Tutorial letter 102/3/2013

Mathematical Modelling

APM1514

Semesters 1 & 2

Department of Mathematical Sciences

CONTENTS:

The Workbook

BAR CODE

Dear student,

This workbook contains exercises dealing with the study material covered in the study guide. The answers, and in some cases full solutions, are given at the end.

You are welcome to contact the lecturer for queries about any of these exercises!

Study Unit 2 EXERCISES

2.1 For each of the following difference equations, classify the equation as either autonomous or not, and as first-order or not.

(a) $a_{n+1} = 2 + a_n$

(b) $a_{n+1} = a_n + (a_{n-1})^n$

(c) $a_{n+1} = na_n$

(d) $a_{n+1} = 2a_{n-1}$

(e) $a_{n+1} = a_n + a_{n-1} + a_{n-2}$

(f) $a_{n+1} = \frac{a_n}{2} + (a_n)^{a_n}$

2.2 Write down a_0, a_1, a_2, a_3 and a_4 when the difference equation and initial value are as given below.

(a) $a_{n+1} = \frac{a_n}{2}, \quad a_0 = 1$

(b) $a_{n+1} = 2a_n + 8, \quad a_0 = 0$

(c) $a_{n+1} = 2a_n (a_n + 3), \quad a_0 = 4$

(d) $a_{n+1} = \sqrt{a_n}, \quad a_0 = 2$

2.3 Write out terms a_0, a_1 and a_2 for the following dynamical systems.

(a) $a_{n+1} = \frac{a_n}{2} - 1, \quad a_0 = 1$

(b) $a_{n+1} = 2a_n + a_n^2, \quad a_0 = 0$

(c) $a_{n+1} = 2a_n - 3, \quad a_0 = 24$

2.4 Find the general solutions to the following difference equations.

(a) $a_{n+1} = \frac{a_n}{2}$

(b) $a_{n+1} = a_n - 8$

(c) $a_{n+1} = \sqrt{a_n}, \quad a_0 = 2$

2.5 Find the solution, that is, an expression for a_n as a function of n, when the difference equation and initial value are as given below.

(a) $a_{n+1} = \frac{a_n}{2}, \quad a_0 = 1$

(b) $a_{n+1} = a_n - 8, \quad a_0 = 0$

(c) $a_{n+1} = \sqrt{a_n}, \quad a_0 = 2$

2.6 For the following systems, find the equilibrium point, if one exists.

(a) $a_{n+1} = 1.1a_n$

(b) $a_{n+1} = -0.7a_n$

(c) $a_{n+1} = a_n^2 - 4a_n$

(d) $a_{n+1} = a_n$

(e) $a_n = 10$

2.7 For the following systems, find the equilibrium point(s), if they exist.

(a) $a_{n+1} = 1.51a_n$

(b) $a_{n+1} = -0.9a_n + 1$

(c) $a_{n+1} = a_n^2 - 4a_n - 2$

(d) $a_{n+1} = a_n^2$

(e) $a_{n+1} = \sqrt{a_n}$

2.8 Find the equilibrium values of and the general solutions to the following difference equations:

(a) $a_{n+1} - a_n = 5$

(b) $a_{n+1} = a_n^2$

(c) $a_{n+1} = (a_n - 1)^2$

2.9 Find the equilibrium points for all possible values of r and b for the system

$$a_{n+1} = ra_n + b.$$

2.10 Find the solution and the equilibrium points for all possible values of k for the system

$$a_{n+1} = a_n + k.$$

2.11 Describe the outcome of the general linear system

$$a_{n+1} = ra_n$$

if we assume that $r < 0.$ is a constant. Investigate all possible negative values of r. You may assume that $a_0 > 0$.

2.12 Write down the solutions to the following linear difference equations.

(a) $a_{n+1} = -2a_n$

(b) $a_{n+1} = 20a_n$

(c) $a_{n+1} = a_n$

(d) $a_{n+1} = 0.1a_n$

(e) $a_{n+1} = -\frac{1}{2}a_n$

2.13 In each of the difference equations given below, with the given initial value, what is the outcome of the solution as n increases?

(a) $a_{n+1} = a_n, \quad a_0 = 1$

(b) $a_{n+1} = 10a_n, \quad a_0 = 0$

(c) $a_{n+1} = \frac{3}{2}a_n, \quad a_0 = -1$

(d) $a_{n+1} = \frac{2}{3}a_n, \quad a_0 = -1$

(e) $a_{n+1} = \frac{1}{2}a_n, \quad a_0 = 0$

(f) $a_{n+1} = 0.99 \cdot a_n, \quad a_0 = 2$

2.14 A difference equation is given by

$$a_{n+1} = a_n^3.$$

(a) If $a_0 = 2$, find a_1 and a_2.

(b) Find all the equilibrium values of the difference equation.

(c) Write down an expression for a_1, a_2 and a_3 as a function of the initial value a_0.

(d) From (c) above, find the general solution to the difference equation.

2.15 Find the equilibrium points of the system

$$x_{n+1} = c - dx_n$$

for all possible values of c and d.

2.16 Consider the difference equation

$$a_{n+1} = 1 - a_n.$$

(a) If $a_0 = 2$, find a_1 and a_2.

(b) Find all the equilibrium values of the difference equation.

(c) Find the general solution to the difference equation.

2.17 A difference equation is given by

$$a_{n+1} = a_n^3.$$

(a) If $a_0 = 2$, find a_1 and a_2.

(b) Find all the equilibrium values of the difference equation.

(c) Is $a_n = a_0^{3n}$ the general solution to the difference equation? Justify your answer!

Study Unit 3 EXERCISES

3.1 Write down the difference equations for the following proportional population models, when time n is measured in years.
(a) A population with 2 000 births and 800 deaths 10 000 individuals per year.

(b) A population with 0.002 births and 0.01 deaths per population member per year.

3.2 For each of the following birth and death rates, write down the difference equation of the corresponding, proportional growth model.
(a) Birth rate $b = 0.2$, death rate $d = 0.05$.

(b) Birth rate $b = 0.01$, death rate $d = 0.1$.

(c) Birth rate $b = 2.0$, death rate $d = 1.0$.

(d) Birth rate $b = 0.2$, death rate $d = 0.2$.

3.3 For each of the following birth and death rates, write down the solution to the corresponding, proportional growth model with initial value $P(0) = 1000$, and explain what the outcome will be as n increases without bound.
(a) Birth rate $b = 0.1$, death rate $d = 0.01$.

(b) Birth rate $b = 0.2$, death rate $d = 0.1$.

(c) Birth rate $b = 1.0$, death rate $d = 1.5$.

(d) Birth rate $b = 0.1$, death rate $d = 0.1$.

3.4 Assume that a proportionally growing population has the death rate $d = 0.1$. What should the birth date be, so that the difference equation describing the growth of the population is

$$P(n+1) = 0.95 \cdot P(n)?$$

3.5 The size of a proportionally growing population at time n is given by

$$P_n = 100 \cdot (1.015)^n ,$$

with n measured in years. If the birth rate was 20 births per thousand individuals per year, how many deaths were there in the population per thousand individuals per year?

3.6 Assume that a population has a birth rate of 3.5 per year and a death rate of 1.5 per year. Find the size of the population after 1, 2 and 5 years, if the initial population size was
(a) 1000

(b) 10 000

3.7 A population grows according to the difference equation

$$P_{n+1} = 0.72 \cdot P_n$$

with an initial value of 20 000.
(a) How big is the population after 5 years?

(b) How many years does it take for the population to reach 2 000?

3.8 A population grows according to the difference equation

$$P_{n+1} = 1.45 \cdot P_n$$

with an initial value of 10 000.
(a) How long does it take for the population to double in size?

(b) How many years does it take for the population to reach 100 000?

3.9 Write down the difference equation to model the following population: The birth rate is $b = 0.8$ per person per year, the death rate is $d = 0.2$ per person per year, and each year 30 percent of the existing population moves out.

3.10 Write down the difference equation to model the following population: The birth rate is $b = 0.1$ per person per year, the death rate is $d = 0.3$ per person per year, and each year 3000 new individuals move into the population.

3.11 Write down the difference equation to model the following population: The birth rate is $b = 0.05$ per person per year, the death rate is $d = 0.01$ per person per year, and each year 100 population members leave the population.

3.12 Write down the difference equation to model the following population: The birth rate is $b = 1.0$ per person per year, the death rate is $d = 0.5$ per person per year, and each year every population member persuades one additional individual to enter the population.

3.13 Consider the following population model with immigration:

$$P_{n+1} = 1.02 \cdot P_n + 1000.$$

Here, P_n denotes the size of the population after n years. Assume that the initial population size, in year $n = 0$, is 10 000.

(a) Calculate the value of P_1 from the value of P_0, the value of P_2 from P_1 and the value of P_3 from P_2.

(b) Use the values you calculated in (a) to prove that the solution to the difference equation is NOT given by $P_n = (1.02)^n P_0 + n * 1000$. *[Finding and using the correct solution does not form a part of this module – if you are interested, it is given later on in this study unit, in the part dealing with the cheque account model!]*

3.14 Assume that the population of a country has a birth rate is $b = 1.1$ per person per year, and a death rate of $d = 0.5$ per person per year.

(a) Assume that 1000 people move out of the country each year. What should the initial population be, to ensure that the population will always increase?

(b) Assume that instead, k percent of the population present at the end of each year moves out. What should k be to guarantee that for any initial population size, the population will forever stay constant?

3.15 Write down the difference equations modelling the following investment accounts. Use A_n to denote the amount of money at the end of month n.

(a) An account paying interest at the rate of 2% for the amount on the account during each month, with interest paid at the end of each month.

(b) An account paying interest at the rate of 10% for the amount on the account during each month, with interest paid at the end of each month.

(c) An account paying interest at the rate of 11% for the amount on the account during each month, with interest paid at the end of each month.

3.16 Write down the solutions to the difference equations in the previous question, 3.15 — that is, an expression for A_n, in terms of n and the initial deposit A_0.

3.17 An investment account pays interest at the rate of 2% per month. If the initial deposit was R10 000, how much money will there be in the account

(a) after 3 months,

(b) after 6 months,

(c) after 24 months?

3.18 Assume that I deposit R50 000 into an investment account which pays interest at the rate of 5% per month. How many months will it take until I will I have more than R60 000 on the account?

3.19 Assume that I deposit R20 000 into an investment account. How much money will I have on the account after 6 months,

(a) if the interest rate is 1% per month,

(b) if the interest rate is 5% per month,

(c) if the interest rate is 10% per month?

3.20 Assume that I deposit R20 000 into an investment account. How many months will it take until I will I have more than R25 000 on the account,

(a) if the interest rate is 1% per month,

(b) if the interest rate is 5% per month,

(c) if the interest rate is 10% per month?

3.21 An amount of R15 000 is deposited into a bank account, which pays interest at the rate of 5% per month. The amount of R500 is withdrawn from the account at the end of each month. Calculate the money in the bank account at the end of month one, month two and month 3.

3.22 An amount of R25 000 is deposited into a bank account, which pays interest at the rate of 10% per month. Assuming that we withdraw R4000 from the account each month, after how many months is the amount of money on the account less than R20 000?

3.23 An amount of R150 000 is deposited into a bank account, which pays interest at the rate of 1% per month. The amount of R2000 is withdrawn at the end of each month.
(a)Write down the difference equation describing the model.

(b)Find the equilibrium point of the model.

(c)Use the equilibrium point to determine whether there will always be money on the bank account.

3.24 Consider an investment in a bank account with a 1% monthly interest rate. Let us assume that the amount of R1000 is withdrawn from the account each month. Calculate the money in the bank account at the end of six months, if the initial investment was:
(a)R99 000

(b)R100 000

(c)R101 000

3.25 Assume that a cheque account pays out interest at the rate of 5% per month. Assuming that we wish to withdraw the amount of R2000 from the account every month, what should the initial deposit amount of money on the account be, so that there will always be money on the account?

3.26 If I deposit R200 000 onto an account which pays out interest at the rate of 1% per month, how much money can I withdraw from the account each months to without ever emptying the account?

3.27 Write down a difference equation to model the following situation: A loan from a bank, where interest is charged at the rate of 1.5% per month, and a repayment of R500 is made at the end of each month. Use A_n to denote the loan amount still owing at the end of the nth month.

3.28 Consider a loan of R150 000, with interest charged at the rate of 2.5% per month, and repayments of R5000 made at the end of each month. Let A_n to denote the loan amount still owing at the end of the nth month.
(a) Write down the difference equation for A.

(b) Find the equilibrium point of the system.

(c) Will the loan ever be paid off? Justify your answer!

3.29 For a home loan of R200 000, with interest charged at 1.1% per month, what should the minimum monthly repayment be to ensure that the loan will eventually be paid off?

3.30 For a loan of R10 000, repayments of R500 are made at the end of each month. What should the interest rate be, if we wish to ensure that the loan will eventually be paid off?

3.31 Formulate a difference equation to model the following situation: A savings account which pays 2% interest at the end of each month, with an initial investment of R5000 and another R200 added at the end of each month.

3.32 A debt of R500, on which interest is charged at a rate of 5% at the end of each month. The debt is being paid back at a rate of R30 per month.

3.33 Formulate a difference equation to model the following situation: A car loan of R40 000 on which interest is charged at the rate of 1.5% at the end of each month, and repayments are made at a rate of R1500 per month.

3.34 If you invest an amount of R10 000 in a bank account which pays you interest at a rate of 15% at the end of each year, what is the fixed sum of money you can withdraw at the end of each year without eventually emptying the account?

3.35 Consider a bank account which pays interest at a monthly rate of 5% on the money on the bank account during that monthly, with interest paid at the end of each month. Assume that the initial deposit is R30 000, and that at the end of each month, after the interest has been paid, one third of all the money then on the account is withdrawn.
(a) Write down the difference equation for the model.

(b) Will the amount of money on the account increase or decrease?

3.36 How do the answers to (a) and (b) in the previous question change, if the one third of all the money is instead withdrawn at the beginning of each month?

3.37 Consider a bank account which pays interest at a monthly rate of 1% on the money on the bank account during that monthly, with interest paid at the end of each month. Assume that the initial deposit is R10 000, and that at the end of each month, after the interest has been paid, the fixed amount of R800 is withdrawn.
(a) Write down the difference equation for the model.

(b) Will the amount of money on the account increase or decrease?

3.38 How do the answers to (a) and (b) in the previous question change, if the R800 is withdrawn at the beginning of the month instead?

3.39 Assume that at the beginning of a year, I deposit R5000 into a bank account. After that, at the beginning of every month (starting in month 1) I deposit another R300 onto the bank account. Also, at the end of each month I withdraw R200. Write down the difference equation for the system, and calculate the amount of money on the account at the end of month 2, in each of the following cases.
(a) If no interest is paid.

(b) If the interest is paid at the rate of 2% at the end of each month.

3.40 The chlorine in a swimming pool breaks down such that only three-quarters of the previous day's quantity remains the following day. If the original quantity was 20 kilograms, how much will be left after 10 days?

3.41 Assume that an unscrupulous petrol station owner dilutes diesel with paraffin as follows: The diesel/paraffin mixture is stored in a 200 litre container. Every day the owner removes 20 litres from the mixture in the container, and adds 20 litres of pure paraffin. Assume that initially the container has pure diesel in it, and that the diesel is at all times well mixed with the paraffin in the container.
(a) Write down a difference equation to model the amount of diesel in the container. Use D_n to denote the amount of diesel (in litres) in the container after n days. Specify also down the initial value D_0

(b) How many days will it take until the container has less than 100 litres of diesel in it?

3.42 Consider a house loan of R$200\ 000$, on which interest is charged at the rate of 1% per month at the end of the month; and a repayment of R$3\ 000$ is made at the end of each month.

Let A_n denote the amount of money still owing at the end of month n, with A_0 denoting the original loan amount.
(a) Write down the difference equation for A_n (that is, an expression for A_{n+1} in terms of A_n).

(b) Find the equilibrium point of the system.

(c) What does the equilibrium point tell us about whether the house loan will eventually be paid off or not?

(d) Does the difference equation in (a) change (and if it does, how), if we make each of the following changes:
i. Make the initial loan amount $300\ 000$.

ii. Change the repayment to $1\ 600$ per month.

iii. Change the interest rate to 2% per month.

3.43 Consider a cheque account. Initially, the amount of R$10\ 000$ is deposited on the account. At the end of each month, interest is added at the rate of 2% per month to the amount of money that was on the account during the month. An additional deposit of N is added to the account at the end of each month, and the fixed amount R500 is withdrawn from the account. Let A_n denote the amount of money in the account at the end of month n, with A_0 denoting the original amount.
(a) Write down the difference equation for A_n (that is, an expression for A_{n+1} in terms of A_n). Write also down the initial value, A_0.

(b) How should the value N be chosen such that there will always be money on the account? Justify your answer!

Study Unit 4 EXERCISES

4.1 Which of the following differential equations are separable?
(a)

$$\frac{dy}{dt} = 2ty - t^2y^2$$

(b)
$$\frac{dy}{dx} = \cos y \sin x$$
(c)
$$\frac{dy}{dx} = 1 - y^2$$
(d)
$$\frac{dy}{dx} = 1 - xy$$
(e)
$$\frac{dy}{dx} = e^{x+y}$$
(f)
$$\frac{dy}{dx} = x - xy$$

4.2 Find the general solutions to the following differential equations.

(a)
$$\frac{dy}{dt} = \frac{1 + t + t^2}{y^2}$$
(b)
$$\frac{dy}{dt} = \frac{5}{y^4}$$
(c)
$$\frac{dy}{dt} = \frac{t}{2y}$$
(d)
$$\frac{dy}{dt} = \frac{2 + t}{y}$$
(e)
$$\frac{dy}{dt} = e^y$$
(f)
$$\frac{dy}{dt} = \frac{e^t}{e^{2y}}$$
(g)
$$\frac{dy}{dt} = y^2$$
(h)
$$\frac{dy}{dt} = 2ty^2 - y^2$$
(i)
$$\frac{dy}{dt} = 4 - 5y$$
(j)
$$\frac{dy}{dt} = yt - y$$
(k)
$$\frac{dy}{dt} = \frac{2y}{5t}$$
(l)
$$\frac{dy}{dt} + y + 1 = 0$$
(m)
$$\frac{dy}{dt} = 1 - y + t - yt$$

(n)

$$\frac{dy}{dt} = 2(1-y)$$

(o)

$$\frac{dy}{dt} = 10 + y$$

4.3 Find the solutions to the following initial value problems.

(a)

$$\frac{dy}{dt} = 1 + y$$
$$y(0) = 3$$

(b)

$$\frac{dy}{dt} = t + t^2$$
$$y(1) = 2$$

(c)

$$\frac{dy}{dt} = \frac{t}{2y}$$
$$y(0) = -2$$

(d)

$$\frac{dy}{dt} = 2 - 3y$$
$$y(0) = -5$$

(e)

$$\frac{dy}{dt} = 10 - 2y$$
$$y(0) = 200$$

(f)

$$\frac{dy}{dt} = y - 5$$
$$y(1) = 1$$

(g)

$$\frac{dy}{dt} = t(1+y)$$
$$y(0) = 2$$

(h)

$$\frac{dy}{dt} = 2 - 8y$$
$$y(0) = -50$$

4.4 Solve the following differential equation:

$$\frac{dy}{dt} = ty$$

4.5 Solve the following differential equation:

$$\frac{dx}{dt} = \frac{t^2}{x^3}$$

4.6 Solve the following differential equation:

$$t - y^2\frac{dy}{dt} = 0$$

4.7 Solve the following differential equation:

$$\frac{dy}{dt} = \frac{-t^3}{(y+1)^2}$$

4.8Solve the following differential equation:

$$\frac{dp}{dt} = \frac{p}{t}$$

4.9 Solve the following differential equation:

$$\frac{dy}{dt} - 2y + 1 = 0$$

4.10 Solve the following initial value problem:

$$\frac{dx}{dt} = 1 - x \quad \text{with initial condition} \quad x(1) = 3$$

4.11 Solve the following initial value problem:

$$\frac{dy}{dt} = \frac{2t}{y} \quad \text{with initial condition} \quad y(1) = -2$$

4.12 Solve the following initial value problem:

$$\frac{dy}{dt} = -\frac{t}{2y} \quad \text{with initial condition} \quad y(1) = 2$$

4.13 Solve the following initial value problem:

$$\frac{dy}{dt} = -\frac{1}{y}t^5 \quad \text{with initial condition} \quad y(1) = -1$$

4.14 Solve the following initial value problem:

$$\frac{dy}{dt} - 24y\left(t^2 - t\right) = 0, \quad y(1) = -e$$

4.15 Solve the following initial value problem:

$$t^3 - y\frac{dy}{dt} = 0, \quad y(1) = -1$$

4.16 Solve the following initial value problem:

$$\frac{dy}{dt} = 2ty, \quad y(2) = 1$$

Study Unit 5 EXERCISES

In the following problems, the population growth is assumed to follow the Malthusian growth model.

5.1 Find the parameter k in the Malthusian model for describing the following situations, and write down the differential equation for the model.

(a) A population where there are 50 births and 20 deaths per 1 000 individuals per year

(b) A population where there are 0.01 births and 0.04 deaths per one individual per year

(c) A population where there are 400 births and 250 deaths per 100 000 individuals per year

(d) A population where there are 2 births and no deaths per 100 individuals per year

5.2 Predict the outcome for each of the following Malthusian models:

(a) $\frac{dP}{dt} = 5P,\ P_0 = 10$

(b) $\frac{dP}{dt} = -0.5P,\ P_0 = 0$

(c) $\frac{dP}{dt} = 100P,\ P_0 = 0$

(d) $\frac{dP}{dt} = -0.01P,\ P_0 = 3$

(e) $\frac{dP}{dt} = -4P,\ P_0 = 500$

5.3 Find the solution functions $P(t)$ for the models in the previous question.

5.4 Find the size of the population at time $t = 10$ for the value $P(t)$ for the models in the previous question.

5.5 Assume that a population grows according to the Malthusian model, with $k = 0.01$ and $P_0 = 10000$ with time measured in years.

(a) Find the values of $P(5)$ and $P(50)$.

(b) Find the value of t for which $P(t) = 100000$.

5.6 Assume that a population grows according to the Malthusian model, with $k = -0.1$ and $P_0 = 100$

(a) Find the values of $P(10)$ and $P(100)$.

(b) How long does it take until $P(t) = 1$?

(c) Prove that we will never have $P(t) = -100$.

5.7 In each of the following Malthusian populations, find the growth constant k from the given values.

(a) $P_0 = 5000$, $P(5) = 20\,000$.

(b) $P_0 = 1000$, $P(10) = 1$.

(c) $P_0 = 1$, $P(10) = 1000$.

5.8 Bacteria grow in a culture. If 100 are present initially and 400 are present after 1 hour, how many bacteria are present after 7 hours?

5.9 A bacterial culture increases from 10 to 6000 in 10 hours. How many bacteria were present after 3 hours?

5.10 The Microtus Arvallis Pall is a species of rodent that reproduces very rapidly. Assume that there are two rodents (male and female) present at time $t = 0$. Let the time units be months and let $k = 0.4$ per month. Calculate the number of rodents at the end of 2, 6 and 10 months respectively, if the Malthusian model applies.

5.11 Fruit flies are being bred in an enclosure that holds a maximum of 640 flies. If the population grows according to the Malthusian model, with a growth constant $k = 0.05$ when time is measured in days, how long will it take for an initial population of 20 flies to multiply until the enclosure is full?

5.12 If the size of a population which grows according to the Malthusian model was 10000 in the year 2000 and 100000 in the year 2020, how big will the population be in the year 2025?

5.13 In 1980 the population of a town was $400\,000$ and in 2000 it was $20\,000$. If the population can be modelled by the Malthusian model, then
(a) what would the population have been in year 2010?

(b) when would the population be $1\,000$?

5.14 Assume that a population of a town was $200\,000$ in the year 1950 and 2000 in the year 2000. Assuming that the Malthusian model is valid,
(a) when will the population size be 200?

(b) when was the population size 20 000 individuals?

5.15 Assume that a population of a town was $200\,000$ in the year 2000 and 2000 in the year 1950. Assuming that the Malthusian model is valid,
(a) when was the population size 200?

(b) when will the population size be 2 000 000?

5.16 We are given the population readings:

$$\begin{aligned} P &= 110\,000 \text{ in } 1980, \\ P &= 175\,000 \text{ in } 1990, \\ P &= 220\,000 \text{ in } 2000. \end{aligned}$$

Based on this information, does the population grow according to a Malthusian model? Justify your answer.

5.17 Find the doubling time of a Malthusian population with $k = 0.006$.

5.18 Find the value of k for a Malthusian population which doubles every 200 years.

5.19 Do the following conversions between the annual growth rate and the growth constant:
(a) Find the value of k if a Malthusian population grows by 1.5% per year.

(b) Find the value of k if a Malthusian population has an annual growth of 30%.

(c) Find the annual growth as a percentage for a Malthusian population with growth constant $k = 0.02$.

(d) By how many percentages does a Malthusian population grow per year, if its growth constant is $k = 0.1$?

5.20 In 1950 the population of a town was 40×10^3 and in 1990 it was 80×10^3. If the population can be modelled by the Malthusian model, then what is the annual growth rate as a percentage?

5.21 If the population size of a Malthusian population at time $t = 10$ is $P(10) = 100\,000$, and if its rate of change at time $t = 10$ is 2500 individuals per time unit, what is the value of the growth constant k?

5.22 If a population grows according to the Malthusian model, with population $P_0 = 2 \times 10^6$ at time $t = 0$, find the value of the growth constant k if the annual growth rate is 2% per year.

5.23 The population of a town grows according to the Malthusian model. The population doubles every 200 years. Find

the value of the growth constant k, and use it to answer the following questions:
(a) How long does it take for the population to grow by 50%?

(b) How long does it take for the population to triple?

5.24 Find the growth constant k for Malthusian population where $P_0 = 600\ 000$ at time $t = 0$ and the population grows by $20\ 000$ from $t = 0$ to $t = 2$

5.25 Find the growth constant k for Malthusian population where $P_0 = 150\ 000$ at time $t = 0$ and the rate of change of the population at time $t = 0$ is -500 per year.

5.26 Assume that the population of a town grows according to the Malthusian model with $k = 0.02$. The population was 150 000 in year 2000. Find the rate of change of the population in year 2000, and in year 2010.

5.27 Find the value of the growth constant k for the following Malthusian populations:
(a) A population which grows by 10% per year.

(b) A population which shrinks to 10% of its original size over each year.

(c) A population which doubles every 3 years.

(d) A population which becomes 10 times larger than before over every 100 years.

(e) A population which decreases by 30% each year.

(f) A population which increases by 30% each year.

5.28 Find the annual growth rate as a percentage for Malthusian populations with the following growth constants:
(a) $k = 0.5$.

(b) $k = -0.5$.

(c) $k = 1$

5.29 Find the growth constant for a population with an annual growth rate of 2% per year.

5.30 If a population grows by 4% each year, how long does it take for it to double?

5.31 If a populations doubles every 50 years, what is its annual growth rate?

5.32 The population of a town grows according to the Malthusian model. If the population doubling time is 50 years, how long does it take for the population to grow by 10%?

5.33 A Malthusian population has an annual growth rate of 1% per year. If its initial value, in year 2000, is 300 000, calculate the following:
(a) The size of the population in year 2001.

(b) The rate of change of the population in year 2000.

(c) The rate of change of the population in year 2001.

(d) The time it takes until the population reaches 1 million.

(e) The doubling time of the population.

5.34 Let $P(t)$ be a population that grows according to the Malthusian model and let $M > 1$ be a fixed number. Prove that the time T it takes for the population to grow from P_0 at time t_0 to $M \cdot P_0$ at time $(t_0 + T)$ is independent of P_0 and t_0.

5.35 Assume that the population of South Africa was 47 million at the beginning of 2007 and that it follows the Malthusian model approximately, with a growth rate of 2.2 percent per year.
(a) When will the population reach 80 million, if the growth rate does not change?

(b) What did this model predict the population of South Africa to be in 2010?

5.36 If a population grows according to the Malthusian model, with population $P_0 = 500\ 000$ at time $t = 0$, find the value of the growth constant k in the following cases:
(a) If the annual growth rate is 2% per year.

(b) If the population grows by $20\ 000$ from $t = 0$ to $t = 2$.

(c) If the rate of change of the population at time $t = 0$ is $2\ 000$ per year. (Remember that the rate of change is the value of dP/dt!)

5.37 The population of a town was $150\ 000$ in year 1980 and $90\ 000$ in year 1990. The population is assumed to obey the Malthusian model.
(a) Find the growth constant k.

(b) When will the population be $10\ 000$?

(c) According to the model, what will the population be in year 2230?

(d) Write down an expression for $P(t)$, the size of the population after t years, if $t = 0$ in year 2000.

5.38 The population of a town grows according to the Malthusian model. The population doubles every 100 years. Find the value of the growth constant k, and use it to answer the following questions:
(a) How long does it take for the population to grow by 30%?

(b) How long does it take for the population to triple?

5.39 If a population grows according to the Malthusian model, with population $P_0 = 2 \times 10^6$ at time $t = 0$, find the value of the growth constant k in the following cases:
(a) If the annual growth rate is 2% per year.

(b) If the population grows by 20 000 from $t = 0$ to $t = 2$.

(c) If the rate of change of the population at time $t = 0$ is 2 000 per year.

5.40 The population of a town was $10\ 000$ in year 1980 and $90\ 000$ in year 1990. The population is assumed to obey the Malthusian model.
(a) Find the growth constant k.

(b) When will the population be $200\ 000$?

(c) According to the model, when was the population size 10?

(d) Write down an expression for $P(s)$, the size of the population after s years, if $s = 0$ in year 2000.

5.41 The population of Russia at the start of 1980 was 255×10^6 with growth constant $k_1 = 0.012$. In the same year the population of the USA was 225×10^6 with growth constant $k_2 = 0.007$. When will the population of Russia be twice that of the USA?

5.42 The populations of two countries, Country A and Country B, are assumed to follow the Malthusian model. In the year 1960, the population of Country A was 1.2×10^7 and the population of Country B was 4×10^5. In the year 1990 the population of Country A was 8×10^6 and the population of Country B was 7×10^5.
(a) When will the population of Country B reach 1×10^8?

(b) When will the populations of the two countries be equal in size?

5.43 The populations of Country A and Country B both grow according to the Malthusian model, Country A with a doubling time 50 years and Country B with a doubling time 80 years. If the sizes of the populations were the same in year 2000, what was the ratio of the population of Country B to the population of Country A in 1950? What will the ratio be in 2050?

5.44 Assume that a radioactive substance has a half-life (see page 117) of 2000 years.
(a) Find the value of the constant of decay, k.

(b) If 10 grams of the substance remains today, when was there 50 g of the substance?

(c) If 10 grams of the substance remains today, when will there be 1 g of the substance left?

5.45 A radioactive substance decays from 8g to 7g in one hour. Find its decay constant k, and then calculate its half-life.

5.46 A certain radioactive substance has a half-life of 3 years. If 10g is present initially, how much of the substance remains after 9 years?

5.47 Radium has a half-life of 1690 years. If 10% of an original quantity y_0 remains today, when was the y_0 formed originally?

5.48 Assume that a radioactive substance has a half-life of 2500 years. If 20 grams of the substance remains today, when was there 100 grams of the substance?

5.49 Assume that a radioactive substance decays with a constant of decay $k = 0.002$ when time is measured in years.
(a) How long does it take for the substance to decay from 5 grams to 2 grams?

(b) If 10% of an original quantity N_0 is present today, when was N_0 formed initially?

(c) Explain why in (b) above, we do not need to know the exact value of N_0.

5.50 Measurements show that 200 kg of radioactive waste removed from a nuclear reactor in 1970 has decayed to 195 kg in 1995.
(a) How much of this radioactive material will be left in the year 2095?

(b) When will there be only 20 kg left?

5.51 Assume that a radioactive substance has a half-life of 2000 years. If 10 grams of the substance remains today,
(a) When was there 50 g of the substance?

(b) When will there be 1 g of the substance left?

5.52 The concentration of a certain chemical in the blood stream is observed, and it is found that the decrease in the concentration is proportional to the concentration itself. Thus, the quantity of the chemical in the blood can be modelled by the decay equation

$$\frac{dC}{dt} = -aC$$

where $C(t)$ is the concentration of the chemical in the blood at time t, and a is a positive constant. If the initial concentration was 0.1 milligrams per millilitre of blood, and 3 days later the concentration was 0.054 milligrams per millilitre, how long will it take for the concentration to reach the acceptable level of 0.0001 milligrams per millilitre?

5.53 Which of the following statements are true, and which are false? You must be able to justify your answers!
(a) The more there is of a radioactive substance, the greater its rate of decay is.

(b) If the growth constants of two Malthusian populations A and B are k_A and k_B, and if $k_A = 2 * k_B$, then after 10 years the population A is two times larger than the population B.

(c) If the growth constants of two Malthusian populations A and B are k_A and k_B, and if $k_A = 2 * k_B$, then the population A takes twice as long to double as population B.

Study Unit 6 EXERCISES

6.1 Write down both equilibrium points of the logistic models with the following parameters.
(a) $a = 0.002$ and $b = 2.4 \cdot 10^{-8}$.

(b) $a = 0.02$ and $b = 0.1$.

(c) $a = 0.0001$ and $b = 0.00005$.

6.2 Draw the phase lines of the logistic modules, if the parameters are as given:
(a) $a = 0.048$ and $b = 0.0002$.

(b) $a = 0.02$ and $b = 0.0001$.

(c) $a = 0.001$ and $b = 0.00002$.

6.3 In each of the following cases of parameter values, draw the phase line of the corresponding logistic model and read from the phase line the outcomes for a solution which starts at (i) $P_0 = 100$, (ii) $P_0 = 1000$.
(a) $a = 0.001$, $b = 0.0005$

(b) $a = 0.1$, $b = 0.0003$

(c) $a = 0.01$, $b = 5 \times 10^{-6}$.

6.4 In each of the following cases of parameter values, use the phase line to predict the outcome. for a solution which starts at (i) $P_0 = 10$, (ii) $P_0 = 50$.
(a) $a = 0.03$, $b = 0.001$

(b) $a = 0.01$, $b = 0.001$

(c) $a = 0.012$, $b = 0.0002$

6.5 Write down the solutions $P(t)$ as functions of time t and the initial value P_0 for the following logistic models.
(a) $a = 0.2$, $b = 0.001$

(b) $a = 0.001$, $b = 0.0005$

6.6 Assume that $P(t)$ obeys the logistic model, with $a = 0.002$ and $b = 2.4 \cdot 10^{-8}$. If $P_0 = 12\ 000$, find $P(10)$ and $P(100)$.

6.7 Assume that $P(t)$ obeys the logistic model, with $a = 0.01$ and $b = 0.0002$. If $P_0 = 100\ 000$, find $P(2)$ and $P(5)$.

6.8 If the human population was estimated to be 3.34×10^9 in 1965 and the growth was assumed to be logistic with $a = 0.029$ and $b = 2.695 \times 10^{-12}$, calculate the population in the year 2000.

6.9 In a logistically growing population, we have

$$P_0 = 190\ 000, \quad P(2) = 170\ 000, \quad a = 0.01.$$

Find the value of b, and the limit population size as $t \to \infty$.

6.10 In a logistic population, find the values of the parameters a and b, if for small parameter values, the population behaves like a Malthusian model with parameter $k = 0.016$, while the population growth approaches zero as the population size $P = 300\ 000$ is approached.

6.11 A population of birds is growing according to the logistic growth model, with parameters a and b. It is known that $a = 1.58$. As the population is observed, it is seen that its size grows fast initially, but the growth slows down until a maximum level of 6000 birds is reached. Find the value of b for the model.

6.12 A logistic model of the human population growth is to be constructed. It is estimated that the constant a is 0.029 if t is measured in years. At the start of a given year the population was $P = 3.34 \times 10^9$, and it was calculated that the actual rate of increase at that moment was 2% of the population per year. Considering that the actual rate of population increase at any given moment is given by the equation $\frac{dP}{dt} = aP - bP^2$, find the value of b and calculate the equilibrium value of $P(t)$.

6.13 A farmer decides to breed fish in a special pond built for the purpose. Suppose he buys 1000 kg of a species of fish. He is told by experts that this species grows according to a logistic population model with $a = 1/10$. If the population doubles in 12 months, calculate the value of b. Then calculate the limiting population size in kilograms.

6.14 A population (Population 1) is known to obey logistic growth with constants a and b. It is known that $a = 2.86$, and when the population is observed, it is seen that as $t \to \infty$, the population approaches the limit value of 2500.
(a) Calculate the value of b.

(b) Assume that another logistically growing population (Population 2) has the same parameter a value as Population 1, but the value of b is 150 times larger than the value of b in Population 1. If Population 2 starts off with an initial population of 30, will the population subsequently increase or decrease?

6.15 Assume that $P(t)$ follows the logistic growth model, with $a = 0.1$ and $b = 0.02$. Assume that the initial population is $P(0) = 100$.
(a) Find $P(10)$ and $P(20)$.

(b) Find the rate of increase of the population at times $t = 0$, $t = 10$ and $t = 20$.

6.16 A population behaves according to the logistic model. If the population is observed, it is seen that it grows, and that the maximum rate of growth is reached when the size of the population is $P = P^*$. What will the limit population size be?

6.17 In a logistically growing population, we have

$$P_0 = 10\ 000, \quad P(1) = 12\ 000, \quad a = 0.25.$$

Find the value of b, and the limit population size as $t \to \infty$.

6.18 Assume that $P(t)$ follows logistic growth, with initial value $P_0 = 200$, $a = 0.5$ and $b = 0.02$.
(a) Find $P(1)$ and $P(2)$.

(b) Find the rate of change of P at the times $t = 0$, $t = 1$ and $t = 2$.

(c) Draw the phase line of this population model.

(d) Plot a sketch of the solution curve $P(t)$ when $P_0 = 200$, over the interval $0 \leq t \leq 3$. Use the information in your answers to (a), (b) and (c) to make the sketch as accurate as possible.

6.19 The population of a country, denoted by $P(t)$ with t measured in years, grows according to the logistic model with $a = 0.3$ and with an initial population size of 6×10^6. Calculate the value of b, in the following two cases:
(a) If the annual increase from $t = 0$ to $t = 1$ is 10%.

(b) If the rate of change of the population at time $t = 0$ is 10% of the population per unit time.

6.20 Let $P(t)$ denote the size of a certain population, with t measured in years. It is known that the population grows according to the logistic model with $a = 0.2$. Two population sizes, namely $P(0) = 100\,000$ and $P(10) = 150\,000$, are known.
(a) Find the value of the parameter b in this population model.

(b) Find the rate of change (dP/dt) at times $t = 0$ and $t = 10$.

(c) From what we know about the logistic model, for what population size P does this population grow fastest? Justify your answer!

(d) At what time t does this population grow fastest? Justify your answer!

6.21 The following measurements were made of the growth of a population of animals:
- At time $t = 0$ days, $P(0) = 10$ and the rate of change, given by the derivative, is 20 per day,
- At time $t = 10$ days, $P(10) = 1500$ and the rate of change is 1 per day.

Based on the information given here, is the population growing according to the Malthusian or the logistic model? Justify your answer!

6.22 Which of the following statements are true, and which are false? Justify your answers!
(a) In a logistic model, if the population is decreasing then $a < 0$.

(b) In a logistic model, if the population is decreasing then $a > b$.

(c) In a logistic model, if $a = b$ then there will be just one equilibrium point.

Study Unit 7 EXERCISES

7.1 Find all the equilibrium points of the following differential equations:
(a) $\frac{dx}{dt} = 1 - 4x$

(b) $\frac{dx}{dt} = 2x + 2x^2$

(c) $\frac{dx}{dt} = 2 - e^x$

(d) $\frac{dx}{dt} = 1 - \cos(x)$

(e) $\frac{dx}{dt} = x^4 - x^3$

7.2 Draw the phase lines of the following differential equations.
(a) $\frac{dx}{dt} = 1 - 4x$

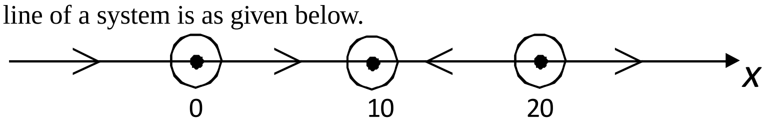
(b) $\frac{dx}{dt} = 2x + 2x^2$

(c) $\frac{dx}{dt} = 2 - e^x$

(d) $\frac{dx}{dt} = 1 - \cos(x)$

(e) $\frac{dx}{dt} = x^4 - x^3$

7.3 Assume that the phase line of a system is as given below.



For each of the following initial points x_0, explain how solution starting from that point will behave: will it increase without bound; or decrease without bound; or increase/decrease asymptotically towards an equilibrium point, or stay at an equilibrium point (if so, which one?).

(a) $x_0 = -5$

(b) $x_0 = 2$

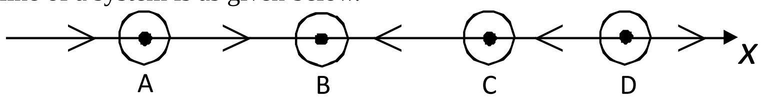
(c) $x_0 = 5$

(d) $x_0 = 10$

(e) $x_0 = 15$

(f) $x_0 = 22$

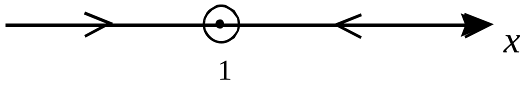
7.4 Assume that the phase line of a system is as given below.



Classify all the equilibrium points as stable or unstable.

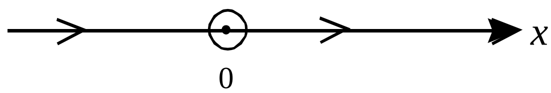
7.5 Use the phase lines of the following systems to draw solution curves starting from the given initial values.

(a) $\frac{dx}{dt} = 1 - x$, Phase line:



initial points $x = 0$, $x = 1$and $x = 3$.

(b) $\frac{dx}{dt} = x^2$, Phase line:



initial points $x = -1$, $x = 0$ and $x = 3$.

7.6 Draw the phase line of the following system, and use the phase line to classify each of the equilibrium points as stable or unstable. The system is:

$$\frac{dx}{dt} = -2\,(x-1)^2\,x\,(x+1)^2\,.$$

7.7 Do a qualitative analysis, by drawing the phase line, for the Malthusian population model in the two cases $k > 0$ and $k < 0$.

7.8 A variable $R(t)$, $t \geq 0$ obeys the differential equation

$$\frac{dR}{dt} = \frac{1}{2}(2 - R^2)(e^R - 1)\,|R+1|\,e^{-5R+19}$$

(a) Find the equilibrium points of the model, and draw a phase line of the model. Indicate the direction of motion in each interval of the phase line.

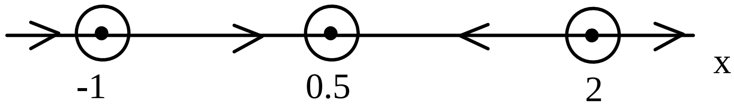
(b) State which equilibrium points are stable and which are unstable.

(c) Analyse, without solving the differential equation, what will happen in the model as $t \to \infty$, when different starting values $R(0)$ are used.

7.9 Assume that a system

$$\frac{dx}{dt} = G(x)$$

has a phase line which looks like this:



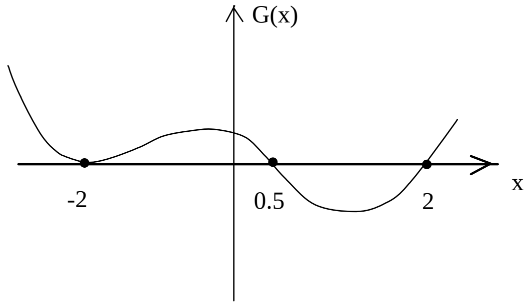
(a) Draw a rough graph of what the function $G(x)$ could look like.

(b) Draw a rough sketch of the solution curves $x(t)$ when the initial value is $x(0) = 1$ and when the initial value is $x(0) = 0$.

7.10 Assume that for the system

$$\frac{dx}{dt} = G(x)$$

a graph of the function G looks like this:



Draw the phase line of the system, and explain what will happen to $x(t)$ as t increases if the initial value is -1 and if the initial value is -3.

7.11 Draw the phase line of the following systems. List all the equilibrium points, and state which are stable and which are unstable. In the systems, a and b are assumed to be positive constants.

