Tutorial letter 101/3/2017

LINEAR ALGEBRA

MAT2611

Semesters 1 & 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains important information about your module.

BARCODE

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CONTENTS

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1 INTRODUCTION

Dear Student

Welcome to the MAT2611 module in the Department of Mathematical Sciences at Unisa. We trust that you will find this module both interesting and rewarding.

Some of this tutorial matter may not be available when you register. Tutorial matter that is not available when you register will be posted to you as soon as possible, but is also available on *my*Unisa.

***my*Unisa**

You must be registered on *my*Unisa (`http://my.unisa.ac.za`) to be able to submit assignments online, gain access to the library functions and various learning resources, download study material, "chat" to your lecturers and fellow students about your studies and the challenges you encounter, and participate in online discussion forums. *my*Unisa provides additional opportunities to take part in activities and discussions of relevance to your module topics, assignments, marks and examinations.

Tutorial matter

A tutorial letter is our way of communicating with you about teaching, learning and assessment. You will receive a number of tutorial letters during the course of the module. This particular tutorial letter contains important information about the scheme of work, resources and assignments for this module as well as the admission requirements for the examination. We urge you to read this and subsequent tutorial letters carefully and to keep it at hand when working through the study material, preparing and submitting the assignments, preparing for the examination and addressing queries that you may have about the course (course content, textbook, worked examples and exercises, theorems and their applications in your assignments, tutorial and textbook problems, etc.) to your MAT2611 lecturers.

2 PURPOSE AND OUTCOMES FOR THE MODULE

2.1 Purpose

This module is a direct continuation of MAT1503. It will be useful to students interested in developing their Linear Algebra techniques and skills in solving problems in the mathematical sciences.

2.2 Outcomes

To understand, compute and apply the following linear algebra concepts:

2.2.1 Vector spaces
(Anton & Rorres, sections 4.1 – 4.5), (Lay, sections 4.1 – 4.5).

2.2.2 Rank of a matrix
(Anton & Rorres, sections 4.7 – 4.8), (Lay, section 4.6).

2.2.3 Change of basis
(Anton & Rorres, section 4.6), (Lay, section 4.7).

2.2.4 Eigenvalues and eigenvectors
(Anton & Rorres, section 5.1), (Lay, sections 5.1 – 5.2).

2.2.5 Diagonalisation of matrices
(Anton & Rorres, section 5.2), (Lay, section 5.3).

2.2.6 Inner products and orthogonality
(Anton & Rorres, sections 6.1 – 6.2), (Lay, sections 6.1 – 6.3).

2.2.7 Gram-Schmidt algorithm
(Anton & Rorres, section 6.3), (Lay, section 6.4).

2.2.8 Orthogonal diagonalisation of symmetric matrices
(Anton & Rorres, sections 7.1 – 7.2), (Lay, section 7.1).

2.2.9 Linear transformations
(Anton & Rorres, chapter 8), (Lay, section 4.2 and section 5.4).

3 LECTURER(S) AND CONTACT DETAILS

3.1 Lecturer(s)

The contact details for the lecturer responsible for this module is

Postal address: The MAT2611 Lecturers
Department of Mathematical Sciences
Private Bag X6
Florida
1709
South Africa

Additional contact details for the module lecturers will be provided in a subsequent tutorial letter.

All queries that are not of a purely administrative nature but are about the content of this module should be directed to your lecturer(s). Tutorial letter 301 will provide additional contact details for your lecturer. Please have your study material with you when you contact your lecturer by telephone. If you are unable to reach us, leave a message with the departmental secretary. Provide your name, the time of the telephone call and contact details. If you have problems with questions that you are unable to solve, please send your own attempts so that the lecturers can determine where the fault lies.

Please note: Letters to lecturers may not be enclosed with or inserted into assignments.

3.2 Department

The contact details for the Department of Mathematical Sciences are:

Departmental Secretary: (011) 670 9147 (SA) +27 11 670 9147 (International)

3.3 University

If you need to contact the University about matters not related to the content of this module, please consult the publication *Study @ Unisa* that you received with your study material. This booklet contains information on how to contact the University (e.g. to whom you can write for different queries, important telephone and fax numbers, addresses and details of the times certain facilities are open). Always have your student number at hand when you contact the University.

4 RESOURCES

4.1 Prescribed books

Prescribed books can be obtained from the University's official booksellers. If you have difficulty locating your book(s) at these booksellers, please contact the Prescribed Books Section at (012) 429 4152 or e-mail vospresc@unisa.ac.za.

Your prescribed textbook for this module is:

Title:	Elementary Linear Algebra with Supplemental Applications
Authors:	Howard Anton and Chris Rorres
Edition:	11th Edition, International Student Version
Publishers:	Wiley
ISBN:	978-1-118-67745-2

However, you may wish to use your copy of

Title:	Linear Algebra and Its Applications
Author:	David C. Lay
Edition:	Pearson New International Edition, 4th edition
Publishers:	Pearson
ISBN:	9781292020556

Students with the textbook by Lay will be accommodated.

Please buy the textbook as soon as possible since you have to study from it directly – you cannot do this module without the prescribed textbook.

4.2 Recommended books

The book "Linear Algebra" by Jim Hefferon is available for free from

`http://joshua.smcvt.edu/linearalgebra/`

with answers to exercises available from the same web site. The concepts are arranged differently to the prescribed book. The relevant chapters and sections are: chapter 2, chapter 3 I-III and V. Some of the terminology is different to the prescribed book.

The book “A First Course in Linear Algebra” by Robert A. Beezer is a free and interactive online book available at

http://linear.ups.edu/

and also has multiple PDF versions available for download. The relevant chapters are “Vectors”, “Matrices” - “Column and Row Spaces”, “Vector Spaces”, “ Eigenvalues”, “Linear Transformations” and “Representations”. Please note that this book assumes that vector spaces are over the field of complex numbers, while the prescribed text book considers only the real numbers.

Finally, the “Book of Proof” (Second Edition) by Richard Hammack, Part I, Chapter 1 (Sets) is recommended for students who need to revise basic set theory and notation. The entire book is available for free from

http://www.people.vcu.edu/~rhammack/BookOfProof/index.html

4.3 Electronic reserves (e-Reserves)

There are no e-Reserves for this module.

4.4 Library services and resources information

For brief information go to:
http://www.unisa.ac.za/brochures/studies

For more detailed information, go to the Unisa website: http://www.unisa.ac.za/, click on **Library**. For research support and services of Personal Librarians, go to:
http://www.unisa.ac.za/Default.asp?Cmd=ViewContent&ContentID=7102

The Library has compiled numerous library guides:

- find recommended reading in the print collection and e-reserves
  - http://libguides.unisa.ac.za/request/undergrad
- request material
  - http://libguides.unisa.ac.za/request/request
- postgraduate information services
  - http://libguides.unisa.ac.za/request/postgrad
- finding , obtaining and using library resources and tools to assist in doing research
  - http://libguides.unisa.ac.za/Research_Skills
- how to contact the Library/find us on social media/frequently asked questions
  - http://libguides.unisa.ac.za/ask

5 STUDENT SUPPORT SERVICES

For information on the various student support services available at Unisa (e.g. student counseling, tutorial classes, language support), please consult the publication *Study @ Unisa* that you received with your study material.

6 STUDY PLAN

The following table provides an outline of the outcomes and ideal dates of completion, and other study activities.

	Semester 1	Semester 2
Outcomes 2.2.1–2.2.3 to be achieved by	17 February 2017	3 August 2017
Outcomes 2.2.3–2.2.5 to be achieved by	16 March 2017	31 August 2017
Outcomes 2.2.6–2.2.9 to be achieved by	13 April 2017	28 September 2017

See the brochure *Study @ Unisa* for general time management and planning skills.

7 PRACTICAL WORK AND WORK INTEGRATED LEARNING

There are no practicals for this module.

8 ASSESSMENT

8.1 Assessment criteria

Specific outcome 1: Understand and apply the definition of a general real vector space, along with the concepts of subspace, linear independence, basis and dimension, row space column space and null space, rank and nullity.

Assessment criteria

You must be able to do the following.

- Decide, with reasons, whether a given set with two given operations defines a vector space.
- Decide, with reasons, whether a given subset of a vector space defines a subspace.
- Find the span of a given set of vectors. Show that a given set of vectors do/do not span a given space, with reasons.
- Test a given set of vectors for linear dependence/independence.
- Find a basis for a given vector space. Find a basis for the span of a given set of vectors. Determine whether or not a given set of vectors forms a basis for a given vector space.
- Find for a given matrix the row space/column space and null space.

- Find, for a given linear system, the general solution.
- Find the rank and nullity of a given matrix.

Specific outcome 2: Understand and be able to apply the basis concepts of inner product spaces.

Assessment criteria

You must be able to do the following.

- Calculate inner products in cases other than the dot product.
- Use the length, angle and distance formulas for arbitrary inner products.
- Test vectors for orthogonality.
- Find orthogonal complements of subspaces.
- Test sets of vectors for orthogonality/orthonormality.
- Use the Gram-Schmidt process to change a basis to an orthogonal/orthonormal one.
- Find the transition matrix between two different bases.
- Find the coordinate vector of a vector with respect to a new basis.
- Decide whether or not a matrix is orthogonal.

Specific outcome 3: Understand and be able to apply the basis concepts of eigenvalues and eigenvectors.

Assessment criteria

You must be able to do the following.

- Test whether a given scalar/vector pair is an eigenvalue–eigenvector pair of a matrix.
- Find the eigenvalues and eigenvectors of a matrix.
- Determine whether or not a given matrix is diagonalisable, giving reasons.
- Find, for a diagonalisable matrix, a diagonalising matrix.
- Determine whether or not a given matrix is orthogonally diagonalisable, giving reasons.
- Find, for an orthogonally diagonalisable matrix, an orthogonal diagonalising matrix.

Specific outcome 4: Understand and be able to apply the concept of linear transformation.

Assessment criteria

The student must be able to:

- Decide, with reasons, whether a given operation on vector space is a linear transformation or not.
- Find the kernel and range of a linear transformation.
- Find the rank and nullity of a linear transformation.
- Determine whether a given linear transformation is one-to-one and/or onto.
- Find, in those cases where is is possible, the inverse of a linear transformation.
- Find the matrix of a linear transformation with respect to a given basis.
- Find the matrices of compositions of transformations and inverse transformations with respect to a given basis.
- Find the matrix of a linear transformation with respect to a basis, given the matrix with respect to a different basis.
- Find the eigenvalues of a linear operator.
- Decide if a given linear transformation is an isomorphism or not, with reasons.

8.2 Assessment plan

A final mark of at least 50% is required to pass the module. If a student does not pass the module then a final mark of at least 40% is required to permit the student access to the supplementary examination. The final mark is composed as follows:

Year mark	
Assignment 01:	30%
Assignment 02:	40%
Assignment 03:	30%

$\longrightarrow$

Final mark	
Year mark:	20%
Exam mark:	80%

8.3 Assignment numbers

8.3.1 General assignment numbers

The assignments for this module are Assignment 01, Assignment 02, etc.

8.3.2 Unique assignment numbers

Please note that each assignment has a unique assignment number which must be written on the cover of your assignment.

8.3.3 Assignment due dates

The dates for the submission of the assignments are:

Semester 1

Assignment 01: Friday, 24 February 2017
Assignment 02: Thursday, 23 March 2017
Assignment 03: Thursday, 20 April 2017

Semester 2

Assignment 01: Thursday, 10 August 2017
Assignment 02: Thursday, 7 September 2017
Assignment 03: Thursday, 5 October 2017

8.4 Submission of assignments

You may submit written assignments either by post or electronically via *my*Unisa. Assignments may **not** be submitted by fax or e-mail.

For detailed information on assignments, please refer to the *Study @ Unisa* brochure which you received with your study package.

Please make a copy of your assignment before you submit!

To submit an assignment via *my*Unisa:

- Go to *my*Unisa.
- Log in with your student number and password.
- Select the module.
- Click on "Assignments" in the menu on the left-hand side of the screen.
- Click on the assignment number you wish to submit.
- Follow the instructions.

8.5 The assignments

Please make sure that you submit the correct assignments for the 1st semester, 2nd semester or year module for which you have registered. For each assignment there is a **fixed closing date**, the date at which the assignment must reach the University. When appropriate, solutions for each assignment will be dispatched, as Tutorial Letter 201 (solutions to Assignment 01) and Tutorial Letter 202 (solutions to Assignment 02) etc., a few days after the closing date. They will also be made available on *my*Unisa. Late assignments **will not** be marked!

Note that at least one assignment must reach us before the due date in order to gain admission to the examination.

8.6 Other assessment methods

There are no other assessment methods for this module.

8.7 The examination

During the relevant semester, the Examination Section will provide you with information regarding the examination in general, examination venues, examination dates and examination times. For general information and requirements as far as examinations are concerned, see the brochure *Study @ Unisa*.

Registered for . . .	Examination period	Supplementary examination period
1st semester module	May/June 2017	October/November 2017
2nd semester module	October/November 2017	May/June 2018
Year module	October/November 2017	January/February 2018

9 FREQUENTLY ASKED QUESTIONS

The *Study @ Unisa* brochure contains an A–Z guide of the most relevant study information.

10 IN CLOSING

We hope that you will enjoy MAT2611 and we wish you all the best in your studies at Unisa!

ADDENDUM A: ASSIGNMENTS – FIRST SEMESTER

ASSIGNMENT 01
Due date: Friday, 24 February 2017
UNIQUE ASSIGNMENT NUMBER: 739590

ONLY FOR SEMESTER 1

This assignment is a multiple choice assignment. Please consult the *Study @ Unisa* brochure for information on how to submit your answers for multiple choice assignments.

Question 1

Consider the set

$$X := \left\{ \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \, : \, a \in \mathbb{R} \right\} \subset M_{22}$$

and the operations $\left(\text{for all } k, a, b \in \mathbb{R},\ \mathbf{u} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in X \text{ and } \mathbf{v} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in X\right)$

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{u} \equiv k \cdot \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} := \begin{bmatrix} ka & 1 \\ 0 & -ka \end{bmatrix},$$

$$+ : X \times X \to X, \qquad \mathbf{u} + \mathbf{v} \equiv \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} b & 1 \\ 0 & -b \end{bmatrix} := \begin{bmatrix} a+b & 1 \\ 0 & -(a+b) \end{bmatrix}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. The zero vector for X is

1. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
2. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$
3. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$
4. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$
5. None of the above.

Question 2

Which of the following are subspaces of P_2 with the usual operations ?

A. $\operatorname{span}\{1,\ x^2\}$

B. $\{a + ax \,:\, a \in \mathbb{R}\}$

C. $\{\, a + \frac{1}{b}x \,:\, a, b \in \mathbb{R},\, b \neq 0 \,\}$

D. $\{\, ax^3 \,:\, a \in \mathbb{R} \,\}$

Select from the following:

1. All of A, B, C and D.
2. Only A, B, and D.
3. Only A and B.
4. Only B and D.
5. None of the above.

Question 3

Which of the following sets are linearly independent?

A. $\{\,(1, 0),\ (1, 1),\ (1, -1)\,\}$ in $\mathbb{R}^2$

B. $\{\,(1, 1, 1),\ (1, -1, 1),\ (-1, 1, 1)\,\}$ in $\mathbb{R}^3$

C. $\{\,1 + x,\ x,\ 2 + 3x\,\}$ in P_2

D. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ in M_{22}

Select from the following:

1. Only A and C.
2. Only B.
3. Only D.
4. Only B and D.
5. None of the above.

Question 4

Which of the following sets are a basis for the following vector subspace of $\mathbb{R}^3$?

$$X = \left\{ \mathbf{x} \in \mathbb{R}^3 \,:\, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{x} \right\}.$$

A. $\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$ B. $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ C. $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$ D. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

Select from the following:

1. Only A and B.
2. Only B and C.
3. Only C and D.
4. Only A and D.
5. None of the above.

Question 5

Which of the following statements are true:

A. $\dim(\operatorname{span}\{\,(1,0,1),(1,0,-1)\,\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\operatorname{span}\{\,(1,0,0),(-1,0,0)\,\}) = 2$ in $\mathbb{R}^3$

C. $\dim(\operatorname{span}\{\,(1,1,1),(1,1,-1),(1,-1,1),(-1,1,1)\,\}) = 4$ in $\mathbb{R}^3$

Select from the following:

1. All of A, B and C.
2. Only A and B.
3. Only A and C.
4. Only A.
5. None of the above.

Question 6

Which of the following sets are a basis for the row space of $\begin{bmatrix} 1 & 3 \\ 1 & 1 \\ 3 & 1 \end{bmatrix}$?

A. $\{\,\begin{bmatrix}1 & 3\end{bmatrix}, \begin{bmatrix}1 & 1\end{bmatrix}, \begin{bmatrix}3 & 1\end{bmatrix}\,\}$

B. $\{\,\begin{bmatrix}1 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\end{bmatrix}\,\}$

C. $\{\,\begin{bmatrix}1 & -1\end{bmatrix}, \begin{bmatrix}1 & 1\end{bmatrix}\,\}$

D. $\{\,\begin{bmatrix}1 & 2\end{bmatrix}, \begin{bmatrix}2 & 1\end{bmatrix}\,\}$

Select from the following:

1. Only A.
2. Only B, C and D.
3. Only B and C.
4. All of A, B, C and D.
5. None of the above.

Question 7

Which of the following sets are a basis for the null space of $\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix} 2 & 0 & 0 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^T \right\}$

D. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & -1 \end{bmatrix}^T \right\}$

Select from the following:

1. Only B.
2. Only D.
3. Only B and C.
4. Only A.
5. None of the above.

Question 8

Which of the following statements are always true for for all $m, n \in \mathbb{N}$ and $m \times n$ matrices A ?

A. $\text{rank}(A) = \text{rank}(A^T)$

B. $\text{rank}(A^T) + \text{nullity}(A^T) = m$

C. $\text{rank}(A^T) + \text{nullity}(A^T) = n$

D. $\text{row space}(A) = \text{column space}(A)$

Select from the following:

1. Only A and C.
2. Only A and B.
3. Only B.
4. Only C.
5. None of the above.

– End of assignment –

ASSIGNMENT 02
Due date: Thursday, 23 March 2017
Total Marks: 40
UNIQUE ASSIGNMENT NUMBER: 701424

ONLY FOR SEMESTER 1

Answer all the questions. Show all your workings.

If you choose to submit via *my*Unisa, note that only PDF files will be accepted.

Question 1: 20 Marks

Let

$$B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right\} \quad \text{and} \quad B_2 = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

be two bases for span(B_1), where the usual left to right ordering is assumed.

(1.1) Find the transition matrix (change of coordinate/change of basis matrix) $P_{B_1 \to B_2}$. (8)

(1.2) Let B_3 be a basis for $\mathbb{R}^3$ and let the transition matrix from B_2 to B_3 be given by

$$P_{B_2 \to B_3} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) Find the transition matrix $P_{B_1 \to B_3}$. (6)

(b) Use $P_{B_2 \to B_3}$ to find B_3. (6)

Question 2: 20 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(2.1) Determine the characteristic equation for A in λ. (4)

(2.2) Find the eigenvalues of A, and their algebraic multiplicities. (4)

(2.3) Find a basis for the eigenspace corresponding to each eigenvalue of A and hence also the geometric multiplicity of each eigenvalue. (12)

– End of assignment –

ASSIGNMENT 03
Due date: Thursday, 20 April 2017
UNIQUE ASSIGNMENT NUMBER: 658165

ONLY FOR SEMESTER 1

This assignment is a multiple choice assignment. Please consult the *Study @ Unisa* brochure for information on how to submit your answers for multiple choice assignments.

Question 1

Let A be an $n \times n$ matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The equation $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$ has the *unique* solution $\mathbf{x} = \mathbf{0}$ if and only if

1. $\lambda = 0$.
2. λ is not an eigenvalue of A.
3. $\lambda = 0$ and 0 is an eigenvalue of A.
4. A is invertible.
5. None of the above.

Question 2

Let A be an $n \times n$ matrix with eigenvalue -1, I_n be the $n \times n$ identity matrix and 0_n be the $n \times n$ zero matrix. Which of the following are true?

A. $(-1)^k$ is an eigenvalue of A^k for all $k \in \mathbb{N}$.

B. $I_n + A$ is singular.

C. $I_n + A = 0_n$.

D. If $\mathbf{x} \in \mathbb{R}^n$ such that $A\mathbf{x} = -\mathbf{x}$, then $\mathbf{x} = 0$.

Select from the following:

1. Only A, B and C.
2. Only A and B.
3. Only A and D.
4. Only B, C and D.
5. None of the above.

Question 3

Which of the following matrices are diagonalizable?

A. $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. B. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}$.

Select from the following:

1. Only A.
2. Only B.
3. Only B and C.
4. Only B, C and D.
5. None of the above.

Question 4

Let A and B be $n \times n$ matrices and let I_n be the $n \times n$ identity matrix. Then

1. $AB + B^T A^T$ is diagonalizable.
2. If A is invertible then A is diagonalizable.
3. If A and B are diagonalizable then $A + B$ is diagonalizable.
4. If $\lambda = 0$ is and eigenvalue of A, then A is not diagonalizable.
5. None of the above.

Question 5

Which one of the following defines an inner product?

1. $\langle A, B \rangle = \operatorname{tr}\left(\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} AB^T \right)$ in M_{22}.
2. $\langle a_1 + b_1 x + c_1 x^2, a_2 + b_2 x + c_2 x^2 \rangle = a_1 b_1 + a_2 b_2$ in P_2.
3. $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + 2x_2 y_2$ in $\mathbb{R}_2$.
4. $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1 y_1 + 2x_2 y_2 - 1$ in $\mathbb{R}_2$.
5. None of the above.

Question 6

Which of the following vectors are unit vectors with respect to the inner product $\langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle = 2x_1y_1 + 2x_2y_2 + 2x_3y_3$ in $\mathbb{R}^3$?

A. $(1, 0, 0)$ B. $(1, 0, 0)/\sqrt{2}$ C. $(1, 0, 1)/\sqrt{2}$ D. $(1, 1, 0)/2$

Select from the following:

1. Only A.
2. Only B and D.
3. Only A and C.
4. All of A, B, C and D.
5. None of the above.

Question 7

Which of the following vectors are orthogonal to each other with respect to the inner product $\langle A, B \rangle = \text{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} A^T B \right)$ in M_{22} ?

A. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. B. $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. All of A, B, C and D are orthogonal to each other.
2. None of A, B, C and D are orthogonal to each other.
3. Only A and B are orthogonal, A and C are orthogonal, B and C are orthogonal.
4. Only A and C are orthogonal, B and C are orthogonal.
5. None of the above.

Question 8

Consider the vector subspace $W = \text{span}\{\, 1 - x,\ 2x^2 \,\}$ of P_2 with the *evaluation inner product* at 0, 1 and -1 (sample points). Which of the following vectors in P_2 lie in the subspace $W^{\perp}$ of P_2?

1. $x^2 - 1$.
2. $x^2 + x + 1$.
3. x.
4. $-2x^2 + x + 2$.
5. None of the above.

– End of assignment –

ADDENDUM B: ASSIGNMENTS – SECOND SEMESTER

ASSIGNMENT 01
Due date: Thursday, 10 August 2017
UNIQUE ASSIGNMENT NUMBER: 707762

ONLY FOR SEMESTER 2

This assignment is a multiple choice assignment. Please consult the *Study @ Unisa* brochure for information on how to submit your answers for multiple choice assignments.

Question 1

Consider the set

$$X := \left\{ \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} : a \in \mathbb{R} \right\} \subset M_{22}$$

and the operations $\left(\text{for all } k, a, b \in \mathbb{R},\ \mathbf{u} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in X \text{ and } \mathbf{v} = \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} \in X\right)$

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{u} \equiv k \cdot \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} := \begin{bmatrix} ka & 1 \\ 0 & -ka \end{bmatrix},$$

$$+ : X \times X \to X, \qquad \mathbf{u} + \mathbf{v} \equiv \begin{bmatrix} a & 1 \\ 0 & -a \end{bmatrix} + \begin{bmatrix} b & 1 \\ 0 & -b \end{bmatrix} := \begin{bmatrix} a+b & 1 \\ 0 & -(a+b) \end{bmatrix}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements are true in this vector space?

1. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}$
2. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$
3. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$
4. $-\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$
5. None of the above.

Question 2

Which of the following are subspaces of P_2 with the usual operations ?

A. span $\{1,\ x^2\}$

B. $\{\, 1 + ax \;:\; a \in \mathbb{R} \,\}$

C. $\{\, a - bx^2 \;:\; a, b \in \mathbb{R} \,\}$

D. $\{\, a \;:\; a \in \mathbb{R},\, a \geq 0 \,\}$

Select from the following:

1. Only A, B and C.
2. Only A, C and D.
3. Only C and D.
4. Only A and C.
5. None of the above.

Question 3

Which of the following sets are linearly independent?

A. $\{\, (1,0),\ (1,1),\ (1,-1) \,\}$ in $\mathbb{R}^2$

B. $\{\, (1,1,1),\ (1,-1,1),\ (2,-3,2) \,\}$ in $\mathbb{R}^3$

C. $\{\, 1+x,\ x,\ 2+3x \,\}$ in P_2

D. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \right\}$ in M_{22}

Select from the following:

1. Only A, B and C.
2. Only B and C.
3. Only B and D.
4. Only D.
5. None of the above.

Question 4

Which of the following sets are a basis for the following vector subspace of M_{22}:

$$X = \left\{ A \in M_{22} \;:\; A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

A. $\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$

Select from the following:

1. Only A and B.
2. Only B and C.
3. Only C and D.
4. Only A and D.
5. None of the above.

Question 5

Which of the following statements are true:

A. $\dim(\text{span}\,\{\,1 + x^2, 1 - x^2\,\}) = 2$ in P_2

B. $\dim(\text{span}\,\{\,x^2, -x^2\,\}) = 2$ in P_2

C. $\dim(\text{span}\,\{\,1 + x + x^2, 1 + x - x^2, 1 - x + x^2, -1 + x + x^2\,\}) = 4$ in P_2

Select from the following:

1. All of A, B, and C.
2. Only A and C.
3. Only A and B.
4. Only A.
5. None of the above.

Question 6

Which of the following sets are a basis for the column space of $\begin{bmatrix} 1 & 1 & 3 \\ 3 & 1 & 1 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$

Select from the following:

1. All of A, B, C and D.
2. Only B, C and D.
3. Only A.
4. Only B and C.
5. None of the above.

Question 7

Which of the following sets are a basis for the null space of $\begin{bmatrix}1 & 1 & -1\\0 & -1 & 1\end{bmatrix}$?

A. $\left\{ \begin{bmatrix}0 & 1 & 1\end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix}0 & 1 & 1\end{bmatrix}^T, \begin{bmatrix}2 & -1 & 1\end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix}1 & 1 & -1\end{bmatrix}^T, \begin{bmatrix}0 & -1 & 1\end{bmatrix}^T \right\}$

D. $\left\{ \begin{bmatrix}1 & 0\end{bmatrix}^T, \begin{bmatrix}1 & -1\end{bmatrix}^T \right\}$

Select from the following:

1. Only A.
2. Only C.
3. Only B.
4. Only A.
5. None of the above.

Question 8

Which of the following statements are always true for for all $m, n \in \mathbb{N}$ and $m \times n$ matrices A ?

A. $\text{rank}(A) = \text{rank}(A^T)$

B. $\text{rank}(A^T) + \text{nullity}(A^T) = m$

C. $\text{rank}(A^T) + \text{nullity}(A^T) = n$

D. row space$(A) =$ column space(A)

Select from the following:

1. Only A and B.
2. Only A and C.
3. Only C.
4. Only A.
5. None of the above.

– End of assignment –

ASSIGNMENT 02
Due date: Thursday, 7 September 2017
Total Marks: 40
UNIQUE ASSIGNMENT NUMBER: 804857

ONLY FOR SEMESTER 2

Answer all the questions. Show all your workings.

If you choose to submit via *my*Unisa, note that only PDF files will be accepted.

Question 1: 20 Marks

Let

$$B_1 = \left\{ \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \quad \text{and} \quad B_2 = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

be two bases for span(B_1) in M_{22}, where the usual left to right ordering is assumed.

(1.1) Find the transition matrix (change of coordinate/change of basis matrix) $P_{B_1 \to B_2}$. (8)

(1.2) Let B_3 be a basis for span(B_1) and let the transition matrix from B_2 to B_3 be given by

$$P_{B_2 \to B_3} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(a) Find the transition matrix $P_{B_1 \to B_3}$. (6)

(b) Use $P_{B_2 \to B_3}$ to find B_3. (6)

Question 2: 20 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}.$$

(2.1) Determine the characteristic equation for A in λ. (4)

(2.2) Find the eigenvalues of A, and their algebraic multiplicities. (4)

(2.3) Find a basis for the eigenspace corresponding to each eigenvalue of A and hence also the geometric multiplicity of each eigenvalue. (12)

– End of assignment –

ASSIGNMENT 03
Due date: Thursday, 5 October 2017
UNIQUE ASSIGNMENT NUMBER: 800799

ONLY FOR SEMESTER 2

This assignment is a multiple choice assignment. Please consult the *Study @ Unisa* brochure for information on how to submit your answers for multiple choice assignments.

Question 1

Let A be an $n \times n$ matrix, $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The equation $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$ has the *unique* solution $\mathbf{x} = \mathbf{0}$ if and only if

1. λ is not an eigenvalue of A.
2. $\lambda = 0$.
3. $\lambda = 0$ and 0 is an eigenvalue of A.
4. A is invertible.
5. None of the above.

Question 2

Let A be an $n \times n$ matrix with eigenvalue -1, I_n be the $n \times n$ identity matrix and 0_n be the $n \times n$ zero matrix. Which of the following are true?

A. 0 is an eigenvalue of $A + I_n$.

B. $A + I_n$ is singular.

C. $A + I_n = 0_n$.

D. 1 is an eigenvalue of A^2.

Select from the following:

1. Only A, B and D.
2. Only A, B and C.
3. Only A, C and D.
4. All of A, B, C and D.
5. None of the above.

Question 3

Which of the following matrices are diagonalizable?

A. $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$. B. $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. D. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only A, C and D.
2. Only A.
3. Only A and B.
4. Only C and D.
5. None of the above.

Question 4

Let A and B be $n \times n$ matrices and let I_n be the $n \times n$ identity matrix. Then

1. If ABB^TA^T is diagonalizable.
2. If A is diagonalizable then A is invertible.
3. If $\lambda = 0$ is an eigenvalue of A, then A is not diagonalizable.
4. If A and B are not diagonalizable then $A + B$ is not diagonalizable.
5. None of the above.

Question 5

Which one of the following defines an inner product?

1. $\langle p(x), q(x) \rangle = p(1)q(1) + 2p(2)q(2) + 3p(3)q(3)$ in P_2.
2. $\langle A, B \rangle = \operatorname{tr}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} AB^T \right)$ in M_{22}.
3. $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_2 + x_2y_1$ in $\mathbb{R}_2$.
4. $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 + x_2y_2 - 1$ in $\mathbb{R}_2$.
5. None of the above.

Question 6

Which of the following vectors are unit vectors with respect to the inner product $\langle (x_1, x_2, x_3), (y_1, y_2.y_3) \rangle = 2x_1y_1 + 2x_2y_2 + x_3y_3$ in $\mathbb{R}^3$?

