in N.

Asymptotic Complexity: As N gets large, concentrate on the highest order term:

- Drop lower order terms such as +3
- Drop the constant coefficient of the highest order term i.e. N

Asymptotic Complexity

- The 5N+3 time bound is said to "grow asymptotically" like N
- This gives us an approximation of the complexity of the algorithm
- Ignores lots of (machine dependent) details, concentrate on the bigger picture

COMPARING FUNCTIONS: ASYMPTOTIC NOTATION

- Big Oh Notation: Upper bound
- Omega Notation: Lower bound
- Theta Notation: Tighter bound

Big Oh Notation [1]

If f(N) and g(N) are two complexity functions, we say

$$f(N) = O(g(N))$$

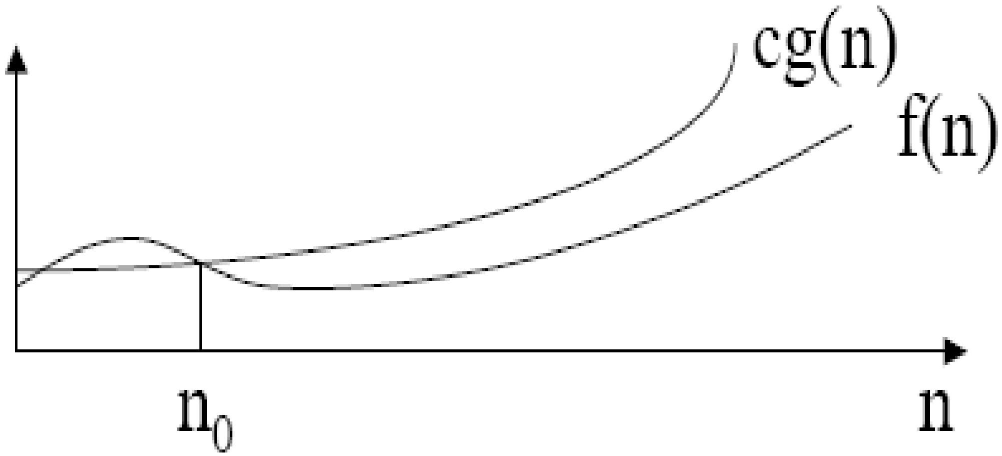
(read "f(N) is order g(N)", or "f(N) is big-O of g(N)")