(a)
$$\frac{dx}{dt} = ax^2 - 1$$

(b)
$$\frac{dx}{dt} = x^2$$

(c)
$$\frac{dx}{dt} = \sin(x)$$

(d)
$$\frac{dx}{dt} = a(x - b)$$

7.12 For each of the following systems, draw a sketch of:

(i)dx/dt as a function x,

(ii)the phase line.

Also, for each system, list all the equilibrium points and classify each of them as stable or unstable; and predict the outcome of the solution if the system starts at $x(0) = 0.5$.

(a) $\dfrac{dx}{dt} = -x$

(b) $\dfrac{dx}{dt} = (x - 1)x$

(c) $\dfrac{dx}{dt} = (x - 2)^2 x(x + 1)$

7.13 Let a system be given by

$$\frac{dx}{dt} = x(x - a)$$

where a is a real number.

(a) How should a be chosen to make $x = 2$ an equilibrium point?

(b) How should a be chosen to make $x = 0$ an unstable equilibrium point?

(c) How should a be chosen to guarantee that any solution starting with $x_0 > 1$ will increase without bound?

(d) Explain why, regardless of the value of a, no solution to this system can ever decrease without bound.

7.14 Consider the model

$$\frac{dP}{dt} = 2P + 4P^2.$$

Here, $P(t)$ denotes the size of a population at time t.

(a) Is this a logistic model? Justify your answer!

(b) Draw the phase line of the model.

(c) Draw a sketch of the solution curve $P(t)$, if $P(0) = 2$.

7.15 For each of the following systems (a), (b) and (c), draw a sketch of:

(a) i. The function dx/dt as a function x,

ii. and the phase line. (Hint: Make sure your answers to (i) and (ii) agree with each other!)

iii. Also, for each system, list all the equilibrium points and classify each of them as stable or unstable;

iv. and predict the outcome of the solution if the system starts at $x(0) = 0.5$.

(b) $\dfrac{dx}{dt} = -2x$

(c) $\dfrac{dx}{dt} = (x-1)\,x$

(d) $\dfrac{dx}{dt} = (x+2)\,x^2\,(x-1)$

7.16 Let a system be given by

$$\frac{dx}{dt} = x\,(a-x)$$

where a is a real number.

(a) How should a be chosen to make $x = 1$ an equilibrium point?

(b) How should a be chosen to guarantee that any solution starting with $x_0 < 0$ will decrease?

(c) Explain why, regardless of the value of a, no solution to this system can ever increase without bound.

Study Unit 8 EXERCISES

8.1 An object with the initial temperature T_0 is placed in a tank of water, the temperature of which is kept at 10°C. After 30 minutes the temperature of the object is 200°C, and after 60 minutes it is 100°C. Assuming that Newton's law of cooling applies, find the values of T_0 and the cooling constant k, and find how long it takes for the object to cool to 30°C.

8.2 An object with the initial temperature T_0 is placed in a tank of water, the temperature of which is kept at 10°C. After 30 minutes the temperature of the object is 200°C, and after 60 minutes it is 100°C.

(a) Assuming that Newton's law of cooling applies, what was the value of T_0?

(b) How long does it take for the object to cool to 50°C?

8.3 An object with an initial temperature of 105$^{\circ}C$ is immersed in a fluid which is held at a temperature of 20$^{\circ}C$. After 15 minutes the temperature of the body has dropped to 65$^{\circ}C$. Write down the differential equation associated with Newton's law of cooling for this situation. Solve the equation using the given initial conditions. Then determine the temperature of the body after 30 minutes.

8.4 A body with an initial temperature of 100$^{\circ}C$ is placed in a tank of water, the temperature of which is held at 30$^{\circ}C$. The body cools down to 70$^{\circ}C$ in 15 minutes. Write down the differential equation for Newton's law of cooling. Solve the equation using the information given and then calculate when the body's temperature will be 40$^{\circ}C$.

8.5 A body with an initial temperature of T_0 is placed in a tank, the temperature of which is held at 20$^{\circ}C$. After 10 minutes the temperature of the body is 60$^{\circ}C$ and after 20 minutes the temperature is 40$^{\circ}C$. What is the temperature of the body after 5 and after 15 minutes?

8.6 A metal cube is dropped into a tank of water, the temperature of which is held at 10°. The initial temperature of the cube is 60°, and after 10 minutes the temperature of the cube has dropped to 30°. Assume that Newton's law of cooling applies.

(a) Write down the differential equation for the temperature of the cube, $T(t)$.

(b) What will the temperature of the cube be after 45 minutes?

(c) Will the temperature of the cube ever reach 5°? Justify your answer!

(d) What is the initial rate of change of the temperature of the cube?

8.7 An object with an initial temperature of 120° is placed in a tank of liquid, the temperature of which is held constant at 80°. Assume that Newton's law of cooling applies. After 10 minutes, the temperature of the object has dropped to 110°.

(a) Write down the differential equation for the temperature of the object.

(b) Write down the solution to the differential equation.

(c) How long does it take for the temperature of the object to reach 85°?

(d) What is the temperature of the object after 2 hours?

8.8 In a factory a small quantity of acid which is to be used in certain processes has to be diluted with water. A tank contains 10 litres of acid at the time $t = 0$. Water is pumped into the tank at a rate of 1 litre per minute. The contents of the tank are pumped out at the same rate. The tank is always kept full and the two liquids in the tank are mixed thoroughly at all times. Let $W(t)$ denote the amount of **water** in the tank at any time t, with t measured in minutes.

(a) Derive a differential equation modelling the amount of water in the tank.

(b) Obtain a formula for the amount of water in the tank at any time $t > 0$.

(c) Calculate the amount of water in the tank at $t = 5$ minutes. Let $W(t)$ denote the amount of water in the tank at time t, in litres.

8.9 A tank initially contains 200 litres of liquid A. Liquid B is pumped into the tank at the rate of 4 litres per minute. The mixture is stirred continuously and the tank is kept full at 200 litres at all times. The contents of the tank is pumped out at the same rate of 4 litres per minute. Let $X(t)$ denote the amount (in litres) of liquid B in the tank after t minutes.

(a) Write down the differential equation for $X(t)$, and its initial value.

(b) Solve the initial value problem in (a) to find the solution $X(t)$ for all t.

(c) How long does it take until the tank contains liquid A and liquid B in equal quantities?

8.10 In a chemical production plant a tank contains S_0 kg of a chemical dissolved in water. The volume of the tank is 200 litres and it is always kept full. Starting at a time $t = 0$, water in which 0.5 kg of the chemical has been dissolved per litre is pumped into the tank at a rate of 4 litres/minute. The contents of the tank are stirred continuously and the resulting solution flows out at the same rate as the inflow. Find the concentration of the chemical at any time $t > 0$.

8.11 A tank with a capacity of 1000 litres contains a solution of salt and water. The tank is always kept full. Initially, the tank contains 10 kg of salt dissolved in the water. Water is pumped into the tank at a constant rate of 200 litres per minute, with 0.1 kg of salt dissolved in each litre of water. The contents of the tank are stirred continuously, and the resulting solution is pumped out at a rate of 200 litres per minute. Let $S(t)$ denote the amount of salt (in kilograms) in the tank after t minutes and let $C(t)$ denote the concentration of salt (in kilograms per litre) in the tank after t minutes.

(a) Write down the differential equations for $S(t)$ and $C(t)$.

(b) Draw the phase lines of the differential equations for the systems for S and C, and draw rough sketches of the values of S and C as functions of time, if their initial values are as specified above.

(c) What will happen to S and C when $t \to \infty$?

8.12 A tank with volume V litres is used to dissolve a chemical in water. At time $t = 0$ the tank contains M_0 kg of the chemical. Water containing C kg of the chemical per litre flows into the tank at a rate of R litres per minute. The mixture in the tank is stirred thoroughly and the tank is kept full at all times. The mixture is pumped out at a rate of R litres per minute. Let $S(t)$ denote the concentration of the chemical (in kg/litre) in the mixture leaving the tank at time t.

(a) Derive a differential equation for $S(t)$.

(b) Find the equilibrium points and draw the phase line of the model. State whether the equilibrium points are stable or not.

(c) Make a rough sketch of $S(t)$ as a function of t. Give examples of solution curves for all the possible choices of M_0.

(d) Explain what will happen to $S(t)$ when $t \to \infty$.

8.13 A tank initially contains 200 litres of liquid A. Liquid B is pumped into the tank at the rate of 4 litres per minute. The mixture is stirred continuously and the tank is kept full at 200 litres at all times. The contents of the tank is pumped out at the same rate of 4 litres per minute. Let $X(t)$ denote the amount (in litres) of liquid B in the tank after t minutes.

(a) Write down the differential equation for $X(t)$, and its initial value.

(b) Solve the initial value problem in (a) to find the solution $X(t)$ for all t.

(c) How long does it take until the tank contains liquid A and liquid B in equal quantities?

8.14 Solve equation (8.7) if $P(0) = 1$ (i.e. if initially only one firm has adopted the new manufacturing process), as follows:

(a) Separate the variables and then use the result

$$\frac{1}{P(N-P)} = \frac{1}{N}\left[\frac{1}{P} + \frac{1}{N-P}\right].$$

(b) Integrate both sides of the equation to get

$$\ln P - \ln(N-P) = Nkt + C$$

here C is a constant of integration.

(c) Use the initial value to find the value C.

(d) Then show that

$$P = \frac{Ne^{Nkt}}{N-1+e^{Nkt}}.$$

8.15 Consider a model describing the spreading of a highly communicable disease in an isolated population of total size N. Let $X(t)$ denote the number of people who have been infected by the time t. The suggested model is as follows:

$$\frac{dX}{dt} = kX(N-X)$$

(a) What are the main assumptions implicit in the model? How reasonable are they?

(b) Draw a graph of dX/dt versus X.

(c) Draw the phase line of the model.

(d) Draw a rough sketch of the solution curve if the initial number of infections is $X_1 < N/2$, and if the initial number of infections is $X_2 > N/2$.

(e) Solve the differential equation to find an expression for $X(t)$ as a function of t.

8.16 In an experiment conducted by the Agricultural Department of a university, it was found that the maximum amount of wheat that could be obtained on the university's experimental farm is 150 kg per hectare. Research had already shown that the amount of wheat can be determined by the equation

$$\frac{dQ}{dx} = k(150-Q)$$

where $Q(x)$ denotes the number of kilograms of wheat obtained per hectare when x kg of fertilizer is used per hectare. Data obtained in the experiment showed that 10 kg of fertilizer per hectare results in 80 kg of wheat per hectare, while 20 kg of fertilizer results in 120 kg of wheat per hectare. Determine the amount of wheat that can be expected if 30 kg of fertilizer is used per hectare.

8.17 During a flu epidemic 5 percent of the 5000 pupils at a school had the illness at a time $t = 0$. It was found that the rate at which pupils were contracting the flu was directly proportional to the product of the number of pupils who had the flu and the remaining pupils who were not infected. If 20 percent of the pupils had contracted the illness by $t = 10$ days, find the number of pupils who had contracted the flu by $t = 13$ days.

8.18 A scientist studies the heating of metal objects by radiation. She suspects that the rate at which the temperature $T(t)$ of the heated object increases is directly proportional to the square of the difference between $T(t)$ and the temperature of the radiation source. To test the model, an experiment is conducted in which a metal object is heated by radiation from a source which is held at a constant temperature of 500°. The following data is obtained:

t (hours)	$T(t)$ (degrees)
0	375
1	400

(a) Write down the differential equation for the model described above.

(b) According to the model, what is the temperature of the object after 3 hours?

(c) Draw a phase line of the model, and sketch possible solution curves starting from various initial values.

(d) What happens if the initial temperature of the object is higher than that of the radiation source?

8.19 In one theory of learning, the rate at which a subject is memorised is assumed to be proportional to the amount that has already been memorised. Suppose that M denotes the total amount of a subject that has to be memorised and $A(t)$ is the amount memorised in time t. Determine a differential equation for the amount $A(t)$.

Study Unit 9 EXERCISES

9.1 South African National Parks Board is planning to issue impala hunting permits. To find out how many permits can be issued, the Board attempts to find a model to describe the impala population in the absence of hunting. It is known that if the impala population falls below a certain level m, the impala will become extinct. It is also known that if the impala population goes above the maximum carrying capacity M, the population will decrease to M. The following model is suggested to describe the growth rate of the impala population as a function of time:

$$\frac{dP}{dt} = kP(M-P)(P-m)$$

where P is the size of the impala population and $k > 0$ is a constant of proportionality.

(a) How does this model differ from the model of logistic growth $dP/dt = kP(M-P)$? Is this model more or less reasonable than the logistic model? Why?

(b) Draw the phase line of this model.

(c) Determine the stability of the equilibrium points of the model.

(d) Find the outcomes of this model for all possible starting points P_0.

(e) Compare the outcomes with the outcomes of the logistic model. Which one is more realistic?

9.2 Consider the impala population in the previous exercise. Assume now that a certain number of hunting permits per unit of time have been issued. This means a constant rate H of "harvesting", and the model of the impala population with hunting becomes

$$\frac{dP}{dt} = kP(M-P)(P-m) - H.$$

Find the maximum value of H, that is, the maximum level of hunting that should be permitted.

9.3 Analyse the following four population models graphically, as follows: First, draw a graph of dP/dt versus P. Then, draw the phase line of the system. Finally, draw a sketch of possible solution curves with different choices of initial populations P_0. Identify and determine the stability of equilibrium points in each model.

(a)

$$\frac{dP}{dt} = a - bP, \quad a, b > 0$$

(b)

$$\frac{dP}{dt} = P\left(a - bP\right), \quad a, b > 0$$

(c)

$$\frac{dP}{dt} = k(M-P)(P-m), \quad k, m, M > 0, \;\; m < M$$

(d)

$$\frac{dP}{dt} = kP(M-P)(P-m), \quad k, m, M > 0 \;\; m < M$$

9.4 A Malthusian type population model which takes immigration into account can be described by the differential equation

$$\frac{dP}{dt} = kP + h,$$

where k and h are positive constants and $P(t)$ denotes the size of the population at time t.

(a) Assuming that the time t is measured in years, what is the meaning of the constant h?

(b) Find the equilibrium points of the model, and draw the phase line of the model.

(c) Find the solution to the differential equation when the initial population is P_0.

(d) Predict the outcome of the system with different values of P_0.

$$\frac{dP}{dt} = kP + h, \qquad k, h > 0$$

9.5 A Malthusian population model which takes emigration into account can be described by the differential equation

$$\frac{dP(t)}{dt} = kP - h,$$

where k and h are positive constants and $P(t)$ denotes the size of the population at time t.
(a)Assuming that the time t is measured in years, what is the meaning of the constant h?

(b)Find the solution to the differential equation when the initial population is P_0.

(c)Find the equilibrium points of the model, and draw the phase line of the model.

(d)Predict the outcome of the system for different values of P_0.

9.6 A flu infection is spreading in a school of 200 pupils. Initially only one pupil has the flu. Let us assume that the rate at which pupils contract the flu is directly proportional to the number of pupils who are not yet infected. Let $X(t)$ denote the number of pupils who have the flu after t days.
(a) Write down the differential equation for the variable X.

(b) After 2 days, 50 students have the flu. How many will have the flu after 7 days?

(c) Plot the phase line of the model.

(d) Find the rate of change of X at $t = 0$, $t = 2$ and $t = 7$.

(e) Use all the information available to plot the solution curve $X(t)$.

(f) How realistic is our assumption about the way that the infection spreads? How realistic is the solution curve showing the spreading of the infection from day to day?

9.7 The following four population models are suggested to model a fish population in a pond:

$$(\text{A}) \quad : \quad \frac{dP}{dt} = k(M - P)$$

$$(\text{B}) \quad : \quad \frac{dP}{dt} = kP(M - P)$$

$$(\text{C}) \quad : \quad \frac{dP}{dt} = k(M - P)(P - m)$$

$$(\text{D}) \quad : \quad \frac{dP}{dt} = kP(M - P)(P - m)$$

In all models, $k > 0$, $M > 0$ and $0 < m < M$.
(a) Draw the phase line of each model.

(b) Explain how the models differ from each other. Compare the outcome each model predicts from different levels of initial population size. Which model is the most realistic one? Why?

9.8 A flu infection is spreading in a school of 2000 pupils. Let $X(t)$ denote the number of pupils who have the flu after t days. The following two alternative epidemic models are suggested to model the spread of the flu infection:

$$(\text{A}) \quad : \quad \frac{dX}{dt} = k(2000)$$

$$(\text{B}) \quad : \quad \frac{dX}{dt} = k(2000 - X)$$

(a) Draw the phase line of each model.

(b) Explain how the models differ from each other. Which model is more realistic? Justify your answer!

9.9 The fish in a fish pond grow according to the logistic model with $a = 0.1$ and $b = 0.002$, when time is measured in months. Assume that each month, X fish are added to the pond, and that 1/3 of all the fish are harvested per month.
(a) Write down a differential equation that describes the model.

(b) What should the value of X be so that $P = 0$ is an equilibrium value of the system?

9.10 In one theory of learning, the rate at which a subject is memorized is assumed to be proportional to the product of the amount already memorised and the amount that is still left to be memorized. Suppose that M denotes the total amount of content that has to be memorized, and $A(t)$ the amount that has been memorized after t hours.

(a) Write down the differential equation for A, using k for the constant of proportionality. Write also down the initial value A_0.

(b) Draw the phase line of the model.

(c) Use the phase line to sketch solution curves when the initial values are $A_0 = M$, $A_0 = M/2$ and $A_0 = 0$.

(d) Use the sketches in (c) to analyse the appropriateness of this model as a theory of learning.

9.11 A flu infection is spreading in a school of 2000 pupils. Initially only one pupil has the flu. Let us assume that the rate at which pupils contract the flu is directly proportional to the number of pupils who are already infected. Let $X(t)$ denote the number of pupils who have the flu after t days.

(a) Write down the differential equation for the variable X.

(b) After 2 days, 5 students have the flu. How many will have the flu after 5 days?

(c) Plot the phase line of the model.

(d) Find the rate of change of X at $t = 0$, $t = 2$ and $t = 5$. (Remember that the rate of change is the derivative!)

(e) Use all the information available to plot the solution curve $X(t)$.

(f) What does the model predict will happen to $X(t)$ as $t \to \infty$?

Study Unit 10 EXERCISES

Investigate each of the following systems as follows: Find the equilibrium points and isoclines. Draw the phase diagram of the system. Determine the stability of the equilibrium points. Interpret the phase diagram to find out what happens with various initial values in the model.

10.1

$$\begin{cases} \frac{dx}{dt} = x \\ \frac{dy}{dt} = y \end{cases}$$

10.2

$$\begin{cases} \frac{dx}{dt} = -x \\ \frac{dy}{dt} = 2y \end{cases}$$

10.3

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -2x \end{cases}$$

10.4

$$\begin{cases} \frac{dx}{dt} = -x + 1 \\ \frac{dy}{dt} = -2y \end{cases}$$

10.5

$$\begin{cases} \frac{dx}{dt} = -1 \\ \frac{dy}{dt} = -xy \end{cases}$$

10.6

$$\begin{cases} \frac{dx}{dt} = y^2 - x \\ \frac{dy}{dt} = -y \end{cases}$$

10.7

$$\begin{cases} \frac{dx}{dt} = -x \\ \frac{dy}{dt} = (y-1)^2 y(x-1) \end{cases}$$

10.8

$$\begin{cases} \frac{dx}{dt} = x^2 - y^2 \\ \frac{dy}{dt} = -y \end{cases}$$

10.9

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = -y \end{cases}$$

10.10

$$\begin{cases} \frac{dx}{dt} = x\left(y-1\right) \\ \frac{dy}{dt} = 2 \end{cases}$$

10.11

$$\begin{cases} \frac{dx}{dt} = x - 1 \\ \frac{dy}{dt} = x \end{cases}$$

10.12

$$\begin{cases} \frac{dx}{dt} = xy \\ \frac{dy}{dt} = x - 1 \end{cases}$$

10.13

$$\begin{cases} \frac{dx}{dt} = y - 1 \\ \frac{dy}{dt} = 1 - x \end{cases}$$

10.14

$$\begin{cases} \frac{dx}{dt} = y\left(1-y\right) \\ \frac{dy}{dt} = -x \end{cases}$$

10.15

$$\begin{cases} \frac{dx}{dt} = x - y \\ \frac{dy}{dt} = x + y \end{cases}$$

10.16

$$\begin{cases} \frac{dx}{dt} = x\left(1-x\right) \\ \frac{dy}{dt} = y\left(1-y\right) \end{cases}$$

10.17

$$\begin{cases} \frac{dx}{dt} = -y \\ \frac{dy}{dt} = x \end{cases}$$

10.18

$$\begin{cases} \frac{dx}{dt} = x - 1 \\ \frac{dy}{dt} = -xy \end{cases}$$

10.19

$$\begin{cases} \frac{dx}{dt} = -x \\ \frac{dy}{dt} = \left(y-1\right)y\left(x-1\right) \end{cases}$$

10.20 Prove that for any solution to the system

$$\begin{cases} \frac{dx}{dt} = y \\ \frac{dy}{dt} = -x \end{cases}$$

$r(t) = \sqrt{x(t)^2 + y(t)^2}$ is constant, by proving that $dr/dt = 0$.

10.21 Find the solution to the system in the previous exercise, by concluding that the solution must be of the form $(x(t), y(t)) = a(\cos\theta(t), \sin\theta(t))$ for some constant a, and by finding the function $\theta(t)$.

Study Unit 11 EXERCISES

11.1 Investigate cases 1 and 2 of the model in Section **??**. In each case, draw and interpret the phase diagram of the system.

11.2 **Herbivore-vegetation model**
Consider a system consisting of a population of impala (the herbivores) grazing on suitable vegetation which is their food source. Let $y = y(t)$ denote the impala population and let $x = x(t)$ denote the amount of vegetation present at time t. Here y represents the predators and x the prey. It has been found that a predator-prey model that describes this system quite successfully is given by the equations below:

$$\begin{cases} \frac{dx}{dt} = a - px - qf(x)y \\ \\ \frac{dy}{dt} = [sf(x) - r]\, y \end{cases}$$

where a, p, q, s and r are positive constants and $f(x)$ is some function of the amount of vegetation. The state of the vegetation will obviously affect the rate at which the impala graze. If the amount $x(t)$ at a time t is small, then the impala may spend more time looking for food and their rate of consumption will drop; on the other hand, if the rains have been plentiful, then $x(t)$ will be large and more vegetation will be eaten in the same time period. The function $f(x)$ is intended to describe this. Although the function $f(x)$ is not specified, we assume that f is a continuous, increasing function (i.e., if $x_1 < x_2$ then $f(x_1) < f(x_2)$) with $f(0) = 0$ and $f(k) = r/s$ for some $k > 0$.

- i. Show that the x-isocline is given by
$$y = \frac{a - px}{qf(x)}$$
for $x > 0$, with the property that $y \to \infty$ as $x \to 0$. Note that this is not a straight line, but is actually a curve for which the positive y-axis is an asymptote and which intersects the x-axis at a/p.

  ii. Show that the y-isoclines are given by
$$f(x) = r/s \quad \text{or} \quad y = 0,$$
that is,
$$x = k \quad \text{or} \quad y = 0.$$

  iii. Show that the two equilibrium points for the system are given by
$$\left(\frac{a}{p}, 0\right) \quad \text{and} \quad \left(k, \frac{s(a - pk)}{qr}\right).$$

  iv. Find the signs of dx/dt and dy/dt in all regions of the phase plane.

  v. Assume that $k < a/p$. Draw the phase diagram, including only a rough sketch of the x-isocline.

  vi. Discuss the stability of the equilibrium points, and interpret the phase diagram in terms of the impala population and the vegetation. We have the model
$$\frac{dx}{dt} = a - px - qf(x)y \quad (1)$$
$$\frac{dy}{dt} = (sf(x) - r)\, y \quad (2)$$
where $x(t)$ is the amount of vegetation present and $y(t)$ is the size of the impala population at time t.[10 %]

10.3 **Cottony cushion scale insects and ladybirds**
During the 1860s citrus crops in the USA were faced with a very serious threat. The citrus trees were attacked by an insect known as cottony cushion scale insects, which we shall refer to as scales. No insecticides were available at the time and the farmers were faced with certain ruin until someone remembered that these insects had a natural enemy, the ladybird beetle. These beetles were imported from Australia and proved to be the answer as they destroyed the scales, thus saving the citrus crops.

We can treat this as a predator-prey system, and attempt to construct a model. Let $P(t)$ denote the scales population at any time t; this will represent the prey. Let $R(t)$ denote the ladybird population; this will represent the predators.

Assume that before the ladybirds arrived, the scales followed a Malthusian growth pattern, i.e.

$$\frac{dP}{dt} = (a - b)P \quad \text{when} \quad R = 0$$

where a and b are positive constants which represent the birth and death rates of the scale insects respectively, with $a > b$.

When the ladybirds are introduced the scales' birth rate remains unchanged, but the death rate increases, and this increase is most likely to be directly proportional to the number of ladybirds present. We can express this as

$$\frac{dP}{dt} = (a - b - cR)P \tag{1}$$

where c is a positive constant. The term $-cR(t)P(t)$ represents the number of scales killed by the ladybirds during one time unit.

Assume that if the prey were not available, the ladybirds would die at some constant rate, which we can express as

$$\frac{dR}{dt} = -dR \quad \text{if} \quad P = 0$$

where d is some positive constant. In the presence of the prey, the ladybirds' death rate will change due to the improved supply of food, and we describe this with the term $f\,P(t)R(t)$, where f is a positive constant. Thus we can model the ladybird population with the equation

$$\frac{dR}{dt} = (fP - d)\,R. \tag{2}$$

Note that the ladybird population increases, provided that $P(t) > d/f$.
Let P be the horizontal axis and R the vertical axis and investigate the behaviour of the system formed by the equations (1) and (2) as follows:

(a) Find the P-isoclines and the R-isoclines.

(b) Calculate the equilibrium points.

(c) Investigate the signs of dP/dt and dR/dt in the regions determined by the P-isoclines and the R-isoclines.

(d) Use this information to make a rough sketch of the phase diagram for P and Q. (Hint: Compare this with the krill-whale population model.)

(e) Discuss the stability of the equilibrium points and interpret the phase diagram.

11.1 Bass and trout are two game fish competing for the same resources. Assume that the two species are forced to share a small pond, with limited resources. Develop a model for the growth of the bass and trout populations, when we assume that in isolation trout demonstrate exponential decay (according to the Malthusian model with growth constant $a < 0$) and that in isolation, the bass population grows logistically, with a population limit M. Write down the differential equations which describe the model, introducing all the necessary parameters. Find the equilibrium points and determine their stability by drawing a phase diagram of the system. Is coexistence of the two species possible?

11.2 Consider the following economic model: Let P be the price of a product. Let Q be the quantity of the product available in the market. Both P and Q are functions of time. They interact with each other, and the following model might be proposed to describe the system:

$$\begin{cases} \frac{dP}{dt} = aP(\frac{b}{Q} - P) \\ \frac{dQ}{dt} = cQ(fP - Q) \end{cases}$$

where a, b, c and f are positive constants. [assume Q not zero]]

(a) If $a = 1$, b = 20 000, c = 1, and f = 30, find the equilibrium points of this system. Classify each equilibrium point in terms of its stability, if possible. If a point cannot be readily classified, explain why.

(b) Draw a phase diagram of the system and determine what happens to the levels of P and Q as time increases.

$$\begin{cases} \dfrac{dP}{dt} = aP\left(\frac{b}{Q} - P\right) \\ \dfrac{dQ}{dt} = cQ\,(fP - Q) \end{cases}$$

11.3 Assume that a model describing two competing species in a closed environment is given by the following system of two differential equations:

$$\begin{aligned}\frac{dx}{dt} &= (2-2x-y)\,x \\ \frac{dy}{dt} &= (2-x-2y)\,y\end{aligned}$$

(a) According to this model, how would each of the species behave in isolation, that is, if the other species is not present?

(b) Find the equilibrium points of the system.

(c) Draw the phase diagram of the system.

(d) According to the phase diagram, what is the outcome of the system?

11.4 Assume that a model describing two competing species in a closed environment is given by the following system of two differential equations:

$$\begin{aligned}\frac{dx}{dt} &= (1-x-2y)\,x \\ \frac{dy}{dt} &= (1-2x-y)\,y\end{aligned}$$

Find the equilibrium points of the system, and draw the phase diagram of the system. Use the phase diagram to analyse the outcome of the system for all possible initial states $x_0 \geq 0,\ \ y_0 \geq 0$. Is coexistence of the two species possible? Is coexistence likely?

11.5 Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 2xy+2x-x^2 \\ \frac{dy}{dt} &= xy-2y\end{aligned}$$

where x and y denote the sizes of two interacting populations.

(a) How does the x and y species, respectively, behave in the absence of the other species?

(b) Describe the type of interaction between the two species (e.g. competition, predator/prey, etc.)

(c) Draw the phase diagram and use it to predict the outcome of the system if initially $x_0 = y_0 = 5$.

11.6 Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 2x+xy-x^2 \\ \frac{dy}{dt} &= 2y-xy\end{aligned}$$

where x and y denote the sizes of two interacting populations.

(a) How does the x and y species, respectively, behave in the absence of the other species?

(b) Describe the type of interaction between the two species (e.g. competition, predator/prey, etc.)

(c) Draw the phase diagram and use it to predict the outcome of the system if initially $x_0 = 2$, $y_0 = 2$.

11.7 The model

$$\begin{aligned}\frac{dx}{dt} &= -4x-2x^2-xy \\ \frac{dy}{dt} &= y-y^2\end{aligned}$$

describes two interacting species, x and y. Which of the following statements are TRUE and which are FALSE? Justify your answers! (You will need to sketch the phase diagram to answer some of the questions!)

(a) This is a predator-prey system, where y is the prey and x is the predator.

(b) Species y can survive without species x.

(c) Species x can survive without species y.

(d) The two species are competing for the same resources.

(e) If the initial populations are large enough, both populations can grow without bound.

(f) Both populations will always die out.

SOLUTIONS

Study Unit 2 SOLUTIONS

2.1 (a) is autonomous, first order

(b) is non-autonomous, first-order

(c) is non-autonomous, first order

(d) is autonomous, not first-order

(e) is autonomous, not first-order

(f) is autonomous, first-order

2.2 (a) $a_0 = 1$ (first term)

$a_1 = \frac{a_0}{2} = \frac{1}{2} = 0.5$
$a_2 = \frac{a_1}{2} = \frac{1/2}{2} = \frac{1}{4} = 0.25$
$a_3 = \frac{a_2}{2} = \frac{1/4}{2} = \frac{1}{8} = 0.125$
$a_4 = \frac{a_3}{2} = \frac{1/8}{2} = \frac{1}{16} = 0.0625$

(b) $a_0 = 0$
$a_1 = 2 \cdot a_0 + 8 = 2 \cdot 0 + 8 = 8$
$a_2 = 2 \cdot a_1 + 8 = 2 \cdot 8 + 8 = 24$
$a_3 = 2 \cdot a_2 + 8 = 2 \cdot 24 + 8 = 56$
$a_4 = 2 \cdot a_3 + 8 = 2 \cdot 56 + 8 = 120$

(c) $a_0 = 4$
$a_1 = 2 \cdot a_0 (a_0 + 3) = 2 \cdot 4 (4 + 3) = 56$
$a_2 = 2 \cdot a_1 (a_1 + 3) = 2 \cdot 56 (56 + 3) = 6\ 608$
$a_3 = 2 \cdot a_2 (a_2 + 3) = 2 \cdot (6\ 608) (6\ 608 + 3) = 87\ 370\ 976$
$a_4 = 2 \cdot a_3 (a_3 + 3) = 2 \cdot (87\ 370\ 976) ((87\ 370\ 976) + 3) \approx 1.53 \times 10^{16}$

(d) $a_0 = 2$
$a_1 = \sqrt{a_0} = \sqrt{2} \approx 1.414\,2$

$a_2 = \sqrt{a_1} = \sqrt{\sqrt{2}} = \sqrt[4]{2} \approx \sqrt{1.414\,2} \approx 1.189\,2$

$a_3 = \sqrt{a_2} = \sqrt{\sqrt{\sqrt{2}}} = \sqrt[8]{2} \approx \sqrt{1.189\,2} \approx 1.090\,5$

$a_4 = \sqrt{a_3} = \sqrt{\sqrt{\sqrt{\sqrt{2}}}} = \sqrt[16]{2} \approx \sqrt{1.090\,5} \approx 1.044\,3$

2.3 (a)

$$\begin{aligned} a_0 &= 1 \\ a_1 &= \frac{a_0}{2} - 1 = \frac{1}{2} - 1 = -\frac{1}{2} \\ a_2 &= \frac{a_1}{2} - 1 = \frac{\left(-\frac{1}{2}\right)}{2} - 1 = -\frac{1}{4} - 1 = -\frac{5}{4} \end{aligned}$$

(b)

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 2a_0 + a_0^2 \;=\; 2\cdot 0 + 0^2 \;=\; 0 \\ a_2 &= 2a_1 + a_1^2 \;=\; 2\cdot 0 + 0^2 \;=\; 0 \end{aligned}$$

[and in fact we will have $a_n = 0$ for all n!]

(c)

$$\begin{aligned} a_0 &= 24 \\ a_1 &= 2\cdot a_0 - 3 \;=\; 2\cdot 24 - 3 \;=\; 48 - 3 \;=\; 45 \\ a_2 &= 2\cdot a_1 - 3 \;=\; 2\cdot 45 - 3 \;=\; 90 - 3 \;=\; 87 \end{aligned}$$

2.4 (a)The general solution is $a_n = \left(\frac{1}{2}\right)^n a_0$.

(b)The general solution to this is $a_n = a_0 - 8n$.

(c)The solution is $a_n = \sqrt[2^n]{a_0} = (a_0)^{\frac{1}{2^n}}$.

2.5 The general solutions were found in Exercise 2.4, so here we just need to substitute the given value for a_0.

(a)$a_n = \left(\frac{1}{2}\right)^n$.

(b)$a_n = -8n$

(c)$a_n = \sqrt[2^n]{2} = (2)^{\frac{1}{2^n}}$.

2.6 To find the equilibrium value a we substitute $a_n = a$ and $a_{n+1} = a$ into the difference equation, and then attempt to find all values of a (if any) for which the equation holds.

(a)Setting $a_n = a_{n+1} = a$, we get

$$\begin{aligned} & a = 1.1a \\ \Leftrightarrow\;& a(1.1-1) = 0 \\ \Leftrightarrow\;& 0.1a = 0 \\ \Leftrightarrow\;& a = 0. \end{aligned}$$

The equilibrium value is $a = 0$.

(b)Set $a_n = a_{n+1} = a$:

$$\begin{aligned} & a = -0.7a \\ \Leftrightarrow\;& (1+0.7)a = 1.7a = 0 \\ \Leftrightarrow\;& a = 0. \end{aligned}$$

The equilibrium value is $a = 0$.

(c)Set $a_n = a_{n+1} = a$:

$$\begin{aligned} & a = a^2 - 4a \\ \Leftrightarrow\;& a^2 - 5a = 0 \\ \Leftrightarrow\;& a(a-5) = 0 \\ \Leftrightarrow\;& a = 0 \text{ or } (a-5) = 0 \end{aligned}$$

There are 2 equilibrium values, $a = 0$ and $a = 5$.

(d)Set $a_n = a_{n+1} = a$:

$$a = a \Leftrightarrow 0 = 0$$

This holds for all values of a, so that every number $a \in \mathbb{R}$ is an equilibrium value.

(e)Set $a_n = a$:

$$a = 10$$

The equilibrium value is $a = 10$.

2.7 (a) $a = 0$

(b) $a = \frac{1}{1.9}$

(c) $a = \frac{5}{2} \pm \frac{1}{2}\sqrt{33}$

(d) $a = 0$ and $a = 1$

(e) $a = 0$ and $a = 1$

2.8 (a)$a_{n+1} - a_n = 5$: Equilibrium values: If we substitute $a_n = a = a_{n+1}$ into the difference equation, we get

$$a - a = 5 \therefore \quad 0 = 5$$

which has no solution; therefore there are no equilibrium values. Solution: We have

$$\begin{aligned} a_1 &= a_0 + 5, \\ a_2 &= a_1 + 5 \ = \ a_0 + 2 \cdot 5, \\ a_3 &= a_2 + 5 \ = \ a_0 + 3 \cdot 5 \end{aligned}$$

and so we see that, generally,

$$a_n = a_0 + 5n.$$

(b)

$$a_{n+1} = a_n^2$$

Equilibrium values: Substituting $a_n = a_{n+1} = a$ gives here

$$\begin{aligned} a &= a^2 \\ \therefore \quad a^2 - a &= 0 \\ \therefore \quad a\,(a-1) &= 0 \end{aligned}$$

so we have the two equilibrium values,

$$a = 0 \quad \text{and} \quad a = 1.$$

Solution: Since

$$a_1 = a_0^2, \;\; a_2 = (a_1)^2 = a_0^{2\cdot 2}, \;\; a_3 = (a_2)^2 = a_0^{2\cdot 2\cdot 2},$$

we see that the general solution is

$$a_n = (a_0)^{(2^n)}.$$

(c)

$$a_{n+1} = (a_n - 1)^2$$

Equilibrium values: Substituting $a_n = a_{n+1} = a$ gives us the equation

$$\begin{aligned} a &= (a-1)^2 \\ \therefore \quad a^2 - 3a + 1 &= 0 \\ \therefore \quad a = \frac{3 \pm \sqrt{g-4}}{2} &= \frac{3 \pm \sqrt{5}}{2} \end{aligned}$$

so the equilibrium values are

$$a = \frac{3 - \sqrt{5}}{2} \approx 0.382$$

and

$$a = \frac{3 + \sqrt{5}}{2} \approx 2.618.$$

Solution: We have

$$\begin{aligned} a_1 &= (a_0 - 1)^2, \\ a_2 &= (a_1 - 1)^2 = \left((a_0-1)^2 - 1\right)^2 \\ a_3 &= (a_2 - 1)^2 = \left(\left((a_0-1)^2 - 1\right)^2 - 1\right)^2 \\ &\vdots \\ a_n &= \left(\left(\cdots\left((a_n-1)^2-1\right)^2 \cdots - 1\right)^2 - 1\right)^2. \end{aligned}$$

[Unfortunately this can not be written in a more neat form!]

2.9 To find the equilibrium values, we set $a_{n+1} = a_n = a$ and find out for which values of a the equation obtained holds. We get

$$a = ra + b \ \therefore \ (1-r)\,a = b. \tag{1}$$

(a)**If r** $\neq$ **1** then we can divide both sides of (1) by $(1-r)$,

and will then get the equilibrium point $a = \dfrac{b}{1-r}$ (whatever the value of b).

(b)**If r** $=$ **1** then (1) becomes, in fact,

$$0 = b \tag{2}$$

The interpretation of this depends on the value of b.

(i)**If b = 0** then (2) is always trivially true. Thus (1) holds for **any** a, so that every a is an equilibrium value. (Note that for $r = 1,\ b = 0$, the original difference equation becomes

$$a_{n+1} = a_n$$

so that clearly, whichever value a_0 we start from, we will always stay at that value!)

(ii)**If b ≠ 0** then (2) is never true. Thus (1) can hold for **no** value of a, so that there are no equilibrium values. (For $r = 1,\ b \neq 0$, the original difference equation becomes

$$a_{n+1} = a_n + b \qquad (b \neq 0)$$

which can never stay at one value!)

2.10 Equilibrium points: If $k = 0$ then all values $a \in \mathbb{R}$ are equilibrium points; if $k \neq 0$ then there are no equilibrium points. The solution is

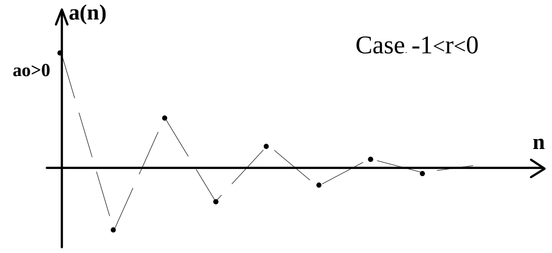
$$a_n = a_0 + nk.$$

2.11 The solution is

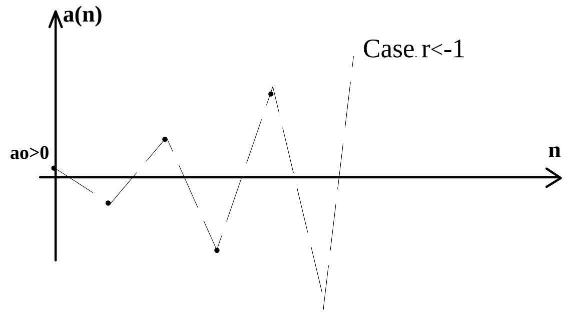
$$a_n = a_0 r^n.$$

If $r < 0$ then r^n will oscillate between positive and negative values.

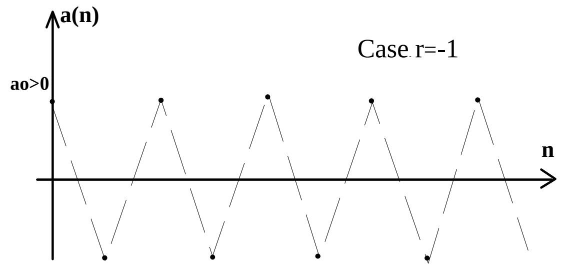
If $-1 < r < 0$ then $a_n \to 0$ while oscillating between positive and negative values.



If $r < -1$ then at limit, a_n will oscillate between $+\infty$ and $-\infty$.



If $r = -1$ then $a_n = (-1)^n a_0$ oscillates between $+a_0$ and $-a_0$.



2.12 (a) $a_n = (-2)^n a_0$

(b) $a_n = 20^n a_0$

(c) $a_n = a_0$

(d) $a_n = 0.1^n a_0$

(e) $a_n = \left(-\frac{1}{2}\right)^n a_0$

2.13 (a) The solution stays at 1.

(b) The solution stays at 0.

(c) The solution decreases without bound, towards -∞.

(d) The solution converges towards zero, from below.

(e) The solution stays at 0.

(f) The solution decreases to zero from above.

2.14 Difference equation

$$a_{n+1} = a_n^3 \qquad (*)$$

(a)If $a_0 = 2$, then

$$\begin{aligned} a_1 &= a_0^3 = 2^3 = 8, \\ a_2 &= a_1^3 = 8^3 = 512. \end{aligned}$$

(b)Equilibrium values: Put $a_n = a$ **and** $a_{n+1} = a$ in the difference equation, to get

$$\begin{aligned} & a = a^3 \\ \therefore\quad & a^3 - a = 0 \\ \therefore\quad & a\left(a^2 - 1\right) = 0 \\ \therefore\quad & a = 0 \quad \text{or} \quad \begin{cases} a^2 = 1 \\ \therefore \\ a = \pm 1 \end{cases} \end{aligned}$$

There are three equilibrium points: $a = 0$, $a = 1$, $a = -1$.

(c)

$$\begin{aligned} a_1 &= a_0^3, \\ a_2 &= a_1^3 = \left(a_0^3\right)^3 = a_0^{3\cdot 3} = a_0^9, \\ a_3 &= a_2^3 = \left(a_0^9\right)^3 = a_0^{3\cdot 3\cdot 3} = a_0^{27} \end{aligned}$$

(d)From (c), the general solution is seen to be $a_n = a_0^{3^n}$.

2.15 The equilibrium points are values x such that if $x_n = x$, then also $x_{n+1} = x$. *[It then follows that the system will stay at the value x forever more!]* So, we put $x_n = x_{n+1} = x$ in the difference equation and find all the values of x for which the obtained equation,

$$x = c - dx$$

holds. This is equivalent to

$$x + dx = c \quad \therefore \quad (1 + d)\, x = c$$

If $1 + d \neq 0$ then we can divide by $(1 + d)$ to get the solution

$$x = \frac{c}{1 + d}$$

If $1 + d = 0$, that is, $d = -1$, then we can't divide by $1 + d$; rather we will need to find the solutions in another way. But, if $1 + d = 0$ then the second–last equation is

$$0 = c$$

If $c = 0$ then this is always true, meaning that every x is an equilibrium point. *[With $d = -1$ and $c = 0$, the original difference equation (1) is $x_{n+1} = x_n$so it will always stay at any initial point!]* But if $c \neq 0$ then

$$0 = c$$

does not hold, and in that case no x is an equilibrium point: There are no equilibrium points. *[For $c \neq 0$ and $d = -1$, the value of x_n, according to the difference equation $x_{n+1} = x_n + c$ increases over every step, so there is no chance of finding an equilibrium point!]*

Summary:

If $d \neq -1$, there is one equilibrium point,

$$x = \frac{c}{1 + d};$$

If $d = -1$, then

– if $c = 0$, every value $x \in \mathbb{R}$ is an equilibrium point;

– if $c \neq 0$, then there are no equilibrium points.