A. $(1, 0, 0)$ B. $(0, 1, 0)/\sqrt{2}$ C. $(1, 1, 1)/\sqrt{3}$ D. $(1, 1, 0)/2$

Select from the following:

1. Only B and D.
2. Only A, C and D.
3. Only A and C.
4. Only A.
5. None of the above.

Question 7

Which of the following vectors are orthogonal to each other with respect to the inner product $\langle A, B \rangle = \mathrm{tr}\left(\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} A^T B\right)$ in M_{22} ?

A. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. B. $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

Select from the following:

1. Only A and C are orthogonal, A and D are orthogonal, C and D are orthogonal.
2. Only B and C are orthogonal.
3. Only A and C are orthogonal, B and D are orthogonal.
4. Only A and C are orthogonal, A and D are orthogonal.
5. None of the above.

Question 8

Consider the vector subspace $W = \mathrm{span}\{\, 1 - x,\, 2x^2 \,\}$ of P_2 with the *standard inner product.* Which of the following vectors in P_2 lie in the subspace $W^\perp$ of P_2?

1. $x^2 + 1$.
2. $x + 1$.
3. $x - 1$.
4. $x^2 - 1$.
5. None of the above.

– End of assignment –

ADDENDUM C: EXAM INFORMATION SHEET

The question papers include an information sheet. Please see *my*Unisa for past papers and their information sheets. An example of an information sheet is reproduced below. The information sheet includes all of the essential concepts and theorems.

INFORMATION SHEET

Vector spaces

Definition (Vector space)**.**
A vector space *is a non-empty set V with vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{R} \times V \to V$ obeying the axioms*

VS1. $\mathbf{u} + \mathbf{v} \in V$ *for all* $\mathbf{u}, \mathbf{v} \in V$,

VS2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *for all* $\mathbf{u}, \mathbf{v} \in V$,

VS3. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

VS4. *there exists* $\mathbf{0} \in V$ *such that* $\mathbf{u} + \mathbf{0} = \mathbf{u}$ *for all* $\mathbf{u} \in V$,

VS5. *for all* $\mathbf{u} \in V$ *there exists* $-\mathbf{u} \in V$ *such that* $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$,

VS6. $a \cdot \mathbf{u} \in V$ *for all* $a \in \mathbb{R}$, $\mathbf{u} \in V$,

VS7. $a \cdot (\mathbf{u} + \mathbf{v}) = a \cdot \mathbf{u} + a \cdot \mathbf{v}$ *for all* $a \in \mathbb{R}$, $\mathbf{u}, \mathbf{v} \in V$,

VS8. $(a + b) \cdot \mathbf{u} = a \cdot \mathbf{u} + b \cdot \mathbf{u}$ *for all* $a, b \in \mathbb{R}$, $\mathbf{u} \in V$,

VS9. $a \cdot (b \cdot \mathbf{u}) = (ab) \cdot \mathbf{u}$ *for all* $a, b \in \mathbb{R}$, $\mathbf{u} \in V$,

VS10. $1 \cdot \mathbf{u} = \mathbf{u}$ *for all* $\mathbf{u} \in V$.

Theorem (VZ)**.** $\mathbf{0} = 0 \cdot \mathbf{u} = a \cdot \mathbf{0}$ *for all* $a \in \mathbb{R}$ *and* $\mathbf{u} \in V$ *in a vector space* V.

Theorem (VN)**.** $(-1) \cdot \mathbf{u} = -\mathbf{u}$ *for all* $\mathbf{u} \in V$ *in a vector space* V.

Definition (Subspace)**.**
A subset $W \subseteq V$ of a vector space V is a subspace *of V if W, with the same vector addition and scalar multiplication as V, is a vector space.*

Theorem (SS)**.**
A subset $W \subseteq V$ of a vector space V is a subspace *of V, with the same vector addition $+$ and scalar multiplication $\cdot$ as V, if and only if*

1. *W is not empty,*

2. $\mathbf{u} + \mathbf{v} \in W$ *for all* $\mathbf{u}, \mathbf{v} \in W$,

3. $a \cdot \mathbf{u} \in W$ *for all* $a \in \mathbb{R}$, $\mathbf{u} \in V$.

Definition (Linear independence)**.**
A subset $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subseteq V$ *in a vector space* V *is* linearly independent *if and only if*

$$c_1 \cdot \mathbf{b}_1 + \cdots + c_n \cdot \mathbf{b}_n = \mathbf{0} \quad \Longleftrightarrow \quad c_1 = \cdots = c_n = 0.$$

Definition (Span)**.**
The span *of a subset* $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subseteq V$ *in a vector space* V *is the subspace of* V *given by*

$$\operatorname{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} = \{c_1 \cdot \mathbf{b}_1 + \cdots + c_n \cdot \mathbf{b}_n \ : \ c_1, \ldots, c_n \in \mathbb{R}\}.$$

Definition (Basis, dimension)**.**
A subset $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subseteq V$ *in a vector space* V *is a* basis *for* V *if and only if*

1. $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ *is linearly independent,*
2. $\operatorname{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} = V$.

If $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\} \subseteq V$ *is a basis for* V *then the dimension of* V *is* n, $\dim(V) = n$.

Definition (Coordinate matrix)**.**
Let $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ *be a basis for* V *and let* $\mathbf{v} \in V$. *Then there exists unique* $c_1, \ldots, c_n \in \mathbb{R}$ *such that* $\mathbf{v} = c_1 \cdot \mathbf{b}_1 + \cdots + c_n \cdot \mathbf{b}_n$. *The column vector*

$$[\mathbf{v}]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is the coordinate matrix *of* $\mathbf{v}$ *relative to* B.

Definition (Transition matrix, change of coordinate matrix)**.**
Let $B_1 = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ *be a basis for the vector space* V, *and* B_2 *be another basis for* V. *The* transition matrix (change of coordinate matrix) $P_{B_1 \to B_2}$ *from* B_1 *to* B_2 *is given by*

$$P_{B_1 \to B_2} = \begin{bmatrix} [\mathbf{b}_1]_{B_2} & \cdots & [\mathbf{b}_n]_{B_2} \end{bmatrix}.$$

Examples (of vector spaces)**.**

- $\mathbb{R}^n$
- *The vector space* $P_n = \{c_0 + c_1 x + \cdots + c_n x^n \ : \ c_0, \ldots, c_n \in \mathbb{R}\}$ *of polynomials of degree* n *or less.*
- *The vector space* M_{mn} *of* $m \times n$ *matrices.*

Inner products

Definition (Inner product)**.**
An inner product *is a function* $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ *on a vector space* V *which obeys the axioms*

IP1. $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$ *for all* $\mathbf{u}, \mathbf{v} \in V$,

IP2. $\langle k \cdot \mathbf{u}, \mathbf{v} \rangle = k\langle \mathbf{u}, \mathbf{v} \rangle$ *for all* $k \in \mathbb{R}$, $\mathbf{u}, \mathbf{v} \in V$,

IP3. $\langle \mathbf{u}, \mathbf{v} + \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle + \langle \mathbf{u}, \mathbf{w} \rangle$ *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$,

IP4. a) $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$, *for all* $\mathbf{u} \in V$,
b) $\langle \mathbf{u}, \mathbf{u} \rangle = 0$ *if and only if* $\mathbf{u} = 0$.

Definition (Orthogonality)**.**
Let $\langle \cdot, \cdot \rangle$ *denote an inner product on a vector space* V. *If* $\langle \mathbf{u}, \mathbf{v} \rangle = 0$, *then* $\mathbf{u}$ *and* $\mathbf{v}$ *are* orthogonal *to each other.*

Definition (Unit vector, normalized)**.**
Let $\langle \cdot, \cdot \rangle$ *denote an inner product on a vector space* V. *If* $\langle \mathbf{u}, \mathbf{u} \rangle = 1$, *then* $\mathbf{u}$ *is a* unit vector (normalized).

Theorem (Cauchy-Schwarz inequality)**.**
Let $\langle \cdot, \cdot \rangle$ *denote an inner product on a vector space* V. *Then*

$$|\langle \mathbf{u}, \mathbf{v} \rangle| \leq \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle \langle \mathbf{v}, \mathbf{v} \rangle}$$

for all $\mathbf{u}, \mathbf{v} \in V$.

Definition (Gram-Schmidt process)**.**
Let $\langle \cdot, \cdot \rangle$ *denote an inner product on a vector space* V *and let* $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ *be a linearly independent set in* V. *The* Gram-Schmidt process *yields an* orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ *for* $\operatorname{span}\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ *as follows*

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{u}_1, \\
\mathbf{v}_2 &= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1, \\
&\vdots \\
\mathbf{v}_m &= \mathbf{u}_m - \frac{\langle \mathbf{u}_m, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \cdots - \frac{\langle \mathbf{u}_m, \mathbf{v}_{m-1} \rangle}{\langle \mathbf{v}_{m-1}, \mathbf{v}_{m-1} \rangle}\mathbf{v}_{m-1}.
\end{aligned}$$

An orthonormal basis $\{\mathbf{v}'_1, \ldots, \mathbf{v}'_m\}$ *is obtained by setting* $\mathbf{v}'_j = \dfrac{\mathbf{v_j}}{\langle \mathbf{v}_j, \mathbf{v}_j \rangle}$.

Linear transformations

Definition (Linear transformation)**.**
A function $T : V \to W$ *between vector spaces* V *and* W *is a* linear transformation *if and only if*

1. $T(k \cdot \mathbf{u}) = k \cdot T(\mathbf{u})$ *for all* $k \in \mathbb{R}$, $\mathbf{u} \in V$
2. $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ *for all* $\mathbf{u}, \mathbf{v} \in V$

Examples (of linear transformations)**.**

- *The trace operation on* M_{nn} *is a linear transformation* $\mathrm{tr} : M_{nn} \to \mathbb{R}$.
- *The transpose operation on* M_{mn} *is a linear transformation.*

Definition (Kernel, nullity)**.**
The kernel *of a linear transformation* $T : V \to W$ *between vector spaces* V *and* W *is the subspace*

$$\ker(T) = \{\mathbf{v} \in V \; : \; T(v) = \mathbf{0}_W\}$$

of V, *where* $\mathbf{0}_W$ *is the zero vector in* W. *The* nullity *of* T *is the dimension of* $\ker(T)$.

Definition (Range, rank)**.**
The range *of a linear transformation* $T : V \to W$ *between vector spaces* V *and* W *is the subspace*

$$R(T) = \{T(\mathbf{v}) \; : \; \mathbf{v} \in V\}$$

of W. *The* rank *of* T *is the dimension of* $R(T)$.

Definition (One-to-one, injective, inverse)**.**
A linear transformation $T : V \to W$ *between vector spaces* V *and* W *is* one-to-one *if and only if*

$$T(\mathbf{u}) = T(\mathbf{v}) \quad \Longleftrightarrow \quad \mathbf{u} = \mathbf{v}.$$

A one-to-one linear transformation $T : V \to W$ *has an inverse linear transformation* $T^{-1} : R(T) \to V$ *satisfying* $T^{-1}(T(\mathbf{u})) = \mathbf{u}$ *for all* $\mathbf{u} \in V$.

Definition (Onto, surjective)**.**
A linear transformation $T : V \to W$ *between vector spaces* V *and* W *is* onto *if and only if* $R(T) = W$.

Theorem (TO)**.** *If* V *and* W *are finite dimensional vector spaces and* $T : V \to W$ *is a linear transformation, then* T *is one-to-one if and only if* $\ker(T) = \{\mathbf{0}\}$. *If* $\dim(V) = \dim(W)$, *then* T *is onto if and only if* T *is one-to-one.*

Definition (Isomorphism, bijection)**.**
A one-to-one and onto linear transformation $T : V \to W$ *between vector spaces* V *and* W *is an* isomorphism (bijection). *If an isomorphism between* V *and* W *exists, then* V *and* W *are* isomorphic.

Theorem (VI)**.** *Every vector space V with* $\dim(V) = n$ *is isomorphic to* $\mathbb{R}^n$*.*

Definition (Matrix representation of a linear transformation)**.**
Let $B_V = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for the vector space V, and B_W be a basis for the vector space W. The matrix representation $[T]_{B_W,B_V}$ *of a linear transformation $T : V \to W$ is given by*

$$[T]_{B_W,B_V} = \left[\, [T(\mathbf{b}_1)]_{B_W} \quad \cdots \quad [T(\mathbf{b}_n)]_{B_W} \,\right].$$

When $V = W$ and $B_V = B_W$, we write $[T]_{B_V} = [T]_{B_V,B_V}$.

Matrices

Definition (Column space, row space, rank)**.**
Let A be an $m \times n$ matrix with columns $A = \begin{bmatrix}\mathbf{c}_1 & \cdots & \mathbf{c}_n\end{bmatrix}$ and rows $A = \begin{bmatrix}\mathbf{r}_1 \\ \vdots \\ \mathbf{r}_m\end{bmatrix}$.
The column space *of A is* $\operatorname{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ *and the* row space *of A is* $\operatorname{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\}$*. The* rank *of A is the dimension of the column and row spaces,* $\operatorname{rank}(A) = \dim(\operatorname{span}\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}) = \dim(\operatorname{span}\{\mathbf{r}_1, \ldots, \mathbf{r}_m\})$*.*

Definition (Null space, nullity)**.**
The null space *of an $m \times n$ matrix A is the subspace*

$$N(A) = \{\mathbf{x} \in \mathbb{R}^n \,:\, A\mathbf{x} = \mathbf{0}\}.$$

The nullity *of T is the dimension of $N(A)$.*

Theorem (RN)**.** $\operatorname{rank}(A) + \operatorname{nullity}(A) = n$ *for every $m \times n$ matrix A.*

Definition (Eigenvalue, eigenvector)**.**
Let A be an $n \times n$ matrix. If $A\mathbf{x} = \lambda\mathbf{x}$, for $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n$ with $\mathbf{x} \neq \mathbf{0}$, then λ is an eigenvalue *of A and $\mathbf{x}$ is an* eigenvector *of A corresponding to the eigenvalue λ.*

Definition (Eigenspace, geometric multiplicity)**.**
Let A be an $n \times n$ matrix, and let λ be an eigenvalue of A. Then

$$E_\lambda = \{\mathbf{x} \in \mathbb{C}^n \,:\, A\mathbf{x} = \lambda\mathbf{x}\}$$

is a vector space, called the eigenspace *for the eigenvalue λ of A. The* geometric multiplicity *of λ is* $\dim(E_\lambda)$*.*

Definition (Characteristic equation, characteristic polynomial)**.**
Let A be an $n \times n$ matrix. Then $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if λ satisfies the characteristic equation $\det(\lambda I_n - A) = 0$*, where I_n is the $n \times n$ identity matrix. The polynomial* $\det(xI_n - A)$ *is the* characteristic polynomial *in the variable x.*

Definition (Algebraic multiplicity).
Let A be an $n \times n$ matrix with eigenvalue λ. The algebraic multiplicity *of λ is the largest number $a \in \mathbb{N}$ such that $(x-\lambda)^a$ is a factor of the characteristic polynomial* $\det(xI_n - A)$.

Definition (Diagonalizable).
An $n \times n$ matrix A is diagonalizable *if and only if A is similar to some $n \times n$ diagonal matrix D, i.e. $A = PDP^{-1}$ for some $n \times n$ diagonal matrix D and non-singular $n \times n$ matrix P.*

Theorem (DI). *An $n \times n$ matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.*

Theorem (DD). *If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.*

Theorem (DS). *If an $n \times n$ matrix A is symmetric, then A is diagonalizable.*

Theorem (DM). *For a square matrix A, the algebraic and geometric multiplicity are equal for each eigenvalue of A if and only if A is diagonalizable.*

Definition (Trace).
The trace *of a square matrix is the sum of its diagonal entries*

$$\operatorname{tr}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} = a_{11} + a_{22} + \cdots + a_{nn}.$$

Theorem (CT). *For all $n \times n$ matrices A, B and C we have* $\operatorname{tr}(ABC) = \operatorname{tr}(CAB)$. *Consequently* $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Definition (Transpose).
The transpose of a matrix is obtained by interchanging corresponding rows and columns

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}.$$

Theorem (TT). *For all $m \times n$ matrices A we have $(A^T)^T = A$.*

Theorem (TI). *For all $n \times n$ matrices A we have* $\operatorname{tr}(A) = \operatorname{tr}(A^T)$.

Determinants

For 2×2 and 3×3 matrices:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc, \qquad \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - afh - bdi - ceg$$

Cofactor expansion along the j-th row:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{k=1}^{n} (-1)^{(j-1)+(k-1)} a_{jk} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,k-1} & a_{j-1,k+1} & \cdots & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,k-1} & a_{j+1,k+1} & \cdots & a_{j+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,k-1} & a_{n,k+1} & \cdots & a_{nn} \end{vmatrix}$$

Cofactor expansion along the k-th column:

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^{n} (-1)^{(j-1)+(k-1)} a_{jk} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1,k-1} & a_{1,k+1} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2,k-1} & a_{2,k+1} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{j-1,1} & a_{j-1,2} & \cdots & a_{j-1,k-1} & a_{j-1,k+1} & \cdots & a_{j-1,n} \\ a_{j+1,1} & a_{j+1,2} & \cdots & a_{j+1,k-1} & a_{j+1,k+1} & \cdots & a_{j+1,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{n,k-1} & a_{n,k+1} & \cdots & a_{nn} \end{vmatrix}$$

Theorem (DC)**.** *For all* $k \in \mathbb{R}$ *and* $n \times n$ *matrices* A *we have* $\det(kA) = k^n \det(A)$.

Theorem (DP)**.** *For all* $n \times n$ *matrices* A *and* B *we have* $\det(AB) = \det(A)\det(B)$.

ADDENDUM D: USEFUL COMPUTER SOFTWARE

It is possible to check the correctness of your calculations by hand. If you are interested in software that may help to check your results please consult the following resources. **Note however that the software will not be available at exam time, so it is recommended to be proficient at checking your own results by hand**.

Maxima:
`http://maxima.sourceforge.net/`
`http://maxima.sourceforge.net/docs/intromax/intromax.html` (section 6).
`http://maxima.sourceforge.net/docs/manual/en/maxima_23.html`

Maxima is also available for Android devices:
`https://sites.google.com/site/maximaonandroid/`

See addendum E for a brief introduction to Maxima for Linear Algebra.

Wolfram Alpha:
`http://www.wolframalpha.com/`
`http://www.wolframalpha.com/examples/Matrices.html`

Please note that the use of software **is not required** for this module.

ADDENDUM E: ELEMENTARY LINEAR ALGEBRA USING MAXIMA

A complete guide to Maxima is beyond the scope of this module. Here we list only the most essential features. Please consult `http://maxima.sourceforge.net/` for documentation on Maxima.

Please note that the use of software **is not required** for this module.

E.1 The `linearalgebra` and `eigen` packages

First we load the packages `eigen` and `linearalgebra`. Type only the line following `(%i1)` in the white boxes, i.e. `load(eigen);`

```
(%i1) load(eigen);
```

```
(%o1)          /usr/pkg/share/maxima/5.27.0/share/matrix/eigen.mac
```

```
(%i2) load(linearalgebra);
```

```
0 errors, 0 warnings
(%o2) /usr/pkg/share/maxima/5.27.0/share/linearalgebra/linearalgebra.mac
```

The output `(%o1)` and `(%o2)` and may be different, but there should be no error messages. Note the semicolon `;` after every command.

E.2 Matrices

Now we input the matrices

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 & -2 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

```
(%i3) A: matrix( [1, 2, 3],
                 [4, 5, 6] );
```

```
                                  [ 1  2  3 ]
(%o3)                             [         ]
                                  [ 4  5  6 ]
```

```
(%i4) B: matrix( [-1, -2],
                 [ 1,  2],
                 [ 0,  0] );
```

```
                                 [ - 1  - 2 ]
                                 [          ]
(%o4)                            [  1    2  ]
                                 [          ]
                                 [  0    0  ]
```

Type carefully to reproduce the input (%i3) and (%i4) correctly. Next we calculate the matrix product $C = AB$. The matrix product is denoted by a full stop between A and B.

```
(%i5) C: A . B;
```

```
                                  [ 1  2 ]
(%o5)                             [      ]
                                  [ 1  2 ]
```

E.3 Eigenvalues and eigenvectors

We can determine the eigenvalues of C, namely 0 and 3 each with algebraic multiplicity 1. The expression `eigenvalues(C)` returns a list of eigenvalues `[0, 3]` and a list of multiplicities for each eigenvalue `[1, 1]` where the multiplicities are in the same order as the eigenvalues.

```
(%i6) eigenvalues(C);
```

```
(%o6)                      [[0, 3], [1, 1]]
```

Similarly the eigenvectors `eigenvectors(C)` can be determined. This returns three lists, the first two are the same as for `eigenvalues(C)` while the last is a list of eigenvectors.

```
(%i7) eigenvectors(C);
```

```
                                               1
(%o7)              [[[0, 3], [1, 1]], [[[1, - -]], [[1, 1]]]]
                                               2
```

i.e. we find the eigenvector

$$\begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix}$$

for the corresponding eigenvalue 0 of C and the eigenvector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for the corresponding eigenvalue 3 of C. The normalized eigenvectors (`uniteigenvectors(C)`) can be determined similarly.

```
(%i8) uniteigenvectors(C);
```

```
                          2          1            1         1
(%o8) [[[0, 3], [1, 1]], [[[-------, - -------]], [[-------, -------]]]]
                       sqrt(5)    sqrt(5)      sqrt(2)   sqrt(2)
```

i.e. the normalized eigenvector corresponding to the eigenvalue 0 of C is

$$\begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \end{bmatrix}.$$

Although you may find a different eigenvector, that does not mean your answer is incorrect!

E.4 Rank, nullity, columnspace and nullspace

The rank of A (appears above) is calculated with `rank(A)`, the nullity with `nullity(A)`, the columnspace with `columnspace(A)` and the nullspace with `nullspace(A)`. Once again, your own answers may differ but still be correct!

```
(%i9) rank(A);
```

```
(%o9)                                2
```

```
(%i10) columnspace(A);
```

```
                                [ 1 ]  [ 2 ]
(%o10)                     span([   ], [   ])
                                [ 4 ]  [ 5 ]
```

```
(%i11) nullspace(A);
```

```
                                  [ - 3 ]
                                  [     ]
(%o11)                       span([  6  ])
                                  [     ]
                                  [ - 3 ]
```

```
(%i12) nullity(A);
```

```
(%o12)                               1
```

E.5 Matrix inverse

The inverse of a matrix (when it exists) is calculated using `invert`. Here we calculate

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}^{-1}.$$

```
(%i13) invert(matrix( [1,1], [1,2] ));
```

```
                               [  2   - 1 ]
(%o13)                         [          ]
                               [ - 1   1  ]
```

E.6 Gram-Schmidt algorithm

The Gram-Schmidt algorithm is easily applied using `gramschmidt`. The vectors for which we want to find an orthogonal basis are specified as *rows* of a matrix. For example, below we apply the gram-Schmidt algorithm for

$$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

with respect to the Euclidean inner product.

```
(%i14) gramschmidt(matrix([1,1],[0,1]));
```

```
                                          1   1
(%o14)                        [[1, 1], [- -, -]]
                                          2   2
```

i.e. we find the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} \right\}.$$

Now consider a non-Euclidean inner product on $\mathbb{R}^2$

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1 y_1 + 2 x_2 y_2, \qquad \mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2), \qquad x_1, x_2, y_1, y_2 \in \mathbb{R}$$

```
(%i15) f(x,y):= x[1]*y[1] + 2*x[2]*y[2];
```

```
(%o15)                    f(x, y) := x  y  + 2 x  y
                                      1  1      2  2
```

we can tell `gramschmidt` to use `f` (our inner product) when applying the Gram-Schmidt algorithm

```
(%i16) ob: gramschmidt(matrix([1,1],[0,1]), f);
```

```
                                          2   1
(%o16)                        [[1, 1], [- -, -]]
                                          3   3
```

i.e. we find the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} \right\}.$$

with respect to our non-Euclidean inner product. To find an orthonormal basis we need to normalize each of these vectors with respect to our non-Euclidean inner product by extracting each vector and divide by its norm. Here we use `first`, `second` and so on to obtain each of the vectors.

```
(%i17) v1: first(ob);
```

```
(%o17)                             [1, 1]
```

```
(%i18) v1 / sqrt(f(v1,v1));
```

```
                          1        1
(%o18)                [-------, -------]
                       sqrt(3)  sqrt(3)
```

```
(%i19) v2: second(ob);
```

```
                                2  1
(%o19)                       [- -, -]
                                3  3
```

```
(%i20) v2 / sqrt(f(v2,v2));
```

```
                        2              1
(%o20)          [- -------------, -------------]
                          4   2          4   2
                   3 sqrt(- + -)  3 sqrt(- + -)
                          9   9          9   9
```

To simplify the rational expressions, use `ratsimp`.

```
(%i21) ratsimp(v2 / sqrt(f(v2,v2)));
```

```
                   sqrt(2)         1
(%o21)          [- -------, ---------------]
                   sqrt(3)  sqrt(2) sqrt(3)
```

ADDENDUM F: Example questions

The following sections provide complete solutions for example past assignment and exam questions.

It is highly recommended to attempt the problems on your own before consulting these solutions. If your own solutions are not consistent with those provided, attempt to identify where (and why) things went awry. Note that some questions have many different, correct, answers.

F.1 Previous assignment questions

Questions

Question 1

Show that the set X with the given operations fails to be a vector space by identifying all axioms that hold *and* fail to hold:

The set $X = \mathbb{R}^3$ with vector addition $\oplus$ defined by

$$(a, b, c) \oplus (x, y, z) = (1, y, c + z)$$

and scalar multiplication $\odot$ defined by $k \odot (a, b, c) = (ka, kb, kc)$.

Question 2

Show that the set X with the given operations fails to be a vector space by identifying all axioms that hold *and* fail to hold:

The set $X = \{(a, 1) : a \in \mathbb{R}\} \subset \mathbb{R}^2$ with vector addition $\oplus$ defined by

$$(a, 1) \oplus (b, 1) = (a + b, 1)$$

and scalar multiplication $\odot$ defined by $k \odot (a, 1) = (k^2a, 1)$.

Question 3

Consider the set

$$X := \{\,(1, x)\ :\ x \in \mathbb{R}\,\}$$

and the operations (for all $k, x, y \in \mathbb{R}$, $\mathbf{a} = (1, x) \in X$ and $\mathbf{b} = (1, y) \in X$)

$$\begin{aligned} \cdot : \mathbb{R} \times X \to X, &\qquad k \cdot \mathbf{a} \equiv k \cdot (1, x) := (1, kx), \\ + : X \times X \to X, &\qquad \mathbf{a} + \mathbf{b} \equiv (1, x) + (1, y) := (1, x + y). \end{aligned}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space.

(3.1) Find the zero vector for X.

(3.2) Prove that each axiom for vector spaces holds for X with the given operations.

(3.3) Is the vector space X a subspace of $\mathbb{R}^2$? Motivate your answer.

Question 4

Consider the set

$$X := \{\, e^x \;:\; x \in \mathbb{R} \,\}$$

and the operations (for all $k, x, y \in \mathbb{R}$, $\mathbf{a} = e^x \in X$ and $\mathbf{b} = e^y \in X$)

$$\begin{aligned} &\odot : \mathbb{R} \times X \to X, && k \odot \mathbf{a} \equiv k \odot e^x := (e^x)^k = e^{kx}, \\ &\oplus : X \times X \to X, && \mathbf{a} \oplus \mathbf{b} \equiv e^x \oplus e^y := e^x \cdot e^y = e^{x+y}. \end{aligned}$$

The set X with these definitions of $\odot$ and $\oplus$ forms a vector space. Here we use $\odot$ instead of $\cdot$ and $\oplus$ instead of $+$ to avoid confusion with the usual arithmetic operations.

(4.1) Find the zero vector for X. Find the zero vector for X.

(4.2) Prove that each axiom for vector spaces holds for X with the given operations.

(4.3) Is the vector space X a subspace of $\mathbb{R}$? Motivate your answer.

Question 5

Show that

$$Y := \{\, (0, y) \;:\; y \in \mathbb{R} \,\},$$

with the usual vector addition and scalar multiplication in $\mathbb{R}^2$, is a subspace of $\mathbb{R}^2$.

Question 6

Let $B \in M_{22}$ be a fixed but arbitrary 2×2 matrix. Show that

$$Y := \{\, A \;:\; A \in M_{22},\ AB = BA \,\},$$

with the usual vector addition and scalar multiplication in M_{22}, is a subspace of M_{22}.

Question 7

Is

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\} ?$$

Question 8

Is

$$\text{span}\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$$

a linearly independent subset of $\mathbb{R}^2$? Motivate your answer.

Question 9

Consider the vector space M_{22} of all 2×2 matrices.

(9.1) Show that $\mathcal{B} = \{A_1, A_2, A_3, A_4\}$ is a basis for M_{22} where

$$A_1 = \begin{bmatrix}3 & 6\\3 & -6\end{bmatrix};\ A_2 = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix};\ A_3 = \begin{bmatrix}0 & -8\\-12 & -4\end{bmatrix};\ \text{and } A_4 = \begin{bmatrix}1 & 0\\-1 & 2\end{bmatrix}.$$

(9.2) Write $A = \begin{bmatrix}6 & 2\\5 & 3\end{bmatrix}$ as a linear combination of vectors from $\mathcal{B}$.

(9.3) Prove that the subset

$$D = \{A \in M_{22} \,:\, A^T + A = 0\}$$

forms a subspace of M_{22}.

Question 10

Consider the vector space M_{22} of all 2×2 matrices.

(10.1) Show that $\mathcal{B} = \{A_1, A_2, A_3, A_4\}$ is a basis for M_{22} where

$$A_1 = \begin{bmatrix}3 & -2\\0 & 1\end{bmatrix};\ A_2 = \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix};\ A_3 = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix};\ \text{and } A_4 = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}.$$

(10.2) Write $A = \begin{bmatrix}6 & 2\\5 & 3\end{bmatrix}$ as a linear combination of vectors from $\mathcal{B}$.

(10.3) Prove that the subset

$$D = \{A \in M_{22} \,:\, \text{tr}(A) = 0\}$$

forms a subspace of M_{22}.

Question 11

Given the matrix

$$A = \begin{bmatrix}1 & 2 & 1\\2 & 1 & 1\end{bmatrix}$$

(11.1) Find the range of A.

(11.2) Find the rank of A.

(11.3) Find the null space of A.

(11.4) Find the nullity of A.

Question 12

Given the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$$

(12.1) Find the range of A.

(12.2) Find the rank of A.

(12.3) Find the nullspace of A.

(12.4) Find the nullity of A.

Question 13

(13.1) Let S and T be subspaces of a vector space V. Prove that $S \cap T$ is also a subspace of V.

(13.2) Let $S = \{(a+b, a, -a, -b) \,:\, a, b \in \mathbb{R}\}$ and $T = \{(x, 2y, -y, -x) \,:\, x, y \in \mathbb{R}\}$.

- **(a)** Show that S and T are subspaces of $\mathbb{R}^4$.
- **(b)** Find the dimension of S and T.
- **(c)** Find $S \cap T$ and hence a basis and the dimension of $S \cap T$.

(13.3) Let

$$S = \{1 - x;\ 5 + 3x - 2x^2;\ 1 + 3x - x^2\} \subset P_2.$$

Find a basis and the dimension for span(S).

Question 14

(14.1) Let $S = \{(a, b, b, a+c) \,:\, a, b, c \in \mathbb{R}\}$ and $T = \{(x, y-x, y+z, z) \,:\, x, y, z \in \mathbb{R}\}$.

- **(a)** Show that S and T are subspaces of $\mathbb{R}^4$.
- **(b)** Find the dimension of S and T.
- **(c)** Find $S \cap T$ and hence a basis and the dimension of $S \cap T$.