if there are constants c and N_0 such that for $N > N_0$,

$$f(N) \leq c * g(N)$$

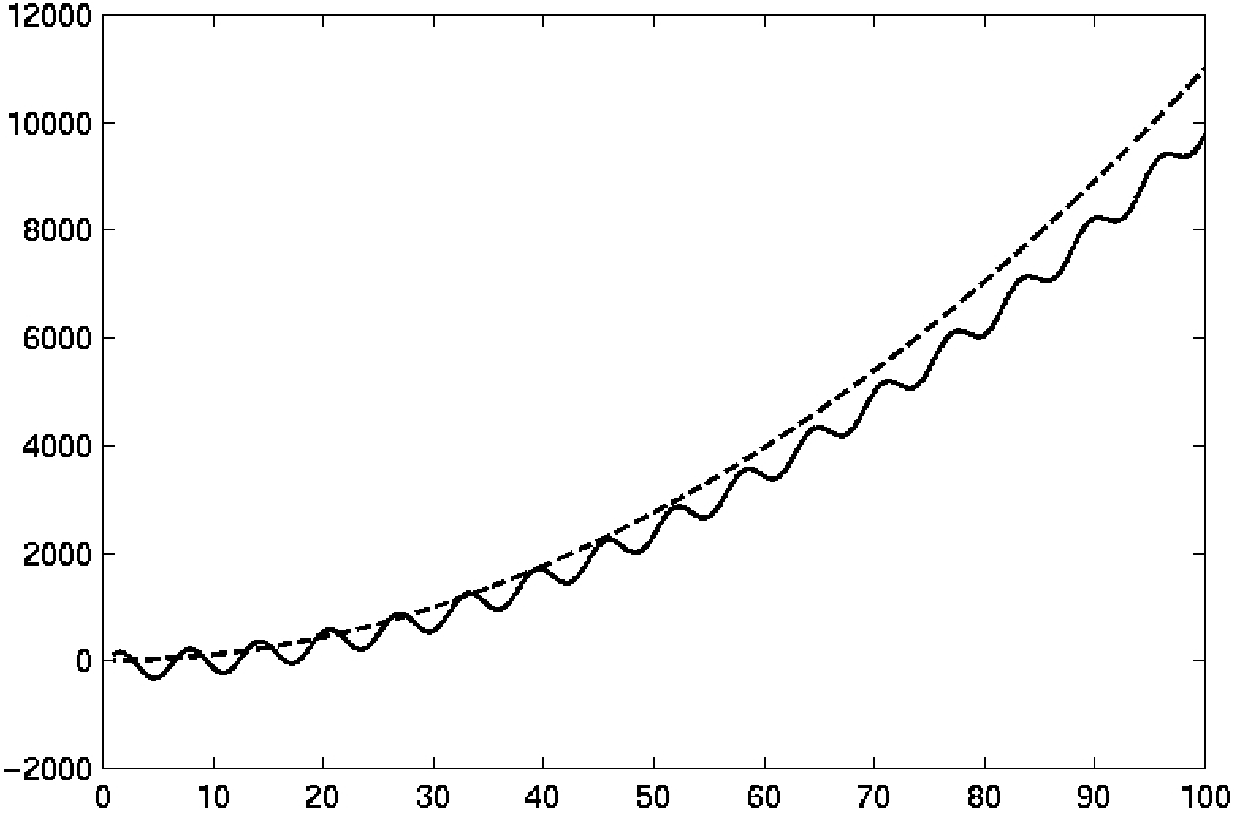
for all sufficiently large N.

Big Oh Notation [2]



- Function cg(n) always dominates f(n) to the right of n_0

O(f(n))