2.16 Difference equation

$$a_{n+1} = 1 - a_n \qquad (*)$$

(a)If $a_0 = 2$, then by applying (*) with $n = 0$ and $n = 1$, we get

$$\begin{aligned} a_1 &= 1 - a_0 = 1 - 2 = -1, \\ a_2 &= 1 - a_1 = 1 - (1) = 2. \end{aligned}$$

(b)To find the equilibrium values, we put $a_n = a$ and $a_{n+1} = a$ in (*), and solve for a. We get

$$\begin{aligned} & a = 1 - a \\ \therefore \quad & 2a = 1 \\ \therefore \quad & a = \frac{1}{2}. \end{aligned}$$

The value $a = 1/2$ is the only equilibrium value for this difference equation.

(c)For any value of a_0, we see that (*) gives us

$$\begin{aligned} a_1 &= 1 - a_0, \\ a_2 &= 1 - a_1 = 1 - (1 - a_0) = a_0, \\ a_3 &= 1 - a_2 = 1 - a_0, \end{aligned}$$

and so on. We see that the value of a_n will alternate between a_0 and $1 - a_0$, and the solution is

$$a_n = \begin{cases} a_0 & \text{for } n \text{ even } (0, 2, 4, \ldots) \\ 1 - a_0 & \text{for } n \text{ odd } (1, 3, 5, \ldots) \end{cases}$$

[Note that the two values calculated in (a) also confirm this!]

2.17 Difference equation

$$a_{n+1} = a_n^3 \qquad (*)$$

(a)If $a_0 = 2$, then

$$\begin{aligned} a_1 &= a_0^3 = 2^3 = 8, \\ a_2 &= a_1^3 = 8^3 = 512. \end{aligned}$$

(b)Equilibrium values: Put $a_n = a$ **and** $a_{n+1} = a$ in (*), to get

$$\begin{aligned} & a = a^3 \\ \therefore \quad & a^3 - a = 0 \\ \therefore \quad & a\left(a^2 - 1\right) = 0 \\ \therefore \quad & a = 0 \quad \text{or} \quad a^2 = 1 \quad \therefore \quad a = \pm 1 \end{aligned}$$

There are three equilibrium points: $a = 0$, $a = 1$, $a = -1$.

(c)No; the solution is in fact $a_n = a_0^{3^n}$. For proof that the claimed answer is not correct, note for instance that $a_n = a_0^{3n}$ would give $a_2 = a_0^6$. With the initial value as in (a), this would give $a_2 = 2^6 = 2^6 = 64$ instead of the value 512 which we already calculated in (a). To see that the solution is $a_n = a_0^{3^n}$, we can do the following calculations:

$$\begin{aligned} a_1 &= a_0^3, \\ a_2 &= a_1^3 = \left(a_0^3\right)^3 = a_0^{3\cdot 3} = a_0^9, \\ a_3 &= a_2^3 = \left(a_0^9\right)^3 = a_0^{3\cdot 3\cdot 3} = a_0^{27}. \end{aligned}$$

from which we can see that the general solution should have 3^n as the exponent.

Study Unit 3 SOLUTIONS

3.1 (a) $P_{n+1} = 1.12 P_n$

(b) $P_{n+1} = 0.992\, P_n$

3.2 (a) $P_{n+1} = 1.15\, P_n$

(b) $P_{n+1} = 0.91\, P_n$

(c) $P_{n+1} = 2\, P_n$

(d) $P_{n+1} = P_n$

3.3 (a) $P_n = 1000 \cdot (1.09)^n$. As n increases, the size of the population P_n will increase without bound. (This can be seen from the fact that the value in the parentheses is greater than one; or you can use the fact that the birth rate is greater than the death rate.)

(b) $P_n = 1000 \cdot (1.1)^n$. As n increases, the size of the population P_n will increase without bound.

(c) $P_n = 1000 \cdot (0.5)^n$. As n increases, the size of the population P_n will decrease towards zero. (This can be seen from the fact that the value in the parentheses is less than one; or you can use the fact that the death rate is greater than the birth rate.)

(d) $P_n = 1000 \cdot (1)^n = 1000$. The size of the population P_n will always stay constant at 1000. (The birth rate and the death rate are equal, so the population size never changes!)

3.4 We must have

$$(1 + b - d) = 0.95.$$

We are given the value $d = 0.1$, so we can find what b should be:

$$\begin{aligned} & (1 + b - 0.1) = 0.95 \\ \therefore \quad & b = 0.95 - 1 + 0.1 = 0.05 \end{aligned}$$

The birth rates should be $b = 0.05$.

3.5 We should have

$$(1 + b - d) = 1.015.$$

The birth rate is given here: We have

$$b = \frac{20}{1000} = 0.02$$

so we can solve for d :

$$\begin{aligned} & (1 + 0.02 - d) = 1.015 \\ \therefore \quad & d = 1.02 - 1.015 = 0.005. \end{aligned}$$

This means $0.005 \cdot 1000 = 5$ deaths per thousand individuals per year.

3.6 The population can be modelled as the proportional growth population model with birth rate $b = 3.5$ per year and death rate $d = 1.5$ per year, so that if $P(n)$ is the size of the population after n years, we have

$$\begin{aligned} P(n+1) &= (1 + 3.5 - 1.5)\, P(n) \\ &= 3P(n)\,. \end{aligned}$$

(a) The solution to this equation is

$$P(n) = 3^n P_0$$

(b) If the initial population is $P_0 = 1\,000$, we will have

$$\begin{aligned} P(1) &= 3 \cdot 1000 &= 3\,000, \\ P(2) &= 3^2 \cdot 1000 &= 9\,000, \\ P(5) &= 3^5 \cdot 1000 &= 243\,000. \end{aligned}$$

(c) If the initial population is $P_0 = 10\ 000$, we will have

$$\begin{aligned} P(1) &= 30\,000, \\ P(2) &= 90\,000, \\ P(5) &= 2\,430\,000. \end{aligned}$$

3.7 The solution to the difference equation, giving the size of the population P_n after n years, will be

$$P_n = 20000 \cdot (0.72)^n\,.$$

(a) After 5 years this gives as the size of the population $P_n = 3870$.

(b) We need to find the value of n for which P_n first reaches (below) 2000; it will be just after $n = 7$.

3.8 We have $P_n = 10000 \cdot (1.45)^n$.
(a) The population has doubled in size when $P_n = 20000$; this happens by time $n = 2$.

(b) It will take 6 years.

3.9

$$\begin{aligned} P_{n+1} &= P_n + 0.8P_n - 0.2P_n - \frac{30}{100}P_n \\ \therefore \qquad P_{n+1} &= 1.3P_n \end{aligned}$$

(This assumes that the 30% leaving is calculated from the number of people present at the beginning of the year — if it were calculated instead based on the number of people present at the beginning of the year, after all the births and the deaths, the difference equation would be different!)

3.10

$$\begin{aligned} P_{n+1} &= P_n + 0.1P_n - 0.3P_n + 3000 \\ \therefore \qquad P_{n+1} &= 0.8P_n + 3000 \end{aligned}$$

3.11

$$\begin{aligned} P_{n+1} &= P_n + 0.05P_n - 0.01P_n - 100 \\ \therefore \qquad P_{n+1} &= 1.04P_n - 100 \end{aligned}$$

3.12

$$\begin{aligned} P_{n+1} &= P_n + 1.0P_n - 0.5P_n + P_n \\ \therefore \qquad P_{n+1} &= 2.5P_n \end{aligned}$$

3.13 Consider the following population model with immigration:

$$P_{n+1} = 1.02 \cdot P_n + 1000.$$

Here, P_n denotes the size of the population after n years. Assume that the initial population size, in year $n = 0$, is 10 000.

(a) With $P_0 = 10\ 000$, we get

$$\begin{aligned} P_1 &= 1.02 \cdot (10000) + 1000 = 11200, \\ P_2 &= 1.02 \cdot (11200) + 1000 = 12424, \\ P_3 &= 1.02 \cdot (12424) + 1000 = 13672.48. \end{aligned}$$

(b) The suggested solution $P_n = (1.02)^n\, P_0 + n * 1000$ would give

$$\begin{aligned} P_1 &= (1.02)^1\,(10000) + 1 * 1000 = 11200, \\ P_2 &= (1.02)^2\,(10000) + 2 * 1000 = 12404, \\ P_3 &= (1.02)^3\,(10000) + 3 * 1000 = 13612.08. \end{aligned}$$

We find that only the first value is correct, the later values are not. Therefore, the suggested solution cannot be correct.

3.14 (a) The model here is

$$\begin{aligned} P_{n+1} &= (1 + 1.1 - 0.5) \cdot P_n - 1000 \\ &\therefore \quad P_{n+1} = 1.6 \cdot P_n - 1000. \end{aligned}$$

The equilibrium point of this difference equation is

$$P \approx 1666.7.$$

If the initial value if larger than this (i.e. if $P_0 \geq 1667$, assuming an integer initial value) then the population will always increase; if $P_0 \leq 1666$ then the population will always decrease. Alternatively we can reason as follows: The change in the population due to births and deaths only during the first year is an increase of $(1.1 - 0.5) \cdot P_0 = 0.6 \cdot P_0$ per year. On the other hand, a total of 1000 population members need to be removed from the population. This means that the nett change in the population is

$$0.6 \cdot P_0 - 1000.$$

For the population to increase we need this to be positive, so that we need

$$0.6 \cdot P_0 - 1000 > 0 \quad \therefore \quad P_0 > \frac{1000}{0.6} \approx 1666.7.$$

(b) The model here is

$$P_{n+1} = (1 + 1.1 - 0.5) \cdot P_n - \frac{k}{100} P_n$$
$$\therefore \quad P_{n+1} = \left(1.6 - \frac{k}{100}\right) \cdot P_n.$$

This is of course a special case of the linear difference equation

$$P_{n+1} = r \cdot P_n$$

with

$$r = 1.6 - \frac{k}{100}.$$

To ensure that such a difference equation stays forever at its initial value, we must have $r = 1$, that is, we must have

$$1.6 - \frac{k}{100} = 1 \quad \therefore \quad \frac{k}{100} = 0.6 \quad \therefore k = 60$$

The population stays constant if 60% of the population moves out at the end of each year.

3.15 Write down the difference equations modelling the following investment accounts. Use A_n to denote the amount of money at the end of month n.

(a) $A_{n+1} = 1.02\, A_n$

(b) $A_{n+1} = 1.1\, A_n$

(c) $A_{n+1} = 1.11\, A_n$

3.16 (a) $A_n = (1.02)^n\, A_0$

(b) $A_n = (1.1)^n\, A_0$

(c) $A_n = (1.11)^n\, A_0$

3.17 The solution is

$$A_n = (1.02)^n \cdot 10000$$

and therefore

(a) after 3 months there will be $A_3 = (1.02)^3 \cdot 10000 \approx 10612$,

(b) after 6 months there will be $A_6 = (1.02)^6 \cdot 10000 \approx 11262$,

(c) after 24 months there will be $A_{24} = (1.02)^{24} \cdot 10000 \approx 16084$.

3.18 The difference equation is $A_{n+1} = 1.05\ A_n$ with initial value $A_0 = 50000$, which has the solution $A_n = (1.05)^n \cdot 50000$. We need to find the value of n for which we first have $A_n \geq 60000$. The value is $n = 4$. We can find it either by trying out $n - 1$, $n = 2$, $n = 3$ and so on until we find the value for which we get $A_n \geq 60000$ (we have $A_3 = 57881.$ but $A_4 = 60775.$) Alternatively, we can solve n directly from

$$(1.05)^n \cdot 50000 = 60000$$
$$\therefore \quad (1.05)^n = \frac{6}{5}.$$

Taking logarithms on both sides, we get

$$\ln\left[(1.05)^n\right] = \ln\left(\frac{6}{5}\right)$$
$$\therefore \quad n \ln(1.05) = \ln\left(\frac{6}{5}\right)$$
$$\therefore \quad n = \frac{\ln\left(\frac{6}{5}\right)}{\ln(1.05)} \approx 3.736\,9.$$

Again the result is that the value R60 000 is reached after 4 full months.

3.19 The answer is $A_6 = \left(1 + \frac{q}{100}\right)^6 \cdot 20000$ where q is the interest rate as a percentage.

(a) For $q = 1$,we get $A_6 = 21230$.

(b) For $q = 5$,we get $A_6 = 26802$.

(c) For $q = 10$,we get $A_6 = 35431$.

3.20 (a) 23 months $(n \approx 22.42)$

(b) 5 months $(n \approx 4.57)$

(c) 3 months $(n \approx 2.34)$

3.21 The difference equation is

$$A_{n+1} = (1.05)\, A_n - 500.$$

Applying this difference equation repeatedly, starting from $A_0 = 15000$ we get

$$\begin{aligned} A_1 &= (1.05)\, A_0 - 500 = 1.05 \cdot 15000 - 500 = 15250, \\ A_2 &= (1.05)\, A_1 - 500 = 1.05 \cdot 15250 - 500 = 15512.5, \\ A_3 &= (1.05)\, A_2 - 500 = 1.05 \cdot 15512.5 - 500 = 15788.13 \end{aligned}$$

as the amounts after month 1, 2 and 3.

3.22 An amount of R25 000 is deposited into a bank account, which pays interest at the rate of 10% per month. Assuming that we withdraw R4000 from the account each month, after how many months is the amount of money on the account less than R20 000? The difference equation is

$$A_{n+1} = (1.03)\, A_n - 1000,$$

so starting with the initial value $A_0 = 25000$, we get $A_1 = 23500$, $A_2 = 21850$, $A_3 = 20035$ and $A_4 = 18038.5$. Thus it takes 4 months for the amount of money to drop to less than R20 000.

3.23 (a)$A_{n+1} = 1.01A_n - 2000$

(b)The equilibrium point is $A = 200\ 000$.

(c)Since the initial value is below the equilibrium point, there is initially too little money to maintain the equilibrium, and thus the money on the account will eventually run out. (Alternatively, note that during the first month interest paid is 1% of 150 000 which is 1500. Since this is less than the amount withdrawn, during the first month the amount of money decreases; this happens in all the subsequent months as well, and thus the money will run out.

3.24 If $A(n)$ denotes the amount of money on the account at the end of month n, then $A(n)$ changes according to the difference equation

$$\begin{aligned} A(n+1) &= A(n) - 1000 \\ &\quad + [\text{interest on the amount } A(n) \text{ for one month}] \\ &= A(n) + \frac{1}{100} A(n) - 1000 \\ &= (1.01)\ A(n) - 1000. \end{aligned}$$

(a) If $A(0) = 99000$ then we have

$$\begin{aligned} A(1) &= (1.01)\ A(0) - 1000 = (1.01)\, 99000 - 1000 = 98990.00, \\ A(2) &= (1.01)\ A(1) - 1000 = (1.01)\, 98990 - 1000 \approx 98979.\, 90, \\ A(3) &= (1.01)\ A(2) - 1000 = (1.01)\, 98979.\, 90 - 1000 \approx 98969.\, 70, \\ A(4) &= (1.01)\ A(3) - 1000 = (1.01)\, 98969.\, 70 - 1000 \approx 98959.\, 40, \\ A(5) &= (1.01)\ A(4) - 1000 = (1.01)\, 98959.\, 40 - 1000 \approx 98948.\, 99, \\ A(6) &= (1.01)\ A(5) - 1000 = (1.01)\, 98948.\, 99 - 1000 \approx 98938.\, 48. \end{aligned}$$

(b) If $A(0) = 100000$ then we have

$$\begin{aligned} A(1) &= (1.01)\ A(0) - 1000 = (1.01)\, 100000 - 1000 = 100000.00, \\ A(2) &= A(3) = A(4) = A(5) = A(6) = 100000.00. \end{aligned}$$

(c) If $A(0) = 101000$ then we have

$$\begin{aligned}
A(1) &= (1.01)\ A(0) - 1000 = (1.01)\,101000 - 1000 = 101010.00, \\
A(2) &= (1.01)\ A(1) - 1000 = (1.01)\,101010.00 - 1000 = 101020.10, \\
A(3) &= (1.01)\ A(2) - 1000 = (1.01)\,101020.10 - 1000 \approx 101030.30, \\
A(4) &= (1.01)\ A(3) - 1000 = (1.01)\,101030.30 - 1000 \approx 101040.60, \\
A(5) &= (1.01)\ A(4) - 1000 = (1.01)\,101040.60 - 1000 \approx 101051.01, \\
A(6) &= (1.01)\ A(5) - 1000 = (1.01)\,101051.01 - 1000 \approx 101061.52.
\end{aligned}$$

Note that R100 000 is the equilibrium point in this model and if the initial amount of money equals this value then there will always be the same amount of money in the account. If $A(0)$ is less than 100 000, such as in (a), then the bank account will eventually run out of money; if $A(0)$ is larger than 100 000 such as in (c) then the amount of money in the bank will keep on increasing.

Note that we could alternatively have found the general solution for the difference equation,and applied it to find $A(6)$ directly. If $A(n+1) = (1.01)\ A(n) - 1000$ then we have

$$\begin{aligned}
A(1) &= (1.01)\ A(0) - 1000, \\
\\
A(2) &= (1.01)\ A(1) - 1000 \\
&= (1.01)^2\ A(0) - (1.01)1000 - 1000, \\
\\
A(3) &= (1.01)\ A(2) - 1000 \\
&= (1.01)^3\ A(0) - (1.01)^2 1000 - (1.01)1000 - 1000
\end{aligned}$$

and more generally,

$$A(n) = (1.01)^n\ A(0) - 1000\ \left[(1.01)^{n-1} + (1.01)^{n-2} + \cdots + (1.01)^2 + (1.01) + 1\right].$$

This gives

$$\begin{aligned}
A(6) &= (1.01)^6\ A(0) - 1000\ \left[(1.01)^5 + (1.01)^4 + (1.01)^3 + (1.01)^2 + (1.01) + 1\right] \\
&= (1.061\,520\,151)\ A(0) - 6152.015\,060\,1
\end{aligned}$$

which gives again the results above if $A\,(0)$ is chosen as in (a), (b) and (c).

3.25 The difference equation here is

$$A_{n+1} = (1.05)\,A_n - 2000,$$

which has the equilibrium point

$$A = 40\ 000.$$

If the initial deposit is $A_0 = 40\ 000$ or larger, then there will always be money on the account. (With that initial amount, during the first month the amount of interest added to the account is 2000, so if that same 2000 is also withdrawn then the amount on the account does not change!)

3.26 The difference equation here is

$$A_{n+1} = (1.01)\,A_n - W,$$

where W is the amount of money monthly withdrawn from the account. The equilibrium point of this model is

$$A = \frac{W}{0.01} = 100W.$$

If the initial amount of money equals this equilibrium point, then the amount of money always stays the same, and if the initial amount of money is greater than this amount, then the amount of money on the account will keep on increasing. So, since the initial amount of money is given as $A_0 = 200000$, the condition that must hold for the account never to become empty is

$$200000 \geq 100W$$

from which we see that we should have

$$W \geq 2000.$$

I can withdraw at most R2000 per month.

3.27 $A_{n+1} = (1.015)\,A_n - 500$

3.28 Consider a loan of R150 000, with interest charged at the rate of 2.5% per month, and repayments of R5000 made at the end of each month. Let $A\,(n)$ to denote the loan amount still owing at the end of the nth month.

(a) $A_{n+1} = (1.025)\,A_n - 5000$

(b) $A = 200000$

(c) The initial value $A_0 = 150000$ is smaller than the equilibrium point which means that the loan will be paid off eventually.

3.29 $W > 2200$

3.30 The difference equation is $A_{n+1} = \left(1 + \frac{q}{100}\right) A_n - 500$ if the interest rate is q %. The equilibrium point of such as system is given by

$$A = \frac{500}{\frac{q}{100}} = \frac{50000}{q}.$$

For the loan to be eventually paid off, the initial loan amount needs to be strictly less than this equilibrium point, that is, we need to have

$$10000 < \frac{50000}{q}$$

from which we find that we should have

$$q < 5.$$

The interest rate has to be less than 5%.

3.31 Let $A(n)$ denote the amount of money on the savings account at the end of month number n. In order to derive a difference equation to model the account, we need to express $A(n+1)$ in terms of $A(n)$, by investigating what happens during the $(n+1)$–st month. At the beginning of the $(n+1)$–st month, the amount of money at the bank account is $A(n)$, the same amount that was there at the end of the previous month. At the end of month $(n+1)$, the bank pays interest on the amount of money which was on the account during that month, that is, on the amount $A(n)$. Since the interest rate is 2%, this means that the amount of interest which the bank adds to the account at the end of the $(n+1)$–st month equals

$$\frac{2}{100} A(n).$$

(a) Also, the fixed amount of R200 is added to the bank account at the end of the month. This means that at the end of the month, the account has the amount

$$A(n+1) = A(n) + \frac{2}{100} A(n) + 200$$

on it. The difference equation describing this account is

$$A(n+1) = (1.02) A(n) + 200, \qquad A(0) = 5000.$$

3.32 Let $D(n)$ denote the remaining amount of the debt, at the end of the n–th month, with $D(0) = 500$ the initial debt value. We will investigate what happens during the $(n+1)$–st month, in order to derive an expression for $D(n+1)$ in terms of $D(n)$. At the beginning of the $(n+1)$–st month, the amount of debt is the same as the end of the n–th month, that is, $D(n)$. At the end of the month, interest is charged on the debt, at the rate of 5% of the amount which was owed during that month. This means that the amount

$$\frac{5}{100} \cdot D(n)$$

is **added** to the debt. Also, a re-payment of R30 is made at the end of the month, which means that the amount R30 is **subtracted** from the debt. Thus, at the end of the $(n+1)$–st month the amount of debt is

$$D(n+1) = D(n) + \frac{5}{100} D(n) - 30,$$

and the difference equation is given by

$$D(n+1) = (1.05) D(n) - 30, \qquad D(0) = 500.$$

[The equilibrium point of this model is

$$D^* = 600.$$

This gives the amount of loan such that if $D(0) = D^*$ then the repayments of 30 rands per month exactly cover the interest charged at 5% per month, with the effect that all the money paid back goes to paying the interest, but the debt itself never gets paid off. Any debt with $D(0)$ below this amount will eventually be paid off, but the closer it is to D^*, the longer the repayment takes. With the loan of R500, after 20 payments the amount owed still stands at R335, although a total amount of R600 has been made in payments! More than 265 rands has been paid for the interest alone. To pay the debt back faster, and to pay less money on interest, the monthly amounts of repayment need to be increased, which has the effect of making the equilibrium point D_* bigger.]

3.33 Let D_n denote the debt (the remaining balance on the on loan) at the end of the n-th month, with $D_0 = 40\,000$ the initial amount of the loan at the beginning of the first month. We will assume that the first repayment is made at the end of the first month, and that interest on the amount D_n is changed at the end of the n-th month. Then during the

$(n+1)$-st month the following happens: The amount on the account during the month is D_n. At the end of the month a repayment of R1500 is made, which means that the amount R1500 is deducted from the loan; and interest of 1.5% is changed on the amount of loan during that month, which means that the amount $\frac{1.5}{100} \cdot D_n$ is added to the loan. After these changes have been made, we are left with a new balance D_{n+1} on the loan. This means that

$$\begin{aligned} D_{n+1} &= D_n - 1500 + \frac{1.5}{100} D_n \\ \therefore \quad D_{n+1} &= 1.015 D_n - 1500. \end{aligned}$$

(a) The difference equation to formulate this situation is therefore

$$\begin{aligned} D_{n+1} &= 1.015 D_n - 1500 \\ D_0 &= 40\,000. \end{aligned}$$

3.34 The situation here is a special case of the cheque account (with time units in years rather than in months). Thus, if $A(n)$ is the amount of money on the account at the end of the n–th year, then $A(n)$ follows the difference equation

$$A(n+1) = \left(1 + \frac{15}{100}\right) A(n) - W, \; A(0) = 10\,000$$

where W is the amount of money withdrawn from the account at the end of each year.

As calculated on page 32, this model has one equilibrium value,

$$A^* = \frac{100W}{15}.$$

As explained on that page, if $A(0) > A_*$ then the amount of money on the account will keep increasing, while if $A(0) < A_*$ then the amount of money on the account will eventually run out. If $A(0) = A_*$ then (since A_* is the equilibrium value) the amount of money on the account will stay constant. Thus we see that to guarantee that the account is **not** eventually emptied, we must have

$$A(0) \geq A_*,$$

that is,

$$\begin{aligned} 10000 &\geq \frac{100W}{15} \\ \therefore \quad & W \leq \frac{15}{100}(10000) \\ \therefore \quad & W \leq 1500. \end{aligned}$$

Conclusion: At most R1500 can be withdrawn at the end of each year.

3.35 (a) Let us look what happens on month $n+1$.If A_n denotes the amount of money on the account at the end of month n, then at the beginning of month $n+1$, the money on the account is A_n (carried over from the previous month). Nothing happens during the $(n+1)$-st month until the end of the month. At that time, the interest is first paid; it consists of 5% of the money on the account during that month, that is, 5% of A_n. After this, the bank account contains

$$A_n + \frac{5}{100} A_n.$$

Next, we remove $1/3$ of this amount. Afterwards, the amount of money will therefore be

$$A_n + \frac{5}{100} A_n - \frac{1}{3}\left(A_n + \frac{5}{100} A_n\right).$$

This is the amount on the account at the end of the $n+1$st month, that is,

$$A_{n+1} = A_n + \frac{5}{100} A_n - \frac{1}{3}\left(A_n + \frac{5}{100} A_n\right).$$

This is our difference equation; we can further tidy this up a bit to get

$$A_{n+1} = \frac{2}{3}\left(A_n + \frac{5}{100} A_n\right) = \frac{2}{3}(1.05) A_n = 0.7 A_n.$$

(b) We see that the difference equation is actually a special case of the linear difference equation, with $r = 0.7 < 1$. This means that the amount of money on the account will decrease.

3.36 If we instead withdraw the third at the beginning of the month, then the order of things will change we will first remove the one-third, and then pay interested on that amount. . The new difference equation will be

$$A_{n+1} = A_n - \frac{1}{3}A_n + \frac{5}{100}\left(A_n - \frac{1}{3}A_n\right).$$

However, this can be rewritten as

$$A_{n+1} = \frac{5}{100}\frac{2}{3}A_n = 0.7A_n$$

again, so we see that nothing has changed in the difference equation. *[However, something has changed: namely the amount of money I get to withdraw each month! To see this, consider the first month, which starts with 30 000 on the bank account. In the situation of Question 3.35, I would withdraw 10000; the rest of the money would stay on the account, and at the month end the amount of interest added to it would be 5% of 20 000, that is,* $1000,$ *and I would then start the next month with 21 000 on the bank account. In the situation in Question 3.36 on the other hand, I would leave the full amount of 30 000 on the bank account, at the end of the month interest would be paid at 5% of 30 000, which amounts to* 1500 *which means that at the end of the month I would have 31 500. I would then withdraw one-third of this, that is, 10 500, leaving again 21 000 on the bank account. Thus the same account will remain on the bank account, but I get more money by withdrawing at the end of the month, rather than at the beginning of the month! This money consists on the extra interest paid by the bank on the money while it stays on the bank account.]*

3.37 (a) The difference equation is

$$A_{n+1} = A_n + \frac{1}{100}A_n - 800 = (1.01)\,A_n - 800.$$

(b) The equilibrium point here is

$$A = 80000.$$

The initial amount $A_0 = 10000$ is smaller than this, so the amount of money on the bank account will decrease.

3.38 (a) Now, the difference equation is

$$\begin{aligned} A_{n+1} &= A_n - 800 + \frac{1}{100}(A_n - 800) = (1.01)(A_n - 800) \\ &= 1.01A_n - 808. \end{aligned}$$

(b) The equilibrium point will now be

$$A = 80800$$

which is still larger than the initial amount $A_0 = 10000$ is smaller than this, so the amount of money on the bank account will again decrease

3.39 Assume that at the beginning of a year, I deposit R5000 into a bank account. After that, at the beginning of every month (starting in month 1) I deposit another R300 onto the bank account. Also, at the end of each month I withdraw R200. Write down the difference equation for the system, and calculate the amount of money on the account at then end of month 2, in each of the following cases.

(a) The difference equation is

$$A_{n+1} = A_n + 300 - 200 = A_n + 100.$$

If $A_0 = 5000$, then at the end of month 1 there will be $A_1 = 5100$ on the bank account, and at the end of month two there will be $A_2 = 5200$ on the account.

(b) If interest is paid at the rate of 2% then the difference equation will be

$$A_{n+1} = A_n + 300 + \frac{2}{100}(A_n + 300) - 200 = 1.02A_n + 106.$$

With $A_0 = 5000$,at the end of month 1 there will be $A_1 = 5206$ and at the end of month 2 there will be $A_2 = 5416.12$ on the bank account.

3.40 If C_n denotes the amount of chlorine in the pool after n days, then the system here is described by

$$C_{n+1} = \frac{3}{4}C_n, \quad C_0 = 20 \text{ kg}.$$

The solution to this system is

$$C_n = 20 \cdot \left(\frac{3}{4}\right)^n,$$

and therefore the amount of chlorine left after 10 days will be approximately $C_n = 1.1$ kg.

3.41 (a)

$$D_{n+1} = \frac{9}{10}D_n \quad D_0 = 200 \text{ litres}.$$

(b) It will take 7 days $(n \approx 6.58)$

3.42 (a)To find the difference equation, we need to calculate A_{n+1} (the money still owing at the end of month $n+1$) from A_n. So let us look at what happens during month $n+1$: The money owing at the beginning of the month, and until the two transactions at the end of month $n+1$, equals A_n. At the end of month $n+1$, interest is added to the load. The interest is calculated as 1% of the amount owed during the month, that is, 1% of A_n. And finally, 3000 is deducted from the loan. So, the difference equation is:

$$A_{n+1} = A_n + \frac{1}{100}A_n - 3000$$

$$\therefore \quad A_{n+1} = 1.01A_n - 3000$$

[**Remark:** Remember that interest adds to the amount owed, therefore needs to be added to the loan amount while the repayment is deducted from the loan amount. Note also that the original loan amount, $A_0 = 200\ 000$, does not appear anywhere in the difference equation!]

(b)Equilibrium point: $A_n = A = A_{n+1}$ holds if

$$A = 1.01A - 3000$$
$$\therefore \quad 0.01A = 3000$$
$$\therefore \quad A = 300\ 000.$$

(c)The equilibrium value of the model is $A^* = 300\ 000$, while the initial value is $A_0 = 200\ 000$. Since $A_0 \subsetneq A^*$, A_n will get smaller and smaller and, eventually zero – which in the exercise means that the loan will eventually be paid off. [If $A_0 = A^*$ then the loan amount will will forever stay at A_0 and will never be paid off; and if $A_0 > A^*$ then the loan still owed will be bigger and bigger each month, and the loan will never be paid off. The value of A^* is the exact loan size for which the interest charged equals the repayment made each month!]

(d) (i)The difference equation does not depend on the initial amount A_0, so it does not change when the initial loan amount is changed to $300\ 000$.

(ii)If the repayment is 1600, then the difference equation will be

$$A_{n+1} = 1.01A_n - 1600$$

(iii)If the interest rate changes to 2% per month, then the difference equation will be

$$A_{n+1} = 1.02A_n - 2000.$$

3.43 (a)To find the difference equation, we need to calculate A_{n+1} (the money in the account at the end of month $n+1$) from A_n. So let us look at what happens during month $n+1$. First of all, The amount of money A_n is carried over from the previous month. This is the money on the account at the beginning of the month, and until the three transactions at the end of month $n+1$. At the end of month $n+1$, first of all, interest is added. The interest is calculated as 2% of the amount in the account during the month, that is, 2% of A_n. After this transaction, we therefore have

$$A_n + \frac{2}{100}A_n$$

on the bank account. Finally, the amount N is added to the account, and 500 is withdrawn from the account. After these are done, the amount on the account is

$$A_n + \frac{2}{100}A_n + N - 500.$$

So, the difference equation is:

$$A_{n+1} = A_n + \frac{2}{100}A_n + N - 500$$

$$\therefore \quad A_{n+1} = 1.02A_n + N - 500.$$

The initial values was given as

$$A_0 = 10\ 000.$$

[Note also that the original loan amount, $A_0 = 400\ 000$, *does not appear anywhere in the difference equation itself!]*

(b)Let us find the equilibrium point: $A_n = A = A_{n+1}$ holds if

$$A = 1.02A + N - 500$$
$$\therefore \quad 0.02A = 500 - N$$
$$\therefore \quad A = \frac{500 - N}{0.02} = 25000 - 50N.$$

How should N be chosen such that the initial value 10 000 coincides with the equilibrium point? we should have

$$\begin{aligned} 25000 - 50N &= 10000 \\ \therefore 50N &= 15000 \\ N &= 300. \end{aligned}$$

So, if $N = 300$ then our initial value 10 000 is the equilibrium point, and therefore there will always be 10 000 on the account. And if N is larger than 300 (in which case we are putting in more money than necessary into the account each month), then 10 000 is larger than the equilibrium point and therefore the amount of money will in fact increase. So the answer to the question is: N should be larger than or equal to R300. *[A quick check: If* A_0 *is 10 000, then the interest paid over the first month is R200; and then if* $N = 300$ *is added and then R500 is removed at the end of the first month, then we are left with* $10\,000 + 200 + 300 - 500 = 10\,000$, *meaning that we still have the same amount of money after the first month.*

Study Unit 4 SOLUTIONS

4.1 (a) Not separable – the expression $2ty - t^2y^2$ cannot be factorised into an expression of the type $g(t) \cdot f(y)$.

(b) Is separable:

$$\frac{dy}{\cos y} = \sin x dx$$

(c) Is separable:

$$\frac{dy}{1 - y^2} = 1dx$$

(d) Not separable

(e) Is separable:

$$\frac{dy}{dx} = e^{x+y} = e^x e^y \quad \therefore \quad e^{-y}dy = e^x dx$$

(f) Is separable:

$$\frac{dy}{dx} = x - xy = x(1 - y) \quad \therefore \quad \frac{dy}{1 - y} = xdx$$

4.2 In the following solutions, C is an arbitrary real value (positive, negative or zero).

(a) $y(t) = \sqrt[3]{3t + \frac{3}{2}t^2 + t^3 + C}$

(b) $y(t) = \sqrt[5]{25t + C}$

(c) $y(t) = \pm\sqrt{\frac{1}{2}t^2 + C}$

(d) $y(t) = \pm\sqrt{t^2 + 4t + C}$

(e) $y(t) = \ln\left(\frac{1}{C-t}\right)$

(f) $y(t) = \frac{1}{2}\ln(2e^t + C)$

(g) $y(t) = 0$ or $y(t) = \frac{1}{C-t}$

(h) $y(t) = 0$ or $y(t) = \frac{1}{C+t-t^2}$

(i) $y(t) = \frac{4}{5} + Ce^{-5t}$

(j) $y(t) = Ce^{\frac{1}{2}t^2 - t}$

(k) $y(t) = Ct^{\frac{2}{5}}$

(l) $y(t) = -1 + Ce^{-t}$

(m) $y(t) = 1 + Ce^{-\frac{1}{2}t^2 - t}$

(n) $y(t) = \left(1 + Ce^{-2t}\right)$

(o) $y(t) = Ce^{t} - 10$

4.3 (a) $y(t) = 4e^{t} - 1$

(b) $y(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 + \frac{7}{6}$

(c) $y(t) = -\frac{1}{2}\sqrt{2}\sqrt{t^2 + 8}$

(d) $y(t) = \frac{1}{3}\left(2 - 17e^{-3t}\right)$

(e) $y(t) = \frac{1}{e^{2t}}\left(5 + 195e^{-2t}\right)$

(f) $y(t) = 5 - 4e^{t-1}$

(g) $y(t) = 3e^{\frac{1}{2}t^2} - 1$

(h) $y(t) = \frac{1}{4}\left(1 - 201e^{-8t}\right)$

4.4

$$\frac{dy}{dt} = ty$$

Separating the variables (assuming that $y \neq 0$, and noting that $y = 0$ is also a solution), we get

$$\frac{dy}{y} = t\,dt.$$

Integrate both sides:

$$\begin{aligned} \int \frac{dy}{y} &= \int t\,dt \\ \therefore \quad \ln|y| &= \frac{1}{2}t^2 + B \\ \therefore \quad |y| &= e^{\frac{1}{2}t^2}\, e^{B} = A\, e^{\frac{1}{2}t^2} \qquad (A = e^{B}) \\ \therefore \quad y &= \pm A\, e^{\frac{1}{2}t^2} \end{aligned}$$

Thus the general solution, which incorporates also the solution $y = 0$, is

$$y(t) = C\, e^{\frac{1}{2}t^2}.$$

4.5

$$\frac{dx}{dt} = \frac{t^2}{x^3}$$

Separate variables:

$$x^3 dx = t^2 dt$$

Integrate:

$$\begin{aligned} \int x^3 dx &= \int t^2 dt \\ \therefore \quad \frac{1}{4}x^4 &= \frac{1}{3}t^3 + C \\ \therefore \quad x^4 &= \frac{4}{3}t^3 + 4C \\ \therefore \quad x &= \pm\sqrt[4]{\frac{4}{3}t^3 + 4C}\,. \end{aligned}$$

The general solution can therefore be written as

$$x(t) = \pm\sqrt[4]{\frac{4}{3}t^3 + C}$$

where C is any constant.

4.6 Rewritten, the differential equation becomes

$$y^2\,\frac{dy}{dt} = t$$

Separate the variables:

$$y^2\,dy = t\,d\,t$$

Integrate:

$$\begin{aligned}\int y^2\,dy &= \int t\,d\,t\\ \therefore\quad \frac{1}{3}y^3 &= \frac{1}{2}t^2 + C\\ \therefore\quad y &= \left(\frac{3}{2}t^2 + 3C\right)^{\frac{1}{3}}\end{aligned}$$

The general solution can be written as

$$y\,(t) = \left(\frac{3}{2}t^2 + C\right)^{\frac{1}{3}}$$

for any constant C.

4.7

$$\frac{dy}{dt} = \frac{-t^3}{(y+1)^2}$$

Separate variables:

$$(y+1)^2\,dy = -t^3dt$$

Integrate:

$$\begin{aligned}\int (y+1)^2\,dy &= \int \left(-t^3\right)dt\\ \therefore\quad \frac{1}{3}(y+1)^3 &= -\frac{1}{4}t^4 + C\\ \therefore\quad y+1 &= \sqrt[3]{-\frac{3}{4}t^4 + 3C}\\ \therefore\quad y &= -1 + \sqrt[3]{-\frac{3}{4}t^4 + 3C}\end{aligned}$$

4.8Separate the variables, assuming that $p \neq 0$:

$$\frac{dp}{p} = \frac{dt}{t}$$

Integrate:

$$\begin{aligned}\int \frac{1}{p}dp &= \int \frac{1}{t}dt\\ \therefore\quad \ln|p| &= \ln|t| + C\\ \therefore\quad \exp(\ln|p|) &= \exp(\ln|t| + C)\\ \therefore\quad |p| &= A\,|t| \qquad \left(A = e^C\right)\\ \therefore\quad p &= \pm A\,|t|\end{aligned}$$

Thus the general solution to the differential equation, also including the solution $p = 0$, is

$$p\,(t) = Ct,$$

where C is any variable. *[Remark: Note that the original differential equation is not defined for $t = 0$, which means that we would not allow t to change sign, hence t will always be either positive or negative. This means that we can ignore the absolute value in $|t|$, and assume that t is always either positive or negative.]*

4.9

$$\frac{dy}{dt} - 2y + 1 = 0$$

Assuming that $2y - 1 \neq 0$, we can separate the variables as follows:

$$\begin{aligned} \frac{dy}{dt} &= 2y - 1 \\ \therefore \quad \frac{dy}{2y-1} &= dt \end{aligned}$$

Integrate:

$$\begin{aligned} \int \frac{1}{2y-1} dy &= \int 1 \cdot dt \\ \therefore \quad \frac{1}{2} \ln |2y-1| &= t + C \\ \therefore \quad |2y-1| &= \exp(2t + 2C) \\ &= A \exp(2t) \qquad \left(A = e^{2C}\right) \\ \therefore \quad 2y - 1 &= \pm A \exp(2t) \\ \therefore \quad y &= \frac{1}{2} \pm B e^{2t} \qquad \left(B = \frac{A}{2}\right). \end{aligned}$$

On the other hand, $2y - 1 = 0$ gives another solution, $y = \frac{1}{2}$. We can combine these possible solution into the following:

$$y(t) = \frac{1}{2} + Ce^{2t}$$

where C is positive, negative or zero.