(14.2) Let

$$S = \{t^3 + t^2 - 2t + 1;\ t^2 + 1;\ t^3 - 2t;\ 2t^3 + 3t^2 - 4t + 3\} \subset P_3.$$

Find a basis and the dimension for span(S).

Question 15

Let $B = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $B' = \{\mathbf{q}_1, \mathbf{q}_2\}$ be basis for P_1, where

$$\mathbf{p}_1 = 1 + 2x, \quad \mathbf{p}_2 = 3 - x, \quad \mathbf{q}_1 = 2 - 2x, \quad \mathbf{q}_2 = 4 + 3x.$$

(15.1) Find the transition matrix $P_{B' \longrightarrow B}$ from the B'-basis to the B-basis

(15.2) Find the transition matrix $Q_{B' \longrightarrow B}$ from the B-basis to the B'-basis

(15.3) Compute $[\mathbf{p}]_{B'}$ if $\mathbf{p} = 5 - x$.

Question 16

Let $B = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $B' = \{\mathbf{q}_1, \mathbf{q}_2\}$ be basis for P_1, where

$$\mathbf{p}_1 = 1 + 3x, \quad \mathbf{p}_2 = 2 - x, \quad \mathbf{q}_1 = 1 - 2x, \quad \mathbf{q}_2 = 1 + x.$$

(16.1) Find the transition matrix $P_{B' \longrightarrow B}$ from the B'-basis to the B-basis

(16.2) Find the transition matrix $Q_{B' \longrightarrow B}$ from the B-basis to the B'-basis

(16.3) Compute $[\mathbf{p}]_{B'}$ if $\mathbf{p} = 3 + x$.

Question 17

Let $B_1 = \{\, 1 + x,\ 1 - x \,\} \subset P_1$ and $B_2 = \{\, 1 + 2x,\ 2 + x \,\} \subset P_1$ be two bases for P_1, where the usual left to right ordering is assumed.

(17.1) Show that B_2 is a basis for P_1.

(17.2) Find the transition matrix $P_{B_1 \to B_2}$.

(17.3) Let B_3 be a basis for P_1 and $P_{B_2 \to B_3}$ be the transition matrix from B_2 to B_3 given by $P_{B_2 \to B_3} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) Find the transition matrix $P_{B_1 \to B_3}$.

(b) Use $P_{B_2 \to B_3}$ to find B_3.

Question 18

Let $B_1 = \{ 1+x,\, 1-x \} \subset P_1$ and $B_2 = \{ 1+2x,\, 2+x \} \subset P_1$ be two bases for P_1, where the usual left to right ordering is assumed.

(18.1) Show that B_1 is a basis for P_1.

(18.2) Find the transition matrix $P_{B_2 \to B_1}$.

(18.3) Let B_3 be a basis for P_1 and $P_{B_1 \to B_3}$ be the transition matrix from B_1 to B_3 given by $P_{B_1 \to B_3} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

(a) Find the transition matrix $P_{B_2 \to B_3}$.

(b) Use $P_{B_1 \to B_3}$ to find B_3.

Question 19

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

(19.1) Determine the characteristic equation for A in λ.

(19.2) Find the eigenvalues of A, and their algebraic multiplicities.

(19.3) Find a basis for the eigenspace corresponding to each eigenvalue of A and hence also the geometric multiplicity of each eigenvalue.

Question 20

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(20.1) Determine the characteristic equation for A in λ.

(20.2) Find the eigenvalues of A, and their algebraic multiplicities.

(20.3) Find a basis for the eigenspace corresponding to each eigenvalue of A and hence also the geometric multiplicity of each eigenvalue.

Question 21

Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

(21.1) Show that A is diagonalizable.

(21.2) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(21.3) Calculate A^{11}.

Question 22

Let

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 5 & 1 & 0 \\ 1 & -1 & 6 \end{bmatrix}$$

(22.1) Show that A is diagonalizable.

(22.2) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(22.3) Calculate A^{10}.

Question 23

Consider the matrix (see 19)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

(23.1) Find an invertible matrix P such that $D := P^{-1}AP$ is diagonal. Determine D.

(23.2) Find the rank of D and hence also the rank of A.

(23.3) Calculate D^n for $n \in \mathbb{N}$ and hence also A^n as a matrix.

(23.4) Show that

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.$$

Use this expression for A to calculate A^2, A^3 etc. and compare with your answer to (24.3).

Question 24

Consider the matrix (see 20)

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(24.1) Find an orthogonal matrix P such that $D := P^T AP$ is diagonal. Determine D.

(24.2) Find the rank of D and hence also the rank of A.

(24.3) Calculate D^n for $n \in \mathbb{N}$ and hence also A^n as a matrix.

(24.4) Let B be a $m \times m$ matrix where $m \in \mathbb{N}$, I be the $m \times m$ identity matrix and $k \in \mathbb{R}$.

- **(a)** Let $\mathbf{x}$ be an eigenvector of B with corresponding eigenvalue λ. Show that $\mathbf{x}$ is an eigenvector of $B + kI$. What is the corresponding eigenvalue of $B + kI$?
- **(b)** Assume that B is diagonalizable, is $B + kI$ diagonalizable?

Question 25

Consider the vector space $\mathbb{R}^3$.

(25.1) Show that

$$\langle \mathbf{x}, \mathbf{y} \rangle := 3x_1y_1 + x_2y_2 + x_3y_3, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

is an inner product on $\mathbb{R}^3$.

(25.2) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

to find an orthonormal basis with respect to the inner product **defined in 2.1** for the span of this subset.

Question 26

Consider the vector space $\mathbb{R}^3$.

(26.1) Show that

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1y_1 + 3x_2y_2 + x_3y_3, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

is an inner product on $\mathbb{R}^3$.

(26.2) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

to find an orthonormal basis with respect to the inner product **defined in 2.1** for the span of this subset.

Question 27

(27.1) Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + 2u_2 v_2 + 3u_3 v_3$$

is an inner product on $\mathbb{R}^3$ for $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

(27.2) Let $\mathbf{u} = (1,1,1)$, $\mathbf{v} = (1,1,0)$ and $\mathbf{w} = (1,0,0)$. Show that $B = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent and spans $\mathbb{R}^3$.

(27.3) Transform B into an orthonormal basis using the inner product in 27.1.

(27.4) Let $\mathbb{R}^3$ have the Euclidean inner product and $W = \text{span}\{\mathbf{u}, \mathbf{v}\}$ where

$$\mathbf{u} = \left(\frac{4}{5}, 0, -\frac{3}{5}\right) \text{ and } \mathbf{v} = (0,1,0).$$

Express $\mathbf{w} = (1,2,3)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^{\perp}$.

Question 28

(28.1) Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1 v_1 + 3u_2 v_2 + u_3 v_3$$

is an inner product on $\mathbb{R}^3$ for $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

(28.2) Let $\mathbf{u} = (1,1,1)$, $\mathbf{v} = (-1,1,0)$ and $\mathbf{w} = (1,2,1)$. Show that $B = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent and spans $\mathbb{R}^3$.

(28.3) Transform B into an orthonormal basis using the inner product in 28.1.

(28.4) Let $\mathbb{R}^3$ have the Euclidean inner product and $W = \text{span}\{\mathbf{u}, \mathbf{v}\}$ where

$$\mathbf{u} = (1,0,-1) \text{ and } \mathbf{v} = (3,1,0).$$

Express $\mathbf{w} = (1,2,3)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^{\perp}$.

Question 29

(29.1) Let

$$A = \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find the bases for the eigenspaces associated with the eigenvalues of A.

(29.2) Let $T : P_2 \to P_2$ be the function defined by $T(p(x)) = p(2x+1)$.

(a) Show that T is a linear transformation.

(b) Find $[T]_B$ with respect to the basis $\{1, x, x^2\}$.

(c) Compute $T(2 - 3x + 4x^2)$.

(29.3) Let $S : P_2 \to P_3$ be defined by $S(p(x)) = xp(x)$.

(a) Show that S is one-to-one.

(b) Find $S^{-1}(p(x))$.

(c) Is S onto? Explain.

Question 30

(30.1) Let

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Find the bases for the eigenspaces associated with the eigenvalues of A.

(30.2) Let $T : P_2 \to P_3$ be the function defined by $T(p(x)) = xp(x-3)$.

(a) Show that T is a linear transformation.

(b) Find $[T]_{B',B}$ with respect to the basis $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$.

(c) Compute $T(1 + x - x^2)$.

(30.3) Let $S : P_3 \to P_3$ be defined by $S(p(x)) = p(x+1)$.

(a) Show that S is one-to-one.

(b) Find $S^{-1}(p(x))$.

(c) Is S onto? Explain.

Question 31

Let $T : P_2 \to P_2$ be defined by

$$T(a_0 + a_1x + a_2x^2) = (2a_0 - a_1 + 3a_2) + (4a_0 - 5a_1)x + (a_1 + 2a_2)x^2.$$

(31.1) Find the eigenvalues of T.

(31.2) Find bases for the eigenspaces of T.

Question 32

Let $T : P_2 \to P_2$ be defined by

$$T(a_0 + a_1x + a_2x^2) = -2a_2 + (a_0 + 2a_1 + a_2)x + (a_0 + 3a_2)x^2.$$

(32.1) Find the eigenvalues of T.

(32.2) Find bases for the eigenspaces of T.

Question 33

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} -x + 2y + 4z \\ 3x + z \\ 2x + 2y + 5z \end{bmatrix}.$$

(33.1) Find a basis B' for $\mathbb{R}^3$ relative to which the matrix T is diagonal using the standard basis B for $\mathbb{R}^3$.

(33.2) Compute $[T]_{B'}$ and verify that $[T]_{B'} = P^{-1}[T]_B P$ where the matrix P diagonalizes $[T]_B$.

Question 34

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4x + z \\ 2x + 3y + 2z \\ x + 4z \end{bmatrix}.$$

(34.1) Find a basis B' for $\mathbb{R}^3$ relative to which the matrix T is diagonal using the standard basis B for $\mathbb{R}^3$.

(34.2) Compute $[T]_{B'}$ and verify that $[T]_{B'} = P^{-1}[T]_B P$ where the matrix P diagonalizes $[T]_B$.

Question 35

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by the matrix $A = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 2 & 5 \end{bmatrix}$. Find

(35.1) a basis for the range of T,

(35.2) a basis for the kernel of T,

(35.3) the rank and nullity of T and

(35.4) the rank and nullity of A.

Question 36

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 4 \\ 1 & 8 & 5 \end{bmatrix}$. Find

(36.1) a basis for the range of T,

(36.2) a basis for the kernel of T,

(36.3) the rank and nullity of T and

(36.4) the rank and nullity of A.

Question 37

Consider $T : P_2 \to M_{22}$ given by $T(a + bx + cx^2) = \frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$ for all $a, b, c \in \mathbb{R}$.

(37.1) Show that T is a linear transform.

(37.2) Find the matrix representation for T relative to the standard basis in P_2 and in M_{22} with the usual ordering.

(37.3) Is T invertible?

(37.4) Show that the range of T is the subspace $\widetilde{M}_{22}$ of M_{22} consisting of symmetric matrices.

(37.5) Let $\widetilde{T} : P_2 \to \widetilde{M_{22}}$ be defined by $\widetilde{T}(p(x)) := T(p(x))$ for all $p(x) \in P_2$. Find the matrix representation for $\widetilde{T}$ relative to the standard basis with the usual ordering in P_2 and the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

for the 2×2 symmetric matrices, ordered left to right.

Question 38

Consider $T : M_{22} \to P_2$ given by

$$T(A) = \begin{bmatrix} 1 & x \end{bmatrix} A \begin{bmatrix} 1 \\ x \end{bmatrix}, \qquad \text{for all } A \in M_{22}.$$

(38.1) Show that T is a linear transform.

(38.2) Find the matrix representation for T relative to the standard basis in M_{22} and in P_2 with the usual ordering.

(38.3) Is T one to one?

(38.4) Let $\widetilde{M_{22}}$ be the subspace of M_{22} consisting of symmetric matrices. Let $\widetilde{T} : \widetilde{M_{22}} \to P_2$ be defined by $\widetilde{T}(A) := T(A)$ for all $A \in \widetilde{M_{22}}$. Find the matrix representation for $\widetilde{T}$ relative to the standard basis with the usual ordering in P_2 and the basis

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

for the 2×2 symmetric matrices, ordered left to right.

Question 39

Consider $T : P_2 \to P_2$ given by $T(a + bx + cx^2) = b + cx + ax^2$ for all $a, b, c \in \mathbb{R}$.

(39.1)

- **(a)** Find the kernel and nullity of T.
- **(b)** Find the range and rank of T.

(39.2) Find the real valued eigenvalues and corresponding eigenspaces of T.

(39.3) Find $T^3 := T \circ T \circ T$.

Solutions

Question 1

Show that the set X with the given operations fails to be a vector space by identifying all axioms that hold *and* fail to hold:

The set $X = \mathbb{R}^3$ with vector addition $\oplus$ defined by

$$(a, b, c) \oplus (x, y, z) = (1, y, c + z)$$

and scalar multiplication $\odot$ defined by $k \odot (a, b, c) = (ka, kb, kc)$.

In the following let $\mathbf{u} = (u_1, u_2, u_3) \in \mathbb{R}^3$, $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, $\mathbf{w} = (w_1, w_2, w_3) \in \mathbb{R}^3$ *be arbitrary elements of* $\mathbb{R}^3$ *and* $k, m \in \mathbb{R}$ *. We have*

1. $\mathbf{u} \oplus \mathbf{v} = (1, v_2, u_3 + v_3) \in \mathbb{R}^3$ ***holds***.

2. $\mathbf{u} \oplus \mathbf{v} = (1, v_2, u_3 + v_3)$.
   $\mathbf{v} \oplus \mathbf{u} = (1, u_2, v_3 + u_3)$.
   Choosing $\mathbf{u} = (1, 0, 0)$ *and* $\mathbf{v} = (0, 1, 0)$ *we see that* $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ ***does not hold*** *in general.*

3. $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = \mathbf{u} \oplus (1, w_2, v_3 + w_3) = (1, w_2, u_3 + v_3 + w_3)$.
   $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = (1, v_2, u_3 + v_3) \oplus \mathbf{w} = (1, w_2, u_3 + v_3 + w_3)$.
   Thus $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$ ***holds***.

4. *Suppose the zero vector* $\mathbf{0} = (a, b, c) \in \mathbb{R}^3$ *exists. Then*
   $\mathbf{u} \oplus \mathbf{0} = (u_1, u_2, u_3) \oplus (a, b, c) = (1, b, u_3 + c) = (u_1, u_2, u_3) \quad \Leftrightarrow \quad u_1 = 1,\ u_2 = b,\ c = 0.$
   The zero vector property obviously ***does not hold****, in particular the equation cannot be satisfied for* $\mathbf{u} = (0, 0, 0)$.

5. *Since the zero vector does not exist, the negative is undefined. Thus the existence of negatives* ***does not hold***.

6. $k \odot \mathbf{u} = (ku_1, ku_2, ku_3) \in \mathbb{R}^3$ ***holds***.

7. $k \odot (\mathbf{u} + \mathbf{v}) = k \odot (1, v_2, u_3 + v_3) = (k, kv_2, ku_3 + kv_3)$.
   $k \odot \mathbf{u} \oplus k \odot \mathbf{v} = (1, kv_2, ku_3 + kv_3)$.
   Choosing $k = 2$*, for example, we find* $k \odot (\mathbf{u} + \mathbf{v}) = k \odot \mathbf{u} + k \odot \mathbf{v}$ ***does not hold*** *in general.*

8. $(k + m) \odot \mathbf{u} = ((k + m)u_1, (k + m)u_2, (k + m)u_3)$.
   $k \odot \mathbf{u} \oplus m \odot \mathbf{u} = (1, mu_2, ku_3 + mu_3)$.
   Choosing $k = m = 0$*, for example, we find* $(k + m) \odot \mathbf{u} = k \odot \mathbf{u} \oplus m \odot \mathbf{u}$ ***does not hold*** *in general.*

9. $k(m \odot \mathbf{u}) = k \odot (mu_1, mu_2, mu_3) = (kmu_1, kmu_2, kmu_3)$.
   $(km) \odot \mathbf{u} = ((km)u_1, (km)u_2, (km)u_3)$.
   Thus $k(m \odot \mathbf{u}) = (km) \odot \mathbf{u}$ ***holds***.

10. $1 \odot \mathbf{u} = (1 \cdot u_1, 1 \cdot u_2, 1 \cdot u_3) = \mathbf{u}$ ***holds***.

Question 2

Show that the set X with the given operations fails to be a vector space by identifying all axioms that hold *and* fail to hold:

The set $X = \{(a, 1) : a \in \mathbb{R}\} \subset \mathbb{R}^2$ with vector addition $\oplus$ defined by

$$(a, 1) \oplus (b, 1) = (a + b, 1)$$

and scalar multiplication $\odot$ defined by $k \odot (a, 1) = (k^2 a, 1)$.
In the following let $\mathbf{u} = (a, 1) \in X$, $\mathbf{v} = (b, 1) \in X$, $\mathbf{w} = (c, 1) \in X$ *be arbitrary elements of* X *and* $k, m \in \mathbb{R}$. *We have*

1. $\mathbf{u} \oplus \mathbf{v} = (a + b, 1) \in \mathbb{X}$ ***holds***.
2. $\mathbf{u} \oplus \mathbf{v} = (a + b, 1)$.
   $\mathbf{v} \oplus \mathbf{u} = (b + a, 1) = (a + b, 1)$.
   Thus $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ ***holds.***
3. $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = \mathbf{u} \oplus (b + c, 1) = (a + (b + c), 1) = (a + b + c, 1)$.
   $(\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w} = (a + b, 1) \oplus \mathbf{w} = ((a + b) + c, 1) = (a + b + c, 1)$.
   Thus $\mathbf{u} \oplus (\mathbf{v} \oplus \mathbf{w}) = (\mathbf{u} \oplus \mathbf{v}) \oplus \mathbf{w}$ ***holds***.
4. *Suppose the zero vector* $\mathbf{0} = (z, 1) \in X$ *exists. Then*
   $\mathbf{u} \oplus \mathbf{0} = (a, 1) \oplus (z, 1) = (a + z, 1) = (a, 1) \quad \Leftrightarrow \quad a + z = a \quad \Leftrightarrow \quad z = 0$.
   The zero vector property ***holds*** *with* $\mathbf{0} = (0, 1)$.
5. *If* X *is a vector space then*
   $\mathbf{u} + (-1) \odot \mathbf{u} = 1 \odot \mathbf{u} + (-1) \odot \mathbf{u} = (1 - 1) \odot \mathbf{u} = 0 \odot \mathbf{u} = \mathbf{0}$,
   i.e. $-\mathbf{u} \equiv (-1) \odot \mathbf{u}$.
   However, here we must find $-\mathbf{u}$ *directly. Let* $-\mathbf{u} := (b, 1) \in X$, $b \in \mathbb{R}$. *Then*
   $-\mathbf{u} + \mathbf{u} = (b, 1) + (a, 1) = (b + a, 1) = (0, 1) = \mathbf{0}$.
   Thus $b = -a$. *From*
   $\mathbf{u} + (-\mathbf{u}) = (a, 1) + (b, 1) = (a + b, 1) = (0, 1) = \mathbf{0}$.
   it also follows that $b = -a$. *Thus* $-\mathbf{u} = -(a, 1) = (-a, 1)$ *exists.*
6. $k \odot \mathbf{u} = (k^2 a, 1) \in X$ ***holds***.
7. $k \odot (\mathbf{u} + \mathbf{v}) = k \odot (a + b, 1) = (k^2(a + b), 1) = (k^2 a + k^2 b, 1)$.
   $k \odot \mathbf{u} + k \odot \mathbf{v} = (k^2 a, 1) + (k^2 b, 1) = (k^2 a + k^2 b, 1)$.
   Thus $k \odot (\mathbf{u} + \mathbf{v}) = k \odot \mathbf{u} + k \odot \mathbf{v}$***holds***.
8. $(k + m) \odot \mathbf{u} = ((k + m)^2 a, 1) = (k^2 a + 2kma + m^2 a, 1)$.
   $k \odot \mathbf{u} \oplus m \odot \mathbf{u} = (k^2 a, 1) \oplus (m^2 a, 1) = (k^2 a + m^2 a, 1)$.
   Choosing $k = m = 1$, *for example, we find* $(k + m) \odot \mathbf{u} = k \odot \mathbf{u} \oplus m \odot \mathbf{u}$ ***does not hold*** *in general.*
9. $k(m \odot \mathbf{u}) = k \odot (m^2 a, 1) = (k^2(m^2)a, 1) = (k^2 m^2 a, 1)$.
   $(km) \odot \mathbf{u} = ((km)^2 a, 1) = (k^2 m^2 a, 1)$.
   Thus $k(m \odot \mathbf{u}) = (km) \odot \mathbf{u}$ ***holds***.
10. $1 \odot \mathbf{u} = (1 \cdot a, 1) = (a, 1) = \mathbf{u}$ ***holds***.

Question 3

Consider the set

$$X := \{ (1, x) \, : \, x \in \mathbb{R} \}$$

and the operations (for all $k, x, y \in \mathbb{R}$, $\mathbf{a} = (1, x) \in X$ and $\mathbf{b} = (1, y) \in X$)

$$\begin{aligned} \cdot : \mathbb{R} \times X \to X, \qquad & k \cdot \mathbf{a} \equiv k \cdot (1, x) := (1, kx), \\ + : X \times X \to X, \qquad & \mathbf{a} + \mathbf{b} \equiv (1, x) + (1, y) := (1, x + y). \end{aligned}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space.

(3.1) Find the zero vector for X.

Let $\mathbf{0} \in X$ be the zero vector, i.e. there exists $z \in \mathbb{R}$ such that $\mathbf{0} = (1, z)$ and for all $\mathbf{a} = (1, x) \in X$ $(x \in \mathbb{R})$

$$\begin{aligned} \mathbf{0} + \mathbf{a} &= (1, z) + (1, x) = (1, z + x) = (1, x) = \mathbf{a}, \\ \mathbf{a} + \mathbf{0} &= (1, x) + (1, z) = (1, x + z) = (1, x) = \mathbf{a}. \end{aligned}$$

Notice that we only use the definition of + given in the question. Since these two equations relate pairs of real numbers, we can equate each component, i.e. $1 = 1$ (holds trivially), $z + x = x$ and $x + z = x$. By subtracting the real number x from both sides of each equation we find that $z = 0$ (since $z + x - x = x + z - x = z$ and $x - x = 0$ for $x, z \in \mathbb{R}$). Thus $\mathbf{0} = (1, 0)$ for X with vector addition and scalar multiplication as defined in the question. (Notice that the zero vector does not depend *on x, and therefore also not on $\mathbf{a}$; otherwise it would not satisfy axiom 4 in Definition 1 on page 172 of the textbook.)*

(3.2) Prove that each axiom for vector spaces holds for X with the given operations.

1. *Let $\mathbf{a} = (1, x) \in X$ and $\mathbf{b} = (1, y) \in X$, i.e. $x, y \in \mathbb{R}$. Then $\mathbf{a} + \mathbf{b} = (1, x + y) \in X$ since the first value of the pair is 1, and the second $x + y \in \mathbb{R}$ is a real number.*

2. *Let $\mathbf{a} = (1, x) \in X$ and $\mathbf{b} = (1, y) \in X$. Then*

$$\begin{aligned} \mathbf{a} + \mathbf{b} &= (1, x) + (1, y) = (1, x + y) \\ \mathbf{b} + \mathbf{a} &= (1, y) + (1, x) = (1, y + x) \end{aligned}$$

so that $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$ (since the addition of real numbers is commutative, i.e. $x + y = y + x$).

3. *Let $\mathbf{a} = (1, x) \in X$, $\mathbf{b} = (1, y) \in X$ and $\mathbf{c} = (1, z) \in X$. Then*

$$\begin{aligned} (\mathbf{a} + \mathbf{b}) + \mathbf{c} &= (1, x + y) + (1, z) = (1, (x + y) + z) = (1, x + y + z) \\ \mathbf{a} + (\mathbf{b} + \mathbf{c}) &= (1, x) + (1, y + z) = (1, x + (y + z)) = (1, x + y + z) \end{aligned}$$

due to the associativity of real numbers, i.e. $(x + y) + z = x + (y + z)$. Thus $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$.

4. *The existence of the zero vector is demonstrated in Question (4.1).*

5. *Let* $\mathbf{a} = (1, x) \in X$. *Assume there exists* $x' \in \mathbb{R}$ *such that* $-\mathbf{a} = (1, x')$, *i.e.*

$$\mathbf{a} + (-\mathbf{a}) = (1, x) + (1, x') = (1, x + x') = (1, 0) = \mathbf{0}$$
$$(-\mathbf{a}) + \mathbf{a} = (1, x') + (1, x) = (1, x' + x) = (1, 0) = \mathbf{0}.$$

Since these two equations relate pairs of real numbers, we can equate each component, i.e. $1 = 1$ *(holds trivially),* $x + x' = 0$ *and* $x' + x = 0$. *Thus* $x' = -x$. *Consequently* $-\mathbf{a}$ *exists and is given by* $-\mathbf{a} = -(1, x) = (1, -x)$.

6. *Let* $k \in \mathbb{R}$ *and* $\mathbf{a} = (1, x) \in X$ *(so that* $x \in \mathbb{R}$*). Then* $k \cdot \mathbf{a} = (1, kx) \in X$ *since the first value of the pair is 1, and the second* $kx \in \mathbb{R}$ *is a real number.*

7. *Let* $k \in \mathbb{R}$, $\mathbf{a} = (1, x) \in X$ *and* $\mathbf{b} = (1, y) \in X$. *Then*

$$k \cdot (\mathbf{a} + \mathbf{b}) = k \cdot ((1, x) + (1, y)) = k(1, x + y) = (1, k(x + y)) = (1, kx + ky)$$
$$(k \cdot \mathbf{a}) + (k \cdot \mathbf{b}) = (1, kx) + (1, ky) = (1, kx + ky)$$

so that $k \cdot (\mathbf{a} + \mathbf{b}) = k \cdot \mathbf{a} + k \cdot \mathbf{b}$.

8. *Let* $k, m \in \mathbb{R}$ *and* $\mathbf{a} = (1, x) \in X$. *Then*

$$(k + m) \cdot \mathbf{a} = (1, (k + m)x) = (1, kx + mx)$$
$$(k \cdot \mathbf{a}) + (m \cdot \mathbf{a}) = (1, kx) + (1, mx) = (1, kx + mx)$$

so that $(k + m) \cdot \mathbf{a} = k \cdot \mathbf{a} + m \cdot \mathbf{a}$.

9. *Let* $k, m \in \mathbb{R}$ *and* $\mathbf{a} = (1, x) \in X$. *Then*

$$(km) \cdot \mathbf{a} = (1, (km)x) = (1, kmx)$$
$$k \cdot (m \cdot \mathbf{a}) = k \cdot (1, mx) = (1, k(mx)) = (1, kmx)$$

due to the associativity of products of real numbers. Thus $(km) \cdot \mathbf{a} = k \cdot (m \cdot \mathbf{a})$.

10. *Let* $\mathbf{a} = (1, x) \in X$. *Then*

$$1 \cdot \mathbf{a} = 1 \cdot (1, x) = (1, 1x) = (1, x) = \mathbf{a}.$$

(3.3) Is the vector space X a subspace of $\mathbb{R}^2$? Motivate your answer.

A subspace X *of a vector space* V *is a subset of* V *together with the* same *vector addition and scalar multiplication as in* V. *Here we use a vector addition for* X *which is* different *from the usual vector addition in* $\mathbb{R}^2$. *Hence* X *is not a subspace of* $\mathbb{R}^2$.

Question 4

Consider the set
$$X := \{\, e^x \,:\, x \in \mathbb{R} \,\}$$
and the operations (for all $k, x, y \in \mathbb{R}$, $\mathbf{a} = e^x \in X$ and $\mathbf{b} = e^y \in X$)

$$\begin{aligned} &\odot : \mathbb{R} \times X \to X, && k \odot \mathbf{a} \equiv k \odot e^x := (e^x)^k = e^{kx}, \\ &\oplus : X \times X \to X, && \mathbf{a} \oplus \mathbf{b} \equiv e^x \oplus e^y := e^x \cdot e^y = e^{x+y}. \end{aligned}$$

The set X with these definitions of $\odot$ and $\oplus$ forms a vector space. Here we use $\odot$ instead of $\cdot$ and $\oplus$ instead of $+$ to avoid confusion with the usual arithmetic operations.

(4.1) Find the zero vector for X. Find the zero vector for X.

Let $\mathbf{0} \in X$ be the zero vector, i.e. there exists $z \in \mathbb{R}$ such that $\mathbf{0} = e^z$ and for all $\mathbf{a} = e^x \in X$ $(x \in \mathbb{R})$

$$\begin{aligned} \mathbf{0} \oplus \mathbf{a} &= e^z \oplus e^x = e^{z+x} = e^x = \mathbf{a}, \\ \mathbf{a} \oplus \mathbf{0} &= e^x \oplus e^z = e^{x+z} = e^x = \mathbf{a}. \end{aligned}$$

Notice that we only use the definition of $\oplus$ given in the question. Since the exponential function is one-to-one we can equate the exponents $z + x = x$ and $x + z = x$. By subtracting the real number x from both sides of each equation we find that $z = 0$ (since $z + x - x = x + z - x = z$ and $x - x = 0$ for $x, z \in \mathbb{R}$). Thus $\mathbf{0} = e^0$ for X with vector addition and scalar multiplication as defined in the question. (Notice that the zero vector does not depend on *x, and therefore also not on $\mathbf{a}$; otherwise it would not satisfy axiom 4 in Definition 1 on page 172 of the textbook.)*

(4.2) Prove that each axiom for vector spaces holds for X with the given operations.

1. *Let $\mathbf{a} = e^x \in X$ and $\mathbf{b} = e^y \in X$, i.e. $x, y \in \mathbb{R}$. Then $\mathbf{a} \oplus \mathbf{b} = e^{x+y} \in X$ since the first value of the pair is 1, and the second $x + y \in \mathbb{R}$ is a real number.*

2. *Let $\mathbf{a} = e^x \in X$ and $\mathbf{b} = e^y \in X$. Then*

$$\begin{aligned} \mathbf{a} \oplus \mathbf{b} &= e^x \oplus e^y = e^{x+y} \\ \mathbf{b} \oplus \mathbf{a} &= e^y \oplus e^x = e^{y+x} \end{aligned}$$

so that $\mathbf{a} \oplus \mathbf{b} = \mathbf{b} \oplus \mathbf{a}$ (since the addition of real numbers is commutative, i.e. $x + y = y + x$).

3. *Let $\mathbf{a} = e^x \in X$, $\mathbf{b} = e^y \in X$ and $\mathbf{c} = e^z \in X$. Then*

$$\begin{aligned} (\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} &= e^{x+y} \oplus e^z = e^{(x+y)+z} = e^{x+y+z} \\ \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c}) &= e^x \oplus e^{y+z} = e^{x+(y+z)} = e^{x+y+z} \end{aligned}$$

due to the associativity of real numbers, i.e. $(x + y) + z = x + (y + z)$. Thus $(\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} = \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c})$.