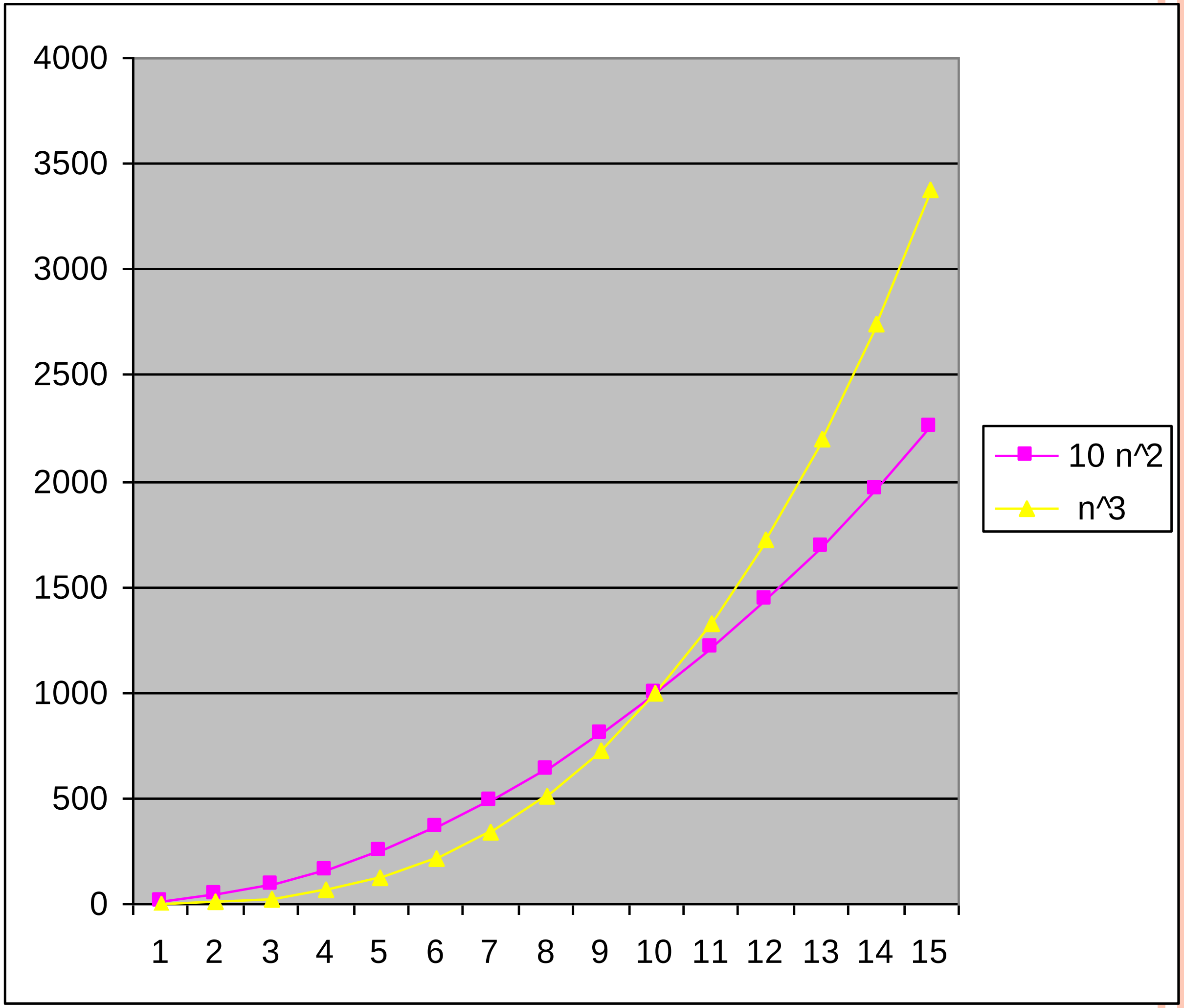


12000
10000
8000
6000
4000
2000
0
−2000
0
10
20
30
40
50
60
70
80
90
100

EXAMPLE (2): COMPARING FUNCTIONS

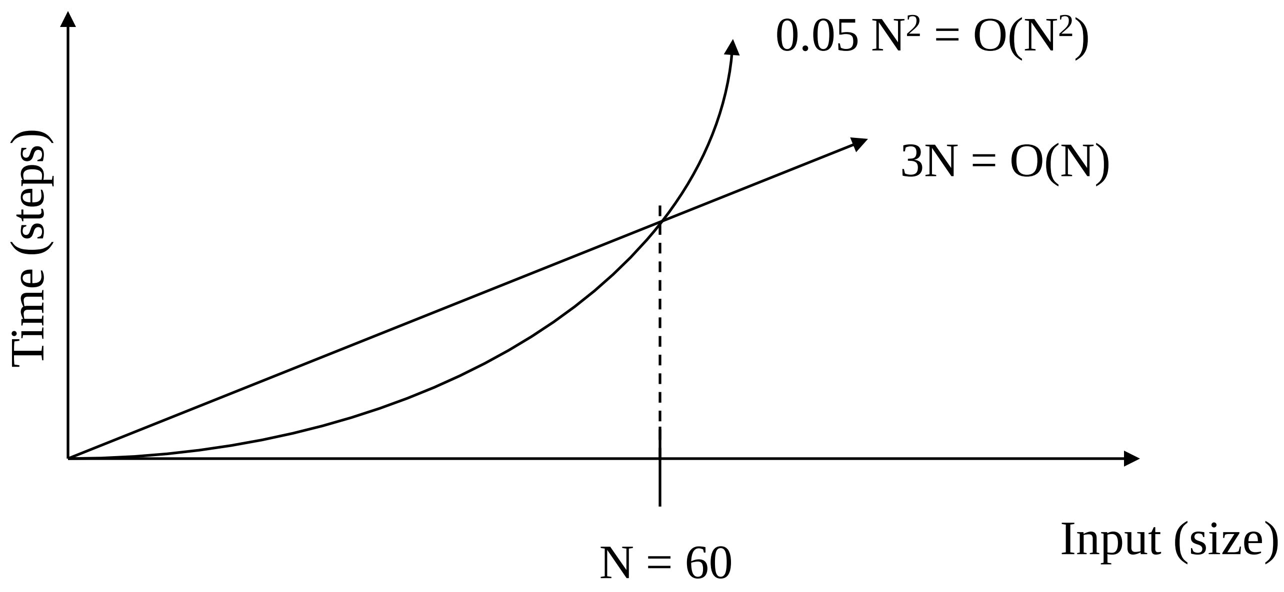
- Which function is better?