4.10

$$\frac{dx}{dt} = 1 - x, \quad x(1) = 3$$

Separate variables, assuming that $x \neq 1$:

$$\frac{dx}{1-x} = d\,t$$

Integrate:

$$\begin{aligned} \int \frac{dx}{1-x} &= \int d\,t \\ -\ln|1-x| &= t + C \\ \ln|1-x| &= -t - C \\ e^{\ln|1-x|} &= e^{-t-C} = e^{-C}e^{-t} \\ |1-x| &= Ae^{-t} \qquad (A = e^{-C}) \\ 1 - x &= \pm Ae^{-t} \\ x &= 1 \mp Ae^{-t}. \end{aligned}$$

The general solution, including the solution $x = 1$, is

$$x(t) = 1 + Ce^{-t}$$

where C is any real value. To obtain the particular solution with $x(1) = 3$, we substitute $t = 1$ and $x = 3$ into the general solution above. We get

$$\begin{aligned} 3 &= 1 + Ce^{-1} \\ Ce^{-1} &= 2 \end{aligned}$$

This can only hold if $C = 2e$. Therefore the particular solution is

$$x(t) = 1 + 2e \cdot e^{-t} = 1 + 2e^{1-t}.$$

4.11

$$\frac{dy}{dt} = \frac{2t}{y}, \quad y(1) = -2$$

To find the general solution, we separate the variables

$$y\,dy = 2t\,dt$$

and then integrate:

$$\begin{aligned} \int y\,dy &= \int 2t\,dt \\ \therefore\ \frac{1}{2}y^2 &= t^2 + C \\ \therefore\ y &= \pm\sqrt{2t^2 + 2C} \end{aligned}$$

To obtain the particular solution with $y(1) = -2$, we substitute $t = 1$ and $y = -2$ into the general solution above. We get

$$\begin{aligned} -2 &= \pm\sqrt{2(1)^2 + 2C} \\ \therefore\ -2 &= \pm\sqrt{2 + 2C}. \end{aligned}$$

We see that the sign must be negative. To find the value of C, we square both sides:

$$\begin{aligned} (-2)^2 &= \left(\pm\sqrt{2 + 2C}\right)^2 \\ \therefore\ 4 &= 2 + 2C \\ \therefore\ C &= 1. \end{aligned}$$

Therefore, the solution to the initial value problem is

$$y(t) = -\sqrt{2t^2 + 2}.$$

4.12

$$\frac{dy}{dt} = -\frac{t}{2y}, \quad y(1) = 2$$

Separate variables:

$$2\,y\,dy = -t\,d\,t$$

Integrate:

$$\begin{aligned} \int 2y\,dy &= -\int t\,d\,t \\ y^2 &= -\frac{1}{2}t^2 + C \\ y &= \pm\sqrt{-\frac{1}{2}t^2 + C} \end{aligned}$$

To obtain the particular solution with $y(1) = 2$, we substitute $t = 1$ and $y = 2$ into the general solution above. We get

$$\begin{aligned} 2 &= \pm\sqrt{-\frac{1}{2}(1)^2 + C} \\ 2 &= \pm\sqrt{C - \frac{1}{2}}. \end{aligned}$$

We see that the sign must be positive. To find the value of C, we can square both sides:

$$\begin{aligned} (2)^2 &= \left(\pm\sqrt{C - \frac{1}{2}}\right)^2 \\ 4 &= C - \frac{1}{2} \\ C &= 4 + \frac{1}{2} = \frac{9}{2}. \end{aligned}$$

Thus, the solution is

$$y(t) = \sqrt{-\frac{1}{2}t^2 + \frac{9}{2}}.$$

4.13

$$\frac{dy}{dt} = -\frac{1}{y}t^5, \quad y(1) = -1$$

Separate variables:

$$y\,dy = -t^5 dt$$

Integrate:

$$\begin{aligned} \int y\,dy &= -\int t^5 dt \\ \therefore\ \frac{1}{2}y^2 &= -\frac{1}{6}t^6 + C \\ \therefore\ y &= \pm\sqrt{-\frac{1}{3}t^6 + 2C} \end{aligned}$$

For the particular solution, substitute $t = 1,\ y = -1$:

$$-1 = \pm\sqrt{-\frac{(1)^6}{3} + 2C}.$$

Thus, the sign in front of the square root must be negative. Squaring both sides gives

$$\begin{aligned} 1 &= -\frac{1}{3} + 2C \\ \therefore\ 2C &= \frac{4}{3} \end{aligned}$$

The solution is

$$y = -\sqrt{-\frac{t^6}{3} + \frac{4}{3}}.$$

4.14

$$\frac{dy}{dt} - 24y\left(t^2 - t\right) = 0, \quad y(1) = e$$

General solution:

$$\begin{aligned} \frac{dy}{dt} - 24y\left(t^2 - t\right) &= 0 \\ \therefore\quad \frac{dy}{dt} &= 24y\left(t^2 - t\right) \end{aligned}$$

Separate variables, assuming that $y \neq 0$:

$$\frac{dy}{y} = 24\left(t^2 - t\right)dt$$

Integrate:

$$\begin{aligned}\int \frac{1}{y}dy &= \int 24\left(t^2-t\right)dt\\ \therefore\quad \ln|y| &= 24\left(\frac{1}{3}t^3-\frac{1}{2}t^2\right)+C\\ &= 8t^3-12t^2+C\\ \therefore\quad |y| &= e^{8t^3-12t^2+C}=Ae^{8t^3-12t^2}\qquad (A=e^c)\\ \therefore\quad y &= \pm Ae^{8t^3-12t^2}\end{aligned}$$

The general solution, including the zero solution, is

$$y(t)=Ce^{8t^3-12t^2}$$

where C is any constant. Solution to initial value problem: We must find the value of A such that

$$y(1)=-e.$$

When we substitute $y=-e$ and $t=1$ into the general solution we get the condition

$$\begin{aligned}-e &= Ce^{8-12}\\ \therefore\quad -e &= Ce^{-4},\end{aligned}$$

from which we see that we should have

$$C=-\frac{e}{e^{-4}}=-e^5.$$

Therefore the solution is

$$y(t)=-e^{8t^3-12t^2+5}.$$

4.15

$$t^3-y\,\frac{dy}{dt}=0,\quad y(1)=-1$$

Separate values:

$$y\,dy=t^3\,dt$$

Integrate:

$$\begin{aligned}\frac{1}{2}y^2 &= \frac{1}{4}t^4+C\\ y^2 &= \frac{1}{2}t^4+2C\\ y &= \pm\sqrt{\frac{1}{2}t^4+2C}\end{aligned}$$

The sign and the value of C are determined from the initial value $y(1)=-1$. We substitute $y=-1$ and $t=1$ into the general solution, to get

$$-1=\pm\sqrt{\frac{1}{2}(1)^4+2C}$$

So, we must choose the negative sign and $C=\frac{1}{4}$. Therefore, the solution to the initial value problem is

$$y(t)=-\sqrt{\frac{1}{2}t^4+\frac{1}{2}}.$$

4.16

$$\frac{dy}{dt}=2ty,\quad y(2)=1$$

First, we will find the general solution by separating the variables and integrating. Here $y=0$ is a solution; while if we assume $y \neq 0$ we get

$$\begin{aligned} \frac{dy}{dt} &= 2ty \\ \therefore \quad \frac{dy}{y} &= 2tdt \\ \therefore \quad \ln|y| &= t^2 + C \\ \therefore \quad y &= \pm A e^{t^2} \quad \left(A = e^C\right) \end{aligned}$$

The general solution is therefore

$$y(t) = Ce^{t^2}$$

where C is any real value. To find to solution which satisfies the initial condition $y(2) = 1$, we substitute $t = 2$ and $y = 1$ into this: We must have

$$1 = Ce^4$$

so we should choose C= e^{-4}. Thus, the solution to the initial value problem is

$$y(t) = e^{t^2-4}.$$

Study Unit 5 SOLUTIONS

5.1 (a) $k = 0.03$ (b) $k = -0.03$ (c) $k = 0.001\,5$ (d) $k = 0.02$

5.2 (a) The population will grow without bound. (b) The population will stay at 0. (c) The population will stay at 0. (d) The population decreases towards 0. (e) The population decreases towards 0.

5.3 (a) $P(t) = 10e^{5t}$ (b) $P(t) = 0$ (c) $P(t) = 0$ (d) $P(t) = 3e^{-0.01t}$ (e) $P(t) = 500e^{-4t}$

5.4 (a) $P(10) = 5.\,184\,7 \times 10^{22}$ (b) $P(10) = 0$ (c) $P(10) = 0$ (d) $P(10) = 2.\,714\,5$ (e) $P(t) = 2.\,124\,2 \times 10^{-15}$

5.5

(a) $P(5) = 10000e^{0.01 \cdot 5} \approx 10513$, $P(50) = 10000e^{0.01 \cdot 50} \approx 16487$.

(b) $t = \frac{1}{0.01} \ln\left(\frac{100000}{10000}\right) \approx 230.\,26$.

5.6 Assume that a population grows according to the Malthusian model, with $k = -0.1$ and $P_0 = 100$

(a) $P(10) = 36.\,788$ and $P(100) = 4.\,540\,0 \times 10^{-3}$.

(b) $t = 46.\,052$

(c) The population decreases asymptotically towards zero, and never gets negative values (since e^x is never negative, not even when x is negative, as will be the case here).

5.7 (a) $k = \frac{1}{5} \ln\left(\frac{20000}{5000}\right) \approx 0.277\,26$ (b) $k = -0.690\,78$ (c) $k = 0.690\,78$

5.8 $1.\,638\,4 \times 10^6$

5.9 $k = 0.639\,69$; $68.\,146$ after 3 hours,

5.10 (a) $P(2) = 2\,e^{(0,4) \cdot 2} = 4.45 \approx 4$ (b) $P(6) = 2\,e^{(0.4) \cdot 6} \approx 22$ (c) $P(10) = 2\,e^{(0.4) \cdot 10} \approx 109$ *(It is interesting to compare the figures obtained here with those actually observed in a laboratory experiment, namely, 5, 20 and 109 – a surprisingly good agreement with the theory!)*

5.11 The enclosure gets filled in day 69.

5.12 177830

5.13 (a) 4472 (b) in 2020

5.14 $k \approx -9.\,210\,3 \times 10^{-2}$, (a) In year 2025, (b) In year 1975

5.15 $k \approx 9.\,210\,3 \times 10^{-2}$, (a) In 1925, (b) In 2025

5.16 The solution to the Malthusian model is given by

$$P(t) = P_0 e^{kt} \qquad (*)$$

This has two constants in it, P_0 and k. So, if we are given just two "readings" along the solution curve (e.g. $P(t_1) = P_1$ and $P(t_2) = P_2$) then the solution can already be uniquely determined. (To put it in another way, only one curve of the type (*) can be plotted through two given points in the (t, P) plane.) So, to answer this question, we will select any two points, find the values of P_0 and k in (*) from them, and then check if the third point satisfies equation (*) for those values of k and P_0. (To put it graphically again, the two points determine a unique solution curve; and then we must check if the third point is also situated on the same curve or not.) We will start with the two first points (although any two would work just as well!!) We will take $t = 0$ in 1980, then we already have $P_0 = 110\ 000$ The second reading gives us (since 1990 $= 1980 + 10$), $P(10) = 175\ 000$. On the other hand (*) is supposed to hold, so we must have

$$175\ 000 = 110\ 000e^{10k}$$

from which we can solve for k:

$$k = \frac{1}{10}\ln\left(\frac{175\ 000}{110\ 000}\right) \approx 4.643\,1 \times 10^{-2}.$$

So, the first two readings tell us that the Malthusian model would need to be one with the solution

$$P(t) = 110\ 000e^{(0.046431)t} \qquad (**)$$

Does the third reading agree with this? According to the third reading, we should have $P(20) = 220\ 000$. Let us see what (**) gives us for $t = 20$:

$$P(20) = 120\ 000e^{(0.046431)\cdot 20} \approx 278\ 410.$$

This is very different from the third reading (220 000), so we conclude that the three readings cannot come from a Malthusian model.

5.17 $T = 115.52$

5.18 $k = 3.465\,7 \times 10^{-3}$

5.19 (a) $k = 1.488\,9 \times 10^{-2}$ (b) $k = 0.262\,36$ (c) $q = 2.020\,1$. (d) $q = 10.517$

5.20 $q = 1.748\%$ per year

5.21 $k = 0.025$

5.22 If a population grows according to the Malthusian model, with population $P_0 = 2 \times 10^6$ at time $t = 0$, find the value of the growth constant k if the annual growth rate is 2% per year.

5.23 $k = 3.465\,7 \times 10^{-3}$; (a) $t = \frac{1}{k}\ln(1.5) \approx 116.99$, (b) $t = \frac{1}{k}\ln(3) \approx 317.00$

5.24 $k = \frac{1}{2}\ln\left(\frac{620000}{600000}\right) \approx 1.639\,5 \times 10^{-2}$

5.25 $k = -3.333\,3 \times 10^{-3}$

5.26 3000 per year; 3664.2 per year.

5.27 (a) $k = 0.095\,31$ (b) $k = -2.302\,6$ (c) $k = 0.231\,05$ (d) $k = 2.302\,6 \times 10^{-2}$ (e) $k = -0.356\,67$. (f) $k = 0.262\,36$

5.28 (a) $q = 64.872$ % (b) $q = -39.347$ % (c) $q = 171.83$ %

5.29 $k = 1.980\,3 \times 10^{-2}$

5.30 $T = 17.673$ years

5.31 $q = 1.395\,9$ % per year.

5.32 6.9 years.

5.33 $k \approx 9.950\,3 \times 10^{-3}$; (a) $P(1) \approx 3.030\,0 \times 10^5$ (b) 2985.1 per year (c) 3014.9 per year (d) 121 years. (e) 69.661 years.

5.34 We need to solve T from

$$P(t + T) = M \cdot P_0 \qquad (*)$$

if $P(t_0) = P_0$. But if $P(0)$ is the population at time $t = 0$ (which is here not necessarily equal to P_0, then we have

$$\begin{aligned} P_0 &= P(t_0) = P(0)\,e^{kt_0}, \\ P(t+T) &= P(0)\,e^{k(T+t_0)} \end{aligned}$$

so that (*) can be written as

$$P(0)\,e^{k(T+t_0)} = M \cdot P(0)\,e^{kt_0}.$$

Solving for T from this we get (note that e^{kt_0} and $P(0)$ cancel out)

$$e^{kT} = M \quad \therefore \quad T = \frac{\ln(M)}{k}$$

which does not depend on P_0 or t_0, but rather only on M and k.

5.35 $k = 2.176\,1 \times 10^{-2}$. (a) In year 2031. (b) 42.7 million.

5.36 (a) An annual growth rate of 2% per year means that $P(1)$ is 2% larger than $P(0) = P_0$. That is,

$$P(1) = P_0 + \frac{2}{100}P_0 = 1.02P_0;$$

on the other hand we must also have $P(1) = P_0e^{kt}$. Combining these and solving for k, we get

$$1.02P_0 = P_0e^{k\cdot 1} \quad \therefore \quad 1.02 = e^k \quad \therefore \quad k = \ln(1.02) \approx 0.0198.$$

Note that the initial population value is irrelevant here!]

(b) If the population grows by 20 000 from $t = 0$ to $t = 2$, we must have $P(2) = P_0 + 20\,000$. Again using the fact that $P(t) = P_0e^{kt}$ must hold, we get the equation $P_0e^{2k} = P_0 + 20\,000$. Note that this time the P_0–value is not going to cancel out! Solving for k, we get $k \approx 0.01961$

(c) The rate of change of the population is given by the differential equation

$$\frac{dP}{dt} = kP$$

So, the rate of change of the population at time $t = 0$ equals

$$k \cdot P_0 = k \cdot (500\,000)$$

We are told that this rate of change equals 2000:

$$k \cdot (500\,000) = 2000$$

from which we can find the value of k:

$$k = \frac{2000}{500000} = \frac{1}{250} = 0.004.$$

5.37 Let us take $t = 0$ to be in year 1980 then we know $P_0 = P(0) = 150\,000$ as well as the population in year $t = 1990 - 1980 = 10$: $P(10) = 90\,000$. If the Malthusian model holds then the population in year t is given by

$$P(t) = P_0e^{kt}$$

With $t = 0$ in 1980, this gives

$$P(t) = 150\,000 \cdot e^{kt} \qquad (*)$$

(a) We are told that $P(10) = 90\,000$; if we apply (*) with $t = 10$ we must therefore have

$$P(10) = 150\,000e^{k\cdot 10} = 90\,000$$

from which we can find the value of k:

$$e^{10k} = \frac{90\,000}{150\,000} = \frac{3}{5} \quad \therefore \quad 10k = \ln\left(\frac{3}{5}\right) \quad \therefore \quad k = \frac{1}{10}\ln\left(\frac{3}{5}\right) = -0.05108$$

[Note that k is negative – as it should be since the population is getting smaller!]

(b) When is $P(t) = 10\,000$? Using the k–value calculated in (a), we have

$$P(t) = 150\,000e^{(-0.05108)\cdot t}$$

so we need to solve t from the equation

$$10\,000 = 150\,000e^{-(0.05108)t}$$

We get $t = \frac{\ln\left(\frac{1}{15}\right)}{-0.05108} = 53 \cdot 02$.The population will be 10 000 during year 1980 $+53 = 2033$.

(c) Year 2230 corresponds to $t = 2230 - 1980 = 50$, so the population is that year will be

$$P(50) = 150\,000 \cdot e^{(-0.05108)\cdot 50} = 11\,665.5 \approx 12\,000$$

[We have rounded the result to the nearest 10 000 since this seems to be the accuracy in the two given population readings!]

(d) If we take $t = 0$ to be in year 2000 then we must use as initial value P_0 the size of the population in year 2000. For $t = 0$ in 1980, 2000 corresponds to $t = 20$, so the size of the population in 2000 is

$$P(20) = 150\,000 \cdot e^{-(0.05108)\cdot 20} = 54\,000$$

Thus is $t = 0$ in 2000, the expression for $P(t)$ is

$$P(t) = 54\,000e^{(-0.05108)t}$$

[Note that the population in 1980, i.e. $t = -20$, is given by this formula as

$$P(-20) = 54\,000e^{(-0.05108)(-20)} = 150\,000,$$

as it should be.]

5.38 $k = \frac{\ln(2)}{100} \approx 0.00693$; (a) 38 years, (b) 15.9 years.

5.39 (a) $k \approx \ln(1.02) \approx 0.0198$ (b) $k \approx 0.04975$ (c) $= \dfrac{2000}{2\,000\,000} = \dfrac{1}{1000} = 0.001$

5.40

(a) $k = \frac{1}{10}\ln(9) \approx 0.21972$.

(b) The population was 200 000 during year $1980 + 13 = 1993$.

(c) The population was 10 approximately 31.4 years before 1980, that is, during year 1948.

(d) If $s = 0$ in 2000, the expression for $P(s)$ is $P(s) = 810\,000e^{(0.21972)s}$.

5.41 Let $P_1(t)$ and $P_2(t)$ denote the sizes of the populations of respectively Russia and USA at the beginning of year t, with $t = 0$ in 1980. Both populations are assumed to grow according to the Malthusian model, and the initial populations and the growth constants of both countries are given. Applying the solution for the Malthusian models,

$$P(t) = P_0e^{kt}$$

for each of the countries, we therefore have

$$\begin{aligned} P_1(t) &= 255 \times 10^6 \cdot e^{0.012t} \\ P_2(t) &= 225 \times 10^6 \cdot e^{0.007t} \end{aligned}$$

We wish to find out when

$$P_1(t) = 2P_2(t).$$

Substituting the functions P_1 and P_2 from above into this, we get

$$\begin{aligned} 255 \times 10^6 \cdot e^{0.012t} &= 2 \cdot 225 \times 10^6 \cdot e^{-0.007} \\ \therefore\ e^{(0.012t-0.007t)} &= \frac{450 \times 10^6}{255 \times 10^6}\ \therefore\ e^{0.005t} = \frac{450}{255} \\ \therefore\ t &= \frac{1}{0.005}\ln\left(\frac{450}{255}\right) \approx 113.597 \text{ years,} \end{aligned}$$

i.e. during the year $1980 + 113 = 2093$.

5.42 Let $P_A(t)$ and $P_B(t)$ denote the population sizes of Country A and Country B after t years, with $t = 0$ in year 1960. Since both populations grow according to the Malthusian model, we know that the populations can be calculated by the formulas

$$\begin{aligned} P_A(t) &= P_A(0)\, e^{k_A t}, \\ P_B(t) &= P_B(0)\, e^{k_B t} \end{aligned} \tag{1}$$

for some constants $k_A,\ k_B$. [*Note that in general, $k_A \neq k_B$ since the populations of two different countries would probably grow at different rates!*] We are given the initial population sizes:

$$\begin{aligned} P_A(0) &= 1.2 \times 10^7 \\ P_B(0) &= 4 \times 10^5 \end{aligned} \tag{2}$$

We do not know the values of k_A and k_B, which we would need to answer the questions, but we can calculate them since we are given the population sizes in year 1990 $(t = 30)$:

$$\begin{aligned} P_A(30) &= 8 \times 10^6 \\ P_B(30) &= 7 \times 10^5 \end{aligned} \tag{3}$$

we get

$$k_A = \frac{1}{30}\ln\left(\frac{2}{3}\right) \approx -0.01352, \qquad k_B = \frac{1}{30}\ln\left(\frac{7}{4}\right) \approx 0.01865.$$

[*Note that the negative value of k_A follows from the fact that the size of Country A is decreasing. In year 1960 it was*

12 million, but in year 1990 it had dropped to 8 million!] Now, we know all we need to know about the populations of Country A and Country B: For all t,

$$P_A(t) = P_A(0)\,e^{k_A t} \text{ with } P_A(0) = 1.2\times 10^7, \quad k_A = -0.01352,$$
$$P_B(t) = P_B(0)\,e^{k_B t} \text{ with } P_B(0) = 4\times 10^5, \quad k_B = 0.01865.$$

(a) When will the population of Country B reach 1×10^8? We put

$$10^8 = P_B(t) = P_B(0)\,e^{k_B t}$$

and solve for t :

$$t = \frac{\ln(250)}{k_B} = \frac{\ln(250)}{0.01865} \approx 296 \text{ years},$$

that is, in year $1960 + 296 = 2256$.

(b) When is $P_A(t) = P_B(t)$? We need to solve t from

$$P_A(t) = P_B(t) \therefore\ P_A(0)\,e^{k_A t} = P_B(0)\,e^{k_B t}$$

$$\therefore\ \frac{e^{k_A t}}{e^{k_B t}} = e^{(k_A - k_B)t} = \frac{P_B(0)}{P_A(0)} \ \therefore\ t = \frac{1}{k_A - k_B}\ln\left(\frac{P_B(0)}{P_A(0)}\right) \approx 105 \text{ years},$$

that is, the populations will be equal in size in year $1960 + 105 = 2065$.

5.43 Let $P_A(t)$ denote the population of country A and $P_B(t)$ the population of country B after t years, and let us take $t = 0$ in year 2000. Then

$$P_A(t) = P_A(0)\,e^{k_A t}$$
$$P_B(t) = P_B(0)\,e^{k_B t}$$

where $P_A(0)$ and $P_B(0)$ are the sizes of the population in year 2000. The values of $P_A(0)$ and $P_B(0)$ are known to be the same, but then exact value is not known.

We can find the growth constant for each population from its doubling time:

$$k_A = \frac{\ln(2)}{50}, \quad k_B = \frac{\ln(2)}{80}$$

Now we can calculate the ratio of $P_B(t)$ and $P_A(t)$ at any time t:

$$\frac{P_B(t)}{P_A(t)} = \frac{P_B(0)\,e^{k_B t}}{P_A(0)\,e^{k_A t}} = \underbrace{\frac{P_B(0)}{P_A(0)}}_{=1}\cdot\frac{e^{k_B t}}{e^{k_B t}} = e^{k_B t - k_A t} = e^{(k_B - k_A)t},$$

but

$$k_B - k_A = \frac{\ln(2)}{50} - \frac{\ln(2)}{80} = \ln(2)\left(\frac{1}{50} - \frac{1}{80}\right) = \ln(2)\cdot\frac{3}{400}$$

and therefore

$$\frac{P_B(t)}{P_A(t)} = e^{\left(\ln(2)\frac{3}{400}\right)t} \approx e^{0.051986t}$$

In particular, in year 1950 $(t = -50)$ the ratio was $e^{(0,051986)\cdot(-50)} \approx 0.8,\ a$ and in year 2050 $(t = +50)$ the ratio will be $e^{(0.051986)\cdot(50)} \approx 1.3$.

5.44

(a) From the half–life, $T = 2000$, we can get the value of the decay constant, k : The half–life is the time until half of the original amount remains. That is, we could have

$$N(t) = \frac{1}{2}N_0 \tag{1}$$

if $N(t)$ denotes the amount left after time t and $N_0 = N(0)$ the original amount. But then we must also always have

$$N(t) = N_0 e^{-kt} \tag{2}$$

So combining (1) and (2) we get the equation

$$N_0 e^{-kT} = \frac{1}{2}N_0$$

from which we can solve k in terms of T :

$$k = -\frac{\ln\left(\frac{1}{2}\right)}{T} = \frac{\ln(2)}{T}$$

So, in this question the value of k is

$$k = \frac{\ln(2)}{2000} \approx 3.4657 \times 10^{-4}$$

(b) If 10 grams remains today, when was there 50 grams? We can either (1) take $t = 0$ to be today and find the time t where $N(t) = 50$ (in which case t will be negative); or (2) taken $t = 0$ to be when there was 50g, and find today's date t such that $N(t) = 10$. **Method 1:** Let $N_0 = N(0) = 10$, and find t such that

$$N(t) = 50$$

But (2) also has to hold, so we should have

$$50 = 10 \cdot e^{-kt}$$

from which we will solve t :

$$T = -\frac{1}{k}\ln\left(\frac{50}{10}\right) \approx -4634 \text{ years} \qquad \text{(using the value of } k \text{ given above)}$$

There was 50 grams about 4600 years ago. **Method 2:** Let $N_0 = N(0) = 50$, and find t such that

$$N(t) = 10 :$$

We must have

$$10 = 50 \cdot e^{-kt}$$

from which we get

$$t = +4634$$

Today must be $t = 4634$ years after there was 50 grams, i.e. there was 50 grams about 4600 years ago.

(c) When will there be 1 gram left, if today there is 10 grams? We can take $t = 0$ to be today, so that

$$N_0 = N(0) = 10$$

We wish to find the t for which

$$N(t) = 1;$$

but this means that

$$1 = 10e^{-kt} \quad \therefore \quad t = -\frac{\ln\left(\frac{1}{10}\right)}{k} \approx 6644 \text{ years}$$

There will be 1 gram left about 6600 years from now.

5.45 5.1921 hours

5.46 $1.25g$

5.47 5614 years ago.

5.48 There was 100g of the substance present approximately 5800 years ago.

5.49 (a) 458 years, (b) 1151 years,
(c)The time t it takes to go from N_0 to another value, e.g. $\hat{N}$, only depends on the ratio of the values $\hat{N}$, and N_0:

$$t = -\frac{1}{k} \cdot \ln\left(\frac{\hat{N}}{N_0}\right)$$

Here, $\hat{N}$ was taken to be $\frac{1}{10}N_0$, so the ratio $\hat{N}/N_0$ in the question is always $1/10$, whatever the value of N_0.

5.50 (a) 176 kg, (b) 20 kg is left in year $1970 + 2274$, i.e. in year 4244.

5.51 $k = \frac{\ln(2)}{2000} \approx 3.4657 \times 10^{-4}$

(a) If 10g remains today, when was there 50g? We can either (Method 1) take $t = 0$ to be today and find the time t where $N(t) = 50$ (in which case t will be negative); or (Method 2) take $t = 0$ to be when there was 50g, and find today's date t such that $N(t) = 10$. **Method 1:** Let $N_0 = N(0) = 10$, and find t such that $N(t) = 50$. But (2) also has to hold, so we should have

$$50 = 10 \cdot e^{-kt}$$

from which we will solve t :

$$\begin{aligned} T &= -\frac{1}{k}\ln\left(\frac{50}{10}\right) \\ &\approx -4644 \text{ years} \qquad \text{(using the value of } k \text{ given above)} \end{aligned}$$

There was 50g about 4600 years ago. **Method 2:** Let $N_0 = N(0) = 50$, and find t such that

$$N(t) = 10:$$

We must have

$$10 = 50 \cdot e^{-kt}$$

from which we get

$$t = +4634$$

Today must be $t = 4634$ years after there was 50g, i.e. there was 50 grams $t = 4634$ years ago.

(b) When will there be 1g left, if today there is 10g? We'll take $t = 0$ to be today, so that $N_0 = N(0) = 10$. We wish to find the t for which $N(t) = 1$; but this means that

$$1 = 10e^{-kt} \therefore \;\; t = -\frac{\ln\left(\frac{1}{10}\right)}{k} = 6644 \text{ years}$$

There will be 1g left about 6600 years from now.

5.52 Let $C(t)$ be the concentration of the chemical in the blood at time t, when t is measured in days and the concentration in milligrams per millimeter. The differential equation modelling how the concentration changes, namely

$$\frac{dC}{dt} = -aC, \tag{1}$$

is the same one as in our model of radioactive decay. The solution is

$$C(t) = C_0\, e^{-at}. \tag{2}$$

We are given the initial concentration, $C_0 = 0.1$. The value of a is not given, but we can calculate it when we know that $C(3) = 0.054$. Applied at $t = 3$, (2) gives

$$0.054 = C(3) = (0.1)\, e^{-a\cdot 3} \therefore \;\; a = -\frac{1}{3}\ln\left(\frac{0.054}{0.1}\right) \approx 0.2054.$$

How long will it take until $C(t) = 0.0001$? Let's put $C(t) = 0.0001$, using (2), and solve for t :

$$0.0001 = (0.1)\, e^{-at} \therefore \;\; t = -\frac{1}{a}\ln\left(\frac{0.0001}{0.1}\right) = -\frac{\ln(0.001)}{0.2054} \approx 33.63.$$

That is, it takes approximately $33\frac{1}{2}$ days for the acceptable concentration level to be reached.

5.53

(a) True. The rate of decay is the value of the derivative dN/dt. According to the differential equation for radioactive decay,

$$\frac{dN}{dt} = -kN,$$

so if N is larger then so is dN/dt (or, rather, $|dN/dt|$.)

(b) False. For one thing, whether this is true or not depends very much on how big the initial populations $P_A(0)$ and $P_B(0)$ are. Even if we assume that they are initially the same $(P_A(0) = P_B(0) = P_0)$ then

$$P_B(10) = P_B(0)\, e^{10k_B} = P_0 e^{10k_B}$$

and

$$P_A(10) = P_A(0)\, e^{10k_A} = P_0 e^{10\cdot 2k_B} = P_0 e^{10k_B} \cdot e^{10k_B}$$

meaning that

$$P_A(10) = e^{10k_B} \times P_B(10)$$

rather than

$$P_A(10) = 2 \times P_B(10).$$

[Note that writing this in another way, we see that what we get is in fact that $P_A(10) = P_B(20)$, and since the growth here is not linear, we cannot expect to have $P_B(20) = 2 * P_B(10)$ which the claim would imply!]

(c) False. The doubling times are

$$T_A = \frac{\ln(2)}{k_A}, \quad T_B = \frac{\ln(2)}{k_B},$$

so if $k_A = 2 * k_B$, we get

$$T_A = \frac{\ln(2)}{2k_B} = \frac{1}{2}\frac{\ln(2)}{k_B} = \frac{1}{2}T_B.$$

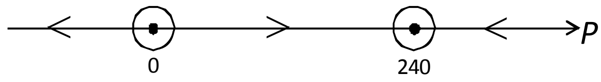
The doubling time for A is half that of B.

Study Unit 6 SOLUTIONS

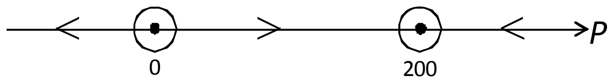
6.1 (a) $P = 0$ and $P = 83333$. (b) $P = 0$ and $P = 0.2$. (c) $P = 0$ and $P = 2$.

6.2

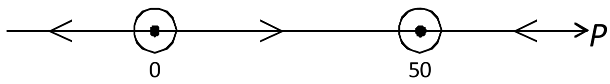
(a)



(b)

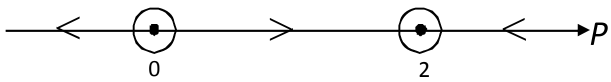


(c)



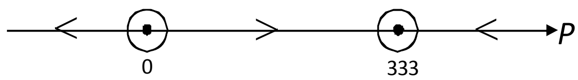
6.3 In each of the following cases of parameter values, draw the phase line of the corresponding logistic model and read from the phase line the outcomes for a solution which starts at (i) $P_0 = 100$, (ii) $P_0 = 1000$.

(a) Here, $a/b = 2$ so the phase line looks as follows:



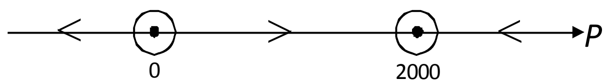
In both (i) and (ii), the initial point (100 and 1000) fall on the portion of the line to the right of the equilibrium point 2, where the motion is towards the left; therefor in both cases the solutions decrease towards 2.

(b) Here, $a/b \approx 333$ so the phase line looks as follows:



In this case (i) the initial value $P_0 = 100$ is on the interval between the two equilibrium points, where motion is towards the right, and therefore the solution will increase towards the equilibrium point 333.33. In (ii), the initial value $P_0 = 1000$ is on the interval to the right of equilibrium point 333, where motion is towards the left , and therefore the solution will decrease towards the equilibrium point 333.33

(c) Here, $a/b = 2000$ so the phase line looks as follows:



Both initial values, in (i) and (ii), are in the interval between the two equilibrium points, where motion is towards the right, so in both cases the solution will increase towards 2000.

6.4

(a) (i): The solution will increase towards 30; (ii) the solution will decrease towards 30.

(b) (i)The solution will stay at 10 (which is an equilibrium point); The solution will decrease towards 10.

(c) (i) and (ii): The solution will increase towards 60.

6.5

$$\text{(a)}\quad P(t) = \frac{0.2}{0.001 + \left[\dfrac{0.2}{P_0} - 0.001\right] e^{-0.2t}}, \quad \text{(b)}\quad P(t) = \frac{0.001}{0.0005 + \left[\dfrac{0.001}{P_0} - 0.0005\right] e^{-0.001t}}$$

6.6 The solution with $P_0 = 12000$ is

$$P(t) = \frac{0.002}{2.4 \cdot 10^{-8} + \left[\frac{0.002}{12000} - 2.4 \cdot 10^{-8}\right] e^{-0.002t}}$$

which gives $P(10) \approx 12207$ and $P(100) \approx 14\,204$.

6.7 $P(2) = 2464$, $P(5) = 1015$. (Note that the population is decreasing! The equilibrium value a/b is here equal to 50.0,and the population is decreasing towards this limit value.

6.8 Let $P(t)$ denote the size of the human population at time t. Here, t is measured in years, and $t = 0$ in year 1965. We assume logistic growth with

$$\begin{aligned} P_0 &= 3.34 \times 10^9 \\ a &= 0.029 \\ b &= 2.695 \times 10^{-12}. \end{aligned}$$

In logistic growth,

$$P(t) = \frac{a}{b + \left[\frac{a}{P_0} - b\right] e^{-at}}.$$

So, the population in the year $2000 = 1965 + 35$ is

$$P(35) = \frac{0.029}{2.695 \times 10^{-12} + \left[\frac{0.029}{3.34 \times 10^9} - 2.695 \times 10^{-12}\right] e^{-0.029 \cdot 35}} \approx 5.9 \times 10^9$$

6.9 $b = 3.6534 \times 10^{-7}$, $P_{\text{limit}} = 27\,372$.

6.10 $a = 0.016$, $b = 5.3333 \times 10^{-8}$

6.11 $b = 2.6333 \times 10^{-4}$

6.12 Let $P(t)$ denote the size of the population in year t. Then if we assume the logistic model holds, P must obey the differential equation

$$\frac{dP}{dt} = aP - bP^2.$$

The value of a is given as $a = 0.029$. We must figure out the value of b from the given information, namely:

”For a certain T, $P(T) = 3.34 \times 10^9$
and at that T, the rate of increase
equals 2% of the population per year”.

But, the rate of increase (or decrease) of the population as a function of the current population size is given by the differential equation. If at a time T the population is $P(T)$ then the differential equation itself tells us that the rate of change (increase or decrease) of the population at that exact moment T equals

$$aP(T) - b(P(T))^2.$$

So, the rate of increase at moment T, as given by the differential equation, is equal to

$$\frac{dP}{dt} = aP(T) - b(P(T))^2 \text{ when } t = T. \tag{1}$$

On the other hand, we are told that at time T, the rate of increase equals 2% of the current population per year, that is, 2% of $P(T)$ which equals $(0.02)P(T)$. That is, we also have

$$\frac{dP}{dt} = (0.02)(P(T)) \quad \text{when } t = T. \tag{2}$$

But (1) and (2) must of course be equal, and therefore we must have

$$a.P(T) - b(P(T))^2 = 0.02P(T)$$

from which we can solve b:

$$b(P(T))^2 = (a - 0.02)P(T) \therefore b = \frac{a - 0.02}{P(T)}.$$

When we substitute the numerical values,

$$\begin{aligned} a &= 0.029, \\ P(T) &= 3.34 \times 10^9 \end{aligned}$$

into this, we get

$$b = \frac{0.029 - 0.02}{3.34 \times 10^9} = \frac{0.009}{3.34 \times 10^9} \therefore b \approx 2.695 \times 10^{-12}.$$

The (non–zero) equilibrium value of the logistic model, towards which the population size will increase, is given by

$$P_{\max} = \frac{a}{b}.$$

In the case of this particular population we therefore get

$$P_{\max} = \frac{0.029}{2.695 \times 10^{-12}}$$

hence

$$P_{\max} \approx 1.1 \times 10^{10}$$

which is approximately 11 billion. (Compare with the current world population of 5 billion!)

6.13 Let $P(t)$ denote the amount of fish in the pond at time t, with t measured in months. Since logistic growth is assumed, the population grows according to the differential equation

$$\frac{dP}{dt} = aP - bP^2$$

We know that the limiting population size in this model always equals a/b. The value of a is given:

$$a = \frac{1}{10} = 0.1$$

so it remains to find the value of b. For this, we will use the given values

$$\begin{aligned} P_0 &= 1000, \\ P(12) &= 2 \cdot P_0 = 2000. \end{aligned}$$

The solution to the equation of logistic growth is given by

$$P = \frac{a}{b + \left[\frac{a}{P_0} - b\right] e^{-at}}$$

We solve b from this:

$$P = \frac{a}{b + \left[\frac{a}{P_0} - b\right] e^{-at}} \quad \therefore \quad \frac{a}{P_0} - b = \left(\frac{a}{P} - b\right) e^{at} \quad \therefore \quad \left(e^{at} - 1\right) b = a \left(\frac{e^{at}}{P} - \frac{1}{P_0}\right)$$

and thus

$$b = a \left(\frac{(e^{at}/P) - (1/P_0)}{e^{at} - 1}\right).$$

We can apply this at time $t = 12$: We substitute $t = 12$, $P = 2000$ and $P_0 = 1000$ and get

$$b = \frac{0.1 \left(e^{12(0.1)}/2000 - 1/1000\right)}{e^{12(0.1)} - 1} \approx 2.845 \times 10^{-5}.$$

The size of the limit population is then

$$P = \frac{a}{b} = \frac{0.1}{2.845 \times 10^{-3}} \approx 3515.$$

6.14

(a) Population 1 obeys logistic growth with growth constants a and b, where $a = 2.86$, but the value of b is unknown. It is known that as $t \to \infty$, the population approaches 2500. But, for logistic growth, the population always approaches the limit population size given by $\frac{a}{b}$. So, $\frac{a}{b} = 2500$, from which we can solve b :

$$b = \frac{a}{2500} = \frac{2.86}{2500} = 0.001144.$$

[Logistic growth always has $\frac{a}{b}$ as its limit value. You should remember this as an important feature of logistic growth. You should also be able to derive it from the differential equation of logistic growth:

$$\frac{dP}{dt} = aP - bP^2.$$

The equilibrium points are $P = 0$ and $P = \frac{a}{b}$; and

$$\begin{aligned} \frac{dP}{dt} &> 0 \quad \text{when } 0 < P < \frac{a}{b} \\ \frac{dP}{dt} &< 0 \quad \text{when } P > \frac{a}{b}. \end{aligned}$$

So, when $0 < P_0 < \frac{a}{b}$, the population will increase towards $\frac{a}{b}$; and when $P_0 > \frac{a}{b}$, the population will decrease towards $\frac{a}{b}$. Thus, for any $P_0 > 0$, the population size will converge towards $\frac{a}{b}$.]

(b) Population 2 obeys logistic growth with $a = 2.86$ and $b = 150 \times 0.001144 = 0.1716$. If the initial population is $P_0 = 30$, then the population will **decrease** in time. This follows from the fact that for Population 2, the limit population size towards which the population size converges is given by

$$P_{\text{limit}} = \frac{a}{b} = \frac{2.86}{0.1716} \approx 16.67.$$

For logistic growth, the population size will increase towards P_{limit} if the initial size was below P_{limit} or it will decrease towards P_{limit} if the initial size was above P_{limit}. Here, $P_0 = 30 > 16.67 = P_{\text{limit}}$, so that the second option holds: The population size will decrease towards 16.67. **Alternatively,** we can simply remember that if $\frac{dP}{dt} > 0$ then the population size increases, and if $\frac{dP}{dt} < 0$ then the population size decreases, so we just need to check the sign of $\frac{dP}{dt}$ at $P = 30$. According to the differential equation of logistic growth, $\frac{dP}{dt}$ can be calculated from P as

$$\frac{dP}{dt} = aP - bP^2.$$

Substitute $a = 2.86$, $b = 0.1716$, $P = 30$ into this to find the value of $\frac{dP}{dt}$ when $P = 30$:

$$\frac{dP}{dt} = (2.86) \cdot 30 - (0.1716)(30)^2 \approx -68.64.$$

This is negative, so that the population size is decreasing at $P = 30$.

6.15 If $P(t)$ follows the logistic model, that is,

$$\frac{dP}{dt} = aP - bP^2$$

then the value of P at time t is given by

$$P(t) = \frac{a}{b + \left(\frac{a}{P_0} - b\right) e^{-at}}.$$

(a) If $a = 0.1$, $b = 0.02$ and $P_0 = P(0) = 100$ then we get

$$P(10) = \frac{0.1}{0.02 + \left(\frac{0.1}{100} - 0.02\right) e^{-(0.1)10}} = 7.686 \approx 8$$

and

$$P(20) = \frac{0.1}{0.02 + \left(\frac{0.1}{100} - 0.02\right) e^{-(0.1)20}} = 5.737 \approx 6.$$

(b) The rate of increase of the population at a given time is the numerical value of the derivative dP/dt at that time, and for the logistic model the value of dP/dt at a given time can be found from the value of P at that time, as described by the differential equation

$$\frac{dP}{dt} = aP - bP^2.$$

At time $t = 0$, $P(0) = 100$ and therefore the rate of change at time $t = 0$ is

$$\frac{dP}{dt} = (0.1)(100) - (0.02)(100)^2 = -190$$

At time $t = 10$, $P(0) = 7.686$ and therefore the rate of change at time $t = 10$ is

$$\frac{dP}{dt} = (0.1)(7.686) - (0.02)(7.686)^2 \approx -0.4.$$

At time $t = 20$, $P(0) = 5.737$ and therefore the rate of change at time $t = 20$ is

$$\frac{dP}{dt} = (0.1)(5.737) - (0.02)(5.737)^2 \approx -0.08.$$

(The negative signs follow from the fact that the population size is actually decreasing, towards the limit population size $a/b = 5$.)

6.16 A population behaves according to the logistic model. If the population is observed, it is seen that it grows, and that the maximum rate of growth is reached when the size of the population is $P = P^*$. What will the limit population size be? We are told that the population behaves according to the logistic model, which means that $P(t)$, the size of the population at time t, is a solution to the differential equation

$$\frac{dP}{dt} = aP - bP^2$$

for some parameters (growth coefficients) a and b. This type of a population was discussed in detail in Chapter 6. Among other things, the following facts were established there:

– The population size always approaches the value $\frac{a}{b}$ (unless the population starts at zero).

– The population increases fastest when the size of the population is $\frac{a}{2b}$.

These facts must hold for every logistic population, including the one in this question. Here, we are told that the maximum rate of growth happens when $P = P^*$. Accordingly, a and b must be such that

$$\frac{a}{2b} = P^*.$$

But then it follows that the limit population size, which we know must equal a/b, must have the value

$$P_{\text{limit}} = \frac{a}{b} = 2 \cdot \frac{a}{2b} = 2P^*.$$

As $t \to \infty$, the size of the population approaches the value $2P^*$.

6.17 $b \approx 6.163\,3 \times 10^{-6}$, $P_{\text{limit}} = 40563$

6.18 Again we have

$$P(t) = \frac{1}{b + \left(\frac{a}{P_0} - b\right) e^{-at}}$$

where $P_0 = 200$, $a = 0.5$, $b = 0.02$.

(a)

$$P(1) = \frac{0.5}{0.02 + \left(\frac{0.5}{200} - 0.02\right) e^{(-0.5)\cdot 1}} \approx 53.27, \quad P(2) = \frac{0.5}{0.02 + \left(\frac{0.5}{200} - 0.02\right) e^{(-0.5)\cdot 2}} \approx 36.87.$$

[Note that we are not rounding these answers, since we will need them in later calculations!]