4. *The existence of the zero vector is demonstrated in Question (4.1).*

5. *Let* $\mathbf{a} = e^x \in X$*. Assume there exists* $x' \in \mathbb{R}$ *such that* $-\mathbf{a} = e^{x'}$*, i.e.*

$$\mathbf{a} \oplus (-\mathbf{a}) = e^x \oplus e^{x'} = e^{x+x'} = e^0 = \mathbf{0}$$
$$(-\mathbf{a}) \oplus \mathbf{a} = e^{x'} \oplus e^x = e^{x'+x} = e^0 = \mathbf{0}.$$

We can equate exponents, i.e. $x + x' = 0$ *and* $x' + x = 0$*. Thus* $x' = -x$*. Consequently* $-\mathbf{a}$ *exists and is given by* $-\mathbf{a} = -e^x = e^{-x}$*.*

6. *Let* $k \in \mathbb{R}$ *and* $\mathbf{a} = e^x \in X$ *(so that* $x \in \mathbb{R}$*). Then* $k \odot \mathbf{a} = e^{kx} \in X$ *since* $kx \in \mathbb{R}$ *is a real number.*

7. *Let* $k \in \mathbb{R}$*,* $\mathbf{a} = e^x \in X$ *and* $\mathbf{b} = e^y \in X$*. Then*

$$k \odot (\mathbf{a} \oplus \mathbf{b}) = k \odot (e^x \oplus e^y) = k \odot e^{x+y} = e^{k(x+y)} = e^{kx+ky}$$
$$(k \odot \mathbf{a}) \oplus (k \odot \mathbf{b}) = e^{kx} \oplus e^{ky} = e^{kx+ky}$$

so that $k \odot (\mathbf{a} \oplus \mathbf{b}) = k \odot \mathbf{a} \oplus k \odot \mathbf{b}$*.*

8. *Let* $k, m \in \mathbb{R}$ *and* $\mathbf{a} = e^x \in X$*. Then*

$$(k + m) \odot \mathbf{a} = e^{(k+m)x} = e^{kx+mx}$$
$$(k \odot \mathbf{a}) \oplus (m \odot \mathbf{a}) = e^{kx} \oplus e^{mx} = e^{kx+mx}$$

so that $(k + m) \odot \mathbf{a} = k \odot \mathbf{a} \oplus m \odot \mathbf{a}$*.*

9. *Let* $k, m \in \mathbb{R}$ *and* $\mathbf{a} = e^x \in X$*. Then*

$$(km) \odot \mathbf{a} = e^{(km)x} = e^{kmx}$$
$$k \odot (m \odot \mathbf{a}) = k \odot e^{mx} = e^{k(mx)} = e^{kmx}$$

due to the associativity of products of real numbers. Thus $(km) \odot \mathbf{a} = k \odot (m \odot \mathbf{a})$*.*

10. *Let* $\mathbf{a} = e^x \in X$*. Then*

$$1 \odot \mathbf{a} = 1 \odot e^x = e^{1x} = e^x = \mathbf{a}.$$

(4.3) Is the vector space X a subspace of $\mathbb{R}$? Motivate your answer.

A subspace X *of a vector space* V *is a subset of* V *together with the same vector addition and scalar multiplication as in* V*. Here we use a vector addition for* X *which is different from the usual vector addition in* $\mathbb{R}$*. Hence* X *is not a subspace of* $\mathbb{R}$*.*

Question 5

Show that

$$Y := \{\, (0, y) \,:\, y \in \mathbb{R} \,\},$$

with the usual vector addition and scalar multiplication in $\mathbb{R}^2$, is a subspace of $\mathbb{R}^2$.

- *First, we show that Y is non-empty (although this is obvious). It is sufficient to check whether the zero vector $(0, 0)$ in $\mathbb{R}^2$ is also in Y (i.e. this condition together with the next two conditions are necessary and sufficient for Y to be a vector subspace of $\mathbb{R}^2$) . Since the first number in the pair is 0, and the second number $0 \in \mathbb{R}$ we have $(0, 0) \in Y$. Thus Y is non-empty.*
- *Second, we show that Y is closed under the usual vector addition in $\mathbb{R}^2$. Let $\mathbf{a}, \mathbf{b} \in Y$, i.e. there exists $x, y \in \mathbb{R}$ such that $\mathbf{a} = (0, x)$ and $\mathbf{b} = (0, y)$. The usual vector addition in $\mathbb{R}^2$ yields*

  $$\mathbf{a} + \mathbf{b} = (0, x) + (0, y) = (0 + 0, x + y) = (0, x + y) \in Y$$

  since $x + y \in \mathbb{R}$.
- *Third, we show that Y is closed under the usual scalar multiplication in $\mathbb{R}^2$. Let $k \in \mathbb{R}$ and $\mathbf{a} \in Y$, i.e. there exists $x \in \mathbb{R}$ such that $\mathbf{a} = (0, x)$. The usual scalar multiplication in $\mathbb{R}^2$ yields*

  $$k \cdot \mathbf{a} = k \cdot (0, x) = (k0, kx) = (0, kx) \in Y$$

  since $kx \in \mathbb{R}$.

Thus by Theorem 4.2.1 of the textbook, Y is a subspace of $\mathbb{R}^2$.

Question 6

Let $B \in M_{22}$ be a fixed but arbitrary 2×2 matrix. Show that

$$Y := \{\, A \,:\, A \in M_{22},\ AB = BA \,\},$$

with the usual vector addition and scalar multiplication in M_{22}, is a subspace of M_{22}.

- *First, we show that Y is non-empty (although this is obvious). It is sufficient to check whether the zero vector $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ in M_{22} is also in Y (i.e. this condition together with the next two conditions are necessary and sufficient for Y to be a vector subspace of M_{22}). Since $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] B = B \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ we have $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \in Y$. Thus Y is non-empty.*
- *Second, we show that Y is closed under the usual vector addition in M_{22}. Let $G, H \in Y$, i.e. $G, H \in M_{22}$, $GB = BG$ and $HB = BH$. The usual vector addition in M_{22} yields $G + H \in M_{22}$ and*

  $$(G + H)B = GB + HB = BG + BH = B(G + H)$$

  so that $G + H \in Y$.
- *Third, we show that Y is closed under the usual scalar multiplication in $\mathbb{R}^2$. Let $k \in \mathbb{R}$ and $G \in Y$, i.e. $G \in M_{22}$ and $GB = BG$. The usual scalar multiplication in M_{22} yields $k \cdot G \in M_{22}$ and*

  $$(k \cdot G)B = k \cdot (GB) = k \cdot (BG) = B(k \cdot G)$$

  so that $k \cdot G \in Y$.

Thus by Theorem 4.2.1 of the textbook, Y is a subspace of M_{22}.

Question 7

Is
$$\operatorname{span}\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\} = \operatorname{span}\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\}?$$

Noting that
$$\begin{bmatrix}1\\2\end{bmatrix} = 1\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}2\\1\end{bmatrix} \qquad \begin{bmatrix}2\\1\end{bmatrix} = 0\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}2\\1\end{bmatrix} \qquad \begin{bmatrix}1\\1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1\\2\end{bmatrix} + \frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix}$$
$$\begin{bmatrix}1\\2\end{bmatrix} = 1\begin{bmatrix}1\\2\end{bmatrix} + 0\begin{bmatrix}2\\1\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix} \qquad \begin{bmatrix}2\\1\end{bmatrix} = 0\begin{bmatrix}1\\2\end{bmatrix} + 1\begin{bmatrix}2\\1\end{bmatrix} + 0\begin{bmatrix}1\\1\end{bmatrix}$$
then by Theorem 4.2.5 of the textbook, the answer is yes.

Alternative:
To answer this question, we recall that for two sets A and B we have $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. We have
$$\operatorname{span}\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\} = \left\{ a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}2\\1\end{bmatrix} + c\begin{bmatrix}1\\1\end{bmatrix} \;:\; a, b, c \in \mathbb{R} \right\}$$
and
$$\operatorname{span}\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix} \right\} = \left\{ \alpha\begin{bmatrix}1\\2\end{bmatrix} + \beta\begin{bmatrix}2\\1\end{bmatrix} \;:\; \alpha, \beta \in \mathbb{R} \right\}.$$
We need to determine:

1. *Given $a, b, c \in \mathbb{R}$ does $\alpha, \beta \in \mathbb{R}$ exist such that*
$$a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}2\\1\end{bmatrix} + c\begin{bmatrix}1\\1\end{bmatrix} = \alpha\begin{bmatrix}1\\2\end{bmatrix} + \beta\begin{bmatrix}2\\1\end{bmatrix}?$$
Thus we have to satisfy $a + 2b + c = \alpha + 2\beta$ and $2a + b + c = 2\alpha + \beta$. Solving for α and β yields $\alpha = a + c/3$ and $\beta = b + c/3$, i.e. we found a solution.

2. *Given $\alpha, \beta \in \mathbb{R}$ does $a, b, c \in \mathbb{R}$ exist such that*
$$a\begin{bmatrix}1\\2\end{bmatrix} + b\begin{bmatrix}2\\1\end{bmatrix} + c\begin{bmatrix}1\\1\end{bmatrix} = \alpha\begin{bmatrix}1\\2\end{bmatrix} + \beta\begin{bmatrix}2\\1\end{bmatrix}?$$
Thus we have to satisfy $a + 2b + c = \alpha + 2\beta$ and $2a + b + c = 2\alpha + \beta$. Solving for a, b and c yields $a = \alpha - c/3$, $b = \beta - c/3$ and $c \in \mathbb{R}$ is free, i.e. we found a solution (many solutions).

Thus the answer is yes.

Question 8

Is
$$\operatorname{span}\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$$
a linearly independent subset of $\mathbb{R}^2$? Motivate your answer.
Since
$$\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\2\end{bmatrix} + \begin{bmatrix}2\\1\end{bmatrix} \in \operatorname{span}\left\{ \begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$$
by Theorem 4.3.1(a) in the textbook, the given set is not a linearly independent subset of $\mathbb{R}^2$. (By this theorem, the span of any set of vectors is not linearly independent.*)*

Question 9

Consider the vector space M_{22} of all 2×2 matrices.

(9.1) Show that $\mathcal{B} = \{A_1, A_2, A_3, A_4\}$ is a basis for M_{22} where

$$A_1 = \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix};\ A_2 = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix};\ A_3 = \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix};\ \text{and } A_4 = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

First we show linear independence of the vectors. Let $c_1, c_2, c_3, c_4 \in \mathbb{R}$ determined by

$$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = 0$$

i.e.

$$\begin{aligned}\begin{bmatrix} 3c_1 & 6c_1 \\ 3c_1 & -6c_1 \end{bmatrix} + \begin{bmatrix} 0 & -c_2 \\ -c_2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -8c_3 \\ -12c_3 & -4c_3 \end{bmatrix} + \begin{bmatrix} c_4 & 0 \\ -c_4 & 2c_4 \end{bmatrix} \\ = \begin{bmatrix} 3c_1 + c_4 & 6c_1 - c_2 - 8c_3 \\ 3c_1 - c_2 - 12c_3 - c_4 & -6c_1 - 4c_3 + 2c_4 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\end{aligned}$$

Thus we obtain the four equations

$$\begin{aligned} 3c_1 + c_4 &= 0 \\ 6c_1 - c_2 - 8c_3 &= 0 \\ 3c_1 - c_2 - 12c_3 - c_4 &= 0 \\ -6c_1 - 4c_3 + 2c_4 &= 0 \end{aligned}$$

or in matrix form

$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row reduction of the augmented matrix yields

$$\begin{matrix} \\ -2R1 \\ -R1 \\ +2R1 \end{matrix} \left[\begin{array}{cccc|c} 3 & 0 & 0 & 1 & 0 \\ 6 & -1 & -8 & 0 & 0 \\ 3 & -1 & -12 & -1 & 0 \\ -6 & 0 & -4 & 2 & 0 \end{array}\right] \to \begin{matrix} \\ \\ -R2 \\ \\ \end{matrix} \left[\begin{array}{cccc|c} 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & -8 & -2 & 0 \\ 0 & -1 & -12 & -2 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{array}\right]$$

$$\to \begin{matrix} \\ \\ \\ -R3 \end{matrix} \left[\begin{array}{cccc|c} 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & -8 & -2 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & -4 & 4 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 3 & 0 & 0 & 1 & 0 \\ 0 & -1 & -8 & -2 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{array}\right]$$

Here $-2R1$ means subtract twice the first row from the second row (appearing right of $-2R1$). Thus $c_4 = 0$, $c_3 = 0$, $c_2 = -8c_3 - 2c_4 = 0$ and $c_1 = -c_4/3 = 0$. Since $c_1 = c_2 = c_3 = c_4 = 0$ is the only solution, the matrices A_1, A_2, A_3 and A_4 are linearly independent.

Next we show that any element of M_{22} can be expressed as a linear combination of A_1, A_2, A_3 and A_4. Let $a, b, c, d \in \mathbb{R}$ and

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a_1A_1 + a_2A_2 + a_3A_3 + a_4A_4 = \begin{bmatrix} 3a_1 + a_4 & 6a_1 - a_2 - 8a_3 \\ 3a_1 - a_2 - 12a_3 - a_4 & -6a_1 - 4a_3 + 2a_4 \end{bmatrix}$$

Thus we obtain the four equations

$$\begin{aligned} 3a_1 + a_4 &= a \\ 6a_1 - a_2 - 8a_3 &= b \\ 3a_1 - a_2 - 12a_3 - c_4 &= c \\ -6a_1 - 4a_3 + 2a_4 &= d \end{aligned}$$

or in matrix form

$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Row reduction of the augmented matrix yields

$$\begin{array}{c} \\ -2R1 \\ -R1 \\ +2R1 \end{array} \left[\begin{array}{cccc|c} 3 & 0 & 0 & 1 & a \\ 6 & -1 & -8 & 0 & b \\ 3 & -1 & -12 & -1 & c \\ -6 & 0 & -4 & 2 & d \end{array}\right] \to \begin{array}{c} \\ \\ -R2 \\ \\ \end{array} \left[\begin{array}{cccc|c} 3 & 0 & 0 & 1 & a \\ 0 & -1 & -8 & -2 & b-2a \\ 0 & -1 & -12 & -2 & c-a \\ 0 & 0 & -4 & 4 & d+2a \end{array}\right]$$

$$\to \begin{array}{c} \\ \\ \\ -R3 \end{array} \left[\begin{array}{cccc|c} 3 & 0 & 0 & 1 & a \\ 0 & -1 & -8 & -2 & b-2a \\ 0 & 0 & -4 & 0 & a-b+c \\ 0 & 0 & -4 & 4 & d+2a \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 3 & 0 & 0 & 1 & a \\ 0 & -1 & -8 & -2 & b-2a \\ 0 & 0 & -4 & 0 & a-b+c \\ 0 & 0 & 0 & 4 & a+b+d-c \end{array}\right]$$

Thus we find

$$a_4 = \frac{a+b+d-c}{4}$$

$$a_3 = \frac{b-a-c}{4}$$

$$a_2 = -8a_3 - 2a_4 - b + 2a = -2b + 2a + 2c - \frac{a+b+d-c}{2} - b + 2a = \frac{7a - 7b + 5c - d}{2}$$

$$a_1 = \frac{a - a_4}{3} = \frac{3a - b + c - d}{12}$$

Since a solution exists, we have $\text{span}(\mathcal{B}) = M_{22}$. *Thus $\mathcal{B}$ is a basis for M_{22}.*

(9.2) Write $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ as a linear combination of vectors from $\mathcal{B}$.

From 9.1 we have (for $a = 6$, $b = 2$, $c = 5$ and $d = 3$)

$$a_4 = \frac{3}{2}, \qquad a_3 = -\frac{9}{4}, \qquad a_2 = 25, \qquad a_1 = \frac{3}{2}$$

so that

$$\begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix} = \frac{3}{2}\begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix} + 25\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} - \frac{9}{4}\begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}.$$

(9.3) Prove that the subset

$$D = \{A \in M_{22} \,:\, A^T + A = 0\}$$

forms a subspace of M_{22}.

First we must show that D is not empty. Note that if D is a subspace of M_{22} then it is a vector space with the same zero vector as for M_{22} (i.e. the 2×2 zero matrix). It suffices to check whether the zero vector is in D:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in D$.

Next we prove the closure under vector addition and under scalar multiplication.

Let $A \in D$ and $k \in \mathbb{R}$. Then, using the properties of the transpose and vector space, we have

$$(kA)^T + (kA) = kA^T + kA = k(A^T + A) = k0 = 0$$

where 0 on the right hand side denotes the 2×2 zero matrix. Thus D is closed under scalar multiplication.

Let $A, B \in D$. Then, using the properties of the transpose and vector space, we find

$$(A + B)^T + (A + B) = (A^T + B^T) + (A + B) = (A^T + A) + (B^T + B) = 0 + 0 = 0$$

where 0 on the right hand side denotes the 2×2 zero matrix. Thus D is closed under vector addition.

Question 10

Consider the vector space M_{22} of all 2×2 matrices.

(10.1) Show that $\mathcal{B} = \{A_1, A_2, A_3, A_4\}$ is a basis for M_{22} where

$$A_1 = \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix}; \; A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}; \; A_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \text{ and } A_4 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

First we show linear independence of the vectors. Let $c_1, c_2, c_3, c_4 \in \mathbb{R}$ determined by

$$c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = 0$$

i.e.

$$\begin{aligned} \begin{bmatrix} 3c_1 & -2c_1 \\ 0 & c_1 \end{bmatrix} + \begin{bmatrix} c_2 & -c_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & c_3 \\ c_3 & 0 \end{bmatrix} + \begin{bmatrix} c_4 & 0 \\ 0 & 0 \end{bmatrix} \\ = \begin{bmatrix} 3c_1 + c_2 + c_4 & -2c_1 - c_2 + c_3 \\ c_3 & c_1 \end{bmatrix} \\ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Thus we obtain the four equations

$$\begin{aligned} 3c_1 + c_2 + c_4 &= 0 \\ -2c_1 - c_2 + c_3 &= 0 \\ c_3 &= 0 \\ c_1 &= 0 \end{aligned}$$

or in matrix form

$$\begin{bmatrix} 3 & 1 & 0 & 1 \\ -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row reduction of the augmented matrix yields

$$\begin{matrix} =R_3 \\ \\ \\ =R_1 \end{matrix} \left[\begin{array}{cccc|c} 3 & 1 & 0 & 1 & 0 \\ -2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{matrix} \\ +2R_1 \\ \\ -3R_1 \end{matrix} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -2 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\rightarrow \begin{matrix} \\ -R_2 \\ \\ +R_2 \end{matrix} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \end{array}\right]$$

$$\rightarrow \begin{matrix} \\ \\ \\ -R_3 \end{matrix} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Here $=R_3$ *means replace row 1 (appearing right of* $=R_3$*) with row 3 and* $+2R_1$ *means add twice the first row from the second row (appearing right of* $+2R_1$*). Thus* $c_4=0$, $c_3=0$, $c_2=c_3=0$ *and* $c_1=0$*. Since* $c_1=c_2=c_3=c_4=0$ *is the only solution, the matrices* A_1, A_2, A_3 *and* A_4 *are linearly independent.*

Next we show that any element of M_{22} *can be expressed as a linear combination of* A_1, A_2, A_3 *and* A_4*. Let* $a,b,c,d\in\mathbb{R}$ *and*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a_1A_1+a_2A_2+a_3A_3+a_4A_4 = \begin{bmatrix} 3a_1+a_2+a_4 & -2a_1-a_2+a_3 \\ a_3 & a_1 \end{bmatrix}$$

Thus we obtain the four equations

$$\begin{aligned} 3a_1+a_2+a_4 &= a \\ -2a_1-a_2+a_3 &= b \\ a_3 &= c \\ a_1 &= d \end{aligned}$$

or in matrix form

$$\begin{bmatrix} 3 & 1 & 0 & 1 \\ -2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

Row reduction of the augmented matrix yields

$$\begin{array}{l} {\scriptstyle =R_3} \\ \\ \\ {\scriptstyle =R_1} \end{array} \left[\begin{array}{cccc|c} 3 & 1 & 0 & 1 & a \\ -2 & -1 & 1 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 1 & 0 & 0 & 0 & d \end{array}\right] \to \begin{array}{l} \\ {\scriptstyle +2R_1} \\ \\ {\scriptstyle -3R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d \\ -2 & -1 & 1 & 0 & b \\ 0 & 0 & 1 & 0 & c \\ 3 & 1 & 0 & 1 & a \end{array}\right]$$

$$\to \begin{array}{l} \\ {\scriptstyle -R_2} \\ \\ {\scriptstyle +R_2} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d \\ 0 & -1 & 1 & 0 & b+2d \\ 0 & 0 & 1 & 0 & c \\ 0 & 1 & 0 & 1 & a-3d \end{array}\right]$$

$$\to \begin{array}{l} \\ \\ \\ {\scriptstyle -R_3} \end{array} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d \\ 0 & 1 & -1 & 0 & -b-2d \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 1 & 1 & a+b-d \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & d \\ 0 & 1 & -1 & 0 & -b-2d \\ 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 1 & a+b-c-d \end{array}\right]$$

Thus we find

$$\begin{aligned} a_4 &= a+b-c-d \\ a_3 &= c \\ a_2 &= -b-2d+a_3 = -b-2d+c \\ a_1 &= d \end{aligned}$$

Since a solution exists, we have $\operatorname{span}(\mathcal{B})=M_{22}$*. Thus* $\mathcal{B}$ *is a basis for* M_{22}*.*

(10.2) Write $A = \begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix}$ as a linear combination of vectors from $\mathcal{B}$.

From 10.1 we have (for $a = 6$, $b = 2$, $c = 5$ and $d = 3$)

$$a_4 = 0, \qquad a_3 = 5, \qquad a_2 = -3, \qquad a_1 = 3$$

so that

$$\begin{bmatrix} 6 & 2 \\ 5 & 3 \end{bmatrix} = 3 \begin{bmatrix} 3 & -2 \\ 0 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + 5 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + 0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

(10.3) Prove that the subset

$$D = \{A \in M_{22} \,:\, \mathrm{tr}(A) = 0\}$$

forms a subspace of M_{22}.

First we must show that D is not empty. Note that if D is a subspace of M_{22} then it is a vector space with the same zero vector as for M_{22} (i.e. the 2×2 zero matrix). It suffices to check whether the zero vector is in D:

$$\mathrm{tr} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0 + 0 = 0$$

so that $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in D$.

Next we prove the closure under vector addition and under scalar multiplication.

Let $A \in D$ and $k \in \mathbb{R}$. Then, using the properties of the trace and vector space, we have

$$\mathrm{tr}(kA) = k\,\mathrm{tr}(A) = k0 = 0$$

where 0 on the right hand side denotes the 2×2 zero matrix. Thus D is closed under scalar multiplication.

Let $A, B \in D$. Then, using the properties of the trace and vector space, we find

$$\mathrm{tr}(A + B) = \mathrm{tr}(A) + \mathrm{tr}(B) = 0 + 0 = 0$$

where 0 on the right hand side denotes the 2×2 zero matrix. Thus D is closed under vector addition.

Question 11

Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

(11.1) Find the range of A.

The range of A is the column space of A

$$\begin{aligned} R(A) &= \left\{ a \begin{bmatrix} 1 \\ 2 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \end{bmatrix} \,:\, a, b, c \in \mathbb{R} \right\} && \textit{(this is sufficient)} \\ &= \left\{ \begin{bmatrix} b + (a+b+c) \\ a + (a+b+c) \end{bmatrix} \,:\, a, b, c \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} b' \\ a' \end{bmatrix} \,:\, a', b' \in \mathbb{R} \right\} && \textit{(e.g. set } c = -a - b, \\ & && a = a',\ b = b') \\ &= \mathbb{R}^2. \end{aligned}$$

(11.2) Find the rank of A.

The rank of A is the dimension of the column space of A (and also the dimension of the row space of A). Since the number of rows is 2, $\text{rank}(A) \leq 2$. *From Question (12.1) it is clear that* $\text{rank}(A) = 2$. *Here we show an* ***alternative*** method *to obtain this answer. The dimension of the row space is the number of non-zero rows obtained after row reduction of A:*

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \end{bmatrix} \qquad (R_2 \leftarrow R_2 - 2R_1)$$

Since the last matrix is in upper triangular form, no further row reduction steps are required. There are two non-zero rows, consequently $\text{rank}(A) = 2$.

(11.3) Find the null space of A.

The null space of A is given by

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \,:\, x, y, z \in \mathbb{R},\ A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

Row reduction of the augmented matrix yields

$$\begin{aligned} \begin{bmatrix} 1 & 2 & 1 & : & 0 \\ 2 & 1 & 1 & : & 0 \end{bmatrix} &\rightarrow \begin{bmatrix} 1 & 2 & 1 & : & 0 \\ 0 & -3 & -1 & : & 0 \end{bmatrix} && (R_2 \leftarrow R_2 - 2R_1) \\ &\rightarrow \begin{bmatrix} 1 & 2 & 1 & : & 0 \\ 0 & 1 & 1/3 & : & 0 \end{bmatrix} && (R_2 \leftarrow -R_2/3) \\ &\rightarrow \begin{bmatrix} 1 & 0 & 1/3 & : & 0 \\ 0 & 1 & 1/3 & : & 0 \end{bmatrix} && (R_1 \leftarrow R_1 - 2R_2) \end{aligned}$$

so that the null space of A is

$$\left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathbb{R},\ A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} -1/3z \\ -1/3z \\ z \end{bmatrix} : z \in \mathbb{R} \right\}$$
$$= \left\{ z \begin{bmatrix} -1/3 \\ -1/3 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}$$

(11.4) Find the nullity of A.

The nullity of A is the dimension of the null space of A (by Theorem 4.8.3: the number of free parameters in Question (12.3)) which is 1. ***Alternatively****, since* $\text{rank}(A) + \text{nullity}(A) = 3$ *and* $\text{rank}(A) = 2$ *then* $\text{nullity}(A) = 1$.

Question 12

Given the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$$

(12.1) Find the range of A.

The range of A is the column space of A

$$R(A) = \left\{ a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

(12.2) Find the rank of A.

The rank of A is the dimension of the column space of A (and also the dimension of the row space of A). Since the number of rows is 2, $\text{rank}(A) \leq 2$. *From Question (12.1) it is clear that* $\text{rank}(A) = 2$. *Here we show an* ***alternative*** *method to obtain this answer. The dimension of the row space is the number of non-zero rows obtained after row reduction of A:*

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 1 \end{bmatrix} \qquad (R_2 \leftarrow R_2 - 2R_1)$$
$$\to \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & -1 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_1)$$
$$\to \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_2)$$

Since the last matrix is in upper triangular form, no further row reduction steps are required. There are two non-zero rows, consequently $\text{rank}(A) = 2$.

(12.3) Find the nullspace of A.

The null space of A is given by

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \; : \; x, y \in \mathbb{R}, \; A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Row reduction of the augmented matrix yields

$$\begin{bmatrix} 1 & 2 & : & 0 \\ 2 & 1 & : & 0 \\ 1 & 1 & : & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & : & 0 \\ 0 & -1 & : & 0 \\ 1 & 1 & : & 0 \end{bmatrix} \qquad (R_2 \leftarrow R_2 - 2R_1)$$

$$\rightarrow \begin{bmatrix} 1 & 2 & : & 0 \\ 0 & -1 & : & 0 \\ 0 & -1 & : & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_1)$$

$$\rightarrow \begin{bmatrix} 1 & 2 & : & 0 \\ 0 & -1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_2)$$

$$\rightarrow \begin{bmatrix} 1 & 2 & : & 0 \\ 0 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \qquad (R_2 \leftarrow -R_2)$$

$$\rightarrow \begin{bmatrix} 1 & 0 & : & 0 \\ 0 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \qquad (R_1 \leftarrow R_1 - 2R_2)$$

so that the null space of A is

$$\left\{ \begin{bmatrix} x \\ y \end{bmatrix} \; : \; x, y \in \mathbb{R}, \; A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

(12.4) Find the nullity of A.

The nullity of A is the dimension of the null space of A (by Theorem 4.8.3: the number of free parameters in Question (12.3)) which is 0. ***Alternatively****, since* $\text{rank}(A) + \text{nullity}(A) = 2$ *and* $\text{rank}(A) = 2$ *then* $\text{nullity}(A) = 0$.

Question 13

(13.1) Let S and T be subspaces of a vector space V. Prove that $S \cap T$ is also a subspace of V.

First we must show that $S \cap T$ is not empty. Note that if S, T and $S \cap T$ are subspaces of V then they are vector spaces with the same zero vector $\mathbf{0}$ *as for V. It suffices to check whether the zero vector is in $S \cap T$. Obviously* $\mathbf{0} \in S$ *and* $\mathbf{0} \in T$ *since S and T are subspaces of V. Thus* $\mathbf{0} \in S \cap T$.

Next we prove the closure under vector addition and under scalar multiplication.

Let $A \in S \cap T$ and $k \in \mathbb{R}$. Then, using the properties of the vector spaces S and T, we have $A, kA \in S$ and $A, kA \in T$ so that $kA \in S \cap T$. Thus $S \cap T$ is closed under scalar multiplication.

Let $A, B \in S \cap T$. Consequently $A, B \in S$ and $A, B \in T$. Using the properties of the vector spaces S and T we find $A + B \in S$ and $A + B \in T$ so that $A + B \in S \cap T$. Thus $S \cap T$ is closed under vector addition.

It follows that $S \cap T$ is also a subspace of V.

(13.2) Let $S = \{(a+b, a, -a, -b) \,:\, a, b \in \mathbb{R}\}$ and $T = \{(x, 2y, -y, -x) \,:\, x, y \in \mathbb{R}\}$.

(a) Show that S and T are subspaces of $\mathbb{R}^4$.

Setting $a = b = 0$ we find that the zero vector of $\mathbb{R}^4$ is in S. Let $a, b, k \in \mathbb{R}$ and $(a+b, a, -a, -b) \in S$, then

$$k(a+b, a, -a, -b) = (ka+kb, ka, -ka, -kb) = (a'+b', a', -a', -b') \in S$$

where $a' := ka$ and $b' := kb$. Thus S is closed under scalar multiplication. Let $a_1, b_1, a_2, b_2 \in \mathbb{R}$ and $(a_1+b_1, a_1, -a_1, -b_1), (a_2+b_2, a_2, -a_2, -b_2) \in S$, then

$$\begin{aligned}(a_1+b_1, a_1, -a_1, -b_1) &+ (a_2+b_2, a_2, -a_2, -b_2)\\ &= (a_1+a_2+b_1+b_2, a_1+a_2, -a_1-a_2, -b_1-b_2)\\ &= (a'+b', a', -a', -b') \in S\end{aligned}$$

where $a' := a_1 + a_2$ and $b' := b_1 + b_2$. Thus S is closed under vector addition. It follows that S is a subspace of $\mathbb{R}^4$.

Setting $x = y = 0$ we find that the zero vector of $\mathbb{R}^4$ is in T. Let $k, x, y \in \mathbb{R}$ and $(x, 2y, -y, -x) \in T$, then

$$k(x, 2y, -y, -x) = (kx, 2ky, -ky, -kx) = (x', 2y', -y', -x') \in T$$

where $x' := kx$ and $y' := ky$. Thus T is closed under scalar multiplication. Let $x_1, y_1, x_2, y_2 \in \mathbb{R}$ and $(x_1, 2y_1, -y_1, -x_1), (x_2, 2y_2, -y_2, -x_2) \in T$, then

$$\begin{aligned}(x_1, 2y_1, -y_1, -x_1) + (x_2, 2y_2, -y_2, -x_2) &= (x_1+x_2, 2(y_1+y_2), -y_1-y_2, -x_1-x_2)\\ &= (x', 2y', -y', -x') \in T\end{aligned}$$

where $x' := x_1 + x_2$ and $y' := y_1 + y_2$. Thus T is closed under vector addition. It follows that T is a subspace of $\mathbb{R}^4$.