  $10\ n^2$ Vs n^3



COMPARING FUNCTIONS

- As inputs get larger, any algorithm of a smaller order will be more efficient than an algorithm of a larger order



Big-Oh Notation

- Even though it is correct to say "7n - 3 is $O(n^3)$", a better statement is "7n - 3 is O(n)", that is, one should make the approximation as tight as possible

- Simple Rule:

  Drop lower order terms and constant factors

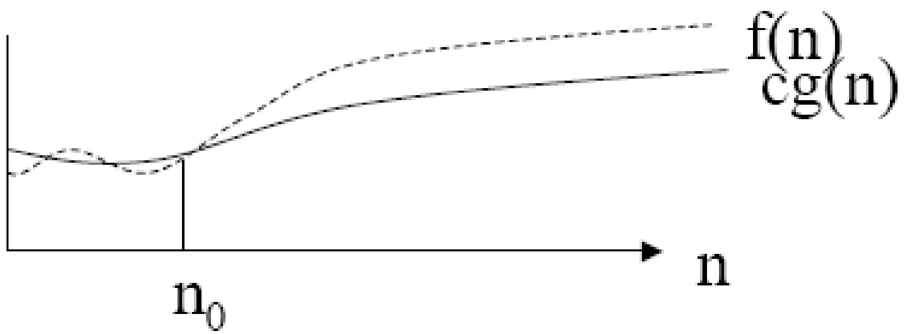
  7n-3 is O(n)

  $8n^2 \log n + 5n^2 + n$ is $O(n^2 \log n)$

Big Omega Notation

- If we wanted to say “running time is at least…” we use Ω
- Big Omega notation, Ω, is used to express the lower bounds on a function.
- If f(n) and g(n) are two complexity functions then we can say:

f(n) is Ω(g(n)) if there exist positive numbers c and n_0 such that 0<=f(n)>=cΩ (n) for all n>=n_0