(b) The rate of change — that is, the derivative of the function P — is given by the differential equation:

$$\frac{dP}{dt} = aP - bP^2.$$

At time $t = 0$, $P = 200$ and therefore the rate of change at that time is

$$\frac{dP}{dt} = 0.5\,(200) - 0.02\,(200)^2 = 100 - 800 = -700.$$

At time $t = 1$, $P \approx 53.27$ and therefore the rate of change at that time is

$$\frac{dP}{dt} = 0.5\,(53.27) - 0.02\,(53.27)^2 \approx -30.1.$$

At time $t = 2$, $P \approx 36.87$ and therefore the rate of change at that time is

$$\frac{dP}{dt} = 0.5\,(36.87) - 0.02\,(36.87)^2 \approx -8.75.$$

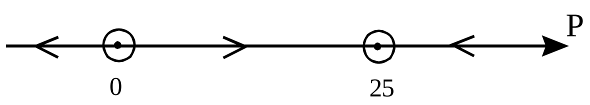
(c) Phase line: The logistic differential equation

$$\frac{dP}{dt} = aP - bP^2 = P\,(a - bP)$$

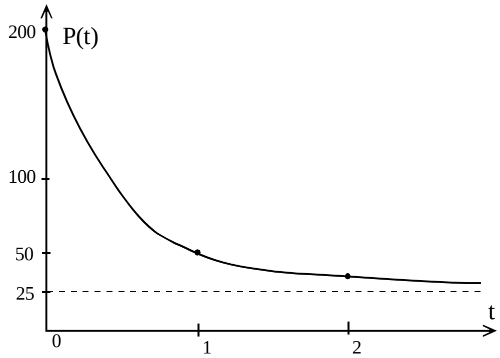
has two equilibrium points, $P = 0$ and $P = \frac{a}{b}$. Also, $dP/dt > 0$ for $0 < P < \frac{a}{b}$ and $dP/dt < 0$ for $P < 0$, $P > \frac{a}{b}$. In this question,

$$\frac{a}{b} = \frac{0.5}{0.02} = 25$$

and therefore the phase line looks like this:



(d) The initial value and (a) tell us that the solution curve $P(t)$ should go through the points $(t = 0,\ P = 200)$, $(t = 1,\ P = 53.27)$ and $(t = 2,\ P = 36.87)$. The answer to (b) further tells us what the slopes of the solution curve should be at these points: -700, -30 and -9, respectively. Finally, since the initial point is larger than the equilibrium point 25, (c) tells that the solution will decrease, approaching asymptotically the value $P = 25$ as $t \to \infty$. In conclusion, the solution curve looks as follows:



6.19

(a) An annual increase of 10% from $t = 0$ to $t = 1$ means that if $P(0) = P_0$, then

$$P(1) = P_0 + \frac{10}{100}P_0 = 1.1P_0.$$

On the other hand we must also have

$$P(1) = \frac{1}{b + \left(\frac{a}{P_0} - b\right)e^{-a\cdot 1}},$$

if the population grows according to the logistic model. This gives us the equation

$$1.1P_0 = \frac{1}{b + \left(\frac{a}{P_0} - b\right)e^{-a}}$$

from which we can solve the value of b, since $P_0 = 6 \times 10^6$ and $a = 0.3$ are known. *[Note that we **do** need the value of P_0; it will not just cancel out as in the Malthusian model!]*

$$1.1P_0 = \frac{a}{b + \left(\frac{a}{P_0} - b\right)e^{-a}} \quad \therefore \quad b\left(1 - e^{-a}\right) + \frac{a}{P_0}e^{-a} = \frac{a}{1.1P_0}$$

$$\therefore \quad b\left(1 - e^{-a}\right) = \frac{a}{1.1P_0} - \frac{a}{P_0}e^{-a} \quad \therefore \quad b = \frac{a}{P_0}\frac{\left(\frac{1}{1.1} - e^{-a}\right)}{1 - e^{-a}}$$

$$= \frac{0.3}{6 \cdot 10^6}\frac{\left(\frac{1}{1.1} - e^{-0.3}\right)}{\left(1 - e^{-0.3}\right)} \approx 3.24 \times 10^{-8}.$$

(b) The rate of change at time t is given by the derivative dP/dt, and according to the differential equation of the logistic model it is therefore equal to

$$\frac{dP}{dt} = aP - bP^2$$

where P is the size of the population at time t. In particular then the rate of change at time $t = 0$ is

$$\left.\frac{dP}{dt}\right|_{t=0} = aP_0 - b\left(P_0\right)^2 \qquad (*)$$

On the other hand, we are told that the rate of change should be 10% of the population size, which at time $t = 0$ is

$$\frac{10}{100}P_0. \qquad (**)$$

Now, (*) and (**) must be equal, which gives us the equation

$$aP_0 - b\left(P_0\right)^2 = \frac{10}{100}P_0$$

from which we can find the value of b:

$$b\left(P_0\right)^2 = aP_0 - \frac{10}{100}P_0 \quad \therefore \quad bP_0 = a - \frac{10}{100}$$

$$\therefore \; b \; = \; \frac{a - \frac{10}{100}}{P_0} = \frac{0.3 - 0.1}{6 \times 10^6} \approx 3.33 \times 10^{-8}.$$

6.20 The solution to the logistic model is given by the equation

$$P(t) = \frac{a}{b + \left(\frac{a}{P_0} - b\right)e^{-at}}. \qquad (*)$$

(a) $b \approx 1.229 \times 10^{-6}$

(b) $7710.0, 2347.5$.

(c) $P = 81367$.

(d) $t \approx -2.331$.

6.21 We are given the following information:

Time (days)	Size of population	Rate of change of population
$t = 0$	$P(0) = 10$	$\frac{dP}{dt} = 20$
$t = 10$	$P(10) = 1500$	$\frac{dP}{dt} = 1$

We see that the population size in increasing, and yet the rate of change is decreasing. This cannot happen in the Malthusian model, since there the differential equation is given by $dP/dt = kP$ and therefore the value of the rate of change is directly proportional to size of the population, and therefore the rate of change must always grow when the population grows. On the other hand, this could happen in a logistic model. We therefore conclude that the population of animals is following the logistic model rather than a Malthusian model.

For a more formal proof, let us assume that the data comes from a logistic model, with parameters a and b. We can now use the supplied data to find the values of a and b; if we get $b = 0$ then the data comes from a Malthusian population but if $b > 0$ then it must come from a logistic population. *[Remember that the Malthusian population model is a special case of the logistic model, when $b = 0$!]* If we assume the logistic model then the differential equation is given by

$$\frac{dP}{dt} = aP - bP^2.$$

Remember that this tells us how to solve for the rate of change, dP/dt, from the value of P. So using just the size of the population, and the corresponding rate of change of population, from the table above, we get the two equations:

$$\begin{cases} 20 = a \cdot 10 - b \cdot 10^2 \\ 1 = a \cdot 1500 - b \cdot 1500^2 \end{cases} \quad \therefore \quad \begin{cases} 10a - 100b = 20 \\ 1500a - 2250\,000b = 1 \end{cases}$$

which has the solution $a \approx 2.013\,4$, $b \approx 1.341\,8 \times 10^{-3}$. So, b is not close enough to 0 for us to call this the Malthusian model. Note that we could go further and check if the logistic model with these values for a and b really gives $P(10) = 1500$ – for that we need the solution to the logistic model. If you do the calculations, you will find that it does give that value, at least close enough.

There are other ways to prove that this data cannot come from the Malthusian model: for instance, you can use the given values of $P(0)$ and $P(10)$ to find the value of k if the Malthusian model is assumed (you would get $k \approx 0.501$) and then note that the rate of change at time $t = 0$ should be approximately 5 rather than 20; or use the fact that dP/dt must be 20 if $P = 10$ to get the value $k = 2$ for the Malthusian growth constant and then note that with this value, $P(10)$ should be much larger than the value given (it should be about 5×10^9).

6.22 Remember that the logistic model is given by

$$\frac{dP}{dt} = aP - bP^2.$$

(a) Actually, a must be positive in the logistic model, so $a < 0$ is not even allowed! But let's assume that $a < 0$ were allowed and see what happens. Compare with the Malthusian model

$$\frac{dP}{dt} = aP$$

where it would certainly hold that the population decreases if and only if $a < 0$ — this is because the population decreases if and only if $dP/dt < 0$, and if P is positive then $aP < 0$ can only hold if $a < 0$. But in the logistic model if the population is decreasing then all we can say is that

$$\frac{dP}{dt} < 0 \quad \therefore \quad aP - bP^2 < 0.$$

This can hold for certain values of P (those above a/b), even if $a > 0$. So, the statement is FALSE.

(b) The reasoning here is as in (a) above: in the logistic model, whether or not the population will increase or decrease does not depend on just the values of the parameters, but also on the current size of the population. Whatever the values of a and b, the population will be decreasing whenever the current population is above the value a/b. The statement is FALSE.

(c) The logistic model always has two equilibrium points, $P = 0$ and $P = a/b$. If $a = b$ it just means that one of the equilibrium points is equal to 1, but there will still be two equilibrium points, 0 and 1. The statement is FALSE.

Study Unit 7 SOLUTIONS

7.1 Find all the equilibrium points of the following differential equations:

(a) $x = 1/4$

(b) $x - -1, x = 0$

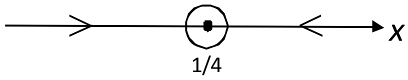
(c) $x = \ln(2)$

(d) $x = 2\pi k, k = 0, \pm 1, \pm 2, \ldots$

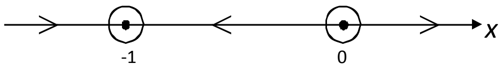
(e) $x = 1, x = 0$

7.2

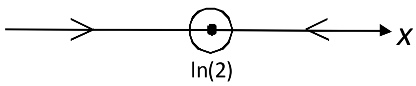
(a)



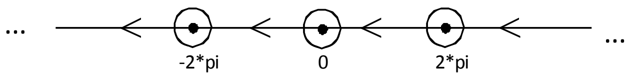
(b)



(c)

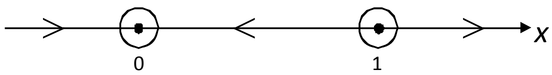


(d)



Note that there are infinitely many equilibrium points here!

(e)

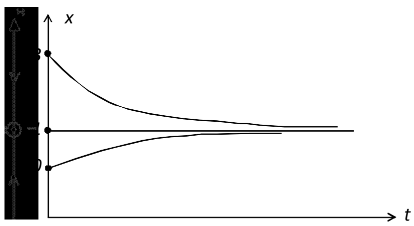


7.3 (a): solution will increase towards 0; (b) and (c): solution will increase towards 10; (d): solution will stay at 10; (e): solution will decrease towards 10; (f): Solution will increase without bound.

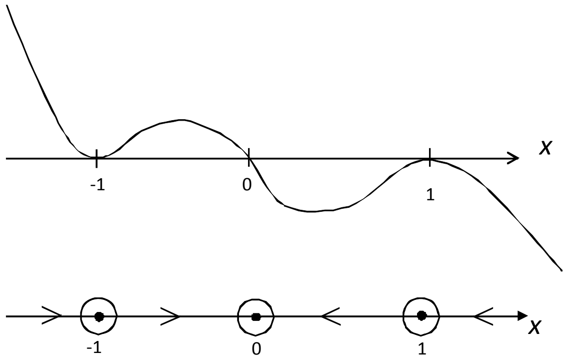
7.4 B is stable, all others are unstable.

7.5

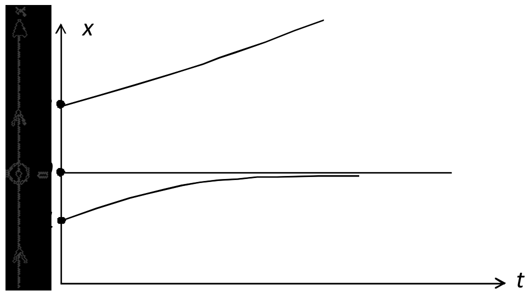
(a)



(b)



1.



7.6 We will start by drawing a rough sketch of the function

$$f(x) = -2(x-1)^2 x (x+1)^2 .$$

The sketch is shown below. Note that we assume in this module that you know how to draw these types of sketches; if you struggle, please go and revise your other mathematics modules and/or school work! From the sketch of the function, we can immediately draw the phase line (You can draw it below the function sketch; draw equilibrium points where the function $f(x)$ has a zero, a leftward arrow where the function is below the x–axis, and a rightward arrow where the function is above the x–axis.)There are three equilibrium points, -1, 0 and +1; we see from the phase line that only $x = 0$ is a stable equilibrium point, the other two are both unstable.

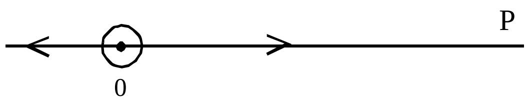
7.7 The differential equation in the Malthusian model is

$$\frac{dP}{dt} = kP.$$

Case $k > 0$: The equilibrium point is $P = 0$, and it can be easily seen that

$$\frac{dP}{dt} > 0 \quad \text{when } P > 0, \quad \frac{dP}{dt} < 0 \quad \text{when } P < 0.$$

Hence the phase line looks as follows:

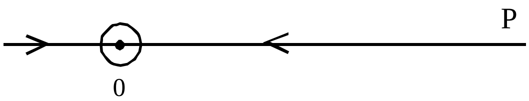


[Since negative population sizes make no sense, we could ignore the point of the phase line to the left of $P = 0$.]
We see that $P = 0$ is an unstable equilibrium point, and whenever the initial population is strictly positive, the population will grow without bound.

Case $k < 0$: The equilibrium point is $P = 0$, and this time

$$\frac{dP}{dt} > 0 \quad \text{when } P < 0, \quad \frac{dP}{dt} < 0 \quad \text{when } P > 0.$$

The phase line now looks like this:



Now, $P = 0$ is a stable equilibrium point. With any initial population size, the population will decline towards zero.
[What about the case $k = 0$*? In this case the differential equation is simply* $\frac{dP}{dt} = 0$*. In particular* $\frac{dP}{dt} = 0$ *holds*

always, for any population size P, so that each and every P is an equilibrium point: whichever population size we start with, the population size will forever stay at that level. In the phase line, since every point is an equilibrium point, there is no motion.]

7.8 We won't even try to solve this equation! Instead, we will use graphical stability analysis, which will give us a fairly good idea of what the solutions would look like. All we need to do is to investigate the sign of $\frac{dR}{dt}$ for various values of R.

(a) The equilibrium points of the model described by the given differential equation are the points R for which $\frac{dR}{dt} = 0$, i.e. for which

$$\frac{1}{2}(2-R^2)(e^R-1)\,|R+1|\,e^{-5R+19} = 0.$$

This happens when either

$$2-R^2 = 0 \quad \therefore\; R = \pm\sqrt{2}$$

or

$$e^R - 1 = 0 \quad \therefore\; R = \ln(1) = 0,$$

or

$$|R+1| = 0 \quad \therefore\; R+1 = 0 \quad \therefore\; R = -1.$$

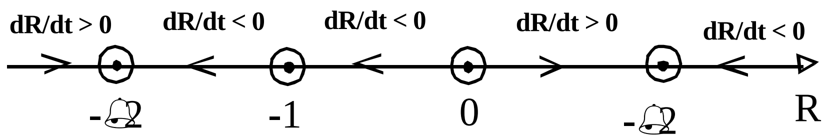
(Note that the last term in the product, e^{-5R+19}, is always strictly positive and never zero). Thus there are 4 equilibrium points: $-\sqrt{2}$, -1, 0, $+\sqrt{2}$.

(b) For the phase line, we need to find the sign of $\frac{dR}{dt}$ for all possible values of R. The easiest way to do this is factorization: We establish the signs of the various components of the product on the left hand side of (2), and then derive the sign of the whole product.

$$\begin{aligned}
(2-R^2) &> 0 \text{ when } -\sqrt{2} < R < +\sqrt{2},\\
(2-R^2) &< 0 \text{ when } R < -\sqrt{2} \text{ or } R > +\sqrt{2};\\
(e^R-1) &> 0 \text{ when } R > 0,\\
(e^R-1) &< 0 \text{ when } R < 0;\\
|R+1| &\geq 0 \text{ always;}\\
e^{-5R+19} &> 0 \text{ always.}
\end{aligned}$$

term	sign of term over each interval				
		$-\sqrt{2}$	-1	0	$\sqrt{2}$
$(2-R^2)$	–	+	+	+	–
(e^R-1)	–	–	–	+	+
$\|R+1\|$	+	+	+	+	+
e^{-5R+19}	+	+	+	+	+
Product	+	–	–	+	–

That is, the product, and thus $\frac{dR}{dt}$, is **positive** when $R < -\sqrt{2}$ or $0 < R < \sqrt{2}$, and **negative** when $-\sqrt{2} < R < 0$ or $R > \sqrt{2}$. The phase line therefore looks as follows:



[The equilibrium points are circled. We have marked in the sign of $\frac{dR}{dt}$ over each interval. The arrows on the phase line indicate the directions of motion in the system: If $\frac{dR}{dt} > 0$ then R increases and motion is towards the right; and if $\frac{dR}{dt} < 0$ then R decreases and motion is towards the left. Note that the equilibrium points are the points where $\frac{dR}{dt} = 0$, and therefore (since the right–hand side of (7) is a continuous function of R) $\frac{dR}{dt}$ can only change its sign at the equilibrium points. Thus, $\frac{dR}{dt}$ is always either positive or negative over the entire length of each interval between two equilibrium points.]

(c) From the phase line, we see immediately that $R = -\sqrt{2}$ and $R = \sqrt{2}$ are stable, and $R = -1$ and $R = 0$ unstable equilibrium points. *[The arrows on both sides of $-\sqrt{2}$ point towards $-\sqrt{2}$. Thus if we start near $-\sqrt{2}$, the motion is always towards $-\sqrt{2}$, so that it must be a stable equilibrium point. Likewise for $+\sqrt{2}$. On the other hand, look at point $R = -1$: Motion on the interval to the right of -1 is towards left, so that for instance if we start with $R = -0.9$, the solutions does move towards -1. On the other hand, if we start at $R = -1.1$ then the direction of motion is again to the left – but that means now that the solution moves away from $R = -1$. We could start at a point $R_0 < -1$ arbitrarily close to -1, and still end up moving away from -1. This means that -1 is an unstable equilibrium point.]*

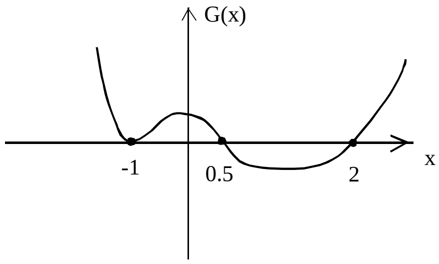
(d) The outcome as $t \to \infty$ for all possible starting values $R(0)$:

$$\begin{aligned} \text{If } R(0) &< -1 \text{ then } R(t) \to -\sqrt{2} \text{ as } t \to \infty, \\ \text{If } -1 &\leq R_0 < 0 \text{ then } R(t) \to -1 \text{ as } t \to \infty, \\ \text{If } R(0) &= 0 \text{ then } R(t) = 0 \text{ for all } t, \\ \text{If } R(0) &> 0 \text{ then } R(t) \to +\sqrt{2} \text{ as } t \to \infty. \end{aligned}$$

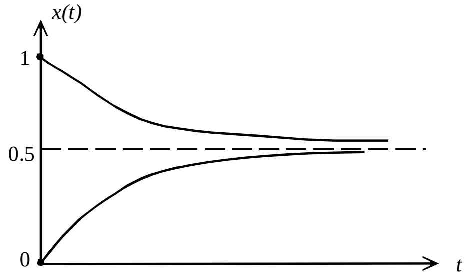
[These results are easily seen from the phase line. Just check where on the phase line a starting value is. If the system starts at an equilibrium point, it stays there. If on the other hand the starting value is on one of the intervals between two equilibrium points, then the direction of the arrow in that interval indicates the outcome!]

7.9 From the phase line we see that dx/dt is zero at $x = -1$, $x = 0.5$ and $x = 2$, and dx/dt is positive for $x < -1$, $-1 < x < 0.5$ and $x > 2$ and negative for $0.5 < x < 2$. (The equilibrium points give the zeroes, and the directions of the arrows the signs of dx/dt.)

(a) Since $dx/dt = G(x)$ holds, we know that the function $G(x)$ must also be a function which is zero at $x = -1$, $x = 0.5$ and $x = 2$, positive for $x < -1$, $-1 < x < 0.5$ and $x > 2$ and negative for $0.5 < x < 2$. Accordingly, $G(x)$ must look roughly as follows:



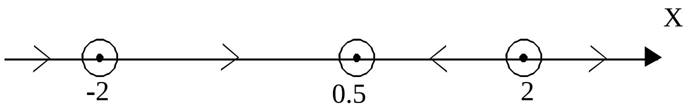
(b) The initial value $x(0) = 1$ falls on the interval between 0.5 and 2, where according to the phase line the solutions should decrease towards 0.5. Therefore a solution curve starting from $x(0) = 1$ should decrease asymptotically towards $x = 0.5$ as t increases. The initial value $x(0) = 0$ on the other hand lies between -1 and 0.5 which is an interval where solutions increase. A solution curve starting from $x(0) = 0$ will increase asymptotically towards $x = 0.5$ as t increases. The two solution curves should look roughly as follows:



7.10 From the graph of the function $G(x)$, we see that $G(x)$, and therefore also dx/dt, is

- zero when $x = -2,\ x = 0.5$ and $x = 2$;
- positive when $x < -2$ or $-2 < x < 0.5$ or $x > 2$
- negative when $0.5 < x < 2$.

Therefore, the phase line of this system would look like this:



[The points $-2,\ 0.5$ and 2, where $\frac{dx}{dt} = 0$, are equilibrium points, and motion along the phase line is to the left on those intervals where $\frac{dx}{dt} < 0$ and to the right on those intervals where $\frac{dx}{dt} > 0$.] The initial value $x_0 = -1$ falls on the interval between -2 and 0.5, where motion is towards the right, towards the equilibrium point $x = 0.5$. The conclusion is that as t increases, the solution $x(t)$ starting from $x_0 = -1$ will increase asymptotically towards the value $x = 0.5$. Reasoning similarly, we see that if $x_0 = -3$ then as t increases, the solution $x(t)$ will increases

asymptotically towards the equilibrium point $x = -2$.

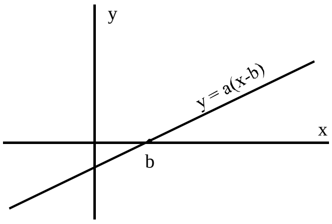
7.11 (d):

$$\frac{dx}{dt} = a\left(x - b\right)$$

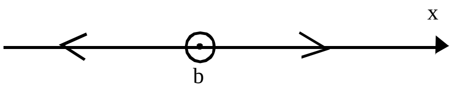
When a and b are positive constants, the function

$$a\left(x - b\right)$$

is a straight line with a positive slope a, which intersects the x–axis at the (positive) value $x = b$.



Therefore $a\left(x - b\right)$, and $\frac{dx}{dt}$, is zero for $x = b$, negative for $x < b$ and positive for $x > b$. The phase line of the system looks like this:

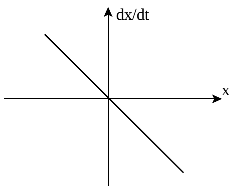


In this system, $x = b$ is the only equilibrium point, and is unstable.

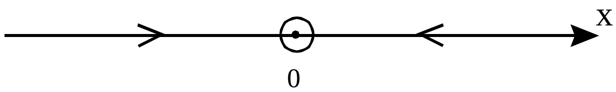
7.12

(a) $\frac{dx}{dt} = -x$

i. dx/dt as a function of x: This is the curve $f\left(x\right) = -x$.



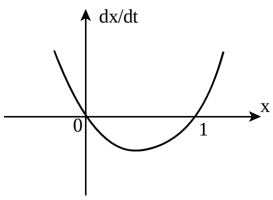
ii. Phase line: $dx/dt = 0$ at $x = 0$, negative for $x > 0$ and positive for $x < 0$, so the phase line looks like this:



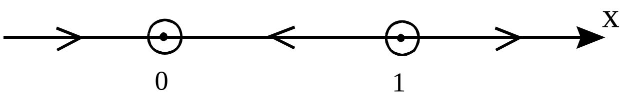
There is one equilibrium point, $x = 0$, which is stable. If $x\left(0\right) = 0.5$ (to the right of the equilibrium point), $x\left(t\right)$ will decrease asymptotically towards $x = 0$.

(b) $\frac{dx}{dt} = \left(x - 1\right)x$

i. The curve $f\left(x\right) = \left(x - 1\right)x$ is an upwards opening parabola, which intersects with the x–axis at $x = 1$ and $x = 0$. Therefore dx/dt as a function of x looks like this:



ii. Phase line:



There are two equilibrium points, $x = 0$ and $x = 1$. Of these, $x = 0$ is stable, and $x = 1$ is unstable.

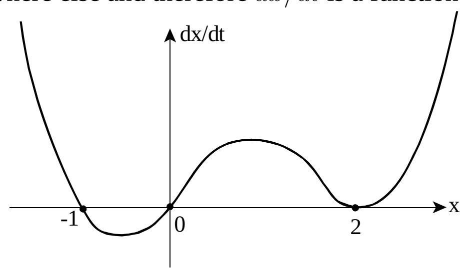
If the system starts at $x\left(0\right) = 0.5$, according to the phase diagram the solution will decrease towards $x = 0$, approaching that value asymptotically as t increases.

(c) $\dfrac{dx}{dt} = (x-2)^2\, x\,(x+1)$:

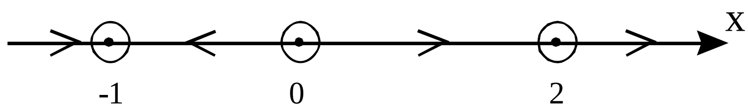
i. The function

$$f(x) = (x-2)^2\, x\,(x+1)$$

is a fourth–order polynomial with zeros at $x = 0$, $x = -1$ and $x = 2$ (double root). It is negative for $-1 < x < 1$ and non–negative everywhere else and therefore dx/dt is a function of x looks like this:



ii. The phase line:



The equilibrium points are $x = -1$ (stable), $x = 0$ (unstable), $x = 2$ (unstable).

If the system starts at $x(0) = 0.5$, then the solution will increase towards 2, approaching that value asymptotically.

7.13 All the questions can be answered by qualitative analysis.

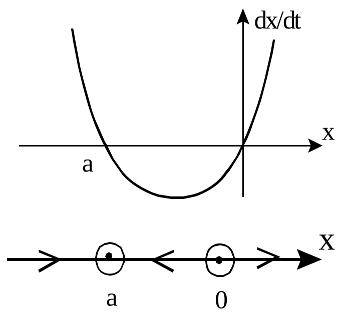
(a) The equilibrium points of system

$$\frac{dx}{dt} = x(x-a)$$

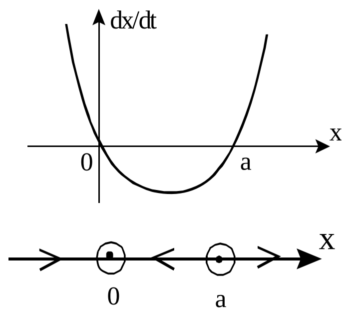
are $x = 0$ and $x = a$. So, if we want one of these to be $x = 2$, we should take $x = 2$.

(b) When will $x = 0$ be stable/unstable? To find out, we should first draw the phase line of the given system. First we will need to analyse where the function $f(x) = x(x-a)$ is positive/negative, which can be done e.g. by plotting it. Here, $f(x)$ is an upwards opening parabola, with zeros at $z = 0$ and $x = a$ – but trying to plot it, you'll soon find that you will get different cases depending on whether $a < 0$ or $a > 0$ or perhaps $a = 0$! Here are the plots of $f(x)$, and the corresponding phase lines, in these three cases:

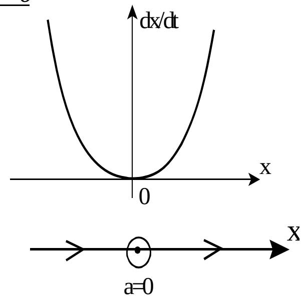
$\underline{a < 0}$



$\underline{a > 0}$



$\underline{a = 0}$



From the phase lines, it is clear that $x = 0$ is unstable if $a = 0$ or $a < 0$. Thus, we should have $a \leq 0$.

(c) Again, the solution can be found from the phase lines in the three cases in (b).

- ∗ If $a < 0$ then any solution starting at $x_0 > 1$ will always grow without bound, since these starting values fall on a portion of the phase line where motion is towards the right.
- ∗ If $a = 0$, the same holds.
- ∗ If $a > 0$, we would need to ensure that the starting points with $x_0 > 1$ all lie to the right of the right–most equilibrium point $x = a$. That is, a must be chosen such that $a \leq 1$.

In conclusion, if $a \leq 1$ then any solution starting at any x_0 with $x_0 > 1$ will increase without bound.

(d) For any solution to decrease without bound (decrease towards $-\infty$) as $t \to \infty$, motion on the left–most part of the phase line would need to be towards the left. This is not true for any of the three possible cases in (b), so it can not happen in this system.

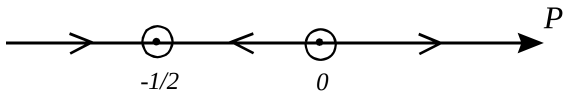
7.14

(a) A logistic model is of the form

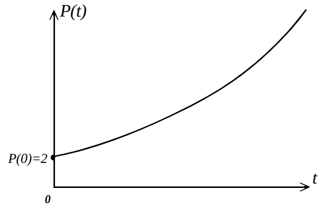
$$\frac{dP}{dt} = aP - bP^2$$

for some positive constants a and b. Here, to fit the model in this question into the above form, we would have $a = 2$, $b = -4$, so the value of b is **not** positive here. Therefore this is not a logistic model. *[It is no good saying that it is a "logistic model with negative b" – the whole point in the logistic population model is that it combines positive proportional growth (the term aP) with a negative term $(-bP^2)$ limiting population growth for large values of P!]*

(b) To draw the phase line, we need to first find where $dP/dt = 0$; and then where dP/dt is positive and where it is negative. Here, the expression for dP/dt is $2P + 4P^2 = 2P(2P + 1)$ which is an upwards opening parabola with zeroes at $P = 0$ and $P = -1/2$. Therefore there are two equilibrium points for this system, at $P = 0$ and $P = -1/2$. Also, dP/dt is negative between the equilibrium points and positive elsewhere. Thus, we get the following phase line.



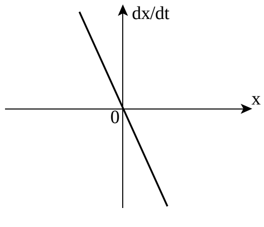
(c) A sketch of a solution curve from any given initial point can be figured out from the phase diagram. We are given the initial point $P(0) = 2$ which is to the right of the bigger equilibrium point, $P = 0$. In this area the arrow of the phase diagram points to the right, meaning that the value of P increases. Also there are no more equilibrium points on the way, which means that the value of P will increase without bound. Therefore a sketch of the solution curve with $P(0) = 2$ would look like this:



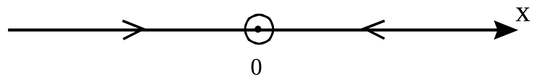
7.15

(a) $\frac{dx}{dt} = -2x$

i. dx/dt as a function of x: This is the curve $f(x) = -2x$.



ii. Phase line: $dx/dt = 0$ at $x = 0$, negative for $x > 0$ and positive for $x < 0$, so the phase line looks like this:

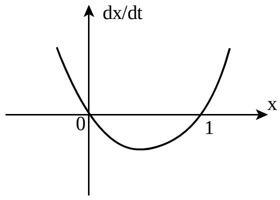


iii. There is one equilibrium point, $x = 0$, which is stable.

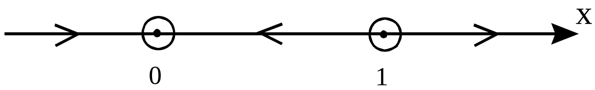
iv. If $x(0) = 0.5$ (in which case the initial point is to the right of the equilibrium point), $x(t)$ will decrease asymptotically towards $x = 0$.

(b) $\dfrac{dx}{dt} = (x-1)\,x$

i. The curve $f(x) = (x-1)\,x$ is an upwards opening parabola, which intersects with the x–axis at $x = 1$ and $x = 0$. Therefore dx/dt as a function of x looks like this:



ii. Phase line:



[This can be read directly from the graph in (i): the equilibrium points are the points where the graph in (i) intersects with the x–axis, and then we put a left arrow on all the intervals on the graph in (i) is negative (below the line) and a right arrow on the intervals where the graph is above the x–axis!]

iii. There are two equilibrium points, $x = 0$ and $x = 1$. Of these, $x = 0$ is stable, and $x = 1$ is unstable.

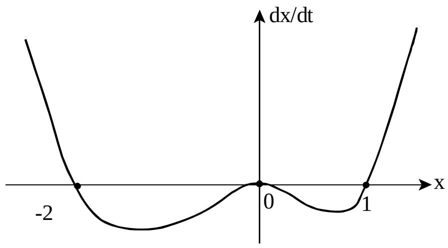
iv. If the system starts at $x(0) = 0.5$, between the two equilibrium points, then according to the phase diagram the solution will decrease towards $x = 0$, approaching that value asymptotically as t increases.

(c) $\dfrac{dx}{dt} = (x+2)\,x^2\,(x-1)$:

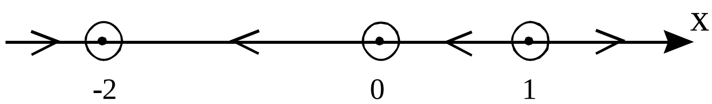
i. The function

$$f(x) = (x+2)^2\,x\,(x-1)$$

is a fourth–order polynomial with zeros at $x = 0$, $x = -2$ and $x = 1$ (with a double root at $x = 0$). The polynomial is negative for $-2 < x < 0$ and for $0 < x < 1$, and non–negative everywhere else and therefore dx/dt is a function of x looks like this:



ii. The phase line:



iii. The equilibrium points are $x = -2$ (stable), $x = 0$ (unstable), $x = 1$ (unstable).

iv. If the system starts at $x(0) = 0.5$, then the solution will decrease towards 0, approaching that value asymptotically.

7.16

(a) $a = 1.$

(b) $a \geq 0.$

(c) For a solution to be able to increase without bound, the direction of motion on the right-most interval of the phase line should be towards the right (so that some solutions can get bigger and bigger values). This does not hold for any values of a, and therefore no solutions can ever increase without bound, never mind what value a has.

Study Unit 8 SOLUTIONS

8.1 The solution in Newton's law of cooling is

$$T(t) - \alpha = (T_0 - \alpha) e^{-kt} \qquad (*)$$

where α is the temperature of the liquid, T_0 is the initial temperature of the object, $T(t)$ is the temperature of the object after time t, and k is the cooling constant. Here $\alpha = 10$ but T_0 and k are not known. However, we are given two readings,

$$\begin{aligned} T(30) &= 200, \\ T(60) &= 100. \end{aligned}$$

So, if we apply (*), we get

$$\begin{cases} 200 - 10 = (T_0 - 10) e^{-30k} \\ 100 - 10 = (T_0 - 10) e^{-60k} \end{cases}$$

We have two equations, from which we can solve the two unknown values, k and T_0 — for instance as follows *[there are many other ways to start solving the values]*: Rewriting the two equations, we get

$$\begin{cases} e^{-30k} = \frac{190}{(T_0 - 10)} \\ \\ e^{-60k} = \frac{90}{(T_0 - 10)} \end{cases}$$

But

$$e^{-60(k)} = \left(e^{-30k}\right)^2,$$

so the following must also hold:

$$\frac{90}{(T_0 - 10)} = \left(\frac{190}{(T_0 - 10}\right)^2 \quad \Big| \cdot (T_0 - 10)^2 \qquad \therefore \ 90(T_0 - 10) = (190)^2$$

$$\therefore \quad T_0 = \frac{(190)^2}{90} + 10 = 411.11 \approx 411;$$

and then

$$e^{-30k} = \frac{190}{\left(\frac{(190)^2}{90}\right)} = \frac{90}{190}$$

$$\therefore \quad k = -\frac{1}{30} \ln\left(\frac{9}{19}\right) \approx 2.4907 \times 10^{-2}.$$

We now know the values of T_0 and k, and can answer the question: How long does it take for the object to cool to 30^o? We will solve t from

$$30 - 10 = (T_0 - 10) e^{-kt}$$

which must hold if $T(t) = 30$. We will get

$$t = -\frac{1}{k} \ln\left(\frac{30 - 10}{T_0 - 10}\right)$$

$$\therefore \quad t = \frac{30}{\ln\left(\frac{9}{19}\right)} \ln\left(\frac{20}{\frac{(190)^2}{90}}\right) = \frac{30}{\ln\left(\frac{9}{19}\right)} \ln\left(\frac{90 * 20}{190^2}\right) \approx 120.30,$$

or,

$$t = -\frac{1}{2.4907 \times 10^{-2}} \ln\left(\frac{30 - 10}{411.11 - 10}\right) \approx 120.39.$$

It will take approximately 2 hours for the object to cool down to 30 degrees.

[Above we measured time t in minutes; alternatively we could have measured t in hours, in which case the value of

k would be different.]

(a) $T_0 = \frac{(190)^2}{90} + 10 = 411.11,\ k = -\frac{1}{30}\ln\left(\frac{9}{19}\right) \approx 2.4907 \times 10^{-2}$.

(b) It will take approximately 93 minutes for the object to cool down to $50°$.

8.2 $k = -\frac{1}{15}\ln\left(\frac{45}{85}\right) \approx 0.0424$, $T(30) = 20 + 85e^{-0.0424\cdot 30} \approx 44$ degrees.

8.3 $T(t) = 30 + 70e^{-kt}$, $k \approx 0.037$, the temperature $40°$ is reached after 52.6 minutes.

8.4 The solution is

$$T(t) = 20 + 80e^{-(0.06931)t}.$$

In particular, we get

$$\begin{aligned} T(5) &= 20 + 80e^{-(0.06931)5} = 76.568, \\ T(5) &= 20 + 80e^{-(0.06931)15} = 48.284 \end{aligned}$$

8.5

(a) $\frac{dT}{dt} = k(10 - T)$

(b) $T(45) = 10 + 50e^{-(0.09163)\cdot 45} \approx 10.8$.

(c) The temperature 5^{o} is never reached.

(d) The initial rate of change is the value of $\frac{dT}{dt}$ at the moment $t = 0$. The differential equation tells us that

$$\frac{dT}{dt} = k(10 - T)$$

always holds; therefore, at time $t = 0$, when $T = 60$, we get

$$\frac{dT}{dt}\,|_{t=0} = k(10 - T)|_{T=60} = k(10 - 60) = -50k = -4.58$$

The initial rate of change is -4.58 degrees per minute.

8.6 Let $T(t)$ denote the temperature of the object after t minutes. We are given the values

$$T(0) = 120,\ \ T(10) = 110.$$

The temperature of the liquid is $80°$.

(a) $\frac{dT}{dt} = -k(T - 80)$ where k is some constant.

(b) The solution to the differential equation is

$$\begin{aligned} & T - 80 = (T(0) - 80)\, e^{-kt} \\ \therefore\ \ & T = 80 + 40e^{-kt}. \end{aligned} \quad (2)$$

(c) To answer this question, we need to solve the value of k. For this, we will apply the given value of the temperature after 10 minutes: $T(10) = 110$. We substitute $T = 110$ and $t = 10$ into (2) and solve for k :

$$110 = 80 + 40e^{-10k} \therefore e^{-10k} = \frac{30}{40} = \frac{3}{4} \therefore -10k = \ln\left(\frac{3}{4}\right)$$

$$\therefore\ \ k = -\frac{1}{10}\ln\left(\frac{3}{4}\right) \approx 0.02877.$$

Now to find out when $t = 85$, we substitute $T = 85$, together with the calculated value of k, into equation (2) and solve for t :

$$85 = 80 + 40e^{-(0.02877)t} \therefore e^{-(0.02877)t} = \frac{5}{40} = \frac{1}{8}$$

$$\therefore\ \ t = -\frac{1}{0.02877}\ln\left(\frac{1}{8}\right) \approx 72.3 \text{ minutes.}$$

(d) After 2 hours = 120 minutes, the temperature will be

$$T(120) = 80 + 40e^{-(0.02877)120} \approx 81.3°.$$

8.7

(a) The differential equation is

$$\begin{aligned} \frac{dW}{dt} &= \text{rate of change of water in tank} \\ &= \text{(rate water enters) – (rate water leaves)} \end{aligned}$$

where

$$\text{rate water enters } = 1 \text{ litre per minute}$$

and

$$\begin{aligned}\text{rate water leaves} \quad &= \quad \text{(rate liquid pumped out)} \\ &\quad \times\text{(concentration of water in liquid pumped out).}\end{aligned}$$

The concentration of water in liquid pumped out equals the concentration of water in the tank, and therefore equals

$$\text{concentration} = \frac{\text{quantity of water}}{\text{volume of tank}} = \frac{W(t)}{10}$$

so that the rate water leaves is

$$\frac{W(t)}{10} \text{ litres per minute.}$$

(Make sure you understand this! Liquid is pumped out of the container at the rate of 1 litre per minute. So, how much water is leaving the tank per minute? That is, how much water does the one litre of the contents of the tank contain? Well, the whole tank of 10 litres contains $W(t)$ litres of water, so one litre of the liquid inside the tank must contain $W(t)/10$ litres of water — and this is the amount of water leaving the tank each minute.)
Therefore, the differential equation of the model is

$$\frac{dW}{dt} = 1 - 0.1W(t).$$

(b) Next, we will solve the differential equation. Separating the variables and then integrating, we get

$$W = 10 \pm Ae^{-0.1t}$$

The value of A can be solved from the initial condition: Initially the tank contains only acid, so $W(0) = 0$. Substituting the values $t = 0$ and $W = 0$ into the equation above, we get $0 = 10 \pm A$ so that we see that we should choose the negative sign and $A = 10$. That is, the only solution satisfying the given initial condition is

$$W(t) = 10 - 10e^{-0.1t}.$$

(c) The amount of water in the tank after 5 minutes is

$$W(5) = 10 - 10e^{(-0.1)\cdot 5} \approx 4 \text{ litres.}$$

8.8

(a) The differential equation:

$$\frac{dX}{dt} = \frac{4 \text{ litres}}{\text{minute}} - \frac{X \text{ litres}}{200 \text{ litres}} \cdot \frac{4 \text{ litres}}{\text{minute}}$$

$$\therefore \quad \frac{dX}{dt} = 4 - \frac{X}{50}$$

Initial value: $X(0) = 0$.

(b) Solution to the differential equation:

$$X - 200 = (X_0 - 200)\, e^{-\frac{1}{50}t},$$

so with $X_0 = 0$, we get the solution

$$X(t) = 200\left(1 - e^{-\frac{1}{50}t}\right)$$

(c) The tank contains equal quantity of both A and B when $X(t) = 100$. To find when this holds, we solve for t from

$$100 = 200\left(1 - e^{-\frac{1}{50}t}\right)$$

to get

$$\therefore \quad t = -50\ln\left(\frac{1}{2}\right) = 50\ln(2) \approx 34.657.$$

After approximately 35 minutes, the tank contains 100 litres of both liquid A and liquid B

8.9 We will first derive and solve a differential equation describing the **quantity** of the chemical in the tank at any time; the **concentration** is then easily calculated since the concentration is simply the quantity of the chemical divided by the volume of the tank. So, let $S(t)$ denote the amount of the chemical (in kilograms) in the tank at time t. Then $\frac{dS}{dt}$

is the rate of change of the chemical in the tank at time t, and can be found by using the fact that

$$\begin{aligned}\frac{dS}{dt} &= \text{rate of change of chemical in tank}\\ &= \text{rate chemical enters – rate chemical leaves.}\end{aligned}$$

But the rate at which the chemical enters is

$$\frac{0.5\text{ kg}}{\text{litre}} \cdot \frac{4\text{ litres}}{\text{minute}} = \frac{S(t)}{50}\frac{\text{kg}}{\text{minute}}$$

and the rate at which the chemical leaves is

$$\frac{S(t)\text{ kg}}{200\text{ litres}} \cdot \frac{4\text{ litres}}{\text{minute}} = 2\frac{\text{kg}}{\text{minute}}.$$

[Make sure that you understand this! The solution from the tank is pumped out at the rate of 4 litres per minute. So, how much of the chemical is leaving the tank per minute? That is, how much of the chemical do the 4 litres of solution pumped out each minute contain? Well, the whole tank of 200 litres contains $S(t)$ kilograms of the chemical, so 1 litre must contain $S(t)/200$ kilograms of the chemical, and 4 litres will contain 4 times this much.]
Therefore, the differential equation for $S(t)$ is

$$\frac{dS}{dt} = 2 - \frac{S}{50}$$

Next, we need to solve $S(t)$ from this differential equation. Separating variables and then integrating, we get

$$|100 - S| = e^{-0.02t+C} = Ae^{-0.02t} \quad (A = e^C)$$

The value of A can be solved from the initial condition, $S(0) = S_0$. Substituting this into the equation above, we get

$$|100 - S_0| = Ae^{-0.02 \cdot 0} = A$$

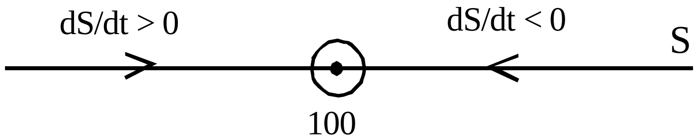
so that we must have

$$|100 - S| = |100 - S_0|\, e^{-0.02t} \qquad (*)$$

How do we remove the absolute value signs from this? Here, it will be very useful to look at the phase line of the differential equation

$$\frac{dS}{dt} = \frac{100 - S}{50}.$$

The equilibrium point is $S = 100$, and it is easy to see that the right–hand side of the equation, and hence the derivative dS/dt, is positive when $S < 100$ and negative when $S > 100$. Therefore the phase line looks like this:



From the phase line we can see that if initially the quantity of the chemical in the tank is **less than 100**, that is, $100 - S_0 > 0$, then at all later times we must also have $100 - S > 0$. Similarly, if initially $100 - S_0 < 0$ then at all later times also $100 - S < 0$. [*This is because no solution can move across the equilibrium point $S = 100$!*] Therefore, the phase line tells us that $(100 - S_0)$ and $(100 - S)$ must always have the same sign. Accordingly, we can simply drop the absolute value signs in (*), and the solution becomes

$$S - 100 = (S_0 - 100)e^{-0.02t}$$

$$\therefore \qquad S(t) = 100 + (S_0 - 100)e^{-0.02t}$$

Hence the **concentration** $c(t)$ of the chemical in the tank at time t is given by

$$\begin{aligned}c(t) &= \frac{S(t)}{200}\\ &= \frac{1}{2} + \left(\frac{S_0}{200} - \frac{1}{2}\right)e^{-0.02t}\end{aligned}$$

[We can re-write this as

$$c(t) = \frac{S_0}{200}e^{-0.02t} + \frac{1}{2}\left(1 - e^{-0.02t}\right).$$

The first term on the right hand side of this formula represents the quantity of the **original** *amount of chemical remaining in the tank at time t, and this becomes smaller and smaller as the original solution flows out of the tank ($e^{-0.02t} \to 0$ as $t \to \infty$). The second term represents the amount due to the flow of the chemical into the tank. At limit as $t \to \infty$, the concentration converges towards $1/2$ kilograms per litre, which is the same as the* *concentration of the chemical in the incoming solution. This agrees with our phase line analysis: The quantity of the chemical,* $S(t)$*, converges towards the equilibrium value* 100*, and therefore the concentration* $c(t)$ *converges towards* $100/200 = 1/2$.]