(b) Find the dimension of S and T.

Note that elements of S can be written in the form

$$(a+b, a, -a, -b) = a(1, 1, -1, 0) + b(1, 0, 0, -1)$$

and that

$$c_1(1,1,-1,0) + c_2(1,0,0,-1) = (0,0,0,0)$$

has only the trivial solution $c_1 = c_2 = 0$ (from the third and fourth components). The two vectors $(1,1,-1,0)$ and $(1,0,0,-1)$ are linearly independent, so the dimension of S is 2.

Note that elements of T can be written in the form

$$(x, 2y, -y, -x) = x(1,0,0,-1) + y(0,2,-1,0)$$

and that

$$c_1(1,0,0,-1) + c_2(0,2,-1,0) = (0,0,0,0)$$

has only the trivial solution $c_1 = c_2 = 0$ (from the first and second components). The two vectors $(1,0,0,-1)$ and $(0,2,-1,0)$ are linearly independent, so the dimension of T is 2.

(c) Find $S \cap T$ and hence a basis and the dimension of $S \cap T$.

Let $a, b, x, y \in \mathbb{R}$ and let $(a+b, a, -a, -b) \in S$ and $(x, 2y, -y, -x) \in T$. The intersection $S \cap T$ is given by a, b, x, y satisfying

$$(a+b, a, -a, -b) = (x, 2y, -y, -x).$$

The third component gives $a = y$ while the second provides $a = 2y = 2a$ so that $a = y = 0$. The first component now gives $a + b = b = x$ and the fourth component $-b = -x$ is identically satisfied. It follows that

$$S \cap T = \{(x, 0, 0, -x) \, : \, x \in \mathbb{R}\}.$$

We have one free parameter (x) describing $S \cap T$. The dimension is 1. A basis is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}.$$

(13.3) Let

$$S = \{1 - x;\ 5 + 3x - 2x^2;\ 1 + 3x - x^2\} \subset P_2.$$

Find a basis and the dimension for span(S).

There are different approaches to determine a basis. First we consider a straightforward method given that $B = \{1,\ x,\ x^2\}$ is a basis for P_2. Then we have the representations (using the ordering $1,\ x,\ x^2$)

$$[1 - x]_B = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad [5 + 3x - 2x^2]_B = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix} \qquad [1 + 3x - x^2]_B = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}$$

Using these column vectors as rows in a matrix and applying row reduction

$$\begin{array}{r} \\ -5R1 \\ -R1 \end{array}\begin{bmatrix} 1 & -1 & 0 \\ 5 & 3 & -2 \\ 1 & 3 & -1 \end{bmatrix} \begin{array}{c} \\ \rightarrow \\ -R2/2 \end{array} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 8 & -2 \\ 0 & 4 & -1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 8 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

we find the basis $\{1 - x;\, 8x - 2x^2\}$ *and dimension 2. Of course, other choices of basis are possible.*

Note: In the absence of an existing basis for P_2 *we need to first determine which subsets are linearly independent and take (one of) the largest linearly independent subsets as a basis. Since* S *is finite, we can do this by enumerating the subsets. First we try all subsets with 3 elements (i.e. the set* S*):*

$$\begin{aligned} 0 &= c_{11}(1 - x) + c_{12}(5 + 3x - 2x^2) + c_{13}(1 + 3x - x^2) = 0 \\ 0 &= c_{11} + 5c_{12} + c_{13} \\ 0 &= -c_{11} + 3c_{12} + 3c_{13} \\ 0 &= -2c_{12} - c_{13} \end{aligned}$$

The last equation yields $c_{13} = -2c_{12}$. *The second last equation becomes* $c_{11} = -3c_{12}$. *For example, setting* $c_{12} = 1$ *so that* $c_{11} = -3$ *and* $c_{13} = -2$ *yields a nontrivial solution. This set is not linearly independent.*

Next we try all subsets with 2 elements until we find a linearly independent set.
First $\{1 - x;\, 5 + 3x - 2x^2\}$*:*

$$\begin{aligned} 0 &= c_{21}(1 - x) + c_{22}(5 + 3x - 2x^2) = 0 \\ 0 &= c_{21} + 5c_{22} \\ 0 &= -c_{21} + 3c_{22} \\ 0 &= c_{22} \end{aligned}$$

which has only the trivial solution $c_{21} = c_{22} = 0$. *Thus a basis is* $\{1 - x;\, 5 + 3x - 2x^2\}$ *and the dimension is 2. (In fact, any two elements from* S *form a basis for* $\operatorname{span}(S)$*; test this yourself.)*

Question 14

(14.1) Let $S = \{(a, b, b, a + c) \,:\, a, b, c \in \mathbb{R}\}$ and $T = \{(x, y - x, y + z, z) \,:\, x, y, z \in \mathbb{R}\}$.

(a) Show that S and T are subspaces of $\mathbb{R}^4$.

Setting $a = b = c = 0$ *we find that the zero vector of* $\mathbb{R}^4$ *is in* S. *Let* $a, b, c, k \in \mathbb{R}$ *and* $(a, b, b, a + c) \in S$, *then*

$$k(a, b, b, a + c) = (ka, kb, kb, ka + kc) = (a', b', b', a' + c') \in S$$

where $a' := ka$, $b' = kb$ and $c' := kc$. Thus S is closed under scalar multiplication. Let $a_1, b_1, a_2, b_2, c_1, c_2 \in \mathbb{R}$ and $(a_1, b_1, b_1, a_1 + c_1), (a_2, b_2, b_2, a_2 + c_2) \in S$, then

$$\begin{aligned}(a_1, b_1, b_1, a_1 + c_1) &+ (a_2, b_2, b_2, a_2 + c_2) \\ &= (a_1 + a_2, b_1 + b_2, b_1 + b_2, a_1 + c_1 + a_2 + c_2) \\ &= (a', b', b', a' + c') \in S\end{aligned}$$

where $a' := a_1 + a_2$, $b' := b_1 + b_2$ and $c' := c_1 + c_2$. Thus S is closed under vector addition. It follows that S is a subspace of $\mathbb{R}^4$.

Setting $x = y = z = 0$ we find that the zero vector of $\mathbb{R}^4$ is in T. Let $k, x, y, z \in \mathbb{R}$ and $(x, y - x, y + z, z) \in T$, then

$$k(x, y - x, y + z, z) = (kx, ky - kx, ky + kz, kz) = (x', y' - x', y' + z', z') \in T$$

where $x' := kx$, $y' = ky$ and $z' := kz$. Thus T is closed under scalar multiplication. Let $x_1, y_1, x_2, y_2, z_1, z_2 \in \mathbb{R}$ and $(x_1, y_1 - x_1, y_1 + z_1, z_1), (x_2, y_2 - x_2, y_2 + z_2, z_2) \in T$, then

$$\begin{aligned}(x_1, y_1 - x_1, y_1 + z_1, z_1) &+ (x_2, y_2 - x_2, y_2 + z_2, z_2) \\ &= (x_1 + x_2, y_1 - x_1 + y_2 - x_2, y_1 + z_1 + y_2 + z_2, z_1 + z_2) \\ &= (x', y' - x', y' + z', z') \in T\end{aligned}$$

where $x' := x_1 + x_2$, $y''' = y_1 + y_2$ and $y' := y_1 + y_2$. Thus T is closed under vector addition. It follows that T is a subspace of $\mathbb{R}^4$.

(b) Find the dimension of S and T.

Note that elements of S can be written in the form

$$(a, b, b, a + c) = a(1, 0, 0, 1) + b(0, 1, 1, 0) + c(0, 0, 0, 1)$$

and that

$$c_1(1, 0, 0, 1) + c_2(0, 1, 1, 0) + c_3(0, 0, 0, 1) = (0, 0, 0, 0)$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$ (from the third and fourth components). The three vectors $(1, 0, 0, 1)$, $(0, 1, 1, 0)$ and $(0, 0, 0, 1)$ are linearly independent, so the dimension of S is 3.

Note that elements of T can be written in the form

$$(x, y - x, y + z, z) = x(1, -1, 0, 0) + y(0, 1, 1, 0) + z(0, 0, 1, 1)$$

and that

$$c_1(1, -1, 0, 0) + c_2(0, 1, 1, 0) + c_3(0, 0, 1, 1) = (0, 0, 0, 0)$$

has only the trivial solution $c_1 = c_2 = c_3 = 0$ (from the first and second components). The three vectors $(1, -1, 0, 0)$, $(0, 1, 1, 0)$ and $(0, 0, 1, 1)$ are linearly independent, so the dimension of T is 3.

(c) Find $S \cap T$ and hence a basis and the dimension of $S \cap T$.

Let $a, b, c, x, y, z \in \mathbb{R}$ and let $(a, b, b, a+c) \in S$ and $(x, y-x, y+z, z) \in T$. The intersection $S \cap T$ is given by a, b, x, y satisfying

$$\begin{aligned}(a, b, b, a+c) &= (x, y-x, y+z, z) \\ &\Rightarrow \quad a = x,\ b = y - x = y + z,\ a + c = z \\ &\Rightarrow \quad a = x,\ b = y - a,\ z = -a,\ c = -2a.\end{aligned}$$

It follows that

$$S \cap T = \{(a, y-a, y-a, -a) \,:\, a, y \in \mathbb{R}\}.$$

We have two free parameters (a and y) describing $S \cap T$. The dimension is 2. A basis is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

(14.2) Let

$$S = \{t^3 + t^2 - 2t + 1;\ t^2 + 1;\ t^3 - 2t;\ 2t^3 + 3t^2 - 4t + 3\} \subset P_3.$$

Find a basis and the dimension for span(S).

There are different approaches to determine a basis. First we consider a straightforward method given that $B = \{1,\ t,\ t^2,\ t^3\}$ is a basis for P_3. Then we have the representations (using the ordering $1,\ t,\ t^2,\ t^3$)

$$[t^3 + t^2 - 2t + 1]_B = \begin{bmatrix} 1 \\ -2 \\ 1 \\ 1 \end{bmatrix} \quad [t^2 + 1]_B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad [t^3 - 2t]_B = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix} \quad [2t^3 + 3t^2 - 4t + 3]_B = \begin{bmatrix} 3 \\ -4 \\ 3 \\ 2 \end{bmatrix}$$

Using these column vectors as rows in a matrix and applying row reduction

$$\begin{matrix} -R_3 \\ \\ \\ \\ \end{matrix} \begin{bmatrix} 1 & -2 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 3 & -4 & 3 & 2 \end{bmatrix} \rightarrow \begin{matrix} \\ -R_1 \\ \\ -3R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 3 & -4 & 3 & 2 \end{bmatrix}$$

$$\rightarrow \begin{matrix} \\ =R_3 \\ =R_2 \\ -2R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & -4 & 0 & 2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

we find the basis $\{1 + t^2,\ -2t + t^3\}$ and dimension 2. Of course, other choices of basis are possible.

Question 15

Let $B = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $B' = \{\mathbf{q}_1, \mathbf{q}_2\}$ be basis for P_1, where

$$\mathbf{p}_1 = 1 + 2x, \quad \mathbf{p}_2 = 3 - x, \quad \mathbf{q}_1 = 2 - 2x, \quad \mathbf{q}_2 = 4 + 3x.$$

(15.1) Find the transition matrix $P_{B' \longrightarrow B}$ from the B'-basis to the B-basis

Solving

$$\mathbf{q}_1 = a\mathbf{p}_1 + b\mathbf{p}_2 \quad \Leftrightarrow \quad 2 - 2x = (a + 3b) + (2a - b)x \quad \Leftrightarrow \quad a + 3b = 2,\ 2a - b = -2$$

for $a, b \in \mathbb{R}$ *yields* $a = -4/7$ *and* $b = 6/7$. *Solving*

$$\mathbf{q}_2 = a\mathbf{p}_1 + b\mathbf{p}_2 \quad \Leftrightarrow \quad 4 + 3x = (a + 3b) + (2a - b)x \quad \Leftrightarrow \quad a + 3b = 4,\ 2a - b = 3$$

for $a, b \in \mathbb{R}$ *yields* $a = 13/7$ *and* $b = 5/7$. *Thus*

$$P_{B' \longrightarrow B} = \frac{1}{7}\begin{bmatrix} -4 & 13 \\ 6 & 5 \end{bmatrix}.$$

(15.2) Find the transition matrix $Q_{B' \longrightarrow B}$ from the B-basis to the B'-basis

Solving

$$\mathbf{p}_1 = a\mathbf{q}_1 + b\mathbf{q}_2 \quad \Leftrightarrow \quad 1 + 2x = (2a + 4b) + (-2a + 3b)x \quad \Leftrightarrow \quad 2a + 4b = 1,\ -2a + 3b = 2$$

for $a, b \in \mathbb{R}$ *yields* $a = -5/14$ *and* $b = 3/7$. *Solving*

$$\mathbf{p}_2 = a\mathbf{q}_1 + b\mathbf{q}_2 \quad \Leftrightarrow \quad 3 - x = (2a + 4b) + (-2a + 3b)x \quad \Leftrightarrow \quad 2a + 4b = 3,\ -2a + 3b = -1$$

for $a, b \in \mathbb{R}$ *yields* $a = 13/14$ *and* $b = 2/7$. *Thus*

$$Q_{B \longrightarrow B'} = \frac{1}{14}\begin{bmatrix} -5 & 13 \\ 6 & 4 \end{bmatrix}.$$

Note that

$$P_{B' \longrightarrow B} Q_{B \longrightarrow B'} = Q_{B \longrightarrow B'} P_{B' \longrightarrow B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Leftrightarrow \quad Q_{B \longrightarrow B'} = P^{-1}_{B' \longrightarrow B}$$

so that an alternative method to find $Q_{B \longrightarrow B'}$ *is to find* $P^{-1}_{B' \longrightarrow B}$.

(15.3) Compute $[\mathbf{p}]_{B'}$ if $\mathbf{p} = 5 - x$.

Solving

$$\mathbf{p} = a\mathbf{q}_1 + b\mathbf{q}_2 \quad \Leftrightarrow \quad 5 - x = (2a + 4b) + (-2a + 3b)x \quad \Leftrightarrow \quad 2a + 4b = 5,\ -2a + 3b = -1$$

for $a, b \in \mathbb{R}$ *yields* $a = 19/14$ *and* $b = 4/7$. *Thus*

$$[\mathbf{p}]_{B'} = \frac{1}{14}\begin{bmatrix} 19 \\ 8 \end{bmatrix}.$$

Question 16

Let $B = \{\mathbf{p}_1, \mathbf{p}_2\}$ and $B' = \{\mathbf{q}_1, \mathbf{q}_2\}$ be basis for P_1, where

$$\mathbf{p}_1 = 1 + 3x, \quad \mathbf{p}_2 = 2 - x, \quad \mathbf{q}_1 = 1 - 2x, \quad \mathbf{q}_2 = 1 + x.$$

(16.1) Find the transition matrix $P_{B' \longrightarrow B}$ from the B'-basis to the B-basis

Solving

$$\mathbf{q}_1 = a\mathbf{p}_1 + b\mathbf{p}_2 \quad \Leftrightarrow \quad 1 - 2x = (a + 2b) + (3a - b)x \quad \Leftrightarrow \quad a + 2b = 1,\ 3a - b = -2$$

for $a, b \in \mathbb{R}$ *yields* $b = 5/7$ *and* $a = -3/7$. *Solving*

$$\mathbf{q}_2 = a\mathbf{p}_1 + b\mathbf{p}_2 \quad \Leftrightarrow \quad 1 + x = (a + 2b) + (3a - b)x \quad \Leftrightarrow \quad a + 2b = 1,\ 3a - b = 1$$

for $a, b \in \mathbb{R}$ *yields* $b = 2/7$ *and* $a = 3/7$. *Thus*

$$P_{B' \longrightarrow B} = \frac{1}{7}\begin{bmatrix} -3 & 3 \\ 5 & 2 \end{bmatrix}.$$

(16.2) Find the transition matrix $Q_{B' \longrightarrow B}$ from the B-basis to the B'-basis

Solving

$$\mathbf{p}_1 = a\mathbf{q}_1 + b\mathbf{q}_2 \quad \Leftrightarrow \quad 1 + 3x = (a + b) + (-2a + b)x \quad \Leftrightarrow \quad a + b = 1,\ -2a + b = 3$$

for $a, b \in \mathbb{R}$ *yields* $a = -2/3$ *and* $b = 5/3$. *Solving*

$$\mathbf{p}_2 = a\mathbf{q}_1 + b\mathbf{q}_2 \quad \Leftrightarrow \quad 2 - x = (a + b) + (-2a + b)x \quad \Leftrightarrow \quad a + b = 2,\ -2a + b = -1$$

for $a, b \in \mathbb{R}$ *yields* $a = 1$ *and* $b = 1$. *Thus*

$$Q_{B \longrightarrow B'} = \begin{bmatrix} -2/3 & 1 \\ 5/3 & 1 \end{bmatrix}.$$

Note that

$$P_{B' \longrightarrow B} Q_{B \longrightarrow B'} = Q_{B \longrightarrow B'} P_{B' \longrightarrow B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \Leftrightarrow \quad Q_{B \longrightarrow B'} = P^{-1}_{B' \longrightarrow B}$$

so that an alternative method to find $Q_{B \longrightarrow B'}$ *is to find* $P^{-1}_{B' \longrightarrow B}$.

(16.3) Compute $[\mathbf{p}]_{B'}$ if $\mathbf{p} = 3 + x$.

Solving

$$\mathbf{p} = a\mathbf{q}_1 + b\mathbf{q}_2 \quad \Leftrightarrow \quad 3 + x = (a + b) + (-2a + b)x \quad \Leftrightarrow \quad a + b = 3,\ -2a + b = 1$$

for $a, b \in \mathbb{R}$ *yields* $a = 2/3$ *and* $b = 7/3$. *Thus*

$$[\mathbf{p}]_{B'} = \frac{1}{3}\begin{bmatrix} 2 \\ 7 \end{bmatrix}.$$

Question 17

Let $B_1 = \{ 1 + x,\ 1 - x \} \subset P_1$ and $B_2 = \{ 1 + 2x,\ 2 + x \} \subset P_1$ be two bases for P_1, where the usual left to right ordering is assumed.

(17.1) Show that B_2 is a basis for P_1.

First we show that $\operatorname{span} B_2 = P_1$*:*

$$\operatorname{span} B_2 = \{ a(1 + 2x) + b(2 + x) \ : \ a, b \in \mathbb{R} \} = \{ (a + 2b) + (2a + b)x \ : \ a, b \in \mathbb{R} \}$$

Now let $\alpha + \beta x \in P_1$ *where* $\alpha, \beta \in \mathbb{R}$ *are arbitrary. Solving* $\alpha + \beta x = (a+2b) + (2a+b)x$ *for* a *and* b *yields* $a + 2b = \alpha$ *and* $2a + b = \beta$ *so that* $a = (2\beta - \alpha)/3$ *and* $b = (2\alpha - \beta)/3$*, i.e.* $\alpha + \beta x \in \operatorname{span} B_2$*. Since* $a + 2b, 2a + b \in \mathbb{R}$ *we also have* $(a + 2b) + (2a + b)x \in P_1$*. Thus* $\operatorname{span} B_2 = P_1$*.*

Next we show that B_2 *is a linearly independent set of vectors. Solving*

$$c_1(1 + 2x) + c_2(2 + x) = (c_1 + 2c_2) + (2c_1 + c_2)x = 0 + 0x$$

for $c_1, c_2 \in \mathbb{R}$ *yields* $c_1 + 2c_2 = 0$ *and* $2c_1 + c_2 = 0$ *(here we used the linear independence of the standard basis* $\{1, x\} \equiv \{1+0x, 0+1x\}$ *in* P_1*). Thus* $c_1 = -2c_2$ *and* $c_2 = -2c_1 = 4c_2$ *so that* $c_1 = c_2 = 0$ *is the only solution. Thus* B_2 *is a linearly independent set of vectors.*

Consequently B_2 *is a basis for* P_1*.*

Alternatively*, since* B_2 *consists of two vectors (and* P_1 *is 2-dimensional) by Theorem 4.5.4 of the textbook we need only show that* B_2 *spans* P_1 *or that* B_2 *is linearly independent, i.e. we can omit either half of the proof above.*

Alternative:
The matrix representations of the elements of B_2 *with respect to the standard basis* B_{P_1} *in* P_1 *are*

$$[1 + 2x]_{B_{P_1}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad [2 + x]_{B_{P_1}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

and taking the determinant of the matrix composed of these vectors as columns yields

$$\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3 \neq 0$$

Consequently B_2 *is a basis for* P_1*.*

(17.2) Find the transition matrix $P_{B_1\to B_2}$.

We express the elements of B_1 in terms of B_2 (using left to right ordering, i.e. $1+x$ first in B_1 and $1+2x$ first in B_2):

$$\begin{aligned}1+x &= p_{11}(1+2x)+p_{21}(2+x) = (p_{11}+2p_{21})+(2p_{11}+p_{21})x\\ 1-x &= p_{12}(1+2x)+p_{22}(2+x) = (p_{12}+2p_{22})+(2p_{12}+p_{22})x\end{aligned}$$

and solving for $p_{11}, p_{12}, p_{21}, p_{22} \in \mathbb{R}$ yields the two sets of equations (once again using linear independence of the standard basis in P_1)

$$\begin{aligned}1 &= p_{11}+2p_{21} & 1 &= p_{12}+2p_{22}\\ 1 &= 2p_{11}+p_{21} & -1 &= 2p_{12}+p_{22}\end{aligned}$$

These equations are straightforward to solve, $p_{11} = 1/3$, $p_{21} = 1/3$, $p_{12} = -1$ and $p_{22} = 1$. The transition matrix is

$$P_{B_1\to B_2} = \begin{bmatrix} p_{11} & p_{12}\\ p_{21} & p_{22}\end{bmatrix} = \begin{bmatrix} 1/3 & -1\\ 1/3 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -3\\ 1 & 3\end{bmatrix}.$$

Alternative:
Using the fact that row reduction of the augmented matrix $\begin{bmatrix} A & : & B\end{bmatrix}$ leads to $\begin{bmatrix} I & : & A^{-1}B\end{bmatrix}$ for square matrices A and B with the same number of rows and A invertible, we have that $\begin{bmatrix} P_{B_2\to B} & : & P_{B_1\to B}\end{bmatrix}$ reduces to

$$\begin{bmatrix} I & : & P_{B_2\to B}^{-1}P_{B_1\to B}\end{bmatrix} = \begin{bmatrix} I & : & P_{B\to B_2}P_{B_1\to B}\end{bmatrix} = \begin{bmatrix} I & : & P_{B_1\to B_2}\end{bmatrix}$$

for any basis B of P_1. Let $B = B_{P_1} = \{1, x\}$ be the standard basis in P_1, then

$$\begin{aligned}P_{B_2\to B} &= \begin{bmatrix}[1+2x]_{B_{P_1}} & [2+x]_{B_{P_1}}\end{bmatrix} = \begin{bmatrix} 1 & 2\\ 2 & 1\end{bmatrix},\\ P_{B_1\to B} &= \begin{bmatrix}[1+x]_{B_{P_1}} & [1-x]_{B_{P_1}}\end{bmatrix} = \begin{bmatrix} 1 & 1\\ 1 & -1\end{bmatrix}\end{aligned}$$

$$\begin{aligned}\begin{bmatrix} 1 & 2 & : & 1 & 1\\ 2 & 1 & : & 1 & -1\end{bmatrix} &\to \begin{bmatrix} 1 & 2 & : & 1 & 1\\ 0 & -3 & : & -1 & -3\end{bmatrix} && (R_2 \leftarrow R_2 - 2R_1)\\ &\to \begin{bmatrix} 1 & 2 & : & 1 & 1\\ 0 & 1 & : & 1/3 & 1\end{bmatrix} && (R_2 \leftarrow -R_2/3)\\ &\to \begin{bmatrix} 1 & 0 & : & 1/3 & -1\\ 0 & 1 & : & 1/3 & 1\end{bmatrix} && (R_1 \leftarrow R_1 - 2R_2).\end{aligned}$$

Thus

$$P_{B_1\to B_2} = \begin{bmatrix} 1/3 & -1\\ 1/3 & 1\end{bmatrix}.$$

(17.3) Let B_3 be a basis for P_1 and $P_{B_2\to B_3}$ be the transition matrix from B_2 to B_3 given by $P_{B_2\to B_3} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

(a) Find the transition matrix $P_{B_1\to B_3}$.

This is a change of basis from B_1 to B_3, which can be achieved by changing basis from B_1 to B_2 and then again changing basis from B_2 to B_3 ($B_1 \to B_3 \equiv B_1 \to B_2 \to B_3$):

$$P_{B_1\to B_3} = P_{B_2\to B_3} P_{B_1\to B_2} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 1 & -3 \\ 1 & 3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}.$$

(b) Use $P_{B_2\to B_3}$ to find B_3.

The columns of $P_{B_2\to B_3}$ give B_2 in terms of B_3. Let $B_3 = \{\, p(x),\, q(x) \,\}$ for some $p(x), q(x) \in P_1$. The first column of $P_{B_2\to B_3}$ provides

$$[1+2x]_{B_3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad 1+2x = 1p(x) + 0q(x) = p(x),$$

i.e. $p(x) = 1+2x$. The second column of $P_{B_2\to B_3}$ provides

$$[2+x]_{B_3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad 2+x = 1p(x) + 1q(x) = p(x) + q(x) = 1+2x+q(x)$$

so that $q(x) = 1-x$. Thus

$$B_3 = \{\, 1+2x,\, 1-x \,\}.$$

Exercise: verify this answer by calculating $P_{B_2\to B_3}$ to see if the matrix you obtain is the same as given in the question. Similarly, verify that $P_{B_1\to B_3}$ is the same as found in Question (18.3(a)).

Alternative: *We use the columns of*

$$P_{B_3\to B_2} = P^{-1}_{B_2\to B_3} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

to find

$$B_3 = \{\, 1(1+2x) + 0(2+x),\, -1(1+2x) + 1(2+x) \,\} = \{\, 1+2x,\, 1-x \,\}.$$

Question 18

Let $B_1 = \{\, 1+x,\ 1-x \,\} \subset P_1$ and $B_2 = \{\, 1+2x,\ 2+x \,\} \subset P_1$ be two bases for P_1, where the usual left to right ordering is assumed.

(18.1) Show that B_1 is a basis for P_1.

First we show that $\operatorname{span} B_1 = P_1$*:*

$$\operatorname{span} B_1 = \{\, a(1+x) + b(1-x) \;:\; a, b \in \mathbb{R} \,\}P_1 = \{\, (a+b) + (a-b)x \;:\; a, b \in \mathbb{R} \,\}$$

Now let $\alpha + \beta x \in P_1$ *where* $\alpha, \beta \in \mathbb{R}$ *are arbitrary. Solving* $\alpha + \beta x = (a+b) + (a-b)x$ *for* a *and* b *yields* $a + b = \alpha$ *and* $a - b = \beta$ *so that* $a = (\alpha + \beta)/2$ *and* $b = (\alpha - \beta)/2$, *i.e.* $\alpha + \beta x \in \operatorname{span} B_1$. *Since* $a + b, a - b \in \mathbb{R}$ *we also have* $(a+b) + (a-b)x \in P_1$. *Thus* $\operatorname{span} B_1 = P_1$.

Next we show that B_1 *is a linearly independent set of vectors. Solving*

$$c_1(1+x) + c_2(1-x) = (c_1 + c_2) + (c_1 - c_2)x = 0 + 0x$$

for $c_1, c_2 \in \mathbb{R}$ *yields* $c_1 + c_2 = 0$ *and* $c_1 - c_2 = 0$ *(here we used the linear independence of the standard basis* $\{1, x\} \equiv \{1 + 0x, 0 + 1x\}$ *in* P_1*). Thus* $c_1 = -c_2$ *and* $c_2 = c_1 = -c_2$ *so that* $c_1 = c_2 = 0$ *is the only solution. Thus* B_1 *is a linearly independent set of vectors.*

Consequently B_1 *is a basis for* P_1.

Alternatively, *since* B_1 *consists of two vectors (and* P_1 *is 2-dimensional) by Theorem 4.5.4 of the textbook we need only show that* B_1 *spans* P_1 *or that* B_1 *is linearly independent, i.e. we can omit either half of the proof above.*

Alternative:
The matrix representations of the elements of B_1 *with respect to the standard basis* B_{P_1} *in* P_1 *are*

$$[1+x]_{B_{P_1}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad [1-x]_{B_{P_1}} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and taking the determinant of the matrix composed of these vectors as columns yields

$$\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2 \neq 0$$

Consequently B_1 *is a basis for* P_1.

(18.2) Find the transition matrix $P_{B_2\to B_1}$.

We express the elements of B_2 in terms of B_1 (using left to right ordering, i.e. $1+x$ first in B_1 and $1+2x$ first in B_2):

$$\begin{aligned} 1+2x &= p_{11}(1+x)+p_{21}(1-x) = (p_{11}+p_{21})+(p_{11}-p_{21})x \\ 2+x &= p_{12}(1+x)+p_{22}(1-x) = (p_{12}+p_{22})+(p_{12}-p_{22})x \end{aligned}$$

and solving for $p_{11}, p_{12}, p_{21}, p_{22} \in \mathbb{R}$ yields the two sets of equations (once again using linear independence of the standard basis in P_1)

$$\begin{aligned} 1 &= p_{11}+p_{21} & 2 &= p_{12}+p_{22} \\ 2 &= p_{11}-p_{21} & -1 &= p_{12}-p_{22} \end{aligned}$$

These equations are straightforward to solve, $p_{11}=3/2$, $p_{21}=-1/2$, $p_{12}=3/2$ and $p_{22}=1/2$. The transition matrix is

$$P_{B_2\to B_1} = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 3/2 & 3/2 \\ -1/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix}.$$

Alternative:
Using the fact that row reduction of the augmented matrix $\begin{bmatrix} A & : & B \end{bmatrix}$ leads to $\begin{bmatrix} I & : & A^{-1}B \end{bmatrix}$ for square matrices A and B with the same number of rows and A invertible, we have that $\begin{bmatrix} P_{B_1\to B} & : & P_{B_2\to B} \end{bmatrix}$ reduces to

$$\begin{bmatrix} I & : & P_{B_1\to B}^{-1}P_{B_2\to B} \end{bmatrix} = \begin{bmatrix} I & : & P_{B\to B_1}P_{B_2\to B} \end{bmatrix} = \begin{bmatrix} I & : & P_{B_2\to B_1} \end{bmatrix}$$

for any basis B of P_1. Let $B = B_{P_1} = \{1, x\}$ be the standard basis in P_1, then

$$\begin{aligned} P_{B_1\to B} &= \begin{bmatrix} [1+x]_{B_{P_1}} & [1-x]_{B_{P_1}} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \\ P_{B_2\to B} &= \begin{bmatrix} [1+2x]_{B_{P_1}} & [2+x]_{B_{P_1}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 1 & 1 & : & 1 & 2 \\ 1 & -1 & : & 2 & 1 \end{bmatrix} &\to \begin{bmatrix} 1 & 1 & : & 1 & 2 \\ 0 & -2 & : & 1 & -1 \end{bmatrix} && (R_2 \leftarrow R_2 - R_1) \\ &\to \begin{bmatrix} 1 & 1 & : & 1 & 2 \\ 0 & 1 & : & -1/2 & 1/2 \end{bmatrix} && (R_2 \leftarrow -R_2/2) \\ &\to \begin{bmatrix} 1 & 0 & : & 3/2 & 3/2 \\ 0 & 1 & : & -1/2 & 1/2 \end{bmatrix} && (R_1 \leftarrow R_1 - R_2). \end{aligned}$$

Thus

$$P_{B_2\to B_1} = \begin{bmatrix} 3/2 & 3/2 \\ -1/2 & 1/2 \end{bmatrix}.$$

(18.3) Let B_3 be a basis for P_1 and $P_{B_1 \to B_3}$ be the transition matrix from B_1 to B_3 given by $P_{B_1 \to B_3} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$.