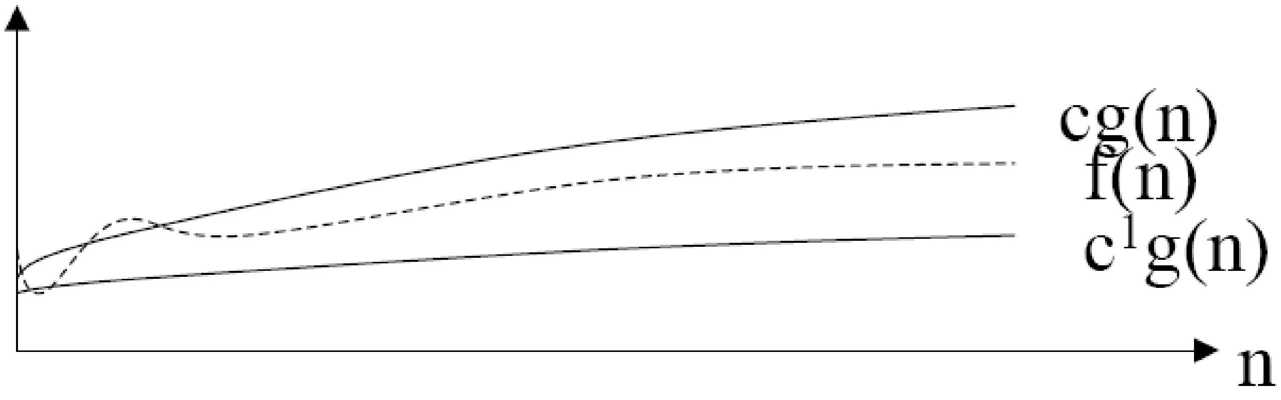


- In this instance, function cg(n) is dominated by function f(n) to the right of n_0

- Example : 3n + 2 = Ω(n)

Big Theta Notation

- If we wish to express tight bounds we use the theta notation, Θ
- $f(n) = \Theta(g(n))$ means that $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$



What does this all mean?

- If $f(n) = \Theta(g(n))$ we say that f(n) and g(n) grow at the same rate, asymptotically

- If $f(n) = O(g(n))$ and $f(n) \neq \Omega(g(n))$, then we say that f(n) is asymptotically slower growing than g(n).

- If $f(n) = \Omega(g(n))$ and $f(n) \neq O(g(n))$, then we say that f(n) is asymptotically faster growing than g(n).

Which Notation do we use?

- To express the efficiency of our algorithms which of the three notations should we use?
- As computer scientist we generally like to express our algorithms as big O since we would like to know the upper bounds of our algorithms.
- Why?
- If we know the worse case then we can aim to improve it and/or avoid it.

Performance Classification

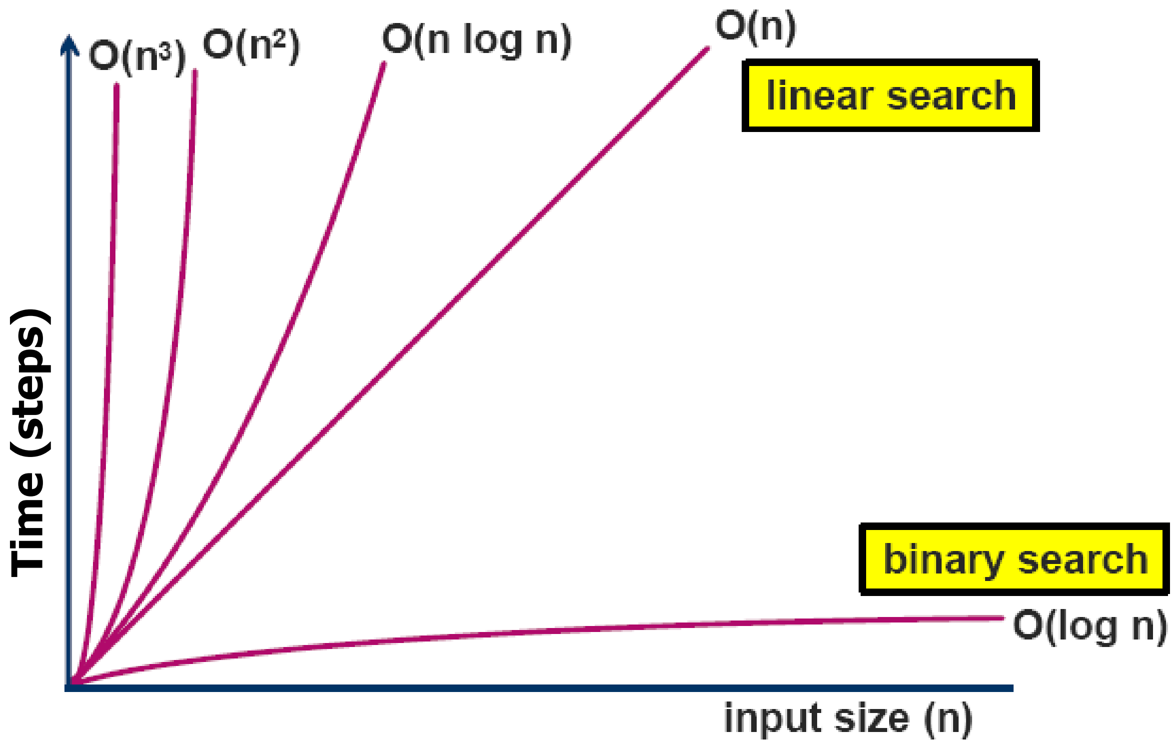
f(n)	Classification
1	***Constant***: run time is fixed, and does not depend upon n. Most instructions are executed once, or only a few times, regardless of the amount of information being processed
log n	***Logarithmic***: when ***n*** increases, so does run time, but much slower. Common in programs which solve large problems by transforming them into smaller problems. Exp : binary Search
n	***Linear***: run time varies directly with ***n***. Typically, a small amount of processing is done on each element. Exp: Linear Search
n log n	**When *n* doubles, run time slightly more than doubles. Common in programs which break a problem down into smaller sub-problems, solves them independently, then combines solutions. Exp: Merge**
n^2	***Quadratic***: when ***n*** doubles, runtime increases fourfold. Practical only for small problems; typically the program processes all pairs of input (e.g. in a double nested loop). Exp: Insertion Search
n^3	***Cubic***: when n doubles, runtime increases eightfold. Exp: Matrix
2^n	***Exponential***: when n doubles, run time squares. This is often the result of a natural, "brute force" solution. Exp: Brute Force. Note: logn, n, nlogn, n^2>> less Input>>Polynomial n^3, 2^n>>high input>> non polynomial