8.10 If $S(t)$ denotes the amount of salt in the tank and $C(t)$ the concentration of salt in the tank then

$$C(t) = \frac{S(t)}{1000}.$$

(a) Differential equation for S:

$$\begin{aligned}\frac{dS}{dt} &= \text{rate of change of the amount of salt in tank}\\ &= \text{(rate salt enters) – (rate salt leaves)}\end{aligned}$$

where

$$\begin{aligned}\text{rate salt enters} &= \text{(rate water pumped in)}\\ &\quad \times\text{(concentration of salt in water pumped in)}\\ &= \text{(200 liters/minute)} \times \text{(0.1 kg/litre)}\\ &= 200 \cdot 0.1 \text{ kg/minute}\\ &= 20 \text{ kg/minute}\end{aligned}$$

and

$$\begin{aligned}\text{rate salt leaves} &= \text{(rate water pumped out)}\\ &\quad \times\text{(concentration of salt in water pumped out)}\\ &= \text{(200 liters/minute)} \times \left(\frac{S(t)\text{ kg}}{1000\text{ litres}}\right)\\ &= \frac{S(t)}{5}\text{ kg/minute.}\end{aligned}$$

Thus, the differential equation is

$$\frac{dS}{dt} = 20 - \frac{1}{5}S \tag{1}$$

or,

$$\frac{dS}{dt} = 20 - 0.2\,S.$$

The differential equation for C is

$$\begin{aligned}\frac{dC}{dt} &= \frac{1}{1000}\frac{dS}{dt}\\ &= \frac{1}{1000}\left(20 - \frac{1}{5}S\right)\\ &= \frac{20}{1000} - \frac{1}{1000 \cdot 5}S\end{aligned}$$

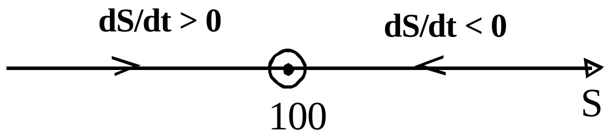
i.e.,

$$\frac{dC}{dt} = \frac{1}{50} - \frac{C}{5} \tag{2}$$

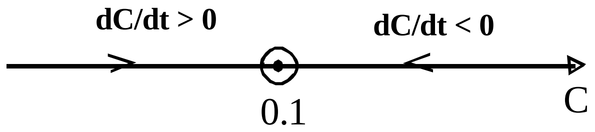
or

$$\frac{dC}{dt} = 0.02 - 0.2C.$$

(b) The equilibrium point of the equation for $S(t)$ is $S = 100$, and the equilibrium point of the equation for $C(t)$ is $C = 0.1$. Phase line for S:



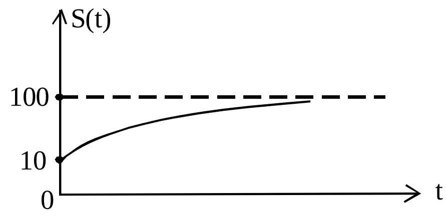
Phase line for C:



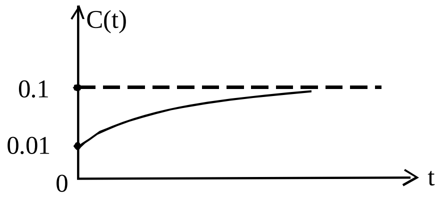
The initial quantity of salt is $S(0) = 10$, and from that we can calculate the initial concentration:

$$C(0) = \frac{S(0)}{1000} = \frac{10}{1000} = 0.01.$$

From the phase line of S we see that since $S(0) = 10$ lies to the left of the equilibrium value $S = 100$, the solution $S(t)$ starting from this value will increase towards 100, approaching that value asymptotically. Thus the solution function $S(t)$ looks roughly as shown below:



Using similar reasoning, we find that the solution function $C(t)$ looks roughly as shown below:



(c) As indicated in the sketches above, we see that as $t \to \infty$, we get

$$\begin{aligned} S(t) &\to 100, \\ C(t) &\to 0.1. \end{aligned}$$

8.11 $S(t)$ denotes the **concentration** of the chemical in the mixture leaving the tank at time t; this is of course the same as the concentration of the chemical in the tank at time t. That is, $S(t)$ equals the amount of chemical in the tank at time t, divided by V, the volume of the tank. Especially the initial concentration is

$$S(0) = \frac{M_0}{V}.$$

(a) To obtain the differential equation for $S(t)$, we note that

$$\begin{aligned} \frac{dS}{dt} &= \text{rate of change of the } \textbf{concentration} \text{ of the chemical in the tank} \\ &= \frac{1}{V} \cdot \text{rate of change of the } \textbf{amount} \text{ of the chemical in the tank} \\ &= \frac{1}{V} \text{(rate chemical enters } - \text{ rate chemical leaves).} \end{aligned}$$

The chemical enters the tank at the rate

$$C\frac{\text{kg}}{\text{litre}} \times R\frac{\text{litres}}{\text{minute}} = C \cdot R\frac{\text{kg}}{\text{minute}}.$$

The chemical leaves the tank at the rate

$$S(t)\frac{\text{kg}}{\text{litre}} \times R\frac{\text{litres}}{\text{minute}} = R \cdot S(t)\frac{\text{kg}}{\text{minute}}.$$

So, the differential equation for $S(t)$ is

$$\frac{dS}{dt} = \frac{1}{V}(CR - RS), \quad S(0) = \frac{M_0}{V}$$

i.e.

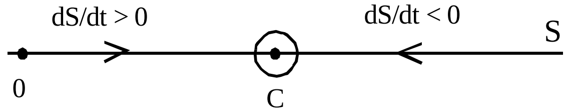
$$\frac{dS}{dt} = \frac{R}{V}(C - S), \quad S(0) = \frac{M_0}{V}.$$

[Remember that S varies as a function of time, while R, V, C and M_0 are all constants!]

(b) The equilibrium points are the values of S such that $dS/dt = 0$. The condition

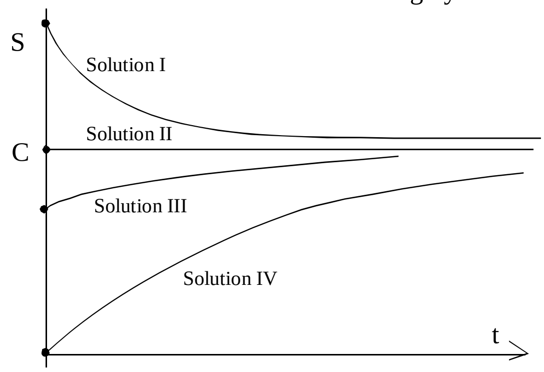
$$\frac{dS}{dt} = \frac{R}{V}(C - S) = 0$$

gives $S = C$ as the only equilibrium point. Also, assuming that R and V are positive, $dS/dt = \frac{R}{V}(C - S)$ is a linear function of S with a negative slope, so that dS/dt is positive when $S < C$ and negative when $S > C$. The phase line looks like this:



We see that the equilibrium point C is stable. *[Note that 0 is not an equilibrium point here!]*

(c) From the phase line, we see that possible solution curves would look roughly as shown below.



Four solution curves are shown here: I, starting with $S(0) > C$ (i.e. $M_0 > CV$); II, starting with $S(0) = C$ (i.e. $M_0 = CV$); and III and IV starting with $S(0) < C$ (i.e. $M_0 < CV$). *[Solution curves starting with $S(0) > C$ (such as I) decrease towards C; solution curves starting with $S(0) < C$ (such as III and IV) increase towards C; and the solution curve II, starting from $S(0) = C$, gives a constant solution. Note that all solutions not starting exactly at the point $S = C$ approach the value C asymptotically, getting closer and closer to that value but never reaching it.]*

(d) Since all solution curves approach the value $S = C$ asymptotically, the equilibrium point C is the steady-state outcome of the system. As $t \to \infty$, $S(t)$ converges towards the value C.

8.12

(a) The differential equation:

$$\frac{dX}{dt} = \frac{4 \text{ litres}}{\text{minute}} - \frac{X \text{ litres}}{200 \text{ litres}} \cdot \frac{4 \text{ litres}}{\text{minute}}$$

$$\therefore \quad \frac{dX}{dt} = 4 - \frac{X}{50}$$

Initial value: $X(0) = 0$.

(b) Solution to the differential equation:

$$\therefore \quad X - 200 = (X_0 - 200)\, e^{-\frac{1}{50}t},$$

so with $X_0 = 0$, we get the solution

$$X(t) = 200\left(1 - e^{-\frac{1}{50}t}\right)$$

(c) The tank contains equal quantity of both A and B when $X(t) = 100$. To find when this holds, we solve for t from

$$100 = 200\left(1 - e^{-\frac{1}{50}t}\right)$$

to get $t = -50\ln\left(\frac{1}{2}\right) = 50\ln(2) \approx 34.657$.

8.13 —

8.14 Model:

$$\frac{dX}{dt} = kX(N - X) \qquad (*)$$

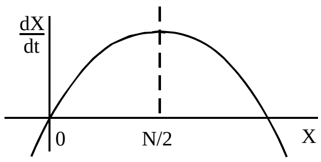
where $N > 0$ is the total population size, $X(t)$ the number of people infected by the time t, and $k > 0$ is a constant.

(a) The differential equation states that the rate at which people get infected (the rate at which X increases) is

directly proportional to the product of the number of infected people and the number of uninfected people. Thus the following assumption has been made: Firstly, that the number of new infections over a small time period is proportional to the number of interactions between infected and uninfected people over that period; and secondly that the number of these interactions is in turn proportional to the product $X(N-X)$. The first assumption seems fairly reasonable for most infectious diseases since some interaction between those infected and those not infected is usually required to pass on a disease. Of course only a certain percentage of these interactions would then lead to an actual infection, but this percentage will be incorporated into the proportionality constant k. As for the second part of the assumption, the product $X(N-X)$ gives the number of **possible** interactions between the infected and uninfected people (since each infected person, of whom there are X, could meet any of the possible $N-X$ uninfected people). We then have to assume that the number of **actual** interactions over a time interval is proportional to the number of **possible** interactions.
The following list some of the possible problems with these assumptions:

- Regarding the last assumption: the number of possible interactions between the infected and uninfected could be quite different from the number of actual interactions. Interaction between infected and uninfected is usually limited as the infected could be at home or hospitals.
- In many diseases the actual infection period is limited, and eventually infected people may become well again.
- It is assumed that the whole population can be classified into just two categories, those infected (who are also actively spreading the disease) and those uninfected (who can all get infected). In practice, there are other categories: e.g. one section of the population could be immune to the disease (e.g. because they have already had it), or could have the disease but not be spreading it. A more sophisticated model could treat the infected and uninfected populations separately, by establishing a suitable system of differential equations as described in Chapter 5.

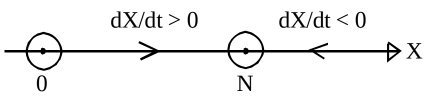
(b) The right hand side of (*) describes a parabola which opens downwards and equals zero at $X=0$ and $X=N$. Hence, the graph of dX/dt versus X looks like this:



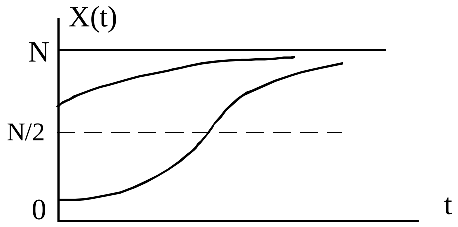
(c) Since we know that the function of X on the right hand side of (*) is a downwards opening parabola with zeroes at $X=0$ and $X=N$, we must have

$$\begin{cases} \frac{dX}{dt} < 0 & \text{for} \quad X<0,\ X>N, \\ \frac{dX}{dt} > 0 & \text{for} \quad 0< X<N. \end{cases}$$

Therefore the phase line will look like this:



(d) From the phase line we see that whatever the initial number of infections, as long as it is between 0 and N, the solutions will converge towards N so that eventually, everyone will be infected. How then do solution curves with initial values $X_1 < N/2$ and with initial values $X_1 > N/2$ differ? Looking at the curve of dX/dt versus X we see that dX/dt has its maximal value at the top of the parabola, i.e. when $X = N/2$. This is therefore the point at which the population grows fastest and the solution curves steepest. Graphically, the solution curves will be convex below $X = N/2$ and concave above $X = N/2$.



(e) See the previous exercise on how to solve this kind of an equation (Note that the model in Section 8.3.2 gives a differential equation identical to (*) above!) The solution is

$$X(t) = X_0 \frac{Ne^{Nkt}}{N - X_0 + X_0 e^{Nkt}}$$

where X_0 is the initial number of infected people.

8.15 Solution to the differential equation:

$$\frac{dQ}{dx} = k\left(150 - Q\right)$$

is:

$$Q\left(x\right) = 150 - Ce^{-kx} \quad \left(C = e^{-C_1}\right)$$

The solution has two unknown constants, C and k. To determine their values, we use the two given values. If we substitute

$$Q\left(10\right) = 80$$

into the solution, we get

$$80 = 150 - Ce^{-10k}$$
$$\therefore \quad C = 70e^{10k} \qquad (1)$$

Similarly,

$$Q\left(20\right) = 120$$

gives

$$120 = 150 - Ce^{-20k}$$
$$\therefore \quad C = 30e^{20k} \qquad (2)$$

By equating (1) and (2), we obtain

$$30e^{20k} = 70e^{10k}$$
$$\therefore \quad e^{10k} = \frac{7}{3}. \qquad (3)$$

Solving k from (3) gives

$$k = \frac{1}{10}\ell n\left(\frac{7}{3}\right) \approx 0.085$$

and substituting (3) into (1) gives

$$C = 70 \cdot \frac{7}{3} = \frac{490}{3} \approx 163.333.$$

Therefore, the solution is

$$Q\left(x\right) = 150 - \frac{490}{3}e^{-0.085x}$$

and

$$Q\left(30\right) = 150 - \frac{490}{3}e^{-0.085 \cdot 30} \approx 137kg.$$

8.16 Let $Q(t)$ denote the number of pupils with flu after t days. Initially 5% of the pupils had the flu; since the school has 5 000 pupils, this means that

$$Q(0) = 5\% \text{ of } 5\,000$$
$$= \frac{5}{100} \cdot 5\,000$$
$$\therefore Q(0) = 250.$$

The rate at which pupils contract the flu is the rate of change of $Q(t)$, that is $dQ(t)/dt$. We are told that this rate is directly proportional to the product of the number of pupils with flu, and the number of pupils without flu. This product is obviously $Q(5\ 000-Q)$. So the differential equation describing the model is

$$\frac{dQ}{dt}=kQ(5\ 000-Q)$$

where k is a constant. To solve the differential equation, we first separate the variables:

$$\frac{dQ}{Q(5\ 000-Q)}=kdt.$$

We then split the left hand side into partial fractions:

$$\frac{1}{Q(5\ 000-Q)}=\frac{1}{5\ 000}\left(\frac{1}{2}+\frac{1}{5\ 000-Q}\right)$$

and multiply both sides by $5\ 000$ to get

$$\left(\frac{1}{Q}+\frac{1}{5\ 000-Q}\right)dQ=5\ 000kdt.$$

Integrate both sides to get

$$\begin{aligned}\ln|Q|-\ln|5\ 000-Q| &= 5\ 000kt+C_1\\ \ln\left(\frac{|Q|}{|5\ 000-Q|}\right) &= 5\ 000kt+C_1\end{aligned}$$

We can drop the absolute value signs because both Q and $5\ 000-Q$ will always be positive. So,

$$\begin{aligned}\frac{Q}{5\ 000-Q} &= e^{5\ 000kt+C_1}\\ \therefore\ Q &= \frac{5\ 000\cdot e^{5\ 000kt+C_1}}{1+e^{5\ 000kt+C_1}}=\frac{5\ 000}{1+e^{-5\ 000kt-C_1}}\\ \therefore\ Q &= \frac{5\ 000}{1+Ce^{-5\ 000kt}}\quad (C=e^{-C_1})\end{aligned}$$

We can solve C from the initial condition $Q(0)=250$:

$$\begin{aligned}Q(0) &= \frac{5\ 000}{1+C}=250\\ \therefore\ C &= 19\end{aligned}$$

Finally, we can solve k from the fact that

$$Q(10)=20\%\text{ of all pupils}=\frac{20}{100}\cdot 5\ 000=1\ 000$$

Substituting this into the solution, we get

$$1\ 000=\frac{5\ 000}{1+19e^{-5\ 000kt\cdot 10}}$$

from which we can obtain

$$e^{-5\ 000k\cdot 10}=\left(\frac{5\ 000}{1\ 000}-1\right)/19\ \therefore\ k=\frac{-1}{50\ 000}\ln(4/19)\approx 3.12\times 10^{-5}$$

So the solution to the differential equation, for which

$$\begin{aligned}Q(0) &= 250\\ Q(10) &= 1\ 000\end{aligned}$$

holds, is

$$Q(t)=\frac{5\ 000}{1+19e^{-0.156t}}$$

which gives as the number of pupils with flu after 13 days,

$$Q(13)=\frac{5\ 000}{1+19e^{-0.156t}}\approx 1428.$$

8.17 Let $T(t)$ be the temperature of the object after t hours.

(a) The rate of change of the temperature is given by dT/dt; and a quantity which is proportional to the square of the difference between the temperature of the object, $T(t)$, and the temperature of the radiation source (500^{o}) can be written as $k(T(t)-500)^2$ where k is some constant of proportionality. Thus, the differential equation for the model is

$$\frac{dT}{dt} = k\left(T - 500\right)^2.$$

(b) We will solve the differential equation by separation of variables:

$$\frac{dT}{dt} = k\left(T-500\right)^2 \therefore \frac{dT}{\left(T-500\right)^2} = kdt$$

$$\therefore \quad -\left(T-500\right)^{-1} = kt + C \qquad \text{[since } \int x^P dx = \tfrac{1}{p+1}x^{p+1} + C \text{ for all } p \neq -1\text{, including } p = -2]$$

$$\therefore \quad T = 500 - \frac{1}{kt+C}. \tag{1}$$

The values of k and C can be solved from the given data, $T\left(0\right) = 375$ and $T\left(1\right) = 400$. Substituting these values into (1), we get

$$375 = 500 - \frac{1}{C}$$

$$400 - 500 - \frac{1}{k+C}$$

from which we can solve k and C: $C = 0.008, \quad k = 0.002$. Thus the solution is

$$T\left(t\right) = 500 - \frac{1}{\left(0.002\right)t + 0.008}.$$

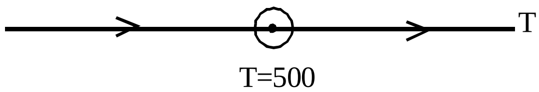
In particular, the temperature after 3 hours is

$$T\left(3\right) = 500 - \frac{1}{\left(0.002\right)3 + 0.008} \approx 429^{\circ}.$$

(c) Equilibrium points:

$$\frac{dT}{dt} = 0 \quad \text{when} \quad k\left(T-500\right)^2 = 0,$$

so that $T = 500$ is the only equilibrium point. Since $k\left(T-500\right)$, as a product of a positive constant and the square of a number, is always non–negative, dT/dt is always non–negative (zero at $T = 500$ and positive everywhere else). Therefore, the phase line looks like this:



(d) What happens if $T\left(0\right) > 500$? According to the phase line, the temperature of the object will increase, both when the initial temperature is lower than the radiation source, but also when the temperature of the object is higher than that of the radiation source. The difference is that starting below the temperature of the radiation source, the temperature of the object will approach the value $T = 500$ so that in this case the temperature is bounded. On the other hand, starting above $T = 500$, the temperature of the object will grow without bound, increasing without bound as t increases. *[This is of course not what we expect to happen in real life, but this is what will happen according to this particular mathematical model!]*

8.18 The differential equation is

$$\frac{dA}{dt} = kA.$$

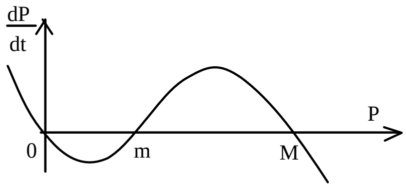
Study Unit 9 SOLUTIONS

9.1

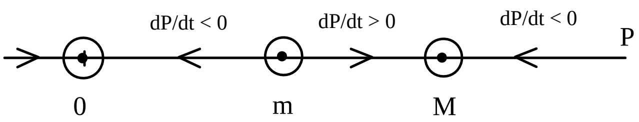
$$\frac{dP}{dt} = kP(M-P)(P-m)$$

(a) The logistic model does **not** have the property that the population becomes extinct if the size of the population is too small – instead, in the logistic model a population can start with an arbitrarily small population (even less than one) and the according to the mathematical model, the size of the population would still increase towards the limit population size. So, if the model given by the differential equation suggested here does indeed have this property, then this is a major point where it differs from the logistic model. The assumption made in the logistic model, that the population can start arbitrarily small and still increase, is of course unreasonable – in any case, we would expect a population with less than one member to be impossible! So, this model is more reasonable than the logistic model.

(b) The right side of $(*)$ as a function of P is a third degree polynomial, which equals zero at $P = 0,\ P = m$ and $P = M$. We can assume that $0 < m < M$, in which case the graph of $dP/dt = kP\,(M - P)\,(P - m)$ as a function of P looks like this:



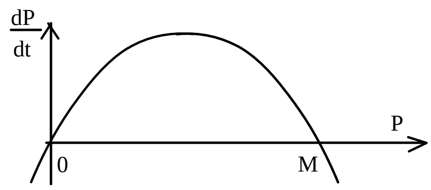
We see that $\frac{dP}{dt}$ is positive when $P < 0$ or $m < P < M$, and dP/dt is negative when $0 < P < m$ or $P > M$. The phase line therefore looks like this:



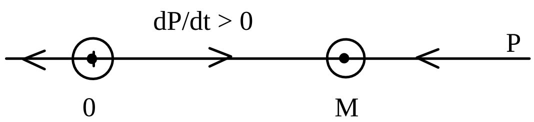
Let us compare this to the phase line of the logistic model. The logistic model with the maximum carrying capacity M would be given by the differential equation

$$\frac{dP}{dt} = kP\,(M - P)\,.$$

The function on the right is now a second degree polynomial of P, so that dP/dt as a function of P looks like this:



and therefore the phase line looks like this:



(c) In this model, 0 and M are stable equilibrium points, while m is unstable.

(d) We will only consider non–negative initial population sizes P_0. From the phase line we see that:

* If $0 \leq P_0 < m$ then the population will die out.
* If $P_0 = m$ then the population will stay at m. (However this is an unstable equilibrium point, which means that if the population shifts however slightly away from m then it will not stay near m anymore, but will rather either decrease to 0 or increase to M.)
* If $P_0 > m$ then the population size will converge towards the value M.

(e) In the logistic model, the population size will converge towards the value M whenever $P_0 > 0$, so that the population can never become extinct unless it started off extinct. In real life, however, this is very unrealistic. For one thing, many species do need at least two individuals to reproduce, and thus if the initial population is one the species will die out. Also, if a population is geographically very wide spread then very small population numbers may lead to a low probability of members of the population meeting each other, which also lowers the rate of reproduction.

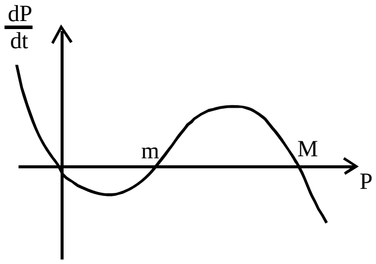
9.2 The model here is described by the differential equation

$$\frac{dP}{dt} = kP\,(M - P)\,(P - m) - H$$

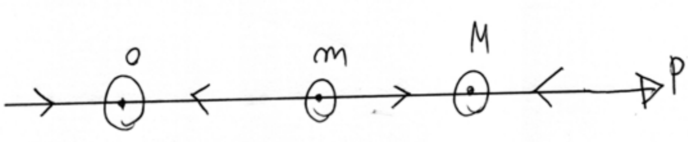
where $P\,(t)$ denotes the size of the impala population at time t, and H is the rate of hunting (with H impalas hunted per time unit). If $H = 0$ (no hunting) then the model is

$$\frac{dP}{dt} = kP\,(M-P)\,(P-m)\,.$$

In this case, the graph of dP/dt looks like this:



and the phase line looks like this:



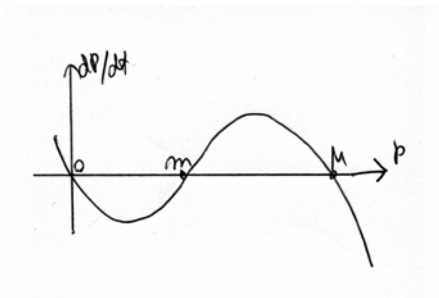
We see that m gives the minimal viable size of the impala population. If the population starts below m then it will die out, while if the population starts anywhere above m, it converges towards the stable equilibrium value M. What about the case $H \neq 0$, where some of the impalas are hunted? An initial difficulty seems to be that even finding the equilibrium points of the equation

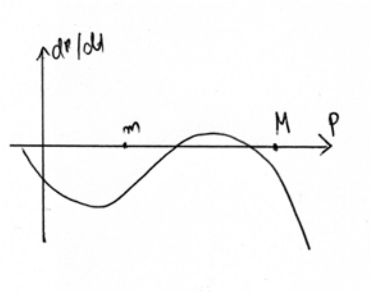
$$\frac{dP}{dt} = kP\,(M-P)\,(P-m) - H \tag{2}$$

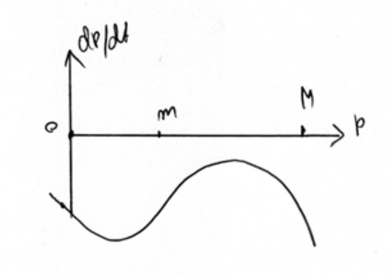
is very difficult since this is a cubic polynomial with no easy factorization, except in the case $H = 0$. However, it turns out that we can figure out the maximum value of H for which the population should survive without finding explicitly the equilibrium values of (2). To see how this goes about, let us consider the graph of the function of P on the right hand side of (2), that is, the graph of the function described by

$$kP\,(M-P)\,(P-m) - H. \tag{3}$$

When $H = 0$, we get the graph shown earlier which crosses the P–axis at the three points 0, m and M. Subtracting the positive value H from the function $kP\,(M-P)\,(P-m)$ to arrive at the function (3) corresponds to moving the graph of the function vertically downwards by the distance H. The following shows the graphs corresponding to three difference cases, $H = 0$, $H_1 > 0$ and $H_2 > H_2$.

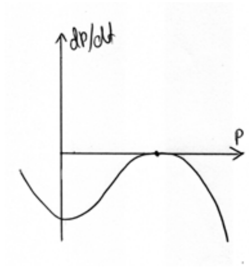




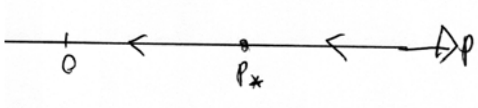


Now, we know what the graph of the function (3) looks like with different values of H.

What kind of a graph are we looking for if we wish the hunted population to survive? Remember that the graph of the function (3) describes how the derivative dP/dt varies as a function of P. Positive values of the function, where the graph lies above the P–axis, mean an increasing population while negative values, where the graph lies above the P–axis, correspond to a decreasing population. Clearly a value of H such as in case (c) of the graph above should not be permitted, as there the value of dP/dt is everywhere negative, meaning that wherever the initial population value lies the population will die out. Case (b) is that graph, or case (a), however, represent possible cases where hunting will not cause the population to die out (at least provided that the population is initially sufficiently large). We conclude that the maximum amount of hunting that can be allowed corresponds to a value of H_* for which the graph dP/dt versus P looks like this:



In this case, the phase line would look like this:



In this case, there are two equilibrium values, a negative one, and a positive one at some value $P = P_*$ (the exact value of which we do not know yet). The value of dP/dt is negative on both sides of P_*, so that the population level is always decreasing, but at least if the initial population was above P_* then the population will not die out but will rather decrease asymptotically towards the value P_*.

How do we find out what value of H_* this corresponds to? The last graph shown above can be obtained by pulling the first graph down by the amount H. which equals the maximum height of the graph between m and M. So we just need to locate the value of P where that maximum is achieved (giving us the value of P_*), and the value of the function $kP(M-P)(P-m)$ at that point (giving us the value of H_*). The maximum point can be found as the point between m and M where the derivative of the function $kP(M-P)(P-m)$ vanishes, i.e. where

$$\frac{d}{dP}\left(kP(M-P)(P-m)\right) = -3P^2 + 2(M+m)P - Mm = 0.$$

There are two solutions to this, given by

$$P = \frac{1}{3}(M+m) \pm \frac{1}{3}\sqrt{M^2 - mM + m^2}.$$

The larger one of these is the one we are looking for:

$$P_* = \frac{1}{3}(M+m) + \frac{1}{3}\sqrt{M^2 - mM + m^2}.$$

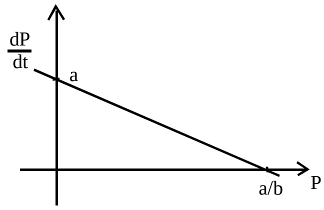
It follows that the maximum value of H which will not lead to the population inevitably dying out is

$$H_* = kP_*(M-P_*)(P_*-m)$$

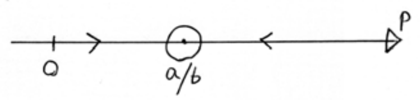
where P_* is as above.

9.3

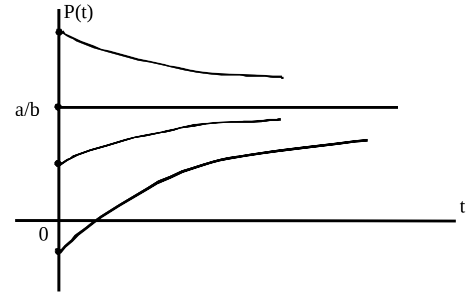
(a) $\dfrac{dP}{dt} = a - bP,\ a, b > 0$. **Graph of** dP/dt **versus** P : The function of P given by $a - bP$ is a straight line which has the value a when $P = 0$ and which crosses the P–axis (i.e. equals zero) at $P = a/b$.



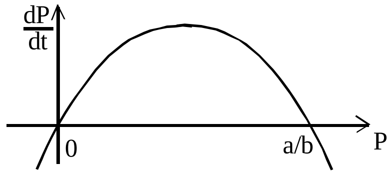
Phase line: From the graph of dP/dt as a function of P we see that $(a - bP)$ and therefore dP/dt is positive when $P < a/b$, negative when $P > a/b$ and zero when $P = a/b$. *[How do we read this information from the graph? We need to find the values of P for which the curve lies above the P–axis (since that is where $(a - bP) = dP/dt$ is positive), and for which the curve lies below the P–axis, and where the curve crosses the P–axis (meaning that $(a - bP) = dP/dt = 0$).]* We see therefore that there is only one equilibrium point, at $P = a/b$; $dP/dt > 0$ (motion towards the right) on the part of the phase line where $P < a/b$, and $dP/dt < 0$ (motion towards the left) when $P > a/b$.



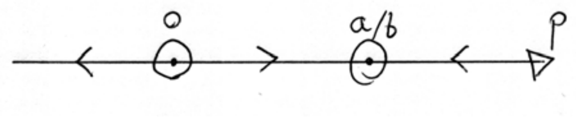
The equilibrium $P = a/b$ point is stable. *[Remark: Note that $P = 0$ is not an equilibrium point in this model! If $P(0) = 0$ then the population will still increase towards the limit value a/b.]* **Sketch of possible solution curves:** We see from the phase line that all solutions converge towards a/b, whether the initial population P_0 is above or below a/b. If $P_0 = a/b$ then P stays at a/b, giving a constant solution, also drawn in the sketch.



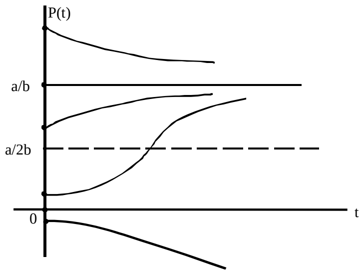
(b) $\dfrac{dP}{dt} = P(a - bP)$, $a, b > 0$ **Graph of dP/dt versus P** : The function of P given by $P(a - bP) = -bP^2 + aP$ is a second–order polynomial with a negative coefficient for P^2, and with zeroes at $P = 0$ and $P = a/b$, so its graph is a parabola which opens downwards and which crosses the P–axis at $P = 0$ and at $P = a/b$.



Phase line: From the graph above, we see that dP/dt is zero when $P = 0$ or $P = a/b$, positive when $0 < P < a/b$ and negative when $P < 0$ or $P > a/b$. Therefore the phase line looks like this:



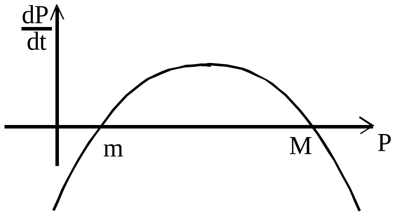
There are two equilibrium points, $P = b$ and $P = a/b$. Of the equilibrium points, $P = 0$ is unstable and $P = a/b$ is stable. **Sketch of solution curves:** According to the phase line, solutions starting below zero decrease without bound; solutions starting between 0 and a/b increase towards a/b, and solutions starting above a/b decrease towards a/b. For further detail we note that the function $dP/dt = aP - bP^2$ has its largest value at $P = a/(2b)$ [the top of the parabola is midway between 0 and a/b], and therefore P grows **fastest** at $P = a/(2b)$.



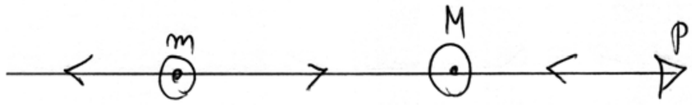
(c) $\frac{dP}{dt} = k(M-P)(P-m)$, $k, m, M > 0$: **Graph of** dP/dt **versus** P : The function of P given by

$$k(M-P)(P-m) = -kP^2 + k(M+m)P - kMm$$

is a second–order polynomial with a negative coefficient for P^2, and with zeroes at $P = m$, $P = M$, so that its graph is a parabola which opens downwards and which crosses the P–axis at $P = m$ and $P = M$. Let us assume that $m < M$, then the graph of dP/dt versus P looks as shown below.



Phase line: From the graph above, we see that dP/dt is zero at $P = m$ and $P = M$, positive when $m < P < M$ and negative elsewhere. The phase line looks like this:

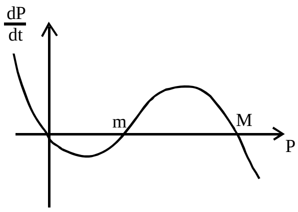


There are two equilibrium points, m (unstable) and M (stable). **Sketch of solution curves:** The solution curves look similar to those in (b) with 0, a/b and $a/(2b)$ replaced by m, M and $(m+M)/2$, respectively.

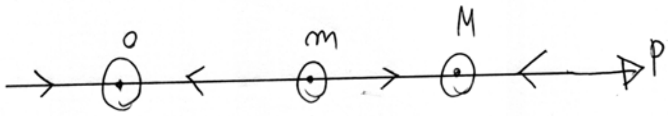
(d) $\frac{dP}{dt} = kP(M-P)(P-m)$, $k, m, M > 0$: **Graph of** dP/dt **versus** P : The function given by

$$kP(M-P)(P-m) = -kP^3 + k(M+m)P^2 - kMmP$$

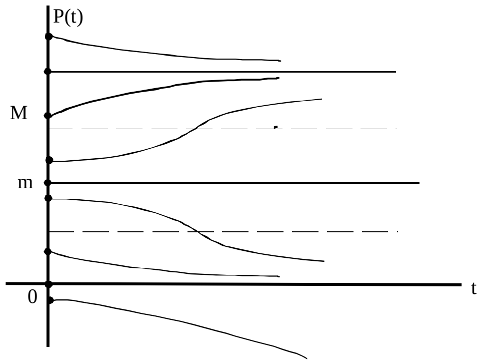
is a third–degree polynomial with a negative coefficient for P^3, and with zeroes at $P = 0$, $P = m$, $P = M$. Therefore (assuming $0 < m < M$) the graph of dP/dt versus P looks like this:



Phase line: This time, $\frac{dP}{dt}$ is zero when $P = 0$ or $P = m$ or $P = M$; these are the three equilibrium points of the model. We also see from the graph that dP/dt is positive when $P < 0$ and on the interval $m < P < M$, and negative on the interval $0 < P < m$ as well as for $P > M$. Therefore the phase line looks like this:



We see from the phase line that $P = 0$ and $P = M$ are stable equilibrium points, while $P = m$ is unstable. **Sketch of solution curves:** Again we see directly from the phase line that solutions starting below 0 increase towards 0, solutions starting between 0 and m decrease towards zero, solutions starting between m and M increase towards M, and solutions starting above M decrease towards M. Note that between 0 and m, the graph of dP/dt has a minimum at some point $P_{\min}$ and the solutions decrease fastest at that point; and dP/dt has a maximum at some point $P_{\max}$ and the solutions increase fastest at that point. Sketches of solution curves could look like this:



9.4

(a) The constant h gives the rate of immigration, so that it gives the number of immigrants entering the population per year.

(b) The equilibrium points are the values of P where $\frac{dP}{dt} = 0$. Since $\frac{dP}{dt} = kP + h$, this means that we set $kP + h = 0$ and solve for P, to get

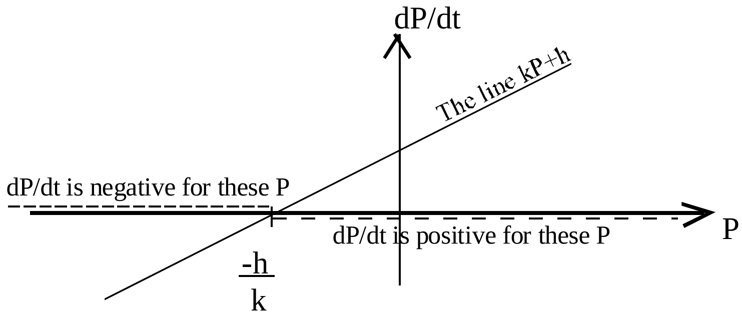
$$P = -\frac{h}{k}.$$

To be able to draw the phase line, we will further need to know the signs of $\frac{dP}{dt}$ on each side of the equilibrium point. But according to $\frac{dP}{dt} = kP + h$, $\frac{dP}{dt}$ as a function of P is a straight line with the positive slope k. Therefore we know that

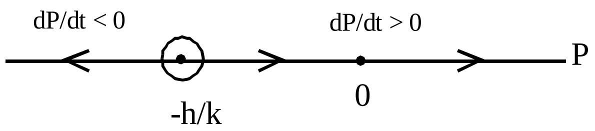
$$\frac{dP}{dt} > 0, \quad p > -\frac{h}{k}$$

$$\frac{dP}{dt} < 0, \quad p < -\frac{h}{k}.$$

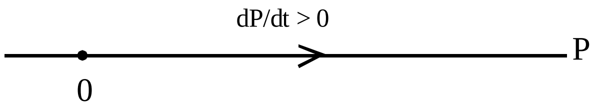
This can be seen immediately from sketch of dP/dt as a function of P :



Therefore the phase line looks as follows:



We have circled the equilibrium point $\frac{-h}{k}$, and indicated the areas where $\frac{dP}{dt} > 0$ by arrows pointing to the right (since increase in the value of P corresponds to motion towards the right on the P–axis) and the areas where $\frac{dP}{dt} < 0$ by arrows pointing to the left (the direction of decreasing P–values on the P–axis). Note that $P = 0$ is **not** an equilibrium point for this differential equation! Since $P(t)$ denotes the size of a population, negative values of P (including the equilibrium point) are not really of interest for us, since the size of the population can not be negative. Therefore, we can leave out the part of the phase line which lies to the left of $P = 0$, and draw the phase line as follows:



(c) First separating the variables and then integrating, we get

$$\frac{dP}{dt} = kP + h$$

$$\therefore \quad \frac{dP}{kP+h} = dt$$

$$\therefore \quad |kP+h| = Ae^{kt} \quad \left(A = e^{C}\right)$$

$$\therefore \quad P = -\frac{h}{k} \pm \frac{A}{k}e^{kt}.$$

The values of the constant A can be solved in terms of the initial value $P(0) = P_0$:

$$P_0 = P(0) = -\frac{h}{k} \pm \frac{A}{k}e^{k\cdot 0} = -\frac{h}{k} \pm \frac{A}{k}$$

$$\therefore \quad \pm A = P_0 k + h$$

where the right–hand side is always positive if we make the reasonable assumption that $P_0 \geq 0$; therefore we should choose the plus sign and thus $A = P_0 k + h$. The solution to the differential equation when the initial value is P_0 is therefore

$$\begin{aligned} P(t) &= -\frac{h}{k} + \frac{(P_0 k + h)}{k}e^{kt} \\ &= P_0 e^{kt} + \frac{h}{k}\left(e^{kt} - 1\right). \end{aligned}$$

(d) From the phase line, we see that for any initial value $P_0 \geq 0$, the population will grow without bound. We can also see this from the solution function,

$$P(T) = P_0 e^{kt} + \frac{h}{k}\left(e^{kt} - 1\right).$$

As $t \to \infty$, the exponential function e^{kt} and hence the entire function $P(t)$ increases without bound.

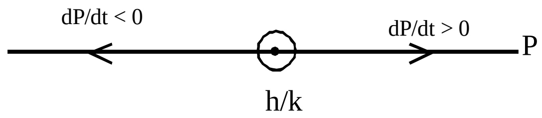
9.5

(a) The constant h gives the rate of emigration, so that it gives the number of individuals leaving the population per year.

(b)

$$P(t) == \frac{h}{k} + \left(P_0 - \frac{h}{k}\right)e^{kt}.$$

(c) One equilibrium point at $P = \frac{h}{k}$, phase line:



(d)If $P_0 > \frac{h}{k}$ then the population will grow without bound, if $P_0 < \frac{h}{k}$ then the population will die out

9.6

(a) The differential equation is

$$\frac{dX}{dt} = k(200 - X) \tag{1}$$

(and the initial value $X(0) = 1$).

(b) To answer this question, we must solve the differential equation in (a); and we need to find the value of k from the given value, $X(2) = 50$. The differential equation can be solved by separating the variables and integrating:

$$\frac{dX}{dt} = k(200 - X)$$

$$\therefore \quad \frac{dX}{200 - X} = kdt$$

$$\therefore \quad -\ln|200 - X| = kt + C$$

$$\therefore \quad |200 - X| = A \cdot e^{-kt} \qquad \left(A = e^{-c}\right).$$

If we express A in terms of $X(0) = X_0$, the solution can be written as

$$(200 - X) = (200 - X_0)\, e^{-kt}.$$

So with $X(0) = 1$, the solution is

$$X(t) = 200 - 199e^{-kt}.$$

If $X(2) = 50$ then we should have

$$50 = 200 - 199e^{-k \cdot 2}$$

from which we can solve k:

$$199e^{-2k} = 200 - 50 = 150$$

$$\therefore \quad e^{-2k} = \frac{150}{199}$$

$$\therefore \; k \;=\; -\frac{1}{2}\ln\left(\frac{150}{199}\right) \approx 0.141335.$$

Now we can find $X(7)$:

$$X(7) \;=\; 200 - 199e^{-7(0.141335)} \approx 126.$$

After 7 days, 126 students have the flu.