(a) Find the transition matrix $P_{B_2 \to B_3}$.

This is a change of basis from B_2 to B_3, which can be achieved by changing basis from B_2 to B_1 and then again changing basis from B_1 to B_3 ($B_2 \to B_3 \equiv B_2 \to B_1 \to B_3$):

$$P_{B_2 \to B_3} = P_{B_1 \to B_3} P_{B_2 \to B_1} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

(b) Use $P_{B_1 \to B_3}$ to find B_3.

The columns of $P_{B_1 \to B_3}$ give B_1 in terms of B_3. Let $B_3 = \{ p(x), q(x) \}$ for some $p(x), q(x) \in P_1$. The first column of $P_{B_1 \to B_3}$ provides

$$[1 + x]_{B_3} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad 1 + x = 1p(x) + 0q(x) = p(x),$$

i.e. $p(x) = 1 + x$. The second column of $P_{B_1 \to B_3}$ provides

$$[1 - x]_{B_3} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad 1 - x = 1p(x) + 2q(x) = p(x) + 2q(x) = 1 + x + 2q(x)$$

so that $q(x) = -x$. Thus

$$B_3 = \{ 1 + x, -x \}.$$

Exercise: verify this answer by calculating $P_{B_1 \to B_3}$ to see if the matrix you obtain is the same as given in the question. Similarly, verify that $P_{B_2 \to B_3}$ is the same as found in Question (18.3(a)).

Alternative: *We use the columns of*

$$P_{B_3 \to B_1} = P_{B_1 \to B_3}^{-1} = \begin{bmatrix} 1 & -1/2 \\ 0 & 1/2 \end{bmatrix}$$

to find

$$B_3 = \{ 1(1 + x) + 0(1 - x), -1/2(1 + x) + 1/2(1 - x) \} = \{ 1 + x, -x \}.$$

Question 19

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

(19.1) Determine the characteristic equation for A in λ.

The characteristic equation is

$$\det \left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \right) = \begin{vmatrix} \lambda - 1 & -2 & -3 \\ -1 & \lambda - 2 & -3 \\ -1 & -2 & \lambda - 3 \end{vmatrix} = \lambda^3 - 6\lambda^2 = 0$$

(19.2) Find the eigenvalues of A, and their algebraic multiplicities.

We solve the characteristic equation for λ*:*

$$\lambda^3 - 6\lambda^2 = 0 \quad \Rightarrow \quad \lambda^2(\lambda - 6) = 0$$

i.e. we find the eigenvalue $\lambda = 0$ *(algebraic multiplicity 2) and the eigenvalue* $\lambda = 6$ *(algebraic multiplicity 1).*

(19.3) Find a basis for the eigenspace corresponding to each eigenvalue of A and hence also the geometric multiplicity of each eigenvalue.

The eigenspace corresponding to the eigenvalue 0 is given by the solutions to the equation

$$\left(0\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1&2&3\\1&2&3\\1&2&3\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-1&-2&-3\\-1&-2&-3\\-1&-2&-3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

where $x, y, z \in \mathbb{R}$*. Row reduction of the augmented matrix yields*

$$\begin{bmatrix}-1&-2&-3&:&0\\-1&-2&-3&:&0\\-1&-2&-3&:&0\end{bmatrix} \to \begin{bmatrix}-1&-2&-3&:&0\\0&0&0&:&0\\0&0&0&:&0\end{bmatrix} \qquad (R_2 \leftarrow R_2 - R_1,\ R_3 \leftarrow R_3 - R_1)$$

so that $x = -2y - 3z$ *where* y *and* z *are free. The eigenspace corresponding to the eigenvalue 0 is*

$$\left\{\begin{bmatrix}-2y-3z\\y\\z\end{bmatrix} : y, z \in \mathbb{R}\right\} = \left\{y\begin{bmatrix}-2\\1\\0\end{bmatrix} + z\begin{bmatrix}-3\\0\\1\end{bmatrix} : y, z \in \mathbb{R}\right\}.$$

The dimension of this eigenspace is 2. The geometric multiplicity for the eigenvalue 0 is 2. A basis for the eigenspace is given by

$$\left\{\begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}-3\\0\\1\end{bmatrix}\right\}.$$

Exercise: verify that the above vectors are linearly independent.

The eigenspace corresponding to the eigenvalue 6 is given by the solutions to the equation

$$\left(6\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}1&2&3\\1&2&3\\1&2&3\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}5&-2&-3\\-1&4&-3\\-1&-2&3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

where $x, y, z \in \mathbb{R}$. Row reduction of the augmented matrix yields

$$\begin{bmatrix} 5 & -2 & -3 & : & 0 \\ -1 & 4 & -3 & : & 0 \\ -1 & -2 & 3 & : & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -3 & : & 0 \\ -1 & 4 & -3 & : & 0 \\ 5 & -2 & -3 & : & 0 \end{bmatrix} \qquad (R_1 \leftarrow -R_3,\ R_3 \leftarrow R_1)$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 & : & 0 \\ 0 & 6 & -6 & : & 0 \\ 0 & -12 & 12 & : & 0 \end{bmatrix} \qquad (R_2 \leftarrow R_2 + R_1,\ R_3 \leftarrow R_3 - 5R_1)$$

$$\rightarrow \begin{bmatrix} 1 & 2 & -3 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \qquad (R_2 \leftarrow R_2/6,\ R_3 \leftarrow R_3 + 2R_2)$$

$$\rightarrow \begin{bmatrix} 1 & 0 & -1 & : & 0 \\ 0 & 1 & -1 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix} \qquad (R_1 \leftarrow R_1 - 2R_2)$$

so that $x = z$ and $y = z$ where z is free. The eigenspace corresponding to the eigenvalue 6 is

$$\left\{ \begin{bmatrix} z \\ z \\ z \end{bmatrix} : z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}.$$

The dimension of this eigenspace is 1. The geometric multiplicity for the eigenvalue 6 is 1. A basis for the eigenspace is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Question 20

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(20.1) Determine the characteristic equation for A in λ.

The characteristic equation is

$$\det \left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right) = \begin{vmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 1 \end{vmatrix} = (\lambda + 1)((\lambda - 1)^2 - 4) = 0.$$

(20.2) Find the eigenvalues of A, and their algebraic multiplicities.

We solve the characteristic equation for λ:

$$(\lambda + 1)((\lambda - 1)^2 - 4) = (\lambda + 1)(\lambda + 1)(\lambda - 3) = 0$$

i.e. we find the eigenvalue $\lambda = -1$ (algebraic multiplicity 2) and the eigenvalue $\lambda = 3$ (algebraic multiplicity 1).

(20.3) Find a basis for the eigenspace corresponding to each eigenvalue of A and hence also the geometric multiplicity of each eigenvalue.

The eigenspace corresponding to the eigenvalue -1 is given by the solutions to the equation

$$\left(-1\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&2&0\\2&1&0\\0&0&-1\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2&-2&0\\-2&-2&0\\0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

where $x, y, z \in \mathbb{R}$. Row reduction of the augmented matrix yields

$$\begin{bmatrix}-2&-2&0&:&0\\-2&-2&0&:&0\\0&0&0&:&0\end{bmatrix}\to\begin{bmatrix}-2&-2&0&:&0\\0&0&0&:&0\\0&0&0&:&0\end{bmatrix}\qquad (R_2 \leftarrow R_2 - R_1)$$

so that $x = -y$ where y and z are free. The eigenspace corresponding to the eigenvalue -1 is

$$\left\{\begin{bmatrix}-y\\y\\z\end{bmatrix} : y, z \in \mathbb{R}\right\}=\left\{y\begin{bmatrix}-1\\1\\0\end{bmatrix}+z\begin{bmatrix}0\\0\\1\end{bmatrix} : y, z \in \mathbb{R}\right\}.$$

The dimension of this eigenspace is 2. The geometric multiplicity for the eigenvalue -1 is 2. A basis for the eigenspace is given by

$$\left\{\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}\right\}.$$

Exercise: verify that the above vectors are linearly independent.

The eigenspace corresponding to the eigenvalue 3 is given by the solutions to the equation

$$\left(3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&2&0\\2&1&0\\0&0&-1\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}2&-2&0\\-2&2&0\\0&0&4\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

where $x, y, z \in \mathbb{R}$. Row reduction of the augmented matrix yields

$$\begin{bmatrix}2&-2&0&:&0\\-2&2&0&:&0\\0&0&4&:&0\end{bmatrix}\to\begin{bmatrix}2&-2&0&:&0\\0&0&0&:&0\\0&0&4&:&0\end{bmatrix}\qquad (R_1 \leftarrow R_2 + R_1)$$

$$\to\begin{bmatrix}2&-2&0&:&0\\0&0&4&:&0\\0&0&0&:&0\end{bmatrix}\qquad (R_2 \leftrightarrow R_3)$$

so that $x = y$ and $z = 0$ where y is free. The eigenspace corresponding to the eigenvalue 3 is

$$\left\{\begin{bmatrix}y\\y\\0\end{bmatrix} : y \in \mathbb{R}\right\}=\left\{y\begin{bmatrix}1\\1\\0\end{bmatrix} : y \in \mathbb{R}\right\}.$$

The dimension of this eigenspace is 1. The geometric multiplicity for the eigenvalue 3 is 1. A basis for the eigenspace is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Question 21

Let

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

(21.1) Show that A is diagonalizable.

First we need the eigenvalues an eigenvectors of A. The characteristic equation is

$$\det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - A \right) = \begin{vmatrix} \lambda+1 & 0 & -1 \\ 0 & \lambda-2 & 0 \\ 0 & 3 & \lambda-1 \end{vmatrix} = (\lambda+1)(\lambda-1)(\lambda-2) = 0.$$

Thus the eigenvalues are -1 , 1 and 2 . Now we need to determine whether the corresponding eigenvectors are linearly independent. However, all the eigenvalues are different (i.e. each has multiplicity 1) so that linear independence follows directly and we conclude that A is diagonalizable.

(21.2) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

We determine the eigenvectors for the eigenvalue -1:

$$\left(-\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & -3 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -3 & 0 \\ 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z \\ -3y \\ 3y-2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Obviously $z = y = 0$. Thus the eigenvectors corresponding to the eigenvalue -1 are

$$\left\{ \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} : x \in \mathbb{R} \setminus \{0\} \right\}.$$

A representative eigenvector is (any one could have been chosen) $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$. Check this by multiplying the vector on the left by A.

We determine the eigenvectors for the eigenvalue 1:

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & -3 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x-z \\ -y \\ 3y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Obviously $y = 0$ and $z = 2x$. Thus the eigenvectors corresponding to the eigenvalue 1 are

$$\left\{ \begin{bmatrix} x \\ 0 \\ 2x \end{bmatrix} : x \in \mathbb{R} \setminus \{0\} \right\}.$$

A representative eigenvector is (any one could have been chosen) $\begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^T$. Check this by multiplying the vector on the left by A.

We determine the eigenvectors for the eigenvalue 2:

$$\left(2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & -3 & 1 \end{bmatrix} \right) \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x - z \\ 0 \\ 3y + z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Obviously $z = 3x = -3y$. Thus the eigenvectors corresponding to the eigenvalue 2 are

$$\left\{ \begin{bmatrix} x \\ -x \\ 3x \end{bmatrix} : x \in \mathbb{R} \setminus \{0\} \right\}.$$

A representative eigenvector is (any one could have been chosen) $\begin{bmatrix} 1 & -1 & 3 \end{bmatrix}^T$. Check this by multiplying the vector on the left by A.

Note that each eigenspace has dimension 1. If the eigenspace had dimension greater than 1, we would choose that many ***linearly independent*** *vectors from that eigenspace.*

Using the representative vectors above we find

$$P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix}.$$

The matrix D is given by

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

The order of the eigenvalues on the diagonal corresponds to the order of the eigenvectors as columns in P.

P^{-1} can be found by row reduction of the matrix P augmented with the 3×3 identity

matrix

$$\begin{array}{l} {\scriptstyle =R3/2} \\ {\scriptstyle =-R2} \end{array}\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \to {\scriptstyle -3R3/2} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 3/2 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$\stackrel{-R2-R3}{\to} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 3/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1/2 & -1/2 \\ 0 & 1 & 0 & 0 & 3/2 & 1/2 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{array}\right]$$

Thus

$$P^{-1} = \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 3/2 & 1/2 \\ 0 & -1 & 0 \end{bmatrix}.$$

Remember to check for calculation errors by ensuring that $P^{-1}AP = D$.

(21.3) Calculate A^{11}.

Since $P^{-1}AP = D$ *we have* $PDP^{-1} = A$ *so that* $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$ *and* $A^3 = AA^2 = PDP^{-1}PD^2P^{-1} = PD^3P^{-1}$ *etc. Thus*

$$\begin{aligned} A^{11} = PD^{11}P^{-1} &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}^{11} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 3/2 & 1/2 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & -1 \\ 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2048 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 3/2 & 1/2 \\ 0 & -1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -1 & -2046 & 1 \\ 0 & 2048 & 0 \\ 0 & -6141 & 1 \end{bmatrix} \end{aligned}$$

Question 22

Let

$$A = \begin{bmatrix} 1 & 5 & 0 \\ 5 & 1 & 0 \\ 1 & -1 & 6 \end{bmatrix}$$

(22.1) Show that A is diagonalizable.

First we need the eigenvalues an eigenvectors of A*. The characteristic equation is*

$$\begin{aligned} \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - A\right) &= \begin{vmatrix} \lambda-1 & -5 & 0 \\ -5 & \lambda-1 & 0 \\ -1 & 1 & \lambda-6 \end{vmatrix} \\ &= (\lambda-1)^2(\lambda-6) - 25(\lambda-6) \\ &= ((\lambda-1)^2 - 25)(\lambda-6) = (\lambda-6)^2(\lambda+4) = 0. \end{aligned}$$

Thus the eigenvalues are 6 *(twice) and* -4 *. Now we need to determine whether three corresponding linearly independent eigenvectors can be found. We determine the eigenvectors for the eigenvalue* -4*:*

$$\left(-4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&5&0\\5&1&0\\-1&1&6\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-5&-5&0\\-5&-5&0\\1&-1&-10\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-5x-5y\\-5x-5y\\-x+y-10z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Obviously $x=-y=-5z$*. Thus the eigenvectors corresponding to the eigenvalue -4 are*

$$\left\{\begin{bmatrix}-5z\\5z\\z\end{bmatrix} : z\in\mathbb{R}\setminus\{0\}\right\}.$$

A representative eigenvector is (any one could have been chosen) $\begin{bmatrix}-5&5&1\end{bmatrix}^T$*. Check this by multiplying the vector on the left by* A*.*

We determine the eigenvectors for the eigenvalue 6*:*

$$\left(6\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&5&0\\5&1&0\\1&-1&6\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5&-5&0\\-5&5&0\\1&-1&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}5x-5y\\-5x+5y\\x-y\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Obviously $x=y$*. (No restriction on* z*.) Thus the eigenvectors corresponding to the eigenvalue 6 are*

$$\left\{\begin{bmatrix}x\\x\\z\end{bmatrix} : (x,z)\in\mathbb{R}^2\setminus\{(0,0)\}\right\}.$$

Two linearly independent representative eigenvectors are (any two could have been chosen) $\begin{bmatrix}1&1&0\end{bmatrix}^T$ *and* $\begin{bmatrix}0&0&1\end{bmatrix}^T$*. Check this by multiplying the vector on the left by* A*.*

Thus we found three linearly independent eigenvectors and A *is diagonalizable.*

(It is not necessary to check that eigenvectors corresponding to different eigenvalues are linearly independent, they must necessarily be.)

(22.2) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP=D$.
Using the representative vectors above we find

$$P=\begin{bmatrix}-5&1&0\\5&1&0\\1&0&1\end{bmatrix}.$$

The matrix D *is given by*

$$D=\begin{bmatrix}-4&0&0\\0&6&0\\0&0&6\end{bmatrix}.$$

The order of the eigenvalues on the diagonal corresponds to the order of the eigenvectors as columns in P.

P^{-1} can be found by row reduction of the matrix P augmented with the 3×3 identity matrix

$$\begin{matrix} +5R_3 \\ +R_1 \\ \end{matrix} \left[\begin{array}{ccc|ccc} -5 & 1 & 0 & 1 & 0 & 0 \\ 5 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \begin{matrix} =R_3 \\ =R_2/2 \\ =R_1 \end{matrix} \left[\begin{array}{ccc|ccc} 0 & 1 & 5 & 1 & 0 & 5 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\to \begin{matrix} \\ \\ -R_2 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 1 & 5 & 1 & 0 & 5 \end{array}\right]$$

$$\to \begin{matrix} -R_3/5 \\ \\ =R_3/5 \end{matrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 5 & 1/2 & -1/2 & 5 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & -1/10 & 1/10 & 0 \\ 0 & 1 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 5 & 1/10 & -1/10 & 1 \end{array}\right]$$

Thus

$$P^{-1} = \begin{bmatrix} -1/10 & 1/10 & 0 \\ 1/2 & 1/2 & 0 \\ 1/10 & -1/10 & 1 \end{bmatrix}.$$

Remember to check for calculation errors by ensuring that $P^{-1}AP = D$.

(22.3) Calculate A^{10}.

Since $P^{-1}AP = D$ we have $PDP^{-1} = A$ so that $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$ and $A^3 = AA^2 = PDP^{-1}PD^2P^{-1} = PD^3P^{-1}$ etc. Thus

$$\begin{aligned} A^{10} = PD^{10}P^{-1} &= \begin{bmatrix} -5 & 1 & 0 \\ 5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}^{10} \begin{bmatrix} -1/10 & 1/10 & 0 \\ 1/2 & 1/2 & 0 \\ 1/10 & -1/10 & 1 \end{bmatrix} \\ &= \begin{bmatrix} -5 & 1 & 0 \\ 5 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1048576 & 0 & 0 \\ 0 & 60466176 & 0 \\ 0 & 0 & 60466176 \end{bmatrix} \begin{bmatrix} -1/10 & 1/10 & 0 \\ 1/2 & 1/2 & 0 \\ 1/10 & -1/10 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 30757376 & 29708800 & 0 \\ 29708800 & 30757376 & 0 \\ 5941760 & -5941760 & 60466176 \end{bmatrix}. \end{aligned}$$

Question 23

Consider the matrix (see 19)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

(23.1) Find an invertible matrix P such that $D := P^{-1}AP$ is diagonal. Determine D.

The eigenvalues of A are 0 (twice) and 6. A basis for the eigenspace corresponding to the eigenvalue 0 is given by

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A basis for the eigenspace corresponding to the eigenvalue 6 is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Thus we may choose

$$P = \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

(23.2) Find the rank of D and hence also the rank of A.

Since D is diagonal, the rank is the number of non-zero diagonal entries of D (i.e. the number of non-zero eigenvalues of A). Thus $\operatorname{rank}(D) = \operatorname{rank}(A) = 1$.

(23.3) Calculate D^n for $n \in \mathbb{N}$ and hence also A^n as a matrix.

Since

$$D^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6^n \end{bmatrix}$$

we have

$$\begin{aligned}
A^n = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6^n \end{bmatrix} P^{-1} &= \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6^n \end{bmatrix} \frac{1}{6} \begin{bmatrix} -1 & 4 & -3 \\ -1 & -2 & 3 \\ 1 & 2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 6^{n-1} & 2 \cdot 6^{n-1} & 3 \cdot 6^{n-1} \end{bmatrix} \\
&= 6^{n-1} \begin{bmatrix} -2 & -3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \end{bmatrix} \\
&= 6^{n-1} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = 6^{n-1}A
\end{aligned}$$

where we used the Adjoint to find P^{-1}. ***Alternatively****, we arrive at the same solution*

$$A^n = P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6^n \end{bmatrix} P^{-1} = 6^{n-1} P \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} P^{-1} = 6^{n-1}A$$

without determining P^{-1}.

(23.4) Show that

$$A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}.$$

Use this expression for A to calculate A^2, A^3 etc. and compare with your answer to (24.3).

Straightforward matrix multiplication yields

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix} = A.$$

We have

$$\begin{aligned} A^2 &= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} 6 \begin{bmatrix} 1 & 2 & 3 \end{bmatrix} = 6A & \qquad A^2 &= 6^{2-1}A \\ A^3 &= (A^2)A = 6A^2 = 6(6A) = 6^2 A & A^3 &= 6^{3-1}A \\ &\vdots & &\vdots \\ A^n &= (A^{n-1})A = 6^{n-2}A^2 = 6^{n-2}(6A) = 6^{n-1}A & A^n &= 6^{n-1}A. \end{aligned}$$

Question 24

Consider the matrix (see 20)

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

(24.1) Find an orthogonal matrix P such that $D := P^T AP$ is diagonal. Determine D.

The eigenvalues of A are -1 (twice) and 3. A basis for the eigenspace corresponding to the eigenvalue -1 is given by

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

A basis for the eigenspace corresponding to the eigenvalue 3 is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Notice that all three basis elements above are already pairwise orthogonal, thus we need only divide each basis element by its Euclidean norm. Thus we may choose

$$P = \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

(24.2) Find the rank of D and hence also the rank of A.

Since D is diagonal, the rank is the number of non-zero diagonal entries of D (i.e. the number of non-zero eigenvalues of A). Thus $\text{rank}(D) = \text{rank}(A) = 3$.

(24.3) Calculate D^n for $n \in \mathbb{N}$ and hence also A^n as a matrix.

Since

$$D^n = \begin{bmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 3^n \end{bmatrix}$$

we have

$$\begin{aligned} A^n = P \begin{bmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} P^T &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} (-1)^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{(-1)^{n+1}}{\sqrt{2}} & \frac{(-1)^n}{\sqrt{2}} & 0 \\ 0 & 0 & (-1)^n \\ \frac{3^n}{\sqrt{2}} & \frac{3^n}{\sqrt{2}} & 0 \end{bmatrix} \\ &= \begin{bmatrix} \dfrac{3^n + (-1)^n}{2} & \dfrac{3^n - (-1)^n}{2} & 0 \\ \dfrac{3^n - (-1)^n}{2} & \dfrac{3^n + (-1)^n}{2} & 0 \\ 0 & 0 & (-1)^n \end{bmatrix} \end{aligned}$$

(24.4) Let B be a $m \times m$ matrix where $m \in \mathbb{N}$, I be the $m \times m$ identity matrix and $k \in \mathbb{R}$.

(a) Let $\mathbf{x}$ be an eigenvector of B with corresponding eigenvalue λ. Show that $\mathbf{x}$ is an eigenvector of $B + kI$. What is the corresponding eigenvalue of $B + kI$?

We have

$$(B + kI)\mathbf{x} = B\mathbf{x} + kI\mathbf{x} = \lambda\mathbf{x} + k\mathbf{x} = (\lambda + k)\mathbf{x}.$$

Since $\mathbf{x} \neq \mathbf{0}$ it follows that $\mathbf{x}$ is an eigenvector of $B + kI$ with corresponding eigenvalue $\lambda + k$.

(b) Assume that B is diagonalizable, is $B + kI$ diagonalizable?

Since B is diagonalizable, there exists an invertible $m \times m$ matrix Q such that $Q^{-1}BQ$ is diagonal. Thus

$$Q^{-1}(B + kI)Q = Q^{-1}BQ + kQ^{-1}IQ = Q^{-1}BQ + kI$$

is diagonal (since the sum of diagonal matrices is diagonal). Consequently, $B + kI$ is diagonalizable.

Question 25

Consider the vector space $\mathbb{R}^3$.

(25.1) Show that

$$\langle \mathbf{x}, \mathbf{y} \rangle := 3x_1y_1 + x_2y_2 + x_3y_3, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

is an inner product on $\mathbb{R}^3$.

We have for $k \in \mathbb{R}$ and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{R}^3$$

1. $\langle \mathbf{x}, \mathbf{y} \rangle = 3x_1y_1 + x_2y_2 + x_3y_3 = 3y_1x_1 + y_2x_2 + y_3x_3 = \langle \mathbf{y}, \mathbf{x} \rangle$

2. $$\begin{aligned} \langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle &= 3(x_1 + z_1)y_1 + (x_2 + z_2)y_2 + (x_3 + z_3)y_3 \\ &= 3x_1y_1 + 3z_1y_1 + x_2y_2 + z_2y_2 + x_3y_3 + z_3y_3 \\ &= 3x_1y_1 + x_2y_2 + x_3y_3 + 3z_1y_1 + z_2y_2 + z_3y_3 = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle \end{aligned}$$

3. $\langle k\mathbf{x}, \mathbf{y} \rangle = 3(kx_1)y_1 + (kx_2)y_2 + (kx_3)y_3 = k(3x_1y_1 + x_2y_2 + x_3y_3) = k\langle \mathbf{x}, \mathbf{y} \rangle$

4. $\langle \mathbf{x}, \mathbf{x} \rangle = 3x_1^2 + x_2^2 + x_3^2 \geq 0$ *so that* $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *if and only if* $x_1 = x_2 = x_3 = 0$ *(since* $x_1^2, x_2^2, x_3^2 \geq 0$*), i.e.* $\mathbf{x} = \mathbf{0}$.

(25.2) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

to find an orthonormal basis with respect to the inner product **defined in 2.1** for the span of this subset.

Let

$$\mathbf{u}_1 := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 := \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 := \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Then the Gram-Schmidt process provides

$$\begin{aligned}
\mathbf{v}_1 &:= \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\
\langle \mathbf{v}_1, \mathbf{v}_1 \rangle &= 3(1^2) + 0^2 + 1^2 = 4 \\
\langle \mathbf{u}_2, \mathbf{v}_1 \rangle &= 3(-1)1 + 0 + 1 = -2 \\
\mathbf{v}_2 &:= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{2}{4}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} \\
\langle \mathbf{v}_2, \mathbf{v}_2 \rangle &= \frac{1}{4}(3(-1)^2 + 0^2 + 3^2) = 3 \\
\langle \mathbf{u}_3, \mathbf{v}_1 \rangle &= (3(-1)1 + 0 + 1^2) = -2 \\
\langle \mathbf{u}_3, \mathbf{v}_2 \rangle &= \frac{1}{2}(3(-1)(-1) + 0 + 3) = 3 \\
\mathbf{v}_3 &:= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 \\
&= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + \frac{2}{4}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3}\cdot\frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.
\end{aligned}$$

Thus we have the orthogonal basis (with respect to the inner product ***defined in 2.1****)*

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Normalizing these vectors yields the orthonormal basis

$$\left\{ \frac{1}{\sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}}\mathbf{v}_1, \frac{1}{\sqrt{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}}\mathbf{v}_2, \frac{1}{\sqrt{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle}}\mathbf{v}_3 \right\} = \left\{ \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2\sqrt{3}}\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Question 26

Consider the vector space $\mathbb{R}^3$.

(26.1) Show that

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1y_1 + 3x_2y_2 + x_3y_3, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

is an inner product on $\mathbb{R}^3$.

We have for $k \in \mathbb{R}$ and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{R}^3$$

1. $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 3x_2y_2 + x_3y_3 = y_1x_1 + 3y_2x_2 + y_3x_3 = \langle \mathbf{y}, \mathbf{x} \rangle$

2. $$\begin{aligned}\langle \mathbf{x}+\mathbf{z}, \mathbf{y} \rangle &= (x_1+z_1)y_1 + 3(x_2+z_2)y_2 + (x_3+z_3)y_3 \\ &= x_1y_1 + z_1y_1 + 3x_2y_2 + 3z_2y_2 + x_3y_3 + z_3y_3 \\ &= x_1y_1 + 3x_2y_2 + x_3y_3 + z_1y_1 + 3z_2y_2 + z_3y_3 = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle\end{aligned}$$

3. $\langle k\mathbf{x}, \mathbf{y} \rangle = (kx_1)y_1 + 3(kx_2)y_2 + (kx_3)y_3 = k(x_1y_1 + 3x_2y_2 + x_3y_3) = k\langle \mathbf{x}, \mathbf{y} \rangle$

4. $\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + 3x_2^2 + x_3^2 \geq 0$ *so that* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and if and only if* $x_1 = x_2 = x_3 = 0$ *(since* $x_1^2, x_2^2, x_3^2 \geq 0$*), i.e.* $\mathbf{x} = \mathbf{0}$.

(26.2) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^3$:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

to find an orthonormal basis with respect to the inner product **defined in 2.1** for the span of this subset.

Let

$$\mathbf{u}_1 := \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 := \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 := \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Then the Gram-Schmidt process provides

$$\begin{aligned}\mathbf{v}_1 &:= \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ \langle \mathbf{v}_1, \mathbf{v}_1 \rangle &= 1^2 + 3 \cdot 0^2 + 1^2 = 2 \\ \mathbf{v}_2 &:= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\ \langle \mathbf{v}_2, \mathbf{v}_2 \rangle &= (-1)^2 + 3 \cdot 0^2 + 1^2 = 2 \\ \mathbf{v}_3 &:= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 \\ &= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{0}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.\end{aligned}$$

Thus we have the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Normalizing these vectors yields the orthonormal basis

$$\left\{ \frac{1}{\sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}}\mathbf{v}_1, \frac{1}{\sqrt{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}}\mathbf{v}_2, \frac{1}{\sqrt{\langle \mathbf{v}_3, \mathbf{v}_3 \rangle}}\mathbf{v}_3 \right\} = \left\{ \frac{1}{\sqrt{2}}\begin{bmatrix}1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}, \frac{1}{\sqrt{3}}\begin{bmatrix}0\\1\\0\end{bmatrix} \right\}.$$

Question 27

(27.1) Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + 2u_2v_2 + 3u_3v_3$$

is an inner product on $\mathbb{R}^3$ for $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.
We have for $k \in \mathbb{R}$ and

$$\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3), \mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3$$

1. $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 2x_2y_2 + 3x_3y_3 = y_1x_1 + 2y_2x_2 + 3y_3x_3 = \langle \mathbf{y}, \mathbf{x} \rangle$
2. $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = (x_1 + z_1)y_1 + 2(x_2 + z_2)y_2 + 3(x_3 + z_3)y_3$
   $= x_1y_1 + z_1y_1 + 2x_2y_2 + 2z_2y_2 + 3x_3y_3 + 3z_3y_3$
   $= x_1y_1 + 2x_2y_2 + 3x_3y_3 + z_1y_1 + 2z_2y_2 + 3z_3y_3 = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$
3. $\langle k\mathbf{x}, \mathbf{y} \rangle = (kx_1)y_1 + 2(kx_2)y_2 + 3(kx_3)y_3 = k(x_1y_1 + 2x_2y_2 + 3x_3y_3) = k\langle \mathbf{x}, \mathbf{y} \rangle$
4. $\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + 2x_2^2 + 3x_3^2 \geq 0$
   so that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $x_1 = x_2 = x_3 = 0$ (since $x_1^2, x_2^2, x_3^2 \geq 0$) , i.e. $\mathbf{x} = \mathbf{0}$.