SIZE DOES MATTER[1]

What happens if we double the input size N?

N	log_2N	5N	$N\ log_2N$	N^2	2^N
8	3	40	24	64	256
16	4	80	64	256	65536
32	5	160	160	1024	$\sim 10^9$
64	6	320	384	4096	$\sim 10^{19}$
128	7	640	896	16384	$\sim 10^{38}$
256	8	1280	2048	65536	$\sim 10^{76}$

Complexity Classes



Size does matter[2]

- Suppose a program has run time O(n!) and the run time for n = 10 is 1 second

  For n = 12, the run time is 2 minutes

  For n = 14, the run time is 6 hours

  For n = 16, the run time is 2 months

  For n = 18, the run time is 50 years

  For n = 20, the run time is 200 centuries

Standard Analysis Techniques

- Constant time statements
- Analyzing Loops
- Analyzing Nested Loops
- Analyzing Sequence of Statements
- Analyzing Conditional Statements

Constant time statements

- Simplest case: O(1) time statements

- Assignment statements of simple data types
  int x = y;

- Arithmetic operations:
  x = 5 * y + 4 - z;

- Array referencing:
  A[j] = 5;

- Array assignment:
  ∀ j, A[j] = 5;

- Most conditional tests:
  if (x < 12) ...

ANALYZING LOOPS[1]

- Any loop has two parts:
  - How many iterations are performed?
  - How many steps per iteration?

```
int sum = 0,j;
for (j=0; j < N; j++)
  sum = sum +j;
```

  - Loop executes N times (0..N-1)
  - 4 = O(1) steps per iteration

- Total time is N * O(1) = O(N*1) = O(N)

ANALYZING LOOPS[2]

- What about this **for** loop?

```
int sum =0, j;
for (j=0; j < 100; j++)
   sum = sum +j;
```

- Loop executes 100 times

- 4 = O(1) steps per iteration

- Total time is 100 * O(1) = O(100 * 1) = O(100) = O(1)

ANALYZING LOOPS – LINEAR LOOPS

- Example (have a look at this code segment):

```
int i = 1;
while (i < n)
{
    <<list of sequence statements>>
    i++;
}
```