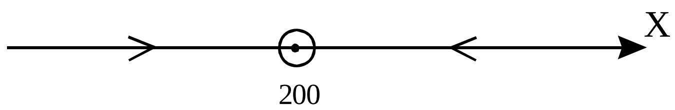
(c) The differential equation was

$$\frac{dX}{dt} = k(200 - X), \qquad k = 0.141335$$

so dX/dt is

- zero if $X = 200$ (this is the only equilibrium point);
- positive if $X < 200$ (since $k > 0$!)
- negative if $X > 200$

Therefore, the phase line looks like this:



(d) The rate of change is the value of the derivative, dX/dt – so according to the differential equation (1), the rate of change is equal to be $(200 - X)$, where X is the current value of $X(t)$. Note that since this is an autonomous equation, the rate of change only depends on $X(t)$ and not on t! At $t = 0$, $X(t) = 1$ and therefore at $t = 0$, the rate of change is

$$(0.141335) * (200 - 1) \approx 28.1$$

At $t = 2$, $X(t) = 50$ and therefore the rate of change is

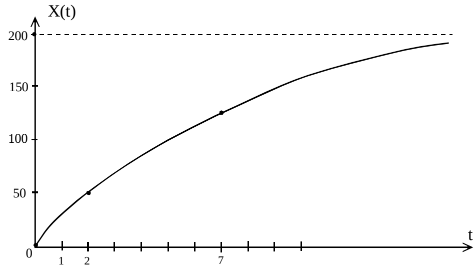
$$(0.141335) * (200 - 50) \approx 21.2$$

At $t = 7$, $X(t) = 126$ and therefore the rate of change is

$$(0.141335) * (200 - 126) \approx 10.5$$

(e) From the problem statement, we know that $X(0) = 1$, and from (b), $X(2) = 50$ and $X(7) = 126$. This specifies 3 points on the (t, X) plane that the solution curve must pass through. Also, according to (d), we know what the slopes of the curve must be at these three points: 28 at $(t, X) = (0, 1)$; 21 at $(t, X) = (2, 50)$ and 10 at $(t, X) = (7, 126)$ .Finally, (c) tells us that the solution curve must increase asymptotically towards $X = 200$.

Thus, our best guess at the solution curve could look like this:



(f) Our assumption about the way the infection spreads was that the rate at which people get sick is directly proportional to the number of people who are not sick yet. This does seem a bit unrealistic: It would mean that the disease spreads fastest initially, even though there is only one sick person, and slower and slower as more and more of the pupils get sick. This does not really agree with the fact that flu spreads from the sick to the healthy: The rate at which I get sick should depend on what percentage of the people I meet are sick. Usually in epidemic models, this is expressed by stating that the rate at which people get sick is proportional to the product $P(M-K)$ of sick and healthy people in the population, which measures the rate of interactions between a sick and a healthy individual, during which interactions the flu could spread.. Indeed in this model, the healthy seem to be getting sick by themselves: If we consider the number of healthy people, $Y = M - X$, then the differential equation for Y would be

$$\frac{dY}{dt} = -kY$$

which exhibits exponential decay!

Of course there are many more ways in which the model is unrealistic. For instance, it does not take into account the following facts:

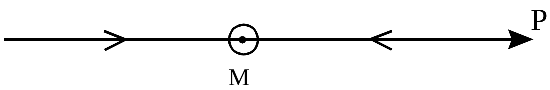
* Some members of the population could be immune to the flu.
* People who are sick would be removed from the population after a day or two, as they stay at home.
* Sick people would only spread the disease for a limited time, so the life time of the epidemic would be limited.

The solution curve grows fast initially, but then slows down and approaches the total population size 200 asymptotically. If we assume that no-one is immune to the flu, then we would expect the disease to spread slowly initially but faster and faster as each healthy individual comes into contact with more sick people.

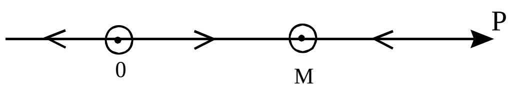
9.7

(a) Phase lines:

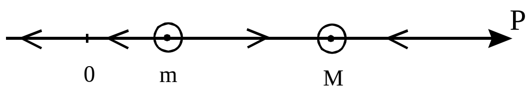
A.



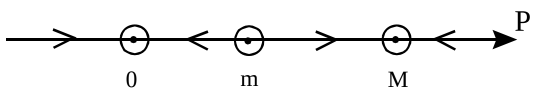
B.



C.



D.



(b) In all the models, populations starting above M decrease towards M so they all have M as the maximum sustainable population size above which for instance competition and resource constrains will cause the population to decrease. The main difference in the models is in what happens to populations starting below the value M. (We will consider only positive initial values in the following.)

* In A, the population values would increase towards M however low the initial value is (even if the populations starts at $P_0 = 0$, or $P_0 < 0$).
* B describes a logistic model: While populations starting at $P_0 = 0$ will stay at 0, for value $P_0 > 0$ the population will increase towards M.
* In C, populations starting with an initial population size above m will increase in size towards M, while populations starting smaller than m will decrease to zero (and indeed reach negative values).
* In D, populations starting with an initial population size above m will increase in size towards M, while populations starting with a non-zero population which is smaller than zero will asymptotically decrease towards zero. Populations starting at zero will stay there.

Which one is most realistic? That really depends on the type of population we are looking at, but for most populations we would really expect there to be a minimum population size below which the population can't drop without dying out. At least, since populations are in fact integer valued quantities (even if our mathematical models for them use real-valued variables), one individual is the smallest possible value, and therefore any model which predicts increase of population from a value $P_0 = 0.0001$ is not realistic from this point of view. So, accordingly, models A and B are not realistic, while in models C and D the population behaves more realistically when starting with small negative values (by dying out). In C and D, the value m can be defined to be the smallest sustainable population size.

Which one of C and D is then more realistic? One possible way of reasoning would be this: In both the models, the population starting below m will decrease. However, in D it will approach $P = 0$ asymptotically, getting close and closer to it but never reaching it, while in C it will reach zero (and die out) in finite time. The scenario in C may be considered more realistic. (Note that we don't have to be too worried about population reaching negative values, as in C: we can interpret negative values to mean that the population has died out.)

I hope you can gain from this account the impression that evaluating which model is "good" and which model is "bad" always depends on what you wish to use the population for. All four models could be very good if we are not concerned about how realistic they are for small values!

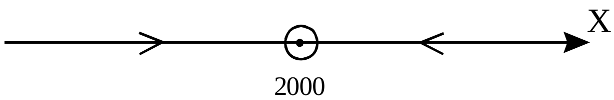
9.8 The differential equations suggested are:

$$\text{(A):} \qquad \frac{dX}{dt} = k\,(2000)\,,$$

$$\text{(B):} \qquad \frac{dX}{dt} = k\,(2000 - X)\,.$$

(a) To be able to draw the phase lines, we need to determine for each model for which values of X the expression on the right, and hence the value of dX/dt, is positive, negative, or zero. Assuming that $k > 0$, we see that in model (A), dX/dt is positive for all values of X (indeed, its value does not even depend on X!) In model (B), we see that dX/dt is zero for $X = 2000$ (which therefore gives the only equilibrium point of the model), and dX/dt is positive for $X < 2000$ and negative for $X > 2000$. We now have all the information we need to draw the phase lines. In model (A), there are no equilibrium points, and direction of motion is always towards the right:

In model (B), there is one equilibrium point at $X = 2000$, and motion is towards right for $X < 2000$ and towards left for $X > 2000$.



(b) The differences in the models can be read directly from the phase lines shown above:

* In model A, the value of X will keep on increasing, up to and beyond the value of 2000.
* In model B, if the value of X is initially less than 2000, it will increase towards 2000, approaching this value asymptotically.

Note also that, as established above:

- $*$ In model A, the rate of change of X, that is, the number of new infections per time unit is constant at the value $k * 2000$.
- $*$ In model B, the rate of change of X depends on the value of $(2000 - X)$. In this model, the rate of change will decrease as X approaches the limit value 2000, so that fewer and fewer new infections happen per time unit as X gets closer and closer to 2000.

So, which model is more realistic?

Model A: We certainly would not want $X(t)$ to reach values of above 2000, when $X(t)$ is supposed to denote the number of sick people out of the 2000. However, this could be dealt with by agreeing that the mathematical model is valid only for $X(t) \leq 2000$, or by taking $X(t) > 2000$ to mean that $X(t) = 2000$, that is, all the students in the school are now infected. What is more unrealistic with model A is the fact that in it, the rate of change is constant, so that from day one of the infection until all students are infected, the number of new infections per day is constant. This would mean that one individual pupil would have exactly the same chance of being infected, whether the total number of already infected students is one or 1999.

Model B: Here, the number of new infections does slow down as the limit value of 2000 is approached, so automatically $X(t)$ always stays below 2000 — this value is approached asymptotically, but never reached. Also, in this model, the rate of change of X, that is, the number of new infections per time unit, varies with X: it is larger when X is small, and gets smaller and smaller as the limit value 2000 is approached (and thus only a few pupils remain to be infected).

We can therefore conclude that model B is more realistic.

9.9

(a) In the logistic model, the differential equation is given by

$$\frac{dP}{dt} = aP - bP^2.$$

In this question, we need to modify this differential equation to take into account the extra fish added to the pond and harvested. Firstly, fish are added to the pond at the rate of X fish per month; this means that we need to add the constant term X to the right hand side of the differential equation. Secondly, the fish are harvested at the rate of $1/3$ of all the current fish per month; this means that we need to deduct the term $\frac{1}{3}P$ from the right hand side of the differential equation. Therefore the differential equation that describes this model is given by

$$\frac{dP}{dt} = aP - bP^2 + X - \frac{1}{3}P$$

which can be rewritten as

$$\frac{dP}{dt} = \left(a - \frac{1}{3}\right)P - bP^2 + X.$$

[Note that this is no longer the logistic system. It is a special case of the logistic model with constant harvesting given in the study guide!]

(b) The equilibrium values of the system are the values of P for which

$$\left(a - \frac{1}{3}\right)P - bP^2 + X = 0. \quad (1)$$

We could now apply the quadratic formula to find the equilibrium values. However, since the request was just to find the value of X for which $P = 0$ is an equilibrium value, we can just substitute $P = 0$ into this equation and see for which value of X the ensuing equation holds. If we put $P = 0$ in (1), we get

$$\left(a - \frac{1}{3}\right) \cdot 0 - b(0)^2 + X = 0. \quad (1)$$

which is only true if $X = 0$. That is, $P = 0$ is only an equilibrium point of the system if $X = 0$.

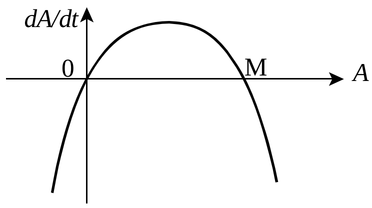
9.10 If $A(t)$ is the amount memorized after t hours and M the total amount of content to be memorized, then the rate at which the subject is memorized is the time derivative of $A(t)$, i.e. $\frac{dA}{dt}$, and the amount still left to be memorized is $M - A$.

(a) So, the differential equation is

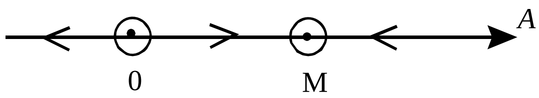
$$\frac{dA}{dt} = kA(M - A),$$

if k denotes the constant of proportionality. The initial value $A_0 = A(0)$ is, by definition, the amount that has been memorized after 0 hours, so we should have $A_0 = 0$.

(b) To draw the phase line, we can first draw a sketch of th function $kA(M - A)$ as a function of A. This is a downwards opening parabola (since k would here be positive), and has zeroes at $A = 0$ and $A = M$. Therefore, the sketch will look as follows.

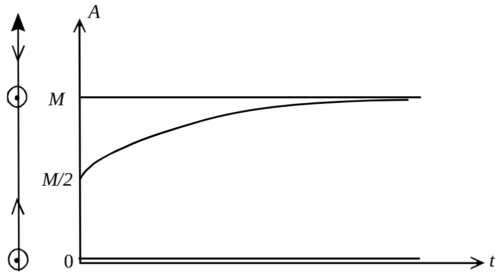


From the sketch we can immediately draw the phase line, which looks as follows.



[The equilibrium points are where the function $kA\,(M-A)$ is zero, that is, where the curve of the parabola crosses the A axis; motion is towards the right where the parabola is above the A axis and towards the left where the parabola is below the A axis.]

(c) The phase line tells us in which intervals of A–values the solutions increase and in which intervals of A–values the solutions decrease. Thus for instance, since $M/2$ falls in the interval between the values 0 and M where the motion is towards the right (towards larger A values), a solution starting at this value would increase, eventually approaching asymptotically the equilibrium point M. The initial values $A_0 = M$ and $A_0 = 0$ on the other hand fall on equilibrium points of the system, so solutions starting at those values are constant solutions, staying forever at those values. Thus we get the following sketches for solutions starting at the given three initial values.



[Drawing the phase line in a vertical position next to the solution sketches helps in drawing the solution curves!]

(d) The three solution curves tell us what happens in this mathematical model if the initial amount of material already memorised is $A_0 = 0$, if it is $A_0 = M/2$, or if it is $A_0 = M$. We will compare the outcomes of what the mathematical model tells us will happen in these three cases, with what we would expect to happen, to analyse whether the model here behaves as it should.

Firstly, if $A_0 = M$ then all the material that needed to be memorised is already memorised; our real-life expectation would be that in that case there should be no change in A since the goal has already been reached. This is what happens in our mathematical model, so in this respect it behaves as we expect.

Secondly, if $A_0 = 0$, our mathematical model predicts that $A\,(t)$ will stay at zero (since A is an equilibrium point in our model). However, in real life, we would expect some progress: if initially nothing is know, learning could and would still happen, meaning that the value of A should increase from 0 rather than staying there. The mathematical model we have here contradicts this, and would be valid only if we have some concrete reason for the assumption that if someone has not already memorised anything at all initially, he or she is unable to memorise anything.

Thirdly, assuming that half of the target of M is initially memorised, the mathematical model predicts that the amount memorised will increase asymptotically towards M. As M is approached, the rate of increase slows down, and the value of M is never actually reached. *[This follows from the fact that M is an equilibrium point in the system!]* Is this what we would expect in real life? That would probably depend on what kind of content needs to be memorised. If the content is simple and straightforward, such as a 10×10 times table, a short poem, or a set of simple instructions, we would definitely expect that all of it should be able to be memorised in finite time. If, on the other hand, the content to be memorised is extensive and complex, we might admit that 100% memorisation is infeasible in finite time.

In summary then, the appropriateness of the model depends on how well the following two assumptions match with the real life situation: (1) that anyone starting with nothing memorised stays there, and (2) that the full amount will never be memorised but will rather be approached asymptotically as time increases.

9.11

(a) The differential equation is

$$\frac{dX}{dt} = kX \tag{1}$$

(and the initial value $X(0) = 1$). Note that this is just the Malthusian model!

(b) The solution is given by

$$X(t) = e^{kt}$$

if the initial value is as given. We can find the value of k from the given value, $X(2) = 5$. We must have

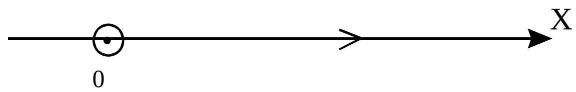
$$e^{2k} = 5 \quad \therefore \quad 2k = \ln(5) \quad \therefore \quad k = \frac{\ln(5)}{2} \approx 0.80472.$$

Using this value for k, we can then find the value of $P(5)$:

$$P(5) = e^{5 \cdot 0.80472} \approx 55.902.$$

Thus, 56 students have the flu after 5 days.

(c) The phase line looks like this:



(d) At $t = 0$, $X(t) = 1$ and therefore at $t = 0$, the rate of change is

$$0.80472 \cdot 1 \approx 0.8$$

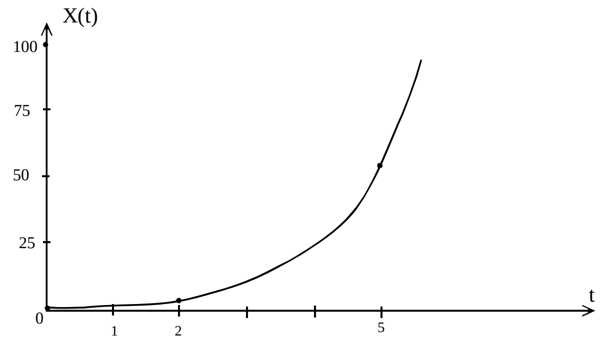
At $t = 2$, $X(t) = 5$ and therefore the rate of change is

$$0.80472 \cdot 5 \approx 4.0$$

At $t = 5$, $X(t) = 56$ and therefore the rate of change is

$$0.80472 \cdot 56 \approx 45$$

(e) From the problem statement, we know that $X(0) = 1$, and from (b), $X(2) = 5$ and $X(5) = 56$. This specifies 3 points on the (t, X) plane that the solution curve must pass through. Also, according to (d), we know what the slopes of the curve must be at these three points: 0.8 at $(t, X) = (0, 1)$; 4 at $(t, X) = (2, 5)$ and 45 at $(t, X) = (5, 56)$ .Thus, our best guess at the solution curve could look like this:



(f) According to the model, the number of people with the flu will grow exponentially: $X(t) \to \infty$ as increases. *[Note that the growth is not even limited by the number of students in the school in this model! This is because the assumptions did not take this number into account at all (e.g. by making the rate of growth also depend on the number of healthy pupils in the school!]*

Study Unit 10 SOLUTIONS

10.1

$$\frac{dx}{dt} = x, \quad \frac{dy}{dt} = y$$

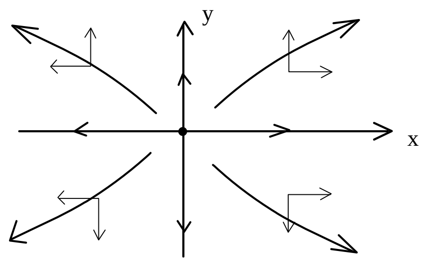
x–isoclines: $\dfrac{dx}{dt} = 0$ when $x = 0$ (y-axis)

y–isoclines: $\dfrac{dy}{dt} = 0$ when $y = 0$ (x–axis).

The equilibrium point is $(0,0)$
Signs of $dx/dt,\ dy/dt$:

$$\begin{aligned} \frac{dx}{dt} &> 0 \quad \text{when} \quad x > 0 \\ \frac{dx}{dt} &< 0 \quad \text{when} \quad x < 0 \\ \frac{dy}{dt} &> 0 \quad \text{when} \quad y > 0 \\ \frac{dy}{dt} &< 0 \quad \text{when} \quad y < 0. \end{aligned}$$

Therefore the phase line looks like this:



The equilibrium point $(0,0)$ is unstable. We can also solve the set of equations to find the exact solutions:

$$\frac{dx}{dt} = x \quad \text{gives} \quad x(t) = x_0 e^t$$

and

$$\frac{dy}{dt} = y \quad \text{gives} \quad y(t) = y_0 e^t.$$

These confirm what we see from the phase diagram, namely that as $t \to \infty$, $x \to \infty$ if $x_0 > 0$ and $x \to -\infty$ if $x_0 < 0$; and $y \to \infty$ if $y_0 > 0$ and $y \to -\infty$ if $y_0 < 0$.

10.2

$$\frac{dx}{dt} = -x, \qquad \frac{dy}{dt} = 2y.$$

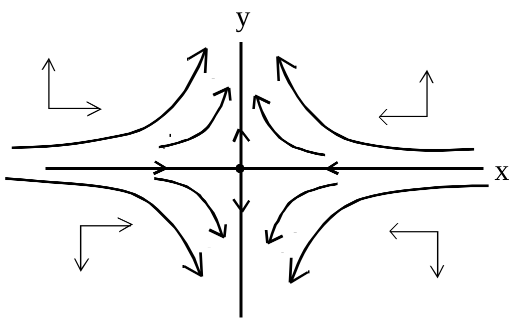
x–isoclines and sign of dx/dt :

$$\begin{aligned} \frac{dx}{dt} &= 0 \quad \text{when} \quad x = 0,\ (y\text{–axis}), \\ \frac{dx}{dt} &> 0 \quad \text{when} \quad x < 0, \\ \frac{dx}{dt} &< 0 \quad \text{when} \quad x > 0 \end{aligned}$$

y–isoclines and sign of dy/dt :

$$\begin{aligned} \frac{dy}{dt} &= 0 \quad \text{when} \quad 2y = 0 \ \therefore\ y = 0 \quad (x\text{–axis}) \\ \frac{dy}{dt} &> 0 \quad \text{when} \quad y > 0, \\ \frac{dy}{dt} &< 0 \quad \text{when} \quad y < 0. \end{aligned}$$

The only x–isocline is the y–axis, and the only y–isocline is the x–axis. The only equilibrium point is at $(0,0)$,where the two isoclines meet. The phase diagram looks like this:



Note that the y-axis is an x–isocline, and therefore motion along it can only be vertically (up or down). Likewise, the x-axis is a y–isocline, and motion along it can only be horizontally (sideways). It follows that neither the x–axis nor the y–axis can be crossed by any of the solution curves. We can also solve the set of equations to find the exact solutions:

$$\frac{dx}{dt} = -x \quad \text{gives} \quad x(t) = x_0 e^{-t}$$

and

$$\frac{dy}{dt} = 2y \quad \text{gives} \quad y(t) = y_0 e^{2t}.$$

The equilibrium point is unstable. As $t \to \infty$, $x \to 0$; and $y \to +\infty$ if $y_0 > 0$, while $y \to -\infty$ if $y_0 < 0$.

10.3

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -2x$$

$$x\text{–isoclines: } \frac{dx}{dt} = 0 \quad \text{when } y = 0 \ (x - \text{axis})$$

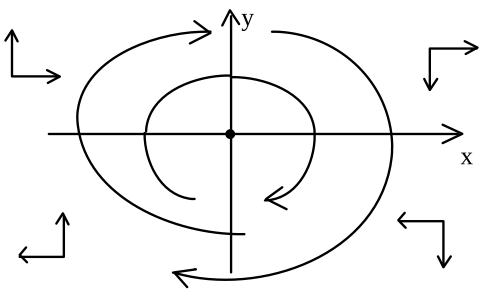
$$y\text{–isoclines: } \frac{dy}{dt} = 0 \quad \text{when } -2x = 0 \quad \therefore \quad x = 0 \ \ (y\text{–axis})$$

The only equilibrium point is $(0,0)$. Signs of $dx/dt, \ \ dy/dt$:

$$\frac{dx}{dt} > 0 \text{ when } y > 0 \text{ and } \frac{dx}{dt} < 0 \text{ when } y < 0.$$

$$\frac{dy}{dt} > 0 \text{ when } x < 0 \text{ and } \frac{dy}{dt} < 0 \text{ when } x > 0.$$

Therefore the phase diagram looks like this:



Please note that the x–axis must be crossed vertically since it is an x–isocline; and the y–axis must be crossed horizontally since it is a y–isocline.

As explained in the study guide, the phase diagram alone can't really tell us in this case what will happen: will the solution spiral towards $(0,0)$, away from it, or will we get periodic solutions, travelling along closed curves around $(0,0)$?

To find out which case we have here, we would need to solve the set of equations to find the exact shape of the trajectory. We will show here how this can be done. But this is a bit beyond this module, so you don't have to know how to do this yourself!

According to the chain rule of differentiation,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-2x}{y}.$$

Separating the variables, we get

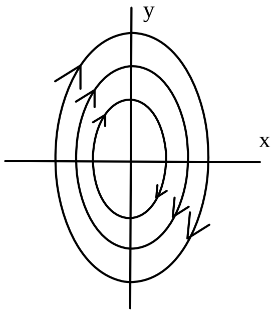
$$y\,dy = -2x\,dx$$

and integrating this,

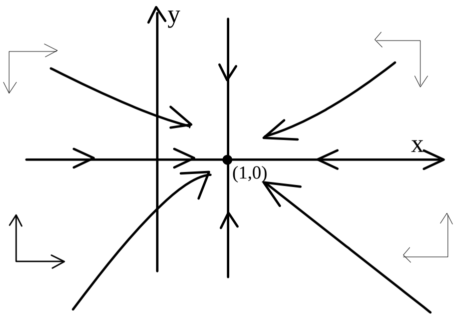
$$\frac{1}{2}y^2 = -x^2 + c$$

$$\therefore \quad \frac{1}{2}y^2 + x^2 = c.$$

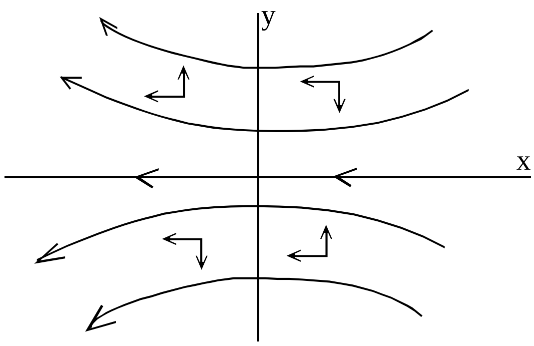
This equation describes **ellipses** in the $x - y$ plane. Therefore, the solutions are in fact closed solutions, and the equilibrium point $(0,0)$ is stable, but not asymptotically stable (since solutions starting near it stay on a closed trajectory near it, but do not converge towards it).



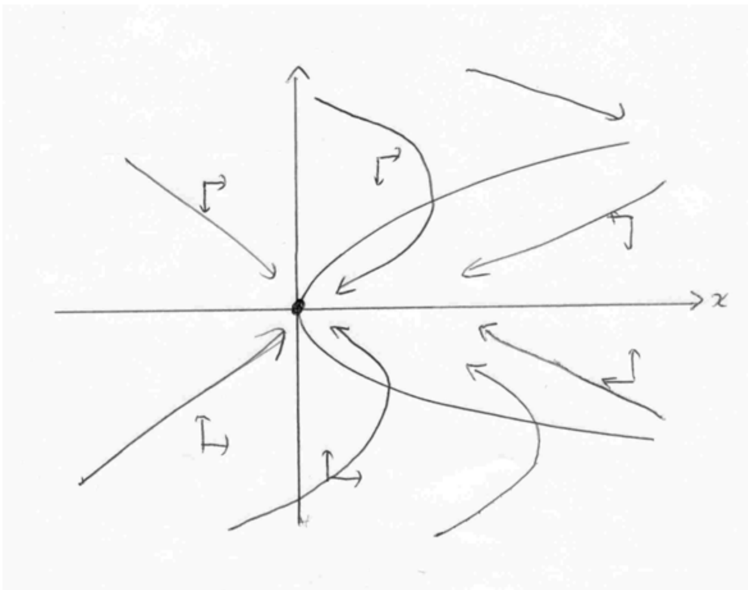
10.4



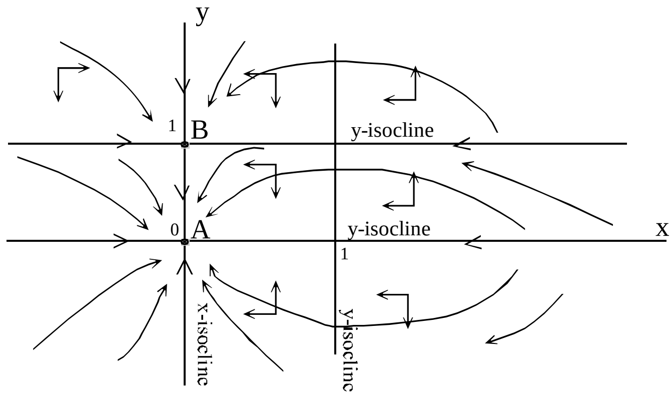
10.5



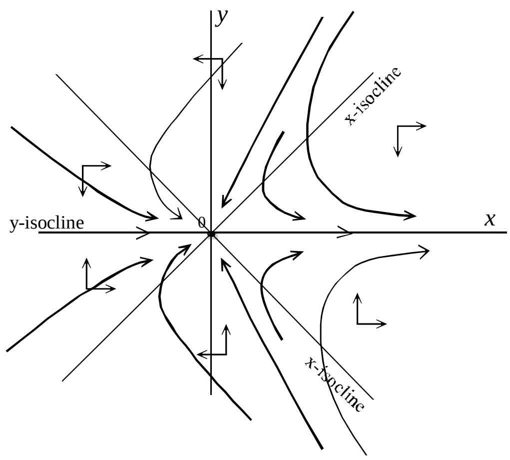
10.6



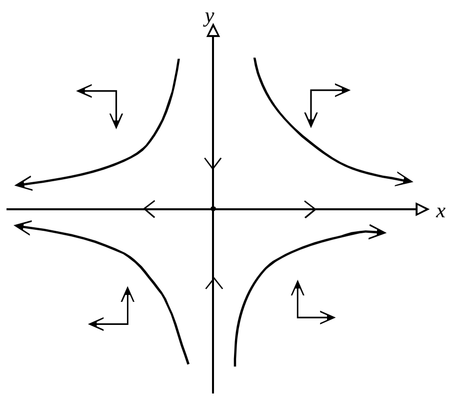
10.7


y
1
B
y-isocline
0
A
y-isocline
x
1
x-isocline
y-isocline

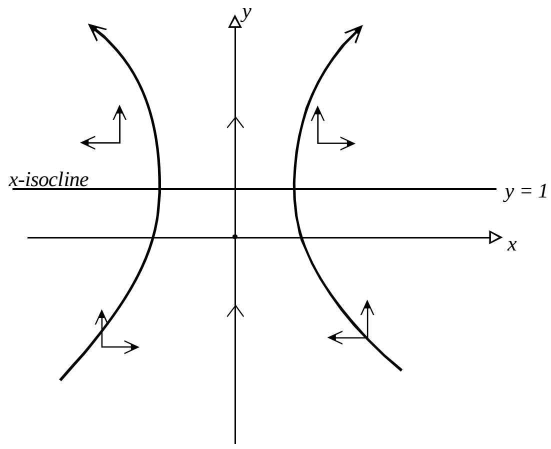
10.8


y
x-isocline
y-isocline
0
x
x-isocline

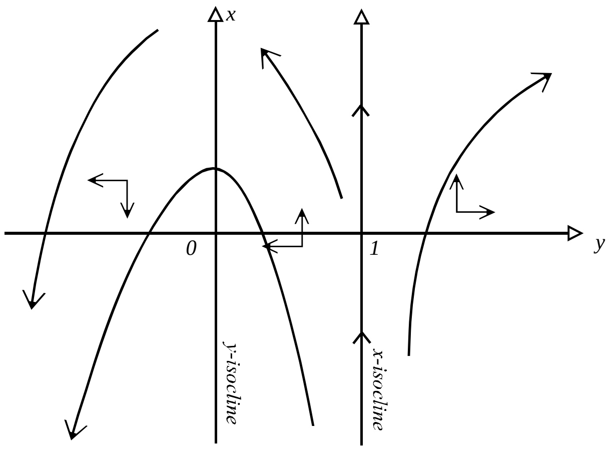
10.9


y
x

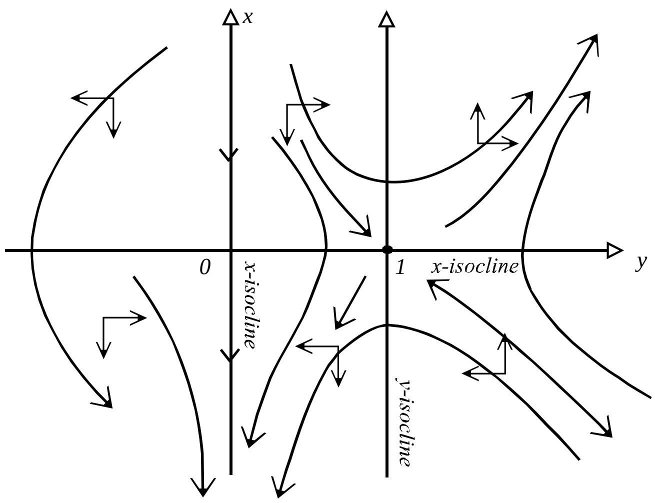
10.10



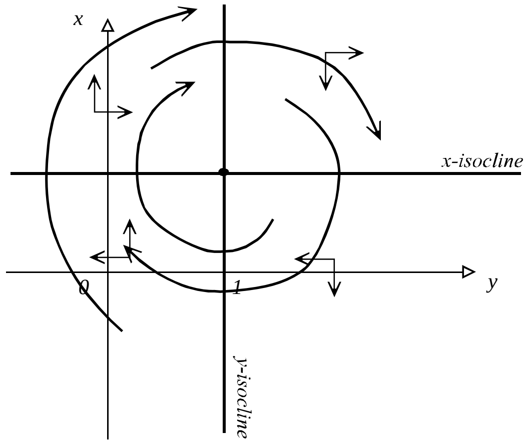
10.11



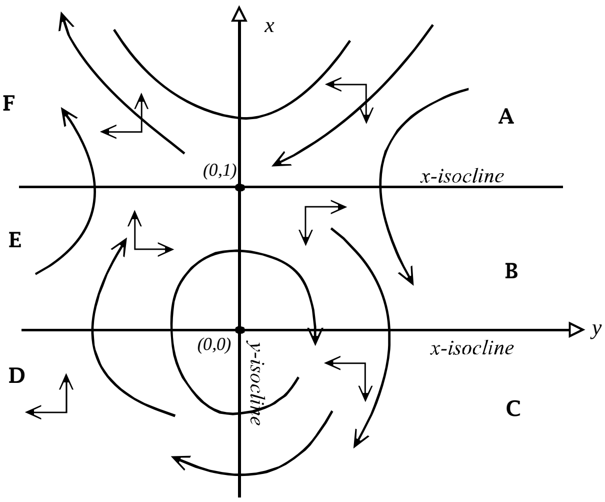
10.12



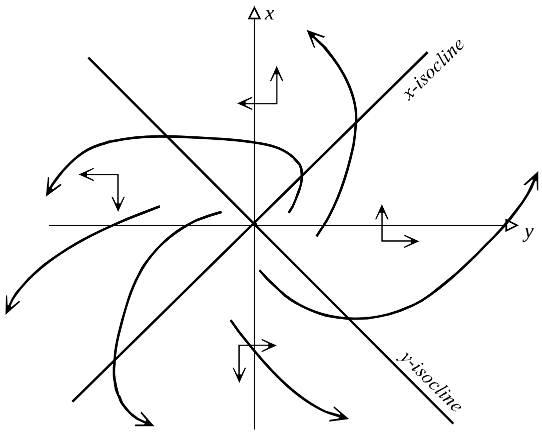
10.13



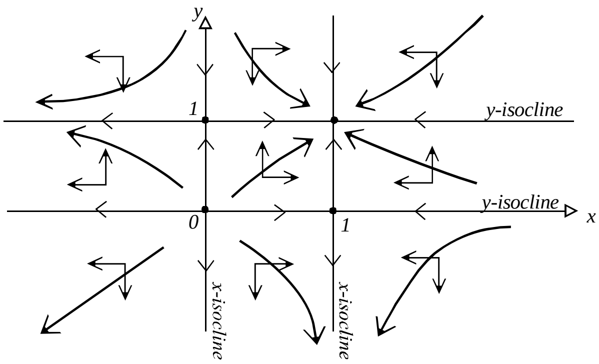
10.14



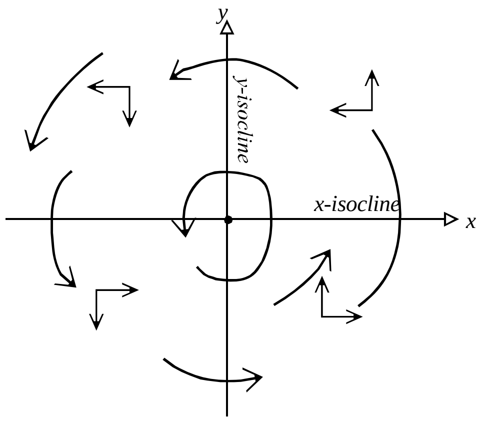
10.15



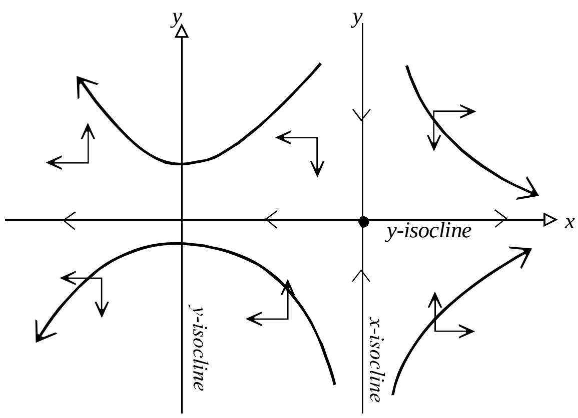
10.16



10.17

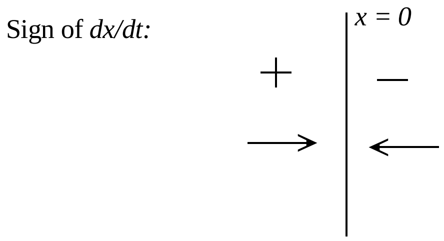


10.18



10.19 **x–isoclines and sign of dx/dt**

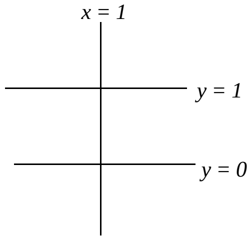
Here, $dx/dt = 0$ when $x = 0$. Also, $dx/dt = -x$ is positive if $x < 0$ and negative if $x > 0$.



y–isoclines and and sign of dy/dt

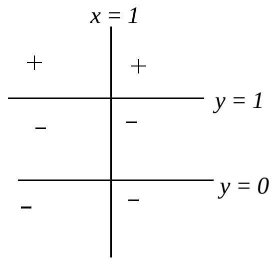
$dy/dt = 0$ holds for $y = 0$, $y = 1$ and $x = 1$. There are thus three y–isoclines. The isoclines are drawn in the diagram below — as you can see, the situation is a bit more complex than what we are used to!

Sign of *dy/dt:*



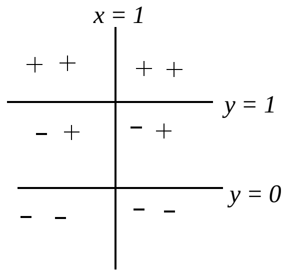
How do we find the signs of the function $(y-1)\,y\,(x-1)$ in the six different regions? One way would be just to pick any point which lies inside one of the regions and see what sign the function has at that particular point. For instance, the point $(2,2)$ would lie in the top-right region, and at that point, $(y-1)\,y\,(x-1) = (2-1)\,2\,(2-1) = 1 \cdot 2 \cdot 2$ is positive, hence motion in this particular region would be to the right. Another way to figure out the signs would be by factorisation; we will illustrate this method in the following. In factorisation, we consider each of the three terms in the expression $(y-1)\,y\,(x-1)$ separately, one by one; we establish the signs of each term in all the regions, and then figure out what the sign of the entire product will be. Firstly, considering the sign of just $y-1$, we easily see that it is positive for $y > 1$ and negative for $y < 1$; we mark this in by adding a plus sign in the regions where this term is positive and a minus sign where it is negative.

Sign of *(y-1)*:



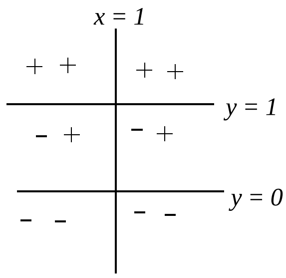
Next we repeat the same for the next term, y, again adding a plus where it is positive and a minus where it is negative.

Signs of *(y-1)* and *y*:

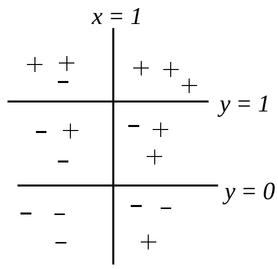


Next, we do the same for the third term, $x-1$.

Signs of *(y-1)* and *y*:

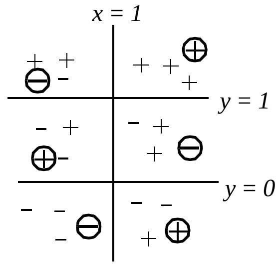


Signs of *(y-1)*, *y* and *(x-1)*:



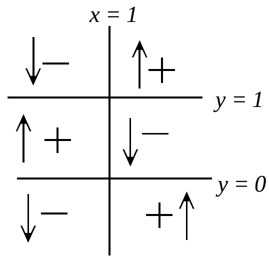
Now there are three signs in each region, one for each term, and what remains is to conclude from these signs what the sign of the product will be — using the facts we know about how the sign of a product is determined (which tell us, for instance, that $[-] \times [-] \times [+] = [+]$, $[-] \times [-] \times [-] = [-]$ and so on.) This gives us the final signs for the entire product

Sign the product:

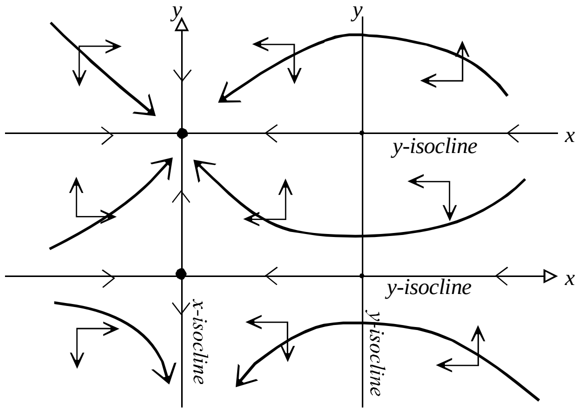


which we can then notate using arrows as follows:

Sign of *dy/dt:*



The phase diagram looks like this:



There is are two equilibrium points, $(x, y) = (0, 0)$ and $(x, y) = (0, 1)$. We see that the outcome of the system is determined on the value of y_0:

– all solutions starting with $y_0 > 1$ or with $0 < y_0 < 1$will eventually converge towards the equilibrium point $(0, 1)$,

– all solutions starting with $y_0 = 1$ will move towards $(0, 1)$ along the y-isocline $y = 1$,

– all solutions starting with $y_0 = 0$ will move towards $(0, 0)$ along the y-isocline $y = 0$,

– all solutions starting with $y_0 < 0$ will approach the negative y-axis asymptotically, so that for them $x \to 0$ while $y \to -\infty$.

The equilibrium point $(0, 0)$ is unstable, while $(0, 1)$ is stable.

10.20 We get, by applying the chain rule of differentiation and the values that the differential equations give for dx/dt and dy/dt,

$$\frac{dr}{dt} = \frac{d}{dt}\left(\sqrt{x(t)^2 + y(t)^2}\right) = -\frac{\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right)}{2\sqrt{x(t)^2 + y(t)^2}} = -\frac{(2x\,(y) + 2y\,(-x))}{2\sqrt{x(t)^2 + y(t)^2}} = 0.$$

Therefore r must be constant.

10.21 Any curve $(x\,(t)\,, y\,(t))$ in two dimensions can equivalently be expressed in the polar coordinate form as $r\,(t)\,(\cos\theta(t), \sin\theta(t))$ where

$$r\,(t) = \sqrt{x(t)^2 + y(t)^2}$$

and

$$\tan\theta(t) = \frac{y\,(t)}{x\,(t)}.$$

But according to the previous exercise, if $(x\,(t)\,, y\,(t))$ solves the given system then $r\,(t)$ is constant, let's call it a.. Therefore the solution must be given by $(x(t), y(t)) = a(\cos\theta(t), \sin\theta(t))$. Substituting these expressions into the differential equations, we see that we must have

$$-a\theta'(t)\sin\theta(t) = \sin\theta(t)$$

$$a\theta'(t)\cos\theta(t) = -\cos\theta(t)$$

which means that we should have

$$a\theta'(t) = -1$$

which give

$$\theta(t) = -\frac{1}{a}t + \theta_0.$$

The solution is

$$(x(t), y(t)) = a(\cos\left(\theta_0 - \frac{1}{a}t\right), \sin\left(\theta_0 - \frac{1}{a}t\right)).$$

These are curves moving with constant angular velocity clockwise motion around a circle with radius a. The initial angle θ_0 specifies where the solution starts from.