(27.2) Let $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (1, 1, 0)$ and $\mathbf{w} = (1, 0, 0)$. Show that $B = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent and spans $\mathbb{R}^3$.
Solving

$$c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0) = (c_1 + c_2 + c_3, c_1 + c_2, c_1) = (0, 0, 0)$$

for c_1, c_2, c_3 we find $c_1 = 0$ from the third component, and consequently $c_2 = 0$ from the second component and finally $c_3 = 0$ This is the only solution, so that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent. Let $x, y, z \in \mathbb{R}$. Solving

$$c_1(1, 1, 1) + c_2(1, 1, 0) + c_3(1, 0, 0) = (c_1 + c_2 + c_3, c_1 + c_2, c_1) = (x, y, z)$$

for c_1, c_2, c_3 we find $c_1 = z$ from the third component, and consequently $c_2 = y - z$ from the second component and finally $c_3 = x - y$. Since we found a solution we have $\operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$.

(27.3) Transform B into an orthonormal basis using the inner product in 27.1.

Let

$$\mathbf{u}_1 := \mathbf{u} = (1,1,1), \quad \mathbf{u}_2 := \mathbf{v} = (1,1,0), \quad \mathbf{u}_3 := \mathbf{w} = (1,0,0).$$

Then the Gram-Schmidt process provides

$$\begin{aligned}
\mathbf{v}_1' &:= \mathbf{u}_1 = (1,1,1) \\
\mathbf{v}_1 &:= \frac{\mathbf{v}_1'}{\sqrt{\langle \mathbf{v}_1', \mathbf{v}_1' \rangle}} = \frac{1}{\sqrt{6}}(1,1,1) \quad \textit{where we used } \langle \mathbf{v}_1', \mathbf{v}_1' \rangle = 6 \\
\mathbf{v}_2' &:= \mathbf{u}_2 - \langle \mathbf{u}_2, \mathbf{v}_1 \rangle \mathbf{v}_1 = (1,1,0) - \frac{1}{2}(1,1,1) = \frac{1}{2}(1,1,-1) \\
\mathbf{v}_2 &:= \frac{\mathbf{v}_2'}{\sqrt{\langle \mathbf{v}_2', \mathbf{v}_2' \rangle}} = \frac{1}{\sqrt{6}}(1,1,-1) \quad \textit{where we used } \langle \mathbf{v}_2', \mathbf{v}_2' \rangle = \frac{3}{2} \\
\mathbf{v}_3' &:= \mathbf{u}_3 - \langle \mathbf{u}_3, \mathbf{v}_1 \rangle \mathbf{v}_1 - \langle \mathbf{u}_3, \mathbf{v}_2 \rangle \mathbf{v}_2 \\
&= (1,0,0) - \frac{1}{6}(1,1,1) - \frac{1}{6}(1,1,-1) = \frac{1}{3}(2,-1,0) \\
\mathbf{v}_3 &:= \frac{\mathbf{v}_3'}{\sqrt{\langle \mathbf{v}_3', \mathbf{v}_3' \rangle}} = \frac{1}{\sqrt{6}}(2,-1,0) \quad \textit{where we used } \langle \mathbf{v}_3', \mathbf{v}_3' \rangle = \frac{6}{9}
\end{aligned}$$

Thus we have the orthonormal basis

$$\left\{ \frac{1}{\sqrt{6}}(1,1,1),\ \frac{1}{\sqrt{6}}(1,1,-1),\ \frac{1}{\sqrt{6}}(2,-1,0) \right\}.$$

(27.4) Let $\mathbb{R}^3$ have the Euclidean inner product and $W = \operatorname{span}\{\mathbf{u}, \mathbf{v}\}$ where

$$\mathbf{u} = \left(\frac{4}{5}, 0, -\frac{3}{5}\right) \text{ and } \mathbf{v} = (0,1,0).$$

Express $\mathbf{w} = (1,2,3)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^\perp$.

Notice that $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ *and* $\langle \mathbf{u}, \mathbf{u} \rangle = \langle \mathbf{v}, \mathbf{v} \rangle = 1$. *Thus* $\{\mathbf{u}, \mathbf{v}\}$ *is an orthonormal basis for* W. *(If this was not the case, we could apply Gram-Schmidt to find such an orthonormal basis). It follows that*

$$\begin{aligned}
\mathbf{w}_1 &= \langle \mathbf{w}, \mathbf{u} \rangle \mathbf{u} + \langle \mathbf{w}, \mathbf{v} \rangle \mathbf{v} = -\mathbf{u} + 2\mathbf{v} = \left(-\frac{4}{5}, 2, \frac{3}{5}\right) \\
\mathbf{w}_2 &= \mathbf{w} - \mathbf{w}_1 = \left(\frac{9}{5}, 0, \frac{12}{5}\right) \\
\mathbf{w} &= \left(-\frac{4}{5}, 2, \frac{3}{5}\right) + \left(\frac{9}{5}, 0, \frac{12}{5}\right).
\end{aligned}$$

Question 28

(28.1) Show that

$$\langle \mathbf{u}, \mathbf{v} \rangle = 2u_1v_1 + 3u_2v_2 + u_3v_3$$

is an inner product on $\mathbb{R}^3$ for $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$.

We have for $k \in \mathbb{R}$ and

$$\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3), \mathbf{z} = (z_1, z_2, z_3) \in \mathbb{R}^3$$

1. $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 3x_2y_2 + x_3y_3 = 2y_1x_1 + 3y_2x_2 + y_3x_3 = \langle \mathbf{y}, \mathbf{x} \rangle$
2. $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = 2(x_1 + z_1)y_1 + 3(x_2 + z_2)y_2 + (x_3 + z_3)y_3$
   $= 2x_1y_1 + 2z_1y_1 + 3x_2y_2 + 3z_2y_2 + x_3y_3 + z_3y_3$
   $= 2x_1y_1 + 3x_2y_2 + x_3y_3 + 2z_1y_1 + 3z_2y_2 + z_3y_3 = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$
3. $\langle k\mathbf{x}, \mathbf{y} \rangle = 2(kx_1)y_1 + 3(kx_2)y_2 + (kx_3)y_3 = k(2x_1y_1 + 3x_2y_2 + x_3y_3) = k\langle \mathbf{x}, \mathbf{y} \rangle$
4. $\langle \mathbf{x}, \mathbf{x} \rangle = 2x_1^2 + 3x_2^2 + x_3^2 \geq 0$
   so that $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *if and only if* $x_1 = x_2 = x_3 = 0$ *(since* $x_1^2, x_2^2, x_3^2 \geq 0$*) , i.e.* $\mathbf{x} = \mathbf{0}$.

(28.2) Let $\mathbf{u} = (1, 1, 1)$, $\mathbf{v} = (-1, 1, 0)$ and $\mathbf{w} = (1, 2, 1)$. Show that $B = \{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is linearly independent and spans $\mathbb{R}^3$.

Solving

$$c_1(1, 1, 1) + c_2(-1, 1, 0) + c_3(1, 2, 1) = (c_1 - c_2 + c_3, c_1 + c_2 + 2c_3, c_1 + c_3) = (0, 0, 0)$$

for c_1, c_2, c_3 and noting that for the determinant of the coefficient matrix

$$\det \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = -1$$

we have a unique solution $c_1 = c_2 = c_3 = 0$. Thus $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.
Let $x, y, z \in \mathbb{R}$. Solving

$$c_1(1, 1, 1) + c_2(-1, 1, 0) + c_3(1, 2, 1) = (c_1 - c_2 + c_3, c_1 + c_2 + 2c_3, c_1 + c_3) = (x, y, z)$$

for c_1, c_2, c_3 and noting that for the determinant of the coefficient matrix

$$\det \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} = -1$$

we have a unique solution. Since we found a solution we have $\operatorname{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\} = \mathbb{R}^3$.

(28.3) Transform B into an orthonormal basis using the inner product in 28.1.

Let

$$\mathbf{u}_1 := \mathbf{u} = (1,1,1), \quad \mathbf{u}_2 := \mathbf{v} = (-1,1,0), \quad \mathbf{u}_3 := \mathbf{w} = (1,2,1).$$

Then the Gram-Schmidt process provides

$$\begin{aligned}
\mathbf{v}_1' &:= \mathbf{u}_1 = (1,1,1) \\
\langle \mathbf{v}_1', \mathbf{v}_1' \rangle &= 2 \cdot 1 \cdot 1 + 3 \cdot 1 \cdot 1 + 1 \cdot 1 = 6 \\
\mathbf{v}_1 &:= \frac{\mathbf{v}_1'}{\sqrt{\langle \mathbf{v}_1', \mathbf{v}_1' \rangle}} = \frac{1}{\sqrt{6}}(1,1,1) \\
\langle \mathbf{u}_2, \mathbf{v}_1 \rangle &= \frac{1}{\sqrt{6}}(2 \cdot (-1) \cdot 1 + 3 \cdot 1 \cdot 1 + 0 \cdot 1) = \frac{1}{\sqrt{6}} \\
\mathbf{v}_2' &:= \mathbf{u}_2 - \langle \mathbf{u}_2, \mathbf{v}_1 \rangle \mathbf{v}_1 = (-1,1,0) - \frac{1}{6}(1,1,1) = \frac{1}{6}(-7,5,-1) \\
\langle \mathbf{v}_2', \mathbf{v}_2' \rangle &= \frac{1}{36}(2 \cdot (-7) \cdot (-7) + 3 \cdot 5 \cdot 5 + (-1) \cdot (-1)) = \frac{29}{6} \\
\mathbf{v}_2 &:= \frac{\mathbf{v}_2'}{\sqrt{\langle \mathbf{v}_2', \mathbf{v}_2' \rangle}} = \frac{1}{\sqrt{6}\sqrt{29}}(-7,5,-1) \\
\langle \mathbf{u}_3, \mathbf{v}_1 \rangle &= \frac{1}{\sqrt{6}}(2 \cdot 1 \cdot 1 + 3 \cdot 2 \cdot 1 + 1 \cdot 1) = \frac{9}{\sqrt{6}} \\
\langle \mathbf{u}_3, \mathbf{v}_2 \rangle &= \frac{1}{\sqrt{6}\sqrt{29}}(2 \cdot 1 \cdot (-7) + 3 \cdot 2 \cdot 5 + 1 \cdot (-1)) = \frac{15}{\sqrt{6}\sqrt{29}} \\
\mathbf{v}_3' &:= \mathbf{u}_3 - \langle \mathbf{u}_3, \mathbf{v}_1 \rangle \mathbf{v}_1 - \langle \mathbf{u}_3, \mathbf{v}_2 \rangle \mathbf{v}_2 \\
&= (1,2,1) - \frac{9}{6}(1,1,1) - \frac{15}{174}(-7,5,-1) = \frac{1}{29}(3,2,-12) \\
\langle \mathbf{v}_3', \mathbf{v}_3' \rangle &= \frac{1}{(29)^2}(2 \cdot 3 \cdot 3 + 3 \cdot 2 \cdot 2 + (-12) \cdot (-12)) = \frac{6}{29} \\
\mathbf{v}_3 &:= \frac{\mathbf{v}_3'}{\sqrt{\langle \mathbf{v}_3', \mathbf{v}_3' \rangle}} = \frac{1}{\sqrt{6}\sqrt{29}}(3,2,-12)
\end{aligned}$$

Thus we have the orthonormal basis

$$\left\{ \frac{1}{\sqrt{6}}(1,1,1), \ \frac{1}{\sqrt{6}\sqrt{29}}(-7,5,-1), \ \frac{1}{\sqrt{6}\sqrt{29}}(3,2,-12) \right\}.$$

(28.4) Let $\mathbb{R}^3$ have the Euclidean inner product and $W = \text{span}\{\mathbf{u}, \mathbf{v}\}$ where

$$\mathbf{u} = (1,0,-1) \text{ and } \mathbf{v} = (3,1,0).$$

Express $\mathbf{w} = (1,2,3)$ in the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$ where $\mathbf{w}_1 \in W$ and $\mathbf{w}_2 \in W^\perp$.

The Gram-Schmidt process yields

$$\begin{aligned}
\mathbf{x} &:= \mathbf{u}/\sqrt{\langle \mathbf{u}, \mathbf{u} \rangle} = \frac{1}{\sqrt{2}}(1,0,-1) \\
\mathbf{y}' &:= \mathbf{v} - \langle \mathbf{v}, \mathbf{x} \rangle \mathbf{x} = (3,1,0) - \frac{3}{2}(1,0,-1) = \frac{1}{2}(3,2,3) \\
\mathbf{y} &:= \mathbf{y}'/\sqrt{\langle \mathbf{y}', \mathbf{y}' \rangle} = \frac{1}{\sqrt{22}}(3,2,3)
\end{aligned}$$

where we used the Euclidean inner product.

$$\begin{aligned}\mathbf{w}_1 &= \langle \mathbf{w}, \mathbf{x}\rangle \mathbf{x} + \langle \mathbf{w}, \mathbf{y}\rangle \mathbf{y} \ = -\sqrt{2}\mathbf{x} + \frac{16}{\sqrt{22}}\mathbf{y} = \frac{1}{11}(13, 16, 35)\\ \mathbf{w}_2 &= \mathbf{w} - \mathbf{w}_1 = \frac{1}{11}(-2, 6, -2)\\ \mathbf{w} &= \frac{1}{11}(13, 16, 35) + \frac{1}{11}(-2, 6, -2).\end{aligned}$$

Question 29

(29.1) Let

$$A = \begin{bmatrix} 0 & 0 & 2 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}.$$

Find the bases for the eigenspaces associated with the eigenvalues of A.
The characteristic equation in λ for A is

$$\begin{aligned}\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} - A\right) &= \begin{vmatrix} \lambda & 0 & -2 & 0\\ -1 & \lambda & -1 & 0\\ 0 & -1 & \lambda+2 & 0\\ 0 & 0 & 0 & \lambda-1\end{vmatrix}\\ &= \lambda \begin{vmatrix} \lambda & -1 & 0\\ -1 & \lambda+2 & 0\\ 0 & 0 & \lambda-1\end{vmatrix} - (-1)\begin{vmatrix} 0 & -2 & 0\\ -1 & \lambda+2 & 0\\ 0 & 0 & \lambda-1\end{vmatrix}\\ &= \lambda(\lambda-1)(\lambda(\lambda+2)-1) - 2(\lambda-1)\\ &= (\lambda-1)(\lambda^3+2\lambda^2-\lambda-2)\\ &= (\lambda-1)(\lambda^2(\lambda+2)-(\lambda+2))\\ &= (\lambda-1)(\lambda+2)(\lambda^2-1)\\ &= (\lambda-1)^2(\lambda+2)(\lambda+1) = 0\end{aligned}$$

where we used cofactor expansion along the first column. We find that the eigenvalues of A are -2, -1 and 1 (twice).

We determine the eigenvectors for the eigenvalue -2:

$$\left(-2\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 & 0\\ 1 & 0 & 1 & 0\\ 0 & 1 & -2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\right)\begin{bmatrix} a\\ b\\ c\\ d\end{bmatrix} = \begin{bmatrix} -2 & 0 & -2 & 0\\ -1 & -2 & -1 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 0 & -3\end{bmatrix}\begin{bmatrix} a\\ b\\ c\\ d\end{bmatrix} = \begin{bmatrix} -2(a+c)\\ -a-2b-c\\ -b\\ -3d\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ 0\\ 0\end{bmatrix}$$

Obviously $b = d = 0$ and $c = -a$. Thus the eigenvectors corresponding to the eigenvalue -2 are

$$\left\{\begin{bmatrix} a\\ 0\\ -a\\ 0\end{bmatrix} : a \in \mathbb{R}\setminus\{0\}\right\}.$$

The eigenspace is

$$\left\{ \begin{bmatrix} a \\ 0 \\ -a \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}.$$

A basis for this eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

We determine the eigenvectors for the eigenvalue -1*:*

$$\left(-\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -1 & 0 & -2 & 0 \\ -1 & -1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} -a-2c \\ -a-b-c \\ -b+c \\ -2d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Obviously $d = 0$ *and* $c = b = -a/2$. *Thus the eigenvectors corresponding to the eigenvalue -1 are*

$$\left\{ \begin{bmatrix} a \\ -a/2 \\ -a/2 \\ 0 \end{bmatrix} : a \in \mathbb{R} \setminus \{0\} \right\}.$$

The eigenspace is

$$\left\{ \begin{bmatrix} a \\ -a/2 \\ -a/2 \\ 0 \end{bmatrix} : a \in \mathbb{R} \right\}.$$

A basis for this eigenspace is

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ -1 \\ 0 \end{bmatrix} \right\}.$$

We determine the eigenvectors for the eigenvalue 1*:*

$$\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a-2c \\ -a+b-c \\ -b+3c \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Obviously $a = 2c$ *and* $b = 3c$. *Thus the eigenvectors corresponding to the eigenvalue 1 are*

$$\left\{ \begin{bmatrix} 2c \\ 3c \\ c \\ d \end{bmatrix} : c \in \mathbb{R} \setminus \{0\} \right\}.$$

The eigenspace is

$$\left\{ \begin{bmatrix} 2c \\ 3c \\ c \\ d \end{bmatrix} : c, d \in \mathbb{R} \right\}.$$

A basis for this eigenspace is

$$\left\{ \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(29.2) Let $T : P_2 \to P_2$ be the function defined by $T(p(x)) = p(2x+1)$.

(a) Show that T is a linear transformation.

We have for $p(x) = p_0 + p_1 x + p_2 x^2, q(x) = q_0 + q_1 x + q_2 x^2 \in P_2$ *and* $k \in \mathbb{R}$:

1.

$$\begin{aligned} T(\,(kp)(x)\,) &= T((kp_0) + (kp_1)x + (kp_2)x^2) = (kp_0) + (kp_1)(2x+1) + (kp_2)(2x+1)^2 \\ &= k(p_0 + p_1(2x+1) + p_2(2x+1)^2) = k\,p(2x+1) = kT(p(x)) \end{aligned}$$

2.

$$\begin{aligned} T(\,(p+q)(x)\,) &= T((p_0+q_0) + (p_1+q_1)x + (p_2+q_2)x^2) \\ &= (p_0+q_0) + (p_1+q_1)(2x+1) + (p_2+q_2)(2x+1)^2 \\ &= p_0 + p_1(2x+1) + p_2(2x+1)^2 + q_0 + q_1(2x+1) + q_2(2x+1)^2 \\ &= p(2x+1) + q(2x+1) = T(p(x)) + T(q(x)) \end{aligned}$$

so that T *is linear.*

(b) Find $[T]_B$ with respect to the basis $\{1, x, x^2\}$.

From

$$\begin{aligned} T(1) &= 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x) &= (2x+1) = 1 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \\ T(x^2) &= (2x+1)^2 = 1 \cdot 1 + 4 \cdot x + 4 \cdot x^2 \end{aligned}$$

and the ordering

$$1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad x \to \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad x^2 \to \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

we find

$$[T]_B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{bmatrix}.$$

(c) Compute $T(2-3x+4x^2)$.

We find

$$T(2-3x+4x^2) = 2-3(2x+1)+4(2x+1)^2 = 2-6x-3+16x^2+16x+4 = 3+10x+16x^2.$$

or equivalently

$$T(2-3x+4x^2) = [T]_B \begin{bmatrix} 2 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \\ 16 \end{bmatrix} \to 3+10x+16x^2.$$

(29.3) Let $S : P_2 \to P_3$ be defined by $S(p(x)) = xp(x)$.

(a) Show that S is one-to-one.

Let $p(x) = p_0 + p_1x + p_2x^2, q(x) = q_0 + q_1x + q_2x^2 \in P_2$. *Then* $S(p(x)) = p_0x + p_1x^2 + p_2x^3$ *and* $S(q(x)) = q_0x + q_1x^2 + q_2x^3$ *so that*

$$\begin{aligned} S(p(x)) - S(q(x)) = (p_0-q_0)x + (p_1-q_1)x^2 + (p_2-q_2)x^3 = 0 \quad &\Leftrightarrow \quad p_0 = q_0, p_1 = q_1, p_2 = q_2 \\ &\Leftrightarrow \quad p(x) = q(x) \end{aligned}$$

since x, x^2, *and* x^3 *are linearly independent.*

(b) Find $S^{-1}(p(x))$.

Note that $S^{-1} : \{ax + bx^2 + cx^3 \ : \ a,b,c \in \mathbb{R}\} \to P_2$. *The inverse is defined by*

$$\begin{aligned} S^{-1}(S(p(x))) = p(x) \quad &\Leftrightarrow \quad S^{-1}(xp(x)) = p(x) \\ S(S^{-1}(q(x))) = q(x) \quad &\Leftrightarrow \quad xS^{-1}(q(x)) = q(x) \end{aligned}$$

where $p(x) \in P_2$ *and* $q(x) \in \{ax + bx^2 + cx^3 \ : \ a,b,c \in \mathbb{R}\}$. *Clearly*

$$S^{-1}(q(x)) = \frac{q(x)}{x}.$$

However, we consider this problem in terms of vector spaces where x *is just a placeholder. Let* $q(x) = q_1x + q_2x^2 + q_3x^3$ *and* $S^{-1}(q(x)) = s_0 + s_1x + s_2x^2$ *where* $q_1, q_2, q_3, s_0, s_1, s_2 \in \mathbb{R}$. *Then*

$$xS^{-1}(q(x)) = s_0x + s_1x^2 + s_2x^3 = q_1x + q_2x^2 + q_3x^3 = q(x)$$

so that $s_0 = q_1$, $s_1 = q_2$, $s_2 = q_3$ *and*

$$S^{-1}(q_1x + q_2x^2 + q_3x^3) = q_1 + q_2x + q_3x^2.$$

(c) Is S onto? Explain.

No. Since $R(S) = \{ax+bx^2+cx^3 \ : \ a,b,c \in \mathbb{R}\}$ *and* $1 \in P_3$ *but* $1 \notin \{ax+bx^2+cx^3 \ : \ a,b,c \in \mathbb{R}\}$.

Question 30

(30.1) Let

$$A = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Find the bases for the eigenspaces associated with the eigenvalues of A.

Since A is upper triangular, the eigenvalues are the entries on the diagonal namely -2 (twice) and 3 (twice).

We determine the eigenvectors for the eigenvalue -2:

$$\left(-2\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -5 & 5 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & -5 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ -5c+5d \\ -5c \\ -5d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Obviously $c = d = 0$. Thus the eigenvectors corresponding to the eigenvalue -2 are

$$\left\{ \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix} : (a,b) \in \mathbb{R}^2 \setminus \{(0,0)\} \right\}.$$

The eigenspace is

$$\left\{ \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix} : a, b \in \mathbb{R} \right\}.$$

A basis for this eigenspace is

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

We determine the eigenvectors for the eigenvalue 3:

$$\left(3\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 5 & -5 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}\right)\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & -5 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 5a \\ 5b-5c+5d \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Obviously $a = 0$ and $b = c - d$. Thus the eigenvectors corresponding to the eigenvalue 3 are

$$\left\{ \begin{bmatrix} 0 \\ c-d \\ c \\ d \end{bmatrix} : (c,d) \in \mathbb{R}^2 \setminus \{(0,0)\} \right\}.$$

The eigenspace is

$$\left\{ \begin{bmatrix} 0 \\ c-d \\ c \\ d \end{bmatrix} : c, d \in \mathbb{R} \right\}.$$

A basis for this eigenspace is

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(30.2) Let $T : P_2 \to P_3$ be the function defined by $T(p(x)) = xp(x-3)$.

(a) Show that T is a linear transformation.

We have for $p(x) = p_0 + p_1 x + p_2 x^2, q(x) = q_0 + q_1 x + q_2 x^2 \in P_2$ and $k \in \mathbb{R}$:

1.

$$\begin{aligned} T(\,(kp)(x)\,) &= T((kp_0) + (kp_1)x + (kp_2)x^2) = x((kp_0) + (kp_1)(x-3) + (kp_2)(x-3)^2) \\ &= k(x(p_0 + p_1(x-3) + p_2(x-3)^2)) = k\,xp(x-3) = kT(p(x)) \end{aligned}$$

2.

$$\begin{aligned} T(\,(p+q)(x)\,) &= T((p_0+q_0) + (p_1+q_1)x + (p_2+q_2)x^2) \\ &= x((p_0+q_0) + (p_1+q_1)(x-3) + (p_2+q_2)(x-3)^2) \\ &= x(p_0 + p_1(x-3) + p_2(x-3)^2) + x(q_0 + q_1(x-3) + q_2(x-3)^2) \\ &= xp(x-3) + xq(x-3) = T(p(x)) + T(q(x)) \end{aligned}$$

so that T is linear.

(b) Find $[T]_{B',B}$ with respect to the basis $B = \{1, x, x^2\}$ and $B' = \{1, x, x^2, x^3\}$.

From

$$\begin{aligned} T(1) &= x \cdot [1]_{x \to x-3} = x = 0 \cdot 1 + 1 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 \\ T(x) &= x \cdot [x]_{x \to x-3} = x(x-3) = 0 \cdot 1 + (-3) \cdot x + 1 \cdot x^2 + 0 \cdot x^3 \\ T(x^2) &= x \cdot [x^2]_{x \to x-3} = x(x-3)^2 = 0 \cdot 1 + 9 \cdot x + (-6) \cdot x^2 + 1 \cdot x^3 \end{aligned}$$

and the ordering

$$1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad x \to \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \qquad x^2 \to \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} x^3 \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

we find (the columns are the coefficients of the elements of B' in $T(1)$, $T(x)$ and $T(x^2)$ above)

$$[T]_{B',B} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 9 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix}.$$

(c) Compute $T(1+x-x^2)$.

We find

$$T(1+x-x^2) = x(1+(x-3)-(x-3)^2) = -11x+7x^2-x^3.$$

or equivalently

$$T(1+x-x^2) = [T]_{B',B}\begin{bmatrix}1\\1\\-1\end{bmatrix} = \begin{bmatrix}0\\-11\\7\\-1\end{bmatrix} \to -11x+7x^2-x^3.$$

(30.3) Let $S : P_3 \to P_3$ be defined by $S(p(x)) = p(x+1)$.

(a) Show that S is one-to-one.

Let $p(x) = p_0+p_1x+p_2x^2+p_3x^3, q(x) = q_0+q_1x+q_2x^2+q_3x^3 \in P_2$. *Then* $S(p(x)) = p_0+p_1(x+1)+p_2(x+1)^2+p_3(x+1)^3$ *and* $S(q(x)) = q_0+q_1(x+1)+q_2(x+1)^2+q_3(x+1)^3$ *so that*

$$\begin{aligned}S(p(x)) - S(q(x)) &= (p_0-q_0)+(p_1-q_1)(x+1)+(p_2-q_2)(x+1)^2+(p_3-q_3)(x+1)^3 = 0\\ &= ((p_0-q_0)+(p_1-q_1)+(p_2-q_2)+(p_3-q_3))\\ &\quad+((p_1-q_1)+2(p_2-q_2)+3(p_3-q_3))x\\ &\quad+((p_2-q_2)+3(p_3-q_3))x^2\\ &\quad+(p_3-q_3)x^3\end{aligned}$$

Setting each coefficient (from x^3 *down to* x^0*) to zero we find* $p_3-q_3=0$, $p_2-q_2=0$, $p_1-q_1=0$ *and* $p_0-q_0=0$ *so that* $p(x)=q(x)$ *since* 1, x, x^2, *and* x^3 *are linearly independent.*

(b) Find $S^{-1}(p(x))$.

The inverse is defined by

$$\begin{aligned}S^{-1}(S(p(x))) = p(x) \quad&\Leftrightarrow\quad S^{-1}(p(x+1)) = p(x)\\ S(S^{-1}(q(x))) = q(x) \quad&\Leftrightarrow\quad S^{-1}(q(x))\Big|_{x\to x+1} = q(x)\end{aligned}$$

where $p(x), q(x) \in P_3$. *Clearly*

$$S^{-1}(q(x)) = q(x-1).$$

However, we consider this problem in terms of vector spaces where x *is just a placeholder. Let* $q(x) = q_0+q_1x+q_2x^2+q_3x^3$ *and* $S^{-1}(q(x)) = s_0+s_1x+s_2x^2+s_3x^3$ *where* $q_0,q_1,q_2,q_3,s_0,s_1,s_2,s_3 \in \mathbb{R}$. *Then*

$$\begin{aligned}S(S^{-1}(q(x))) &= s_0+s_1(x+1)+s_2(x+1)^2+s_3(x+1)^3 = q_0+q_1x+q_2x^2+q_3x^3 = q(x)\\ &\Leftrightarrow s_0+s_1+s_2+s_3 = q_0,\; s_1+2s_2+3s_3 = q_1,\; s_2+3s_3 = q_2,\; s_3 = q_3 \qquad (\star)\end{aligned}$$

so that $s_3 = q_3$, $s_2 = q_2 - 3q_3$, $s_1 = q_1 - 2q_2 + 3q_3$ and $s_0 = q_0 - q_1 + q_2 - q_3$ and

$$\begin{aligned} S^{-1}(q_0 + q_1 x + q_2 x^2 + q_3 x^3) &= (q_0 - q_1 + q_2 - q_3) + (q_1 - 2q_2 + 3q_3)x + (q_2 - 3q_3)x^2 + q_3 x^2 \\ &= q_0 + q_1(x-1) + q_2(x-1)^2 + q_3(x-1)^3 = q(x-1). \end{aligned}$$

(c) Is S onto? Explain.

*Yes. Since from (**??**) for every $q(x) \in P_3$ there exists $s_0 + s_1 x + s_2 x^2 + s_3 x^3 \in P_3$ (where $s_0, s_1, s_2, s_3 \in \mathbb{R}$) such that $S(s_0 + s_1 x + s_2 x^2 + s_3 x^3) = q(x)$.*

Alternatively, since S is one-to-one and the kernel of S is $\{0\}$, we find that S is onto.

Question 31

Let $T : P_2 \to P_2$ be defined by

$$T(a_0 + a_1 x + a_2 x^2) = (2a_0 - a_1 + 3a_2) + (4a_0 - 5a_1)x + (a_1 + 2a_2)x^2.$$

(31.1) Find the eigenvalues of T.

We determine the matrix representation for T with respect to the basis $B = \{1, x, x^2\}$ in that order. We have

$$\begin{aligned} T(1) &= 2 \cdot 1 + 4 \cdot x + 0 \cdot x^2 \\ T(x) &= -1 \cdot 1 - 5 \cdot x + 1 \cdot x^2 \\ T(x^2) &= 3 \cdot 1 + 0 \cdot x + 2 \cdot x^2 \end{aligned}$$

and the ordering

$$1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad x \to \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad x^2 \to \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so that

$$[T]_B = \begin{bmatrix} 2 & -1 & 3 \\ 4 & -5 & 0 \\ 0 & 1 & 2 \end{bmatrix}.$$

The characteristic equation is

$$\begin{aligned} \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - [T]_B \right) &= \begin{vmatrix} \lambda - 2 & 1 & -3 \\ -4 & \lambda + 5 & 0 \\ 0 & -1 & \lambda - 2 \end{vmatrix} \\ &= (\lambda + 5)(\lambda - 2)^2 - 12 + 4(\lambda - 2) = \lambda^3 + \lambda^2 - 12\lambda = \lambda(\lambda + 4)(\lambda - 3). \end{aligned}$$

Thus the eigenvalues are -4, 0 and 3.

(31.2) Find bases for the eigenspaces of T.