Executed 1 times

n comparisons

Lets presume there is constant c steps present here; so we have c*n steps

Executed n times

- Efficiency is proportional to the number of iterations.
- Efficiency time function is :

$$f(n) = 1 + (n-1) + c*(n-1) +(n-1)$$

$$= (c+2)*(n-1) + 1$$

$$= (c+2)n – (c+2) +1$$

- Asymptotically, efficiency is : O(n)

Analyzing Nested Loops[1]

- Treat just like a single loop and evaluate each level of nesting as needed:

```
int j,k;
for (j=0; j<N; j++)
  for (k=N; k>0; k--)
    sum += k+j;
```

- Start with outer loop:
  - How many iterations? N
  - How much time per iteration? Need to evaluate inner loop

- Inner loop uses O(N) time

- Total time is N * O(N) = O(N*N) = $O(N^2)$

ANALYZING NESTED LOOPS[2]

- What if the number of iterations of one loop depends on the counter of the other?

```
int j,k;
for (j=0; j < N; j++)
  for (k=0; k < j; k++)
    sum += k+j;
```

- Analyze inner and outer loop together:

- Number of iterations of the inner loop is:
- $0 + 1 + 2 + ... + (N-1) = O(N^2)$

How Did We Get This Answer?

- When doing Big-O analysis, we sometimes have to compute a series like: 1 + 2 + 3 + ... + (n-1) + n
- i.e. Sum of first n numbers. What is the complexity of this?
- Gauss figured out that the sum of the first n numbers is always:

$$\sum_{i=1}^{n} i = \frac{n * (n+1)}{2} = \frac{n^2 + n}{2} = O(n^2)$$

Sequence of Statements

- For a sequence of statements, compute their complexity functions individually and add them up

```
for (j=0; j < N; j++)                       O(N^2)
   for (k =0; k < j; k++)
      sum = sum + j*k;
for (l=0; l < N; l++)                       O(N)
   sum = sum -l;
System.out.print("sum is now"+sum);         O(1)
```

- Total cost is $O(n^2) + O(n) + O(1) = O(n^2)$

Conditional Statements

- What about conditional statements such as

```
if (condition)
        statement1;
else
        statement2;
```

- where statement1 runs in O(n) time and statement2 runs in $O(n^2)$ time?

- We use "worst case" complexity: among all inputs of size n, what is the maximum running time?

- The analysis for the example above is $O(n^2)$

Deriving a Recurrence Equation

- So far, all algorithms that we have been analyzing have been non recursive
- Example : Recursive power method

```
double power( double x, int n) {
    if ( n == 0)
      return 1.0;                    // base calse
    //else
      return power(x, n-1)*x;      // recursive case
}
```

- If N = 1, then running time T(N) is 2
- However if $N \geq 2$, then running time T(N) is the cost of each step taken plus time required to compute power(x,n-1). (i.e. $T(N) = 2+T(N-1)$ for $N \geq 2$)
- How do we solve this? One way is to use the iteration method.

Iteration Method

- This is sometimes known as "Back Substituting".
- Involves expanding the recurrence in order to see a pattern.
- Solving formula from previous example using the iteration method :
- ***Solution*** : Expand and apply to itself :

  Let T(1) = n0 = 2

  T(N) = 2 + T(N-1)

  = 2 + 2 + T(N-2)

  = 2 + 2 + 2 + T(N-3)

  = 2 + 2 + 2 + ……+ 2 + T(1)

  = 2N + 2 remember that T(1) = n0 = 2 for N = 1
- So T(N) = 2N+2 is O(N) for last example.

SUMMARY

- Algorithms can be classified according to their complexity => O-Notation
    - only relevant for large input sizes
- "Measurements" are machine independent
    - worst-, average-, best-case analysis

REFERENCES

Introduction to Algorithms by Thomas H. Cormen

Chapter 3 (Growth of Functions)