Study Unit 11 SOLUTIONS

11.1 The model here is defined by the equations

$$\frac{dx}{dt} = (a - bx - cy)x$$

$$\frac{dy}{dt} = (d - ex - fy)y$$

where a, b, c, d, e, and f are positive constants. As described in the study guide, the x-isoclines are

$$x = 0 \tag{1}$$

and

$$a - bx - cy = 0 \tag{2}$$

while the y-isoclines are

$$y = 0 \tag{3}$$

and

$$d - ex - fy = 0. \tag{4}$$

Here. the x–isocline (1) is the y–axis and (2) is a line which meets y–axis at $y = a/c$ and x–axis at $x = a/b$; y–isocline (3) is the x–axis and (4) is a line that meets y–axis at $y = d/f$ and x–axis at $x = d/e$. The four possible possible equilibrium points are

$$A = (0, 0), \;\; B = (0, \frac{d}{f}), \;\; C = (\frac{a}{b}, 0) \;\; D = \left(\frac{af - dc}{bf - ec}, \frac{bd - ae}{bf - ec}\right)$$

The phase space diagram depends on the values of the parameters a, b, c, d, e, and f . There are four possibilities according to the relative sizes of $\frac{d}{f}$ and $\frac{a}{c}$, and the relative sizes of $\frac{a}{b}$ and $\frac{d}{e}$:

Case 1

$$\frac{d}{f} > \frac{a}{c}; \quad \frac{a}{b} > \frac{d}{e}$$

Case 2

$$\frac{d}{f} > \frac{a}{c}; \quad \frac{a}{b} < \frac{d}{e}$$

Case 3

$$\frac{d}{f} < \frac{a}{c}; \quad \frac{a}{b} < \frac{d}{e}$$

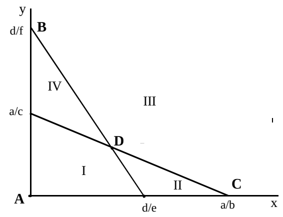
Case 4

$$\frac{d}{f} < \frac{a}{c}; \quad \frac{a}{b} > \frac{d}{e}$$

Cases 3 and 4 were investigated in the study guide; here we will see what happens in cases 1 and 2.

Case 1

In Case 1, the x–isocline (2) crosses the y–axis below the y–isocline (4), and the x–axis to the right of the y–isocline (4). The x– and y–isoclines are indicated in the diagram below. We have also marked in the four equilibrium points A, B, C and D, each of which lies in the intersection of an x–isocline and a y–isocline.



The x– and y– isoclines again divide the $x - y$ plane into 4 regions, denoted (counterclockwise) by I, II, III, IV. Note that now, I and II lie below x–isocline (2) and I and IV lie below y–isocline (4).

We will next investigate the signs of dx/dt and dy/dt in each of these regions.

In regions I and II, below x–isocline (2),

$$(a - bx - cy) < 0 \ \text{ so } \frac{dx}{dt} > 0.$$

In regions III and IV, above x–isocline (2),

$$(a - bx - cy) < 0 \ \text{ so } \frac{dx}{dt} < 0.$$

In regions I and IV, below y–isocline (4),

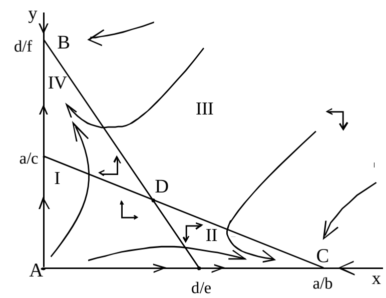
$$(d - ex - fy) > 0 \ \text{ so } \frac{dy}{dt} > 0.$$

In regions II and III, above y–isocline (4),

$$(d - ex - fy) < 0 \ \text{ so } \frac{dy}{dt} < 0.$$

These results specify the permitted directions of motion in the different regions of the $x - y$ plane. We indicate these directions of motion in the phase diagram by drawing in each region an upward arrow if $dy/dt > 0$ or alternatively a downward arrow if $dy/dt < 0$; as well as a leftward arrow if $dx/dt < 0$ or a rightward arrow if $dx/dt > 0$.

Examples of permitted trajectories are also drawn into the diagram. Remember that x-isoclines can only be crossed vertically, and y–isoclines can only be crossed horizontally. The final phase diagrams looks like this :



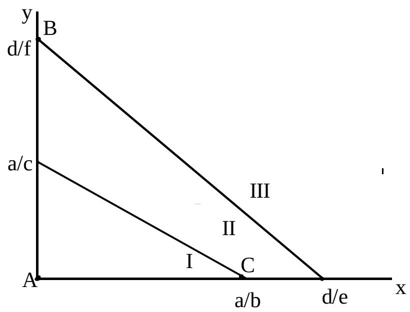
We see from the diagram that D is **not stable**. This is because there are trajectories starting arbitrarily close to D (in regions II and IV) which regardless lead away from D. $A = (0, 0)$ is also unstable (for example, motion along the x and y axes is away from A!). The equilibrium points B and C, on the other hand, are stable since all paths starting near them do converge towards them.

In conclusion, depending on where the trajectory starts, it could converge either towards B or C. (The only exceptions are when the trajectory starts exactly either at A or D. However, these cases are unlikely to happen in practice.)

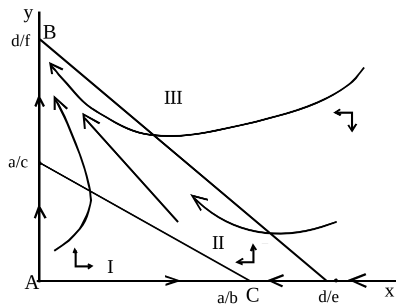
The interpretation is that only one species will survive. Which one will be the dominant, surviving one, will depend on what the initial populations were.

Case 2

In Case 2, as in Case 4, the x–isocline (2) and the y–isocline (4) do not intersect in the quadrant $x > 0, \ y > 0$. However, this time the y–isocline is above the x–isocline.



As in Case 4, A, B and C are the only equilibrium points. The signs of dx/dt and dy/dt are obtained just like before. The phase diagram looks as follows :



We see that B is stable, while A and C are unstable. Population y will always dominate, approaching the limiting population size $y = d/f$, while population x will always become extinct in the long term. The only exception is when $y = 0$ initially; in that case y stays at zero and x–population converges, all by itself, towards its limiting value $x = a/b$.

11.1 (a) The x–isocline is found as follows:

$$\frac{dx}{dt} = 0$$

$$\therefore \quad a - px - qf(x)y$$

$$\therefore \quad y = \frac{a - px}{qf(x)} \quad (x \neq 0). \tag{3}$$

Note that since we assume that $f(0) = 0$ and f is a strictly increasing function, we have $f(x) > 0$ for all $x > 0$. From the assumptions on f, we see that the curve described by (3) is decreasing, and crosses the x–axis at the point $x = a/p$. Let us consider the behaviour of the curve when x is close to zero: As $x \to 0$, we have $a - px \to a$ while $qf(x) \to qf(0) = 0$ since f is continuous. Hence as $x \to 0$, we must have $y \to \infty$. This means that the y–axis is a vertical asymptote for the x–isocline.

(b) The y–isoclines are found as follows:

$$\frac{dx}{dt} = 0$$

$$\therefore \quad (sf(x) - r)\, y = 0$$

$$\therefore \quad y = 0 \quad \text{or} \quad f(x) = r/s. \tag{3}$$

We know that $f(k) = r/s$, and since $f(x)$ is strictly increasing, $x = k$ is the only point where $f(x) = r/s$ holds. Hence the y–isoclines are the lines

$$x = k, \tag{4}$$
$$y = 0 \tag{5}$$

Note that the first line is parallel to the y–axis and the second line is the x–axis.

(c) Finding the simultaneous solutions to (3) and (5) we get the equilibrium point

$$A = \left(\frac{a}{p}, 0\right)$$

while solving (3) and (4) yields the equilibrium point

$$B = \left(k, \frac{s\,(a - pk)}{qr}\right)$$

(remember that $f(k) = r/s$).

(d) If $x > k$, then $f(x) > f(k) = r/s$ since f is strictly increasing, and thus it follows from (2) that dy/dt is positive if $y > 0$. Similarly, if $x < k$ and $y > 0$, we find that dy/dt is negative. Therefore, $dy/dt > 0$ in the area to the right of the line $x = k$; and $dy/dt < 0$ in the area to left of the line $x = k$.

If

$$y > \frac{a - px}{qf(x)}$$

(that is, above the curve described by the isocline (3)), then it follows from (1) that

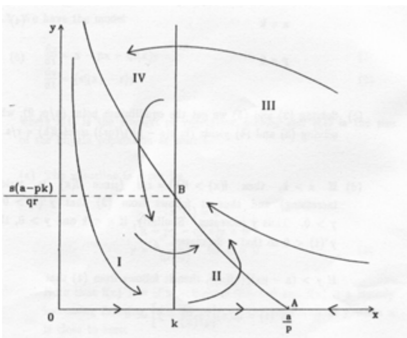
$$\frac{dx}{dt} = a - px - qf(x)y = qf(x)\left[\frac{a - px}{qf(x)} - y\right] < 0.$$

By the same argument, if

$$y < \frac{a - px}{qf(x)}$$

(below the curve) then $dx/dt > 0$.

(e) Since x and y denote sizes of populations, we can assume that only non-negative values of x and y are of interest. Thus we are justified to only consider the first quadrant of the xy–plane. If we assume that $k < a/p$ then the two isoclines (3) and (4) intersect each other within the first quadrant. Therefore in this case, the isoclines (3) and (4) divide the first quadrant of the phase plane into four regions, shown below numbered I to IV. From the considerations in (d) above, we conclude that, $dy/dt > 0$ (y increases) in regions II and III; and $dy/dt > 0$ (y decreases) in regions I and IV. Likewise, $dx/dt < 0$ (x decreases) in the regions III and IV; and $dx/dt > 0$ (x increases) in the regions I and II.



As explained on page 116 of the study guide, we do not know the precise forms of the trajectories around B. However, the geometry of the phase plane as a whole, in particular the fact that all the trajectories lead towards B, suggests that the trajectories near B spiral inwards to it.

(f) Point A is an unstable equilibrium point, because the trajectories starting at points with $y > 0$ near A lead away from A. Point B is clearly a stable equilibrium point since all the trajectories lead towards it. Hence the sizes of the vegetation and impala population will always converge to the levels $x = k$ and $y = s(a - pk)/(qr)$, a state of stable coexistence.

Note that according to this model, this is the case even for trajectories starting on the y–axis, i.e. if there is initially no vegetation. The impala population then initially decreases, but does not die out and begins to increase again once the vegetation reaches the level $x = k$.

From the above analysis, a conservation official could deduce that the number of impala in a reserve should be kept near $y = s(a - pk)/(qr)$ to ensure a stable impala – vegetation ecosystem without large fluctuations in the levels of impala or vegetation. In practice it may, however, be very difficult, if not possible, to determine the values of the constants a, k, p, q, r, s. This could possibly require more data than is known about the impala and the vegetation. In situations where the necessary data is available, suitable mathematical techniques must be used to fit the model to the data, i.e. to determine the values of the constants. This aspect of modelling is however beyond the scope of this module.

11.2 The equations modelling the growth of the scale population $P(t)$ and the ladybird population $R(t)$ are

$$\frac{dP}{dt} = (a-b-cR)P \qquad (1)$$

$$\frac{dR}{dt} = (fP-d)R \qquad (2)$$

The P-isoclines are the lines

$$R = \frac{a-b}{c} \qquad (3)$$

$$P = 0 \qquad (4)$$

The R-isoclines are the lines

$$P = d/f \qquad (5)$$

$$R = 0 \qquad (6)$$

Solving (3) and (5) gives $(d/f;\ (a-b)/c)$, and solving (4) and (6) we get (0; 0). The system of equations (3), (6) has no solution since $a-b>0$. Similarly (4), (5) has no solution. Hence there are only two equilibrium points:

$$(0;\ 0) \quad \text{and} \quad A = (d/f; (a-b)/c)$$

The signs of dP/dt and dR/dt are as follows:

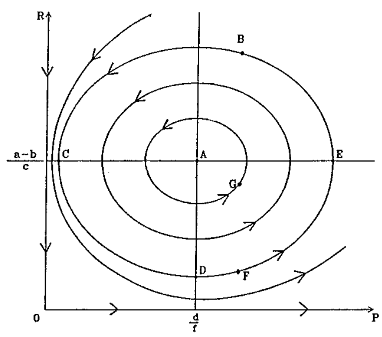
$P'(t) > 0$ if $P > 0$ and $0 \le R < (a-b)/c$.

$P'(t) < 0$ if $P > 0$ and $R > (a-b)/c$.

$R'(t) > 0$ if $P > d/f$ and $R > 0$.

$R'(t) < 0$ if $P < d/f$ and $R > 0$.

By taking x, y, a, b, m, n in (10.12) equal to P, R, $a-b$, c, d, f, respectively, in (1), (2), we see that the two models have exactly the same form. The results derived for (10.12) may therefore be applied to (1), (2). It follows that the phase plane diagram looks like this:



The equilibrium point (0; 0) is unstable. Point A is stable in the sense that all the trajectories starting near it always remain near it, although they do not approach A.

An interesting point is that according this model the scales can never be eliminated by the ladybird beetles alone, no matter how many are imported. For example, if we start with a large number of beetles (say at B in the phase plane) then the scale population does initially decrease (to point C), but this causes a decline in the beetle population (point D) and consequently a resurgence of the scale population (point E). One could therefore just as well have started with a small number of beetles ($e.g.$ at F). Moreover, this large oscillation between very few scales (at C) and a scales plague (at E) is probably undesirable from a farming point of view. By introducing a number of beetles close to the level $R = (a-b)/c$ (i.e. starting from e.g. G), one could keep the scales population more or less constant near the level $P = d/f$. This may however be too high for profitable farming.

The discussion above highlights a shortcoming of the model, namely that the scales population will always recover no matter how close to the y-axis the trajectory turns (even if $P < 1$ at point C). In practice a population will die out if the remaining individuals are too widely dispersed for mating, etc. This is what happened in the 1860's, with the corresponding trajectory in our model probably starting from a point such as F.

11.3 Let $x(t)$ be the size of the bass population, and $y(t)$ the size of the trout population. Then in isolation we have

$$\frac{dx}{dt} = ax(M-x)$$

since in isolation the bass behaves according to the logistic model with population limit M. (The general logistic model is

$$\frac{dx}{dt} = a_0 x - b_0 x^2,$$

but the population limit must be $\dfrac{a_0}{b_0} = M$. Hence $b_0 = \dfrac{a_0}{M}$, so

$$\begin{aligned} dx/dt &= a_0 x - a_0 \frac{1}{M} x^2 = a_0 x \left(1 - \frac{x}{M}\right) \\ \therefore \quad \frac{dx}{dt} &= ax(M-x) \end{aligned}$$

when $a = a_0/M$.) Adding competition between the trout and the bass, we get

$$dx/dt = ax(M-x) - cxy.$$

In isolation, the trout population decays exponentially, so that

$$dy/dt = -by,$$

so adding the competition term we get

$$dy/dt = -by - dxy.$$

So, the model is described by the differential equations

$$\begin{cases} \dfrac{dx}{dt} = ax(M-x) - cxy \\ \dfrac{dy}{dt} = -by - dxy \end{cases}$$

where $M > 0,\ a,\ b,\ c,\ d > 0$. *[See the examples in the study guide for the reasoning behind the competition terms.]* The x–isoclines:

$$\begin{aligned} \frac{dx}{dt} &= 0 \quad \text{when} \quad ax(M-x) - cxy = 0 \\ \therefore \quad x &= 0 \quad \text{or} \quad y = \frac{a}{c}(M-x). \end{aligned}$$

The first x–isocline is thus the y–axis, and the second one is the line $y = \frac{a}{c}(M-x)$ which crosses the x–axis at $x = M$ and the y–axis at $y = aM/c$. (Note that $M > 0$, $aM/c > 0$.) The y–isoclines:

$$\frac{dy}{dt} = 0 \quad \text{when} \quad -by - dxy = 0$$

$$\begin{aligned} \therefore \quad & -y(b + dx) = 0 \\ \therefore \quad & y = 0 \quad (x\text{–axis}) \end{aligned}$$

$$\text{or } x = -\frac{b}{d} \quad \text{(line parallel to } y\text{–axis).}$$

Since x and y represent population sizes, we only need to investigate what happens in the first quadrant of the xy–plane, with $x, y \geq 0$. Note that the second y–isocline $x = -b/d$ is not in the first quadrant, when $b, d > 0$. For the equilibrium points we take every possible combination of a y– and x–isocline:

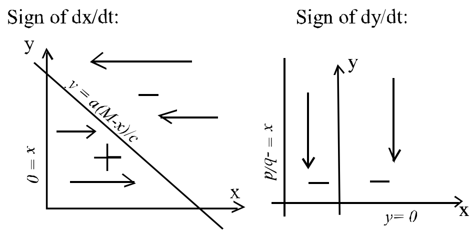
$$\begin{cases} x = 0 \\ y = 0 \end{cases} \quad \text{gives the point } (0,0) = A$$

$$\begin{cases} x = 0 \\ x = -\frac{b}{d} \end{cases} \quad \text{gives no equilibrium points,}$$

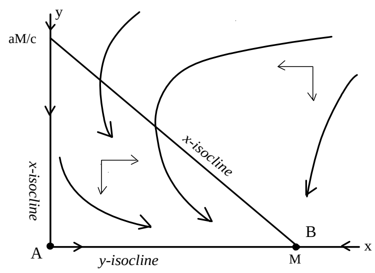
$$\begin{cases} y = \frac{a}{c}(M-x) \\ y = 0 \end{cases} \quad \text{gives the point } (M,0) = B$$

$$\begin{cases} y = \frac{a}{c}(M-x) \\ x = -\frac{b}{d} \end{cases} \quad \text{gives the point } \left(-\frac{b}{d}, \frac{a}{c}\left(M + \frac{b}{d}\right)\right) = C.$$

The last equilibrium point C is outside the area $x \geq 0$, $y \geq 0$ and will therefore be ignored. The following sketches summarize the signs of dx/dt and dy/dt in different parts of the first quadrant of the xy–plane:



Hence the phase diagram looks like this:



(Note that the point $\left(0, \frac{aM}{c}\right)$ is not an equilibrium point.) A is unstable and B is stable. We see that the trout population (y) will always die out, so that coexistence will not happen. The bass population (x) will converge towards its limiting value M if there are initially any bass present, however few.

(a) We will first find the P– and Q–isoclines of the system.

P–isoclines: The equation $\frac{dP}{dt} = 0$holds when

$$aP\left(\frac{b}{Q} - P\right) = 0,$$

which gives us the P–isoclines

$$P = 0 \tag{1}$$

and

$$P = \frac{b}{Q} \tag{2}$$

Q–isoclines: The equation

$$\frac{dQ}{dt} = 0$$

holds when

$$cQ\left(fP - Q\right) = 0$$

which gives us the Q–isoclines

$$Q = 0 \tag{3}$$

and

$$Q = fP \tag{4}$$

An equilibrium point must be on both a P–isocline and a Q–isocline, so we must look at all possible intersections of a P–isocline and a Q–isocline.

Combining (1) and (3) gives us the point $(P, Q) = (0, 0)$.

Combining (1) and (4) again gives $(0, 0)$.

Combining (2) and (3) we get no solution (if $Q = 0$ then $P = \frac{b}{Q}$ is not defined).

Combining (2) and (4) we get

$$\begin{cases} P = \frac{b}{Q} \\ Q = fP \end{cases} \therefore \begin{cases} PQ = b \\ Q = fP \end{cases} \therefore \begin{cases} fP^2 = b \\ Q = fP \end{cases}$$

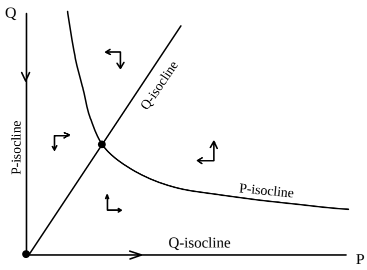
$$\therefore \begin{cases} P^2 = \frac{b}{f} \\ Q = fP \end{cases} \therefore \begin{cases} P = \pm\sqrt{\frac{b}{f}} \\ Q = fP \end{cases}$$

We can assume that $P \geq 0$, since P is the price of a product. Thus we can reject the negative solution, and end up with the equilibrium point

$$(P, Q) = \left(\sqrt{\frac{b}{f}}, \sqrt{fb}\right).$$

We will classify the equilibrium points after sketching the phase diagram.

(b) The phase diagram will now be a sketch in the PQ–plane. We will take the P–axis to be the horizontal axis and the Q–axis to be the vertical axis. We will first draw the P and Q isoclines into the PQ–plane. Isoclines (1) and (3) correspond to the Q and P–axes, respectively. Isocline (4) is a straight line through the origin of the PQ–plane; its (positive) slope is the constant f. Isocline (2) is the hyperbola $Q = \frac{b}{P}$, which goes through the points $(P,Q)=(1,b)$ and $(P,Q)=(b,1)$. The four isoclines are shown below. Note that since we can assume that $P\geq 0$, $Q\geq 0$ (why?), we are showing only the first quadrant of the PQ–plane.



We see that as expected, a P–isocline intersects a Q–isocline at the two equilibrium points $(0,0)$ and $\left(\sqrt{\frac{b}{f}},\sqrt{fb}\right)$, and at only these points. To determine the directions of motion in the regions of the phase plane, we note that

$$\frac{dP}{dt}>0 \qquad \text{if} \quad \left(aP>0 \quad \textbf{and} \quad \frac{b}{Q}-P>0\right)$$
$$\text{or if} \quad \left(aP<0 \quad \textbf{and} \quad \frac{b}{Q}-P<0\right)$$

and

$$\frac{dP}{dt}<0 \qquad \text{otherwise.}$$

Since $aP>0$ holds whenever $P>0$, we find that in the first quadrant,

$$\frac{dP}{dt} > 0 \text{ for points below the hyperbola } Q=\frac{b}{P},$$
$$\frac{dP}{dt} < 0 \text{ for points above the hyperbola } Q=\frac{b}{P}.$$

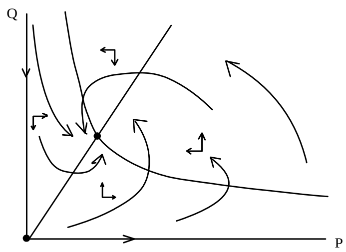
Similarly we see that

$$\frac{dQ}{dt}>0 \qquad \text{if} \quad (Q>0 \quad \textbf{and} \quad fP-Q>0)$$
$$\text{or if} \quad (Q<0 \quad \textbf{and} \quad fP-Q<0).$$

Since $Q\geq 0$ holds in the first quadrant, this means that

$$\frac{dQ}{dt} > 0 \text{ below the line } Q=fP,$$
$$\frac{dQ}{dt} < 0 \text{ above the line } Q=fP.$$

The phase diagram therefore looks like as shown below. We have also drawn into the diagram possible solution curves.



We conclude that the equilibrium point $(P,Q)=(0,0)$ is unstable, since motion along the P–axis is away from that point. (Note that since one of the original differential equations was

$$\frac{dP}{dt}=aP\left(\frac{b}{a}-P\right),$$

where the right hand side is, in fact, not defined when $Q=0$, we should not really consider the point $(0,0)$ at all, nor any values on the P–axis!) What about the other equilibrium point,

$$(P,Q)=\left(\sqrt{\frac{b}{f}},\sqrt{fb}\right)?$$

This point cannot be classified with absolute certainty, based just on the phase diagram. It does seem, however, that all solutions not starting on either the P or Q axis spiral towards

$$(P,Q)=\left(\sqrt{\frac{b}{f}},\sqrt{fb}\right)$$

so that this point seem to be an asymptotically stable equilibrium point.
Note that up to now, we have worked with general constants a,b,c,f rather than the given values – this also means that all our conclusions up to now have been valid in the more general case! With the given values

$$\begin{aligned} a &= c=1 \\ b &= 20\,000 \\ f &= 30, \end{aligned}$$

the equilibrium point becomes

$$(P,Q)=(25.8,\ 774.6)\,.$$

Assuming that we can conclude that

$$(P,Q)=(25.8,\ 774.6)$$

is indeed an asymptotically stable equilibrium point, we can make the following conclusions: According to this model, if initially the price of the product is zero $(P=0)$ then the quantity of the product available will drop to zero. If the price and quantity of the product are initially nonzero, there will follow a series of price increase followed by an increase in the quantity of the product (e.g. by increased production) followed by a drop in price followed by a drop in the quantity of the product (e.g. less production when prices are low). This sequence of events will repeat itself over and over again, but the price and quantity changes will get smaller and smaller over time, and the price and quantity will converge towards the equilibrium values $P=25.8$, $Q=774,6$.

11.4

$$\begin{aligned} \frac{dx}{dt} &= (2-2x-y)\,x \\ \frac{dy}{dt} &= (2-x-2y)\,y \end{aligned}$$

(a)If the y–species is not present, that is, if $y=0$ then the equation for the x–species becomes

$$\frac{dx}{dt}=(2-2x)\,x=2x-2x^2$$

which is the logistic model with $a=b=2$. In isolation, the x–species behaves according to the logistic model, which implies that as $t\to\infty$, for any initial value $x(0)>0$ the x–population converges to 1.
If the x–species is not present, then the y–species follows the equation

$$\frac{dy}{dt}=(2-2y)\,y=2y-2y^2,$$

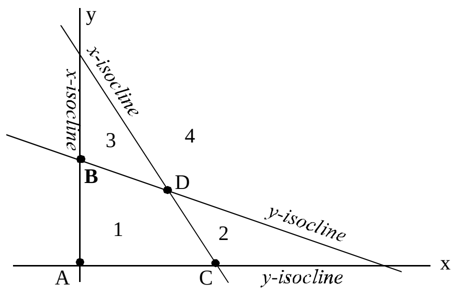
that is, the logistic model with $a=b=2$. Again, this means a limit population size of $y=1$.

(b)The x–isoclines are:

* $x=0$ (y–axis),
* $2-2x-y=0 \therefore y=-2x+2$ (a line with stop (-2), which intersects the y–axis at $y=2$ and x–axis at $x=1$)

The y–isoclines are:

* $y=0$ (x–axis)
* $2-x-2y=0 \therefore y=-\frac{1}{2}x+1$ (a line with slope $\left(-\frac{1}{2}\right)$, which intersects y–axis at $y=1$ and x–axis at $x=2$)



The equilibrium points lie at the intersections of an x–isocline and a y–isocline, so there are four of them:

$$\begin{aligned} A &= (0,0), \\ B &= (0,1), \\ C &= (1,0), \\ D &= \left(\frac{2}{3},\frac{2}{3}\right). \end{aligned}$$

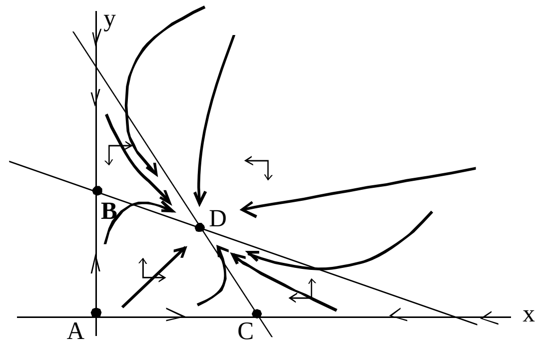
Note that $(0,2)$ and $(2,0)$ are **not** equilibrium points!

(c)The isoclines divide the first quadrant of the xy–place into four regions, $1,\ 2,\ 3$ and 4.

Below the x–isocline $y=-2x+2$,in regions 1 and 3 , we have $2-2x-y>0$ and therefore $dx/dt>0$. Above this x–isoclines, in regions 2 and 4, $dx/dt<0$.

Below the y–isocline $y=-\frac{1}{2}x+1$, in regions 1 and 2, we have $2-x-2y>0$ and therefore $dy/dt>0$; and above this isocline, in regions 3 and 4, we have $dy/dt<0$.

Marking these facts into the phase diagram, we get the following:



(d)We see that all solutions, except those starting on either x or y axis (which would mean that only one species is present), will converge towards the stable equilibrium point $D=\left(\frac{2}{3},\frac{2}{3}\right)$.

The exceptions are as follows: If both species are initially extinct, they will stay extend (these are solutions starting at, and staying at, the (unstable) equilibrium point A).

If only species x is initially present, then the system will converge towards the (unstable) equilibrium point C.

If only species y is initially present, then the system will converge towards the (unstable) equilibrium point B.

11.5 $\dfrac{dx}{dt}=(1-x-2y)\,x,\ \dfrac{dy}{dt}=(1-2x-y)\,y$

x**–isoclines and sign of** dx/dt

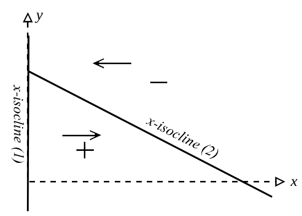
$dx/dt=0$ holds when $(1-x-2y)\cdot x=0$, that is, when either

$$x=0 \qquad (y\text{–axis}) \tag{1}$$

or

$$\begin{aligned} 1-x-2y&=0 \\ \therefore \quad y&=\frac{1}{2}(1-x) \end{aligned} \tag{2}$$

This is a line with slope $-\frac{1}{2}$, which intersects with the x–axis at $x=1$ and with the y–axis at $y=\frac{1}{2}$.



The diagram above shows the two x–isoclines. Since x and y are supposed to be the population sizes of two species, negative values make no sense and therefore we are only interested in the region $x \geq 0, y \geq 0$. The second isocline (2) divides this region with two areas. The sign of dx/dt is positive below line (2) and negative above it. For dx/dt, positive sign means motion towards the right and negative towards the left; we have indicated the direction of motion also with horizontal arrows.

y–isoclines and sign of dy/dt

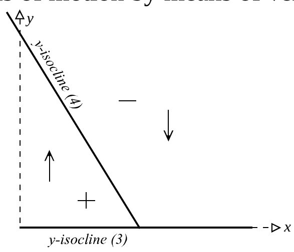
$dy/dt = 0$ holds when $(1 - 2x - y) \cdot y = 0$, that is, when either

$$y = 0 \qquad (x\text{–axis}) \tag{3}$$

or

$$\begin{aligned} & 1 - 2x - y = 0 \\ \therefore \quad & y = 1 - 2x \end{aligned} \tag{4}$$

The first y–isocline (3) is the x–axis, and the second one, (4), is a line with slope (-2) which intersects the x–axis at $x = \frac{1}{2}$ and the y–axis at $y = 1$. The diagram below shows the y–isoclines, the sign of dy/dt at the two regions in the first quadrant, and the corresponding directions of motion by means of vertical arrows.



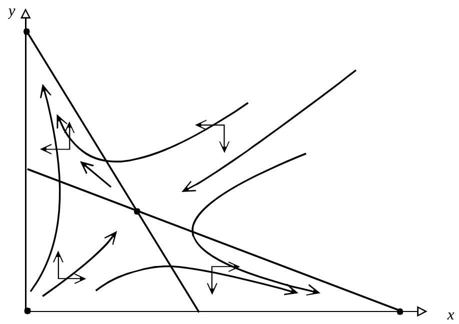
Equilibrium points

Equilibrium points are found where an x–isocline and a y–isocline intersect. Lines (1) and (3) intersect at the point $(0, 0)$; Lines (1) and (4) intersect at the point $(0, 1)$; Lines (2) and (3) intersect at $(1, 0)$; Lines (2) and (4) intersect at $\left(\frac{1}{3}, \frac{1}{3}\right)$.

These four points are the equilibrium points.

Phase diagram

When we combine the x– and y–isoclines and the directions of motion, we get the following diagram which contains all the information about the system.



We have drawn in possible solution curves. Note that the following rules must be obeyed:

–on each region, the direction arrows dictate the possible directions of motion. We are not allowed to contradict the allowed directions.

–the x–isoclines must be crossed vertically and the y–isoclines horizontally.

These rules ensure that the trajectories shown above are essentially the only possible ones. Apart from the "trivial" solutions which start on x–axis or y–axis, we see that there are 3 possible outcomes. Firstly, solutions starting close to the y–axis (i.e. if x–species is sufficiently much smaller than the y–species) will converge towards the equilibrium point $(0, 1)$, meaning that species x will die out while y converges towards its limiting value $y = 2$. Secondly, if y–population is sufficiently small compared to x, the opposite will happen: y–species will die out while x converges to 1, so that the solutions converge towards the equilibrium point $(1, 0)$.

The third possibility, is as follows: if the proportion of the x and y–species is just right, then we could end up on a trajectory which converges towards the equilibrium point $\left(\frac{1}{3}, \frac{1}{3}\right)$. If this does happen then the two species will co–exist (in all the other cases, at least one species will die out!) So, the co–existence of the two species is possible.

However, although co–existence is possible, it is not likely. There is really only one solution trajectory which leads to the equilibrium point $\left(\frac{1}{3}, \frac{1}{3}\right)$. Even the slightest deviation from this trajectory – or, correspondingly, even a slight change in the initial point (x_0, y_0) will cause the solution to end up instead of either $(1, 0)$ or $(0, 1)$ (the two stable equilibrium points).

11.6 $\dfrac{dx}{dt} = 2xy + 2x - x^2$, $\dfrac{dy}{dt} = xy - 2y$

(a) If y–species is not present (i.e. $y = 0$) then the differential equation for the x–species is

$$\frac{dx}{dt} = 2x - x^2,$$

which is the logistic model with $a = 2, b = 1$. So, in the absence of y, x behaves as a logistically growing species with limit population size $x_{\text{limit}} = a/b = 2$. If x–species is not present, the differential equation for the y–species will be

$$\frac{dy}{dt} = -2y$$

which is the Malthusian model with negative growth constant $k = -2$: Species y will die out if x is not present.

(b) To see how the two species interact, we need to investigate the sign of the "interaction" term xy in the differential equation for each species. The differential equation for the x–species,

$$\frac{dx}{dt} = 2xy + 2x - x^2$$

has the interaction term which a positive coefficient: $+2xy$. This means that interaction with the y–species increases the growth rate dx/dt of the x–species, so x–species benefits from the interaction with the y–species. What about the y–species, then? The differential equation of the y–species,

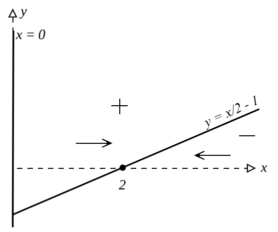
$$\frac{dy}{dt} = xy - 2y,$$

also has a positive coefficient of the interaction term: $+xy$. Therefore, the y–species also benefits from interaction with the x–species. The type of interaction is therefore one of mutual cooperation, with both species benefiting from the other. Note however that, as found out in (a), species x will survive without y, while y will die out if x is not present! So, y needs x while x just benefits from y. An example of this kind of situation could be one where x is the bee while y is a plant species which relies on bees for propagation. If there are plenty of other plants around to produce nectar for the bees then x benefits if there is lots of y–plants around, but if y is not there then the bees will survive with just the other plant species. On the other hand, if the bees are not present then the y–species will suffer and eventually die out.

(c) Phase diagram: x–isoclines and the sign of dx/dt : $dx/dt = 0$ when

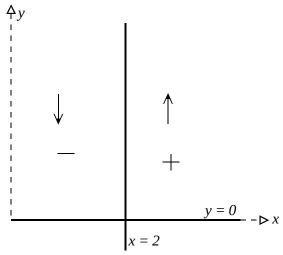
$$2xy + 2x - x^2 = 0 \therefore\ x(2y + 2 - x) = 0 \therefore\ x = 0 \quad \text{or} \quad y = \frac{1}{2}x - 1$$

The x–isoclines, and the sign of dx/dt as well as the corresponding directions of motion, are shown below. Only the first quadrant is of interest.

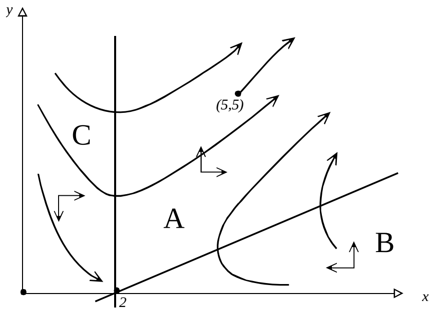


y–isoclines and the sign of dy/dt : $dy/dt = 0$ when

$$xy - 2y = 0 \therefore\ y(x - 2) = 0 \therefore\ y = 0 \quad \text{or} \quad x = 2$$



The y–isoclines and the directions of motion in the two different points of the first quadrant are shown above. Combining the information, we get the following phase diagram.



Note that there are 2 equilibrium points in the first quadrant: $(x, y) = (0, 0)$ and $(x, y) = (2, 0)$. If initially $(x_0, y_0) = (5, 5)$, we get a solution trajectory in which both x and y grow without bound. *[Indeed this is the eventual outcome for MOST non–trivial solution curves (non–trivial meaning ones where one or both of both species are not already extinct). If $x_0 > 2$ and $y > \frac{1}{2}x - 1$ (region A) both species will grow. If $x_0 > 0$ but $y_0 < \frac{1}{2}x - 1$ (region B), initially the x–species will get smaller. However, the y–species is growing in this region, which means that eventually the trajectory will reach and cross over the line $y = \frac{1}{2}x - 1$, moving to region A where both species will grow. For trajectories starting in region C $(x_0 < 2)$ species y will decrease. However, species x will grow, and most trajectories will reach and cross over (horizontally!) the y–isocline $x = 2$, after which both populations will grow without bound. The exception is some trajectories for which y_0 is too small for this to happen – for those trajectories, the solutions will converge towards the equilibrium point $(2, 0)$, meaning that y dies out and x converges to its limit value, $x = 2$. The division of which paths in region C move towards $(2, 0)$ and which cross over to region A is beyond this module, and will be taught in higher–level modules! To put it briefly, there is a very special solution trajectory called a separatrix which leads to the equilibrium point $(2, 0)$. All solution curves starting on one side of the separatrix will cross over to region A, while ones starting on the other side converge to $(2, 0)$.]*

11.7

$$\frac{dx}{dt} = 2x + xy - x^2, \quad \frac{dy}{dt} = 2y - xy$$

(a) If y is absent then the equation for x is

$$\frac{dx}{dt} = 2x - x^2,$$

which is the logistic model with $a = 2,\ b = 1$. Thus is y is not there, $x \to 2$ (unless initially $x = 0$ in which case x stays at 0.) If x is absent then the equation for y is

$$\frac{dy}{dt} = 2y,$$

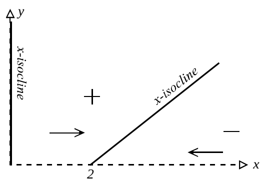
which is unlimited Malthusian growth (unless $y(0) = 0$ in which case y stays at 0.)

(b) The type of interaction is determined by the signs of the interaction term, xy, in both differential equations. In the equation for x (that is, the equation $dx/dt = ...$), the $xy-$term has a positive coefficient, which means that x-species benefits from each interaction. In the equation for y (that is, the equation $dy/dt = ...$) the xy–term has a negative coefficient, which means that y-species suffers from each interaction. This means that the interaction could be that of a predator–prey system, with x as predator and y as prey).

(c) x–isoclines and signs of dx/dt :

$$\frac{dx}{dt} = 2x + xy - x^2 = x(2 + y - x)$$

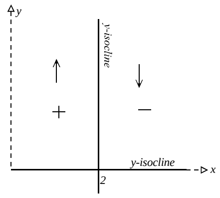
is zero when $x = 0$ or $y = x - 2$. These are the x–isoclines. The sign of dx/dt in various regions in the first quadrant, and the corresponding directions of motion, are shown below:



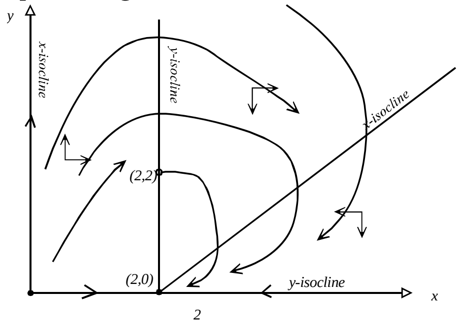
y–isoclines and signs of dy/dt :

$$\frac{dy}{dt} = 2y - xy = y(2 - x)$$

is zero when $y = 0$ or $x = 2$ (the y–isoclines). The signs of dy/dt in various regions of the first quadrant are shown below.



Combining this information, we get the phase diagram shown below.



The system has two equilibrium points $(0, 0)$ and $(2, 0)$.

If initially $(x_0, y_0) = (2, 2)$ then the solution moves initially towards right (since it is on a y-isocline and thus must move horizontally). It then moves down and right, until it crosses over the x–isocline $y = x - 2$ vertically. After that motion is left-and-down; and since the solution cannot cross over the y–isocline on the x–axis, it will have to converge towards the equilibrium point $(2, 0)$.

11.8 The model is given by

$$\frac{dx}{dt} = -4x - 2x^2 - xy \qquad (1)$$

$$\frac{dy}{dt} = y - y^2. \qquad (2)$$

We will go through the statements one by one.

(a) **This is a predator-prey system, where y is the prey and x is the predator.** This statement is a claim about the type of interaction between the two species, and to find whether it is true or not, we must consider the "interaction" terms between the two species, that is, the terms which have the expression xy in it, in the differential equations. Where this term has a negative coefficient in a differential equation for a species, let's say x (that is the differential equation which gives an expression for dx/dt), it means that species x loses out in each interaction. If the coefficient is positive, then species x gains from each interaction. Please note that if x loses out, it does not automatically follow that y benefits – there are many possible interactions, and for instance they could both benefit from all interactions! To find out how y fares in each interaction, we must then investigate the signs of the terms xy in the differential equation for y, that is, the differential equation which gives an expression for dy/dt. Now, what we would expect to see in a predator-prey system is that the predator should benefit, and the prey should lose out, in each interaction – that is, the coefficient of the term xy should be negative in the differential equation for the prey, and positive in the differential equation for the predator. In the model here, the coefficient of the interaction term is negative for the x–species (in equation (1)) so x loses out in the interactions (so x could not be a predator – if anything, it would be a prey animal). On the other hand, there is no interaction term at all for the y species: species x makes no difference to y, so y can be neither predator nor prey here. We conclude that this is not a predator-prey system, and the statement is FALSE.

(b) **Species y can survive without species x.** The differential equation (2) for the y–species does not include an x term anywhere in it, so the behaviour of y–species does not depend on the x–species at all. Further, we can recognise the differential equation (2) as being the one for logistic growth: Species y follows the logistic equation with parameters $a = 1$ and $b = 1$. The outcome for species y is therefore as follows: as long as it does not actually start off already extinct, it will survive, and in the long run the size of y converges towards the limit value $a/b = 1$. *[Note that rather than y being the number of individuals, it could also denote the size of a population measured in thousands!]* In particular, y can certainly can survive without species x, and the statement is TRUE.

(c) **Species x can survive without species y.** To answer this, let us see how the x– species behaves without y, that is, if $y = 0$. If we put $y = 0$ in equation (1), it becomes

$$\frac{dx}{dt} = -4x - 2x^2.$$

Note that this is neither the Malthusian nor the logistic model! However, if x is positive then clearly the right hand side is always negative, so that the population size will always decrease towards zero. (Zero is an equilibrium point here; the other equilibrium point is -2.) In particular, species x cannot survive without species y. (It cannot survive with y either since the coefficient of xy is negative in (1): species x will die out even faster if y is present!) The statement is FALSE.

(d) **The two species are competing for the same resources.** Again we look at the interaction terms: If the species were competing for the same resources, then both species should always lose out in any interactions, and therefore the coefficient of xy should be negative in both differential equations. That is not the case here: the coefficient of xy is indeed negative in the differential equation for x, but there is no interaction term in the differential equation y (its coefficient is zero rather than negative). So, we conclude that the two species do not compete for the same resources, and the statement is FALSE.

(e) **If the initial populations are large enough, both populations can grow without bound.** We have already stated that whatever happens with species x, species y will behave according to the logistic model so it will never grow without bound; while on the other hand x will always die out. So this statement is FALSE.

(f) **Both populations will always die out.** It is true that the x–species will always die out. On the other hand, y will not die out since it behaves according to the logistic model, and will converge towards the limit value 1. So this statement is FALSE.

Note that we were able to answer all the questions just by looking at the differential equations. The phase diagram looks as shown below, and you will see that the outcomes you can read from there do agree with our answers to (e) and (f). Also answers to (b) and (c) follow by checking on the behaviour of the solutions on the positive x and y axes. *[Note that we have only completed the first quadrant of the xy–plane since that is the only region of interest to us!]*