The eigenvectors of T have matrix representations given by the eigenvectors of $[T]_B$. We determine the eigenvectors for the eigenvalue -4 of $[T]_B$:

$$\left(-4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}2&-1&3\\4&-5&0\\0&1&2\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-6&1&-3\\-4&1&0\\0&-1&-6\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-6x+y-3z\\-4x+y\\-y-6z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Obviously $4x = y = -6z$. Thus the eigenvectors corresponding to the eigenvalue -4 are

$$\left\{\begin{bmatrix}-3z/2\\-6z\\z\end{bmatrix} : z \in \mathbb{R}\setminus\{0\}\right\}.$$

Setting $z = 2$, for example, yields the basis

$$\{-3-12x+2x^2\}.$$

We determine the eigenvectors for the eigenvalue 0 of $[T]_B$:

$$\left(0\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}2&-1&3\\4&-5&0\\0&1&2\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2&1&-3\\-4&5&0\\0&-1&-2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-2x+y-3z\\-4x+5y\\-y-2z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Obviously $y = -2z$, $x = 5y/4 = -5z/2$. Thus the eigenvectors corresponding to the eigenvalue 0 are

$$\left\{\begin{bmatrix}-5z/2\\-2z\\z\end{bmatrix} : z \in \mathbb{R}\setminus\{0\}\right\}.$$

Setting $z = 2$, for example, yields the basis

$$\{-5-4x+2x^2\}.$$

We determine the eigenvectors for the eigenvalue 3 of $[T]_B$:

$$\left(3\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}2&-1&3\\4&-5&0\\0&1&2\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1&1&-3\\-4&8&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}x+y-3z\\-4x+8y\\-y+z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Obviously $y = z$, $x = 2y = 2z$. Thus the eigenvectors corresponding to the eigenvalue 3 are

$$\left\{\begin{bmatrix}2z\\z\\z\end{bmatrix} : z \in \mathbb{R}\setminus\{0\}\right\}.$$

Setting $z = 1$, for example, yields the basis

$$\{2+x+x^2\}.$$

Question 32

Let $T : P_2 \to P_2$ be defined by

$$T(a_0 + a_1x + a_2x^2) = -2a_2 + (a_0 + 2a_1 + a_2)x + (a_0 + 3a_2)x^2.$$

(32.1) Find the eigenvalues of T.

We determine the matrix representation for T with respect to the basis $B = \{1, x, x^2\}$ in that order. We have

$$\begin{array}{lll} (a_0 = 1, a_1 = 0, a_2 = 0) & T(1) = x + x^2 = & 0 \cdot 1 + 1 \cdot x + 1 \cdot x^2 \\ (a_0 = 0, a_1 = 1, a_2 = 0) & T(x) = 2x = & 0 \cdot 1 + 2 \cdot x + 0 \cdot x^2 \\ (a_0 = 0, a_1 = 0, a_2 = 1) & T(x^2) = -2 + x + 3x^2 = & -2 \cdot 1 + 1 \cdot x + 3 \cdot x^2 \end{array}$$

and the ordering

$$1 \to \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad x \to \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad x^2 \to \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

so that

$$[T]_B = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}.$$

The characteristic equation is

$$\begin{aligned} \det\left(\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - [T]_B \right) &= \begin{vmatrix} \lambda & 0 & 2 \\ -1 & \lambda - 2 & -1 \\ -1 & 0 & \lambda - 3 \end{vmatrix} \\ &= \lambda(\lambda - 2)(\lambda - 3) + 2(\lambda - 2) \\ &= (\lambda(\lambda - 3) + 2)(\lambda - 2) = (\lambda - 1)(\lambda - 2)^2. \end{aligned}$$

Thus the eigenvalues are 1 and 2 (twice).

(32.2) Find bases for the eigenspaces of T.

The eigenvectors of T have matrix representations given by the eigenvectors of $[T]_B$. We determine the eigenvectors for the eigenvalue 1 of $[T]_B$:

$$\left(1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix} \right) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2 \\ -1 & -1 & -1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + 2c \\ -a - b - c \\ -a - 2c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Obviously $a = -2b = -2c$. Thus the eigenvectors corresponding to the eigenvalue 1 are

$$\left\{ \begin{bmatrix} -2c \\ c \\ c \end{bmatrix} : c \in \mathbb{R} \setminus \{0\} \right\}.$$

Setting $c = 1$, for example, yields the basis

$$\{-2 + x + x^2\}.$$

We determine the eigenvectors for the eigenvalue 2 of $[T]_B$:

$$\left(2\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&-2\\1&2&1\\1&0&3\end{bmatrix}\right)\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}2&0&2\\-1&0&-1\\-1&0&-1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix} = \begin{bmatrix}2a+2c\\-a-c\\-a-c\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Obviously $a = -c$. Thus the eigenvectors corresponding to the eigenvalue 2 are

$$\left\{\begin{bmatrix}a\\b\\-a\end{bmatrix} : b \in \mathbb{R}\setminus\{0\}\right\}.$$

Setting $a = 0$, $b = 1$, and then $a = 1$, $b = 0$ for example, yields the basis

$$\{x,\ 1 - x^2\}.$$

Question 33

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by

$$T\left(\begin{bmatrix}x\\y\\z\end{bmatrix}\right) = \begin{bmatrix}-x+2y+4z\\3x+z\\2x+2y+5z\end{bmatrix}.$$

(33.1) Find a basis B' for $\mathbb{R}^3$ relative to which the matrix T is diagonal using the standard basis B for $\mathbb{R}^3$.

A set of 3 linearly independent eigenvectors of T will diagonalize T. We find the representation of the eigenvectors in the standard basis. For the standard basis we have

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = -1\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = 4\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\1\end{bmatrix}$$

so that (with the usual ordering of the standard basis)

$$[T]_B = \begin{bmatrix}-1&2&4\\3&0&1\\2&2&5\end{bmatrix}.$$

First we need the eigenvalues an eigenvectors of $[T]_B$. The characteristic equation is

$$\det\left(\begin{bmatrix}\lambda & 0 & 0\\ 0 & \lambda & 0\\ 0 & 0 & \lambda\end{bmatrix} - [T]_B\right) = \begin{vmatrix}\lambda+1 & -2 & -4\\ -3 & \lambda & -1\\ -2 & -2 & \lambda-5\end{vmatrix}$$
$$= \lambda^3 - 4\lambda^2 - 21\lambda = \lambda(\lambda-7)(\lambda+3).$$

Thus the eigenvalues are 0, -3 and 7 . We determine the eigenvectors for the eigenvalue 0:

$$\left(0\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}-1 & 2 & 4\\ 3 & 0 & 1\\ 2 & 2 & 5\end{bmatrix}\right)\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}1 & -2 & -4\\ -3 & 0 & -1\\ -2 & -2 & -5\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}x-2y-4z\\ -3x-z\\ -2x-2y-5z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Obviously $z = -3x$ and $y = (x-4z)/2 = 13x/2$. Thus the eigenvectors corresponding to the eigenvalue 0 are

$$\left\{\begin{bmatrix}x\\ 13x/2\\ -3x\end{bmatrix} : x \in \mathbb{R}\setminus\{0\}\right\}.$$

A representative eigenvector is (any one could have been chosen)

$$\begin{bmatrix}2\\ 13\\ -6\end{bmatrix}.$$

We determine the eigenvectors for the eigenvalue -3:

$$\left(-3\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}-1 & 2 & 4\\ 3 & 0 & 1\\ 2 & 2 & 5\end{bmatrix}\right)\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-2 & -2 & -4\\ -3 & -3 & -1\\ -2 & -2 & -8\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}-2x-2y-4z\\ -3x-3y-z\\ -2x-2y-8z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

We have $-2x-2y-4z = -2x-2y-8z = 0$ so that $z = 0$. It follows that $-3x-3y-z = -3x-3y = 0$ and $x = -y$. Thus the eigenvectors corresponding to the eigenvalue -3 are

$$\left\{\begin{bmatrix}x\\ -x\\ 0\end{bmatrix} : x \in \mathbb{R}\setminus\{0\}\right\}.$$

A representative eigenvector is (any one could have been chosen)

$$\begin{bmatrix}1\\ -1\\ 0\end{bmatrix}.$$

We determine the eigenvectors for the eigenvalue 7:

$$\left(7\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} - \begin{bmatrix}-1 & 2 & 4\\ 3 & 0 & 1\\ 2 & 2 & 5\end{bmatrix}\right)\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}8 & -2 & -4\\ -3 & 7 & -1\\ -2 & -2 & 2\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} = \begin{bmatrix}8x-2y-4z\\ -3x+7y-z\\ -2x-2y+2z\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Row reduction yields

$$\begin{array}{r}+4R_3\\-3R_3/2\\-R_3/2\end{array}\left[\begin{array}{ccc|c}8&-2&-4&0\\-3&7&-1&0\\-2&-2&2&0\end{array}\right]\to\begin{array}{r}=R_3\\+R_1\\=R_2\end{array}\left[\begin{array}{ccc|c}0&-10&4&0\\0&10&-4&0\\1&1&-1&0\end{array}\right]$$

$$\to\left[\begin{array}{ccc|c}1&-1&1&0\\0&0&0&0\\0&10&-4&0\end{array}\right].$$

Obviously $y = 2z/5$ *and* $x = -y + z = 3z/5$*. Thus the eigenvectors corresponding to the eigenvalue 7 are*

$$\left\{\begin{bmatrix}3z/5\\2z/5\\z\end{bmatrix} : z \in \mathbb{R}\setminus\{0\}\right\}.$$

A representative eigenvector is (any one could have been chosen)

$$\begin{bmatrix}3\\2\\5\end{bmatrix}.$$

Thus a basis B' *relative to which the matrix for* T *is diagonal is*

$$B' = \left\{\begin{bmatrix}2\\13\\-6\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}3\\2\\5\end{bmatrix}\right\}.$$

(33.2) Compute $[T]_{B'}$ and verify that $[T]_{B'} = P^{-1}[T]_B P$ where the matrix P diagonalizes $[T]_B$.

For the basis B' *above we have*

$$T\left(\begin{bmatrix}2\\13\\-6\end{bmatrix}\right) = \begin{bmatrix}0\\0\\0\end{bmatrix} = 0\begin{bmatrix}2\\13\\-6\end{bmatrix} + 0\begin{bmatrix}1\\-1\\0\end{bmatrix} + 0\begin{bmatrix}3\\2\\5\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}\right) = \begin{bmatrix}-3\\3\\0\end{bmatrix} = 0\begin{bmatrix}2\\13\\-6\end{bmatrix} - 3\begin{bmatrix}1\\-1\\0\end{bmatrix} + 0\begin{bmatrix}3\\2\\5\end{bmatrix}$$

$$T\left(\begin{bmatrix}3\\2\\5\end{bmatrix}\right) = \begin{bmatrix}21\\14\\35\end{bmatrix} = 0\begin{bmatrix}2\\13\\-6\end{bmatrix} + 0\begin{bmatrix}1\\-1\\0\end{bmatrix} + 7\begin{bmatrix}3\\2\\5\end{bmatrix}$$

so that (with the ordering in the set B' *above)*

$$[T]_{B'} = \begin{bmatrix}0&0&0\\0&-3&0\\0&0&7\end{bmatrix}.$$

Using the representative vectors above we find

$$P = \begin{bmatrix} 2 & 1 & 3 \\ 13 & -1 & 2 \\ -6 & 0 & 5 \end{bmatrix}.$$

P^{-1} can be found by row reduction of the matrix P augmented with the 3×3 identity matrix

$$\begin{array}{r} {\scriptstyle =R_1/2} \\ {\scriptstyle -13R_1/2} \\ {\scriptstyle +3R_1} \end{array}\left[\begin{array}{ccc|ccc} 2 & 1 & 3 & 1 & 0 & 0 \\ 13 & -1 & 2 & 0 & 1 & 0 \\ -6 & 0 & 5 & 0 & 0 & 1 \end{array}\right] \rightarrow {\scriptstyle +5R_3/2} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & -15/2 & -35/2 & -13/2 & 1 & 0 \\ 0 & 3 & 14 & 3 & 0 & 1 \end{array}\right]$$

$$\rightarrow {\scriptstyle =2R_2/35} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 0 & 35/2 & 1 & 1 & 5/2 \\ 0 & 3 & 14 & 3 & 0 & 1 \end{array}\right]$$

$$\rightarrow \begin{array}{r} {\scriptstyle -3R_2/2} \\ \\ {\scriptstyle -14R_2} \end{array} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 2/35 & 2/35 & 1/7 \\ 0 & 3 & 14 & 3 & 0 & 1 \end{array}\right]$$

$$\rightarrow \begin{array}{r} {\scriptstyle -R_3/6-3R_2/2} \\ {\scriptstyle =R_3/3} \\ {\scriptstyle =R_2/3} \end{array} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 3/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 2/35 & 2/35 & 1/7 \\ 0 & 3 & 0 & 11/5 & -4/5 & -1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/21 & 1/21 & -1/21 \\ 0 & 1 & 0 & 11/15 & -4/15 & -1/3 \\ 0 & 0 & 1 & 2/35 & 2/35 & 1/7 \end{array}\right].$$

Thus

$$P^{-1} = \frac{1}{2}\begin{bmatrix} 1/21 & 1/21 & -1/21 \\ 11/15 & -4/15 & -1/3 \\ 2/35 & 2/35 & 1/7 \end{bmatrix}.$$

It follows that

$$\begin{aligned} P^{-1}[T]_B P &= \begin{bmatrix} 1/21 & 1/21 & -1/21 \\ 11/15 & -4/15 & -1/3 \\ 2/35 & 2/35 & 1/7 \end{bmatrix} \begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 13 & -1 & 2 \\ -6 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix} = [T]_{B'}. \end{aligned}$$

Question 34

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear operator given by

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 4x + z \\ 2x + 3y + 2z \\ x + 4z \end{bmatrix}.$$

(34.1) Find a basis B' for $\mathbb{R}^3$ relative to which the matrix T is diagonal using the standard basis B for $\mathbb{R}^3$.

A set of 3 linearly independent eigenvectors of T will diagonalize T. We find the representation of the eigenvectors in the standard basis. For the standard basis we have

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = 4\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}0\\0\\1\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = 0\begin{bmatrix}1\\0\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\\0\end{bmatrix} + 0\begin{bmatrix}0\\0\\1\end{bmatrix}$$
$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\\0\end{bmatrix} + 4\begin{bmatrix}0\\0\\1\end{bmatrix}$$

so that (with the usual ordering of the standard basis)

$$[T]_B = \begin{bmatrix}4 & 0 & 1\\2 & 3 & 2\\1 & 0 & 4\end{bmatrix}.$$

First we need the eigenvalues an eigenvectors of $[T]_B$. The characteristic equation is

$$\det\left(\begin{bmatrix}\lambda & 0 & 0\\0 & \lambda & 0\\0 & 0 & \lambda\end{bmatrix} - [T]_B\right) = \begin{vmatrix}\lambda-4 & 0 & -1\\-2 & \lambda-3 & -2\\-1 & 0 & \lambda-4\end{vmatrix}$$
$$= (\lambda-3)(\lambda-4)^2 - (\lambda-3) = (\lambda-3)((\lambda-4)^2-1) = (\lambda-3)^2(\lambda-5).$$

Thus the eigenvalues are 3 *(twice)* *and* 5 . *We determine the eigenvectors for the eigenvalue* 3*:*

$$\left(3\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}4 & 0 & 1\\2 & 3 & 2\\1 & 0 & 4\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-1 & 0 & -1\\-2 & 0 & -2\\-1 & 0 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}-x-z\\-2x-2z\\-x-z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Obviously $x = -z$. Thus the eigenvectors corresponding to the eigenvalue 3 are

$$\left\{\begin{bmatrix}x\\y\\-x\end{bmatrix} : (x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}\right\}.$$

Two representative eigenvectors are (any two linearly independent eigenvectors could have been chosen)

$$\begin{bmatrix}1\\0\\-1\end{bmatrix}, \quad \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

We determine the eigenvectors for the eigenvalue 5*:*

$$\left(5\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}4 & 0 & 1\\2 & 3 & 2\\1 & 0 & 4\end{bmatrix}\right)\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1 & 0 & -1\\-2 & 2 & -2\\-1 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}x-z\\-2x+2y-2z\\-x+z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Obviously $x = z$ and $y = 2x$. Thus the eigenvectors corresponding to the eigenvalue 5 are

$$\left\{ \begin{bmatrix} x \\ 2x \\ x \end{bmatrix} : x \in \mathbb{R} \setminus \{0\} \right\}.$$

A representative eigenvector is (any one could have been chosen)

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Thus a basis B' relative to which the matrix for T is diagonal is

$$B' = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \right\}.$$

(34.2) Compute $[T]_{B'}$ and verify that $[T]_{B'} = P^{-1}[T]_B P$ where the matrix P diagonalizes $[T]_B$.
For the basis B' above we have

$$T\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 10 \\ 5 \end{bmatrix} = 0\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

so that (with the ordering in the set B' above)

$$[T]_{B'} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Using the representative vectors above we find

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix}.$$

P^{-1} can be found by row reduction of the matrix P augmented with the 3×3 identity matrix

$$\begin{matrix} \\ \\ +R1 \end{matrix}\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \begin{matrix} -R3/2 \\ -R3 \\ =R3/2 \end{matrix}\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & \frac{1}{2} & 0 & \frac{1}{2} \end{array}\right].$$

Thus

$$P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}.$$

It follows that

$$\begin{aligned} P^{-1}[T]_B P &= \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 0 & -1 \\ -2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 0 & 5 \\ 0 & 3 & 10 \\ -3 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix} = [T]_{B'}. \end{aligned}$$

Question 35

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by the matrix $A = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 2 & 5 \end{bmatrix}$. Find

(35.1) a basis for the range of T,

The range of T is the column space of A. Applying row reduction to A^T we find

$$\begin{matrix} \\ +2\,R1 \\ +4\,R1 \end{matrix}\begin{bmatrix} -1 & 3 & 2 \\ 2 & 0 & 2 \\ 4 & 1 & 5 \end{bmatrix} \to \begin{matrix} \\ =R2/6 \\ =R3/13 \end{matrix}\begin{bmatrix} -1 & 3 & 2 \\ 0 & 6 & 6 \\ 0 & 13 & 13 \end{bmatrix} \to \begin{matrix} \\ \\ -R2 \end{matrix}\begin{bmatrix} -1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus a basis for the range of T is

$$\left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(35.2) a basis for the kernel of T,

The kernel of T is the nullspace of A. Let $x, y, z \in \mathbb{R}$ satisfy

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row reduction yields

$$\begin{matrix} \\ +3\,R1 \\ +2\,R1 \end{matrix}\begin{bmatrix} -1 & 2 & 4 \\ 3 & 0 & 1 \\ 2 & 2 & 5 \end{bmatrix} \to \begin{matrix} \\ \\ -R2 \end{matrix}\begin{bmatrix} -1 & 2 & 4 \\ 0 & 6 & 13 \\ 0 & 6 & 13 \end{bmatrix} \to \begin{matrix} \\ =R2/6 \\ =R3/13 \end{matrix}\begin{bmatrix} -1 & 2 & 4 \\ 0 & 6 & 13 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus we have the nullpsace of A $y = -13z/6$ *and* $x = 2y + 4z = -z/3$.

$$\left\{ \frac{z}{6} \begin{bmatrix} -2 \\ -13 \\ 6 \end{bmatrix}, \ z \in \mathbb{R} \right\}$$

which has a basis

$$\left\{ \begin{bmatrix} -2 \\ -13 \\ 6 \end{bmatrix} \right\}.$$

(35.3) the rank and nullity of T and

From the basis found in 35.1 we find the rank of T *is 2.*
From the basis found in 35.2 we find the nullity of T *is 1.*

(35.4) the rank and nullity of A.

From the row reduction in 35.2 we find the rank of A *is 2.*
From 35.2 we find the nullity of A *is 1.*

Question 36

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be multiplication by the matrix $A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 4 \\ 1 & 8 & 5 \end{bmatrix}$. Find

(36.1) a basis for the range of T,

The range of T *is the column space of* A. *Applying row reduction to* A^T *we find*

$$\begin{matrix} \\ \\ -3\,R_1 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 3 & 4 & 5 \end{bmatrix} \to \begin{matrix} \\ = R_2/2 \\ -\,R_2/2 \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 8 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{bmatrix}.$$

Thus a basis for the range of T *is*

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \right\}.$$

In fact, any basis for $\mathbb{R}^3$ *will do, including the standard basis (or the same three columns), since the three columns are linearly independent.*

(36.2) a basis for the kernel of T,

Since A is invertible, the kernel consists only of the zero vector.

Alternative:
The kernel of T is the nullspace of A. Let $x, y, z \in \mathbb{R}$ satisfy

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 4 \\ 1 & 8 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row reduction yields

$$\begin{matrix} \\ -R_1 \\ -R_1 \end{matrix}\begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 4 \\ 1 & 8 & 5 \end{bmatrix} \rightarrow \begin{matrix} \\ \\ -4R_1 \end{matrix}\begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 8 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -2 \end{bmatrix}.$$

$z = 0$, $y = -z/2 = 0$ *and* $x = -3z = 0$. *Thus we have the nullpsace of A*

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

which has no basis.

(36.3) the rank and nullity of T and

From the basis found in 36.1 we find the rank of T is 3.
From the basis found in 36.2 we find the nullity of T is 0.

(36.4) the rank and nullity of A.

From the row reduction in 36.2 we find the rank of A is 3.
From 36.2 we find the nullity of A is 0.

Question 37

Consider $T : P_2 \to M_{22}$ given by $T(a + bx + cx^2) = \frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}$ for all $a, b, c \in \mathbb{R}$.

(37.1) Show that T is a linear transform.

- *For all $a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$*

$$\begin{aligned} T((a + bx + cx^2) + (\alpha + \beta x + \gamma x^2)) &= T((a + \alpha) + (b + \beta)x + (c + \gamma)x^2) \\ &= \frac{1}{2}\begin{bmatrix} 2(a + \alpha) & b + \beta \\ b + \beta & 2(c + \gamma) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2a + 2\alpha & b + \beta \\ b + \beta & 2c + 2\gamma \end{bmatrix} \\ T(a + bx + cx^2) + T(\alpha + \beta x + \gamma x^2) &= \frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 2\alpha & \beta \\ \beta & 2\gamma \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2a + 2\alpha & b + \beta \\ b + \beta & 2c + 2\gamma \end{bmatrix} \end{aligned}$$

i.e. $T((a + bx + cx^2) + (\alpha + \beta x + \gamma x^2)) = T(a + bx + cx^2) + T(\alpha + \beta x + \gamma x^2)$.

- *For all* $a, b, c, k \in \mathbb{R}$

$$\begin{aligned} T(k(a+bx+cx^2)) &= T((ka)+(kb)x+(kc)x^2) \\ &= \frac{1}{2}\begin{bmatrix} 2(ka) & kb \\ kb & 2(kc) \end{bmatrix} = k\left(\frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}\right) = kT(a+bx+cx^2). \end{aligned}$$

Thus T *is a linear transform.*

(37.2) Find the matrix representation for T relative to the standard basis in P_2 and in M_{22} with the usual ordering.

Applying T *to the elements of the standard basis* B_{P_2} *of* P_2 *yields*

$$\begin{aligned} T(1) = T(1+0x+0x^2) &= \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ T(x) = T(0+1x+0x^2) &= \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \\ T(x^2) = T(0+0x+1x^2) &= \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

The coefficients of the elements of the standard basis $B_{M_{22}}$ *in* M_{22} *provide the columns of the matrix representation:*

$$[T]_{B_{P_2},B_{M_{22}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(37.3) Is T invertible?

We have

$$R(T) = \left\{ T(a+bx+cx^2) \ : \ a,b,c \in \mathbb{R} \right\} = \left\{ \frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix} \ : \ a,b,c \in \mathbb{R} \right\}.$$

If we can solve for $T^{-1} : R(T) \to P_2$ *satisfying*

$$\begin{aligned} (T^{-1} \circ T)(a+bx+cx^2) &= T^{-1}(T(a+bx+cx^2)) = a+bx+cx^2, \\ (T \circ T^{-1})\left(\frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}\right) &= T\left(T^{-1}\left(\frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}\right)\right) = \frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}. \end{aligned} \tag{TI}$$

for all $a,b,c \in \mathbb{R}$ *then* T *is invertible. Suppose*

$$T^{-1}\left(\frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}\right) = \alpha + \beta x + \gamma x^2$$

then

$$(T \circ T^{-1})\left(\frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}\right) = T(\alpha+\beta x+\gamma x^2) = \frac{1}{2}\begin{bmatrix} 2\alpha & \beta \\ \beta & 2\gamma \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2a & b \\ b & 2c \end{bmatrix}.$$

Clearly $\alpha = a$, $\beta = b$ and $\gamma = c$. Thus

$$T^{-1}\left(\frac{1}{2}\begin{bmatrix}2a & b\\ b & 2c\end{bmatrix}\right) = a + bx + cx^2.$$

Also,

$$(T^{-1} \circ T)(a + bx + cx^2) = T^{-1}(T(a + bx + cx^2)) = T^{-1}\left(\frac{1}{2}\begin{bmatrix}2a & b\\ b & 2c\end{bmatrix}\right) = a + bx + cx^2.$$

Since both equations (TI) *are satisfied, T is invertible.*

(37.4) Show that the range of T is the subspace $\widetilde{M}_{22}$ of M_{22} consisting of symmetric matrices.

We have

$$\begin{aligned}R(T) &= \left\{\, T(a + bx + cx^2) \;:\; a, b, c \in \mathbb{R} \,\right\}\\ &= \left\{ \begin{bmatrix}a & b/2\\ b/2 & c\end{bmatrix} \;:\; a, b, c \in \mathbb{R} \right\}\\ &= \left\{ \begin{bmatrix}a & b'\\ b' & c\end{bmatrix} \;:\; a, b', c \in \mathbb{R} \right\} && \textit{(set } b' = b/2\textit{)}\\ &= \widetilde{M}_{22}.\end{aligned}$$

(37.5) Let $\widetilde{T} : P_2 \to \widetilde{M}_{22}$ be defined by $\widetilde{T}(p(x)) := T(p(x))$ for all $p(x) \in P_2$. Find the matrix representation for $\widetilde{T}$ relative to the standard basis with the usual ordering in P_2 and the basis

$$\left\{ \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} \right\}$$

for the 2×2 symmetric matrices, ordered left to right.

Applying $\widetilde{T}$ to the elements of the standard basis B_{P_2} of P_2 yields

$$\begin{aligned}\widetilde{T}(1) = \widetilde{T}(1 + 0x + 0x^2) &= \frac{1}{2}\begin{bmatrix}2 & 0\\ 0 & 0\end{bmatrix} = 1\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + 0\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\ \widetilde{T}(x) = \widetilde{T}(0 + 1x + 0x^2) &= \frac{1}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix} = 0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 0\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\\ \widetilde{T}(x^2) = \widetilde{T}(0 + 0x + 1x^2) &= \frac{1}{2}\begin{bmatrix}0 & 0\\ 0 & 2\end{bmatrix} = 0\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + 1\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix} + 0\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}.\end{aligned}$$

The coefficients of the elements of the basis $B_{\widetilde{M}_{22}}$ in $\widetilde{M}_{22}$ provide the columns of the matrix representation:

$$[\widetilde{T}]_{B_{P_2}, B_{\widetilde{M}_{22}}} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1/2 & 0\end{bmatrix}.$$

Question 38

Consider $T : M_{22} \to P_2$ given by

$$T(A) = \begin{bmatrix} 1 & x \end{bmatrix} A \begin{bmatrix} 1 \\ x \end{bmatrix}, \qquad \text{for all } A \in M_{22}.$$

(38.1) Show that T is a linear transform.

- *By distributivity of the matrix product, we have for all $A, B \in M_{22}$*

$$\begin{aligned} T(A+B) &= \begin{bmatrix} 1 & x \end{bmatrix} (A+B) \begin{bmatrix} 1 \\ x \end{bmatrix} \\ &= \left(\begin{bmatrix} 1 & x \end{bmatrix} A + \begin{bmatrix} 1 & x \end{bmatrix} B\right) \begin{bmatrix} 1 \\ x \end{bmatrix} \\ &= \begin{bmatrix} 1 & x \end{bmatrix} A \begin{bmatrix} 1 \\ x \end{bmatrix} + \begin{bmatrix} 1 & x \end{bmatrix} B \begin{bmatrix} 1 \\ x \end{bmatrix} \\ &= T(A) + T(B). \end{aligned}$$

- *For all $k \in \mathbb{R}$ and $A \in M_{22}$ we have*

$$T(A) = \begin{bmatrix} 1 & x \end{bmatrix} kA \begin{bmatrix} 1 \\ x \end{bmatrix} = k \begin{bmatrix} 1 & x \end{bmatrix} A \begin{bmatrix} 1 \\ x \end{bmatrix} = kT(A).$$

Thus T is a linear transform.

(38.2) Find the matrix representation for T relative to the standard basis in M_{22} and in P_2 with the usual ordering.
Applying T to the elements of the standard basis $B_{M_{22}}$ of M_{22} yields

$$\begin{aligned} T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) &= 1 + 0x + 0x^2 \\ T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) &= 0 + 1x + 0x^2 \\ T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) &= 0 + 1x + 0x^2 \\ T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 0 + 0x + 1x^2. \end{aligned}$$

The coefficients of the elements of the standard basis B_{P_2} in P_2 provide the columns of the matrix representation:

$$[T]_{B_{M_{22}}, B_{P_2}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(38.3) Is T one to one?

Since

$$T\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right) = 2x$$

and

$$T\left(\begin{bmatrix}0 & 2\\0 & 0\end{bmatrix}\right) = 2x$$

T is not one to one.

(38.4) Let $\widetilde{M_{22}}$ be the subspace of M_{22} consisting of symmetric matrices. Let $\widetilde{T} : \widetilde{M_{22}} \to P_2$ be defined by $\widetilde{T}(A) := T(A)$ for all $A \in \widetilde{M_{22}}$. Find the matrix representation for $\widetilde{T}$ relative to the standard basis with the usual ordering in P_2 and the basis

$$\left\{ \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} \right\}$$

for the 2×2 symmetric matrices, ordered left to right.

Applying T to the elements of the given basis B of $\widetilde{M_{22}}$

$$\begin{aligned}T\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\right) &= 1 + 0x + 0x^2\\T\left(\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right) &= 0 + 0x + 1x^2\\T\left(\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\right) &= 0 + 2x + 0x^2.\end{aligned}$$

The coefficients of the elements of the standard basis B_{P_2} in P_2 provide the columns of the matrix representation:

$$[T]_{B,B_{P_2}} = \begin{bmatrix}1 & 0 & 0\\0 & 0 & 2\\0 & 1 & 0\end{bmatrix}.$$

Question 39

Consider $T : P_2 \to P_2$ given by $T(a + bx + cx^2) = b + cx + ax^2$ for all $a, b, c \in \mathbb{R}$.

(39.1)

(a) Find the kernel and nullity of T.

The kernel of T is given by

$$\begin{aligned}\ker(T) &= \{\, a + bx + cx^2 \;:\; a, b, c \in \mathbb{R},\, T(a + bx + cx^2) = 0 + 0x + 0x^2 \,\}\\&= \{\, a + bx + cx^2 \;:\; a, b, c \in \mathbb{R},\, b + cx + ax^2 = 0 + 0x + 0x^2 \,\}\\&= \{\, a + bx + cx^2 \;:\; a = 0, b = 0, c = 0 \,\} = \{\, 0 + 0x + 0x^2 \,\}.\end{aligned}$$

The nullity of T is the dimension of $\ker(T)$ which is 0.

(b) Find the range and rank of T.

The range of T is given by

$$\begin{aligned} R(T) &= \{\, T(a+bx+cx^2) \,:\, a,b,c \in \mathbb{R} \,\} \\ &= \{\, b+cx+ax^2 \,:\, a,b,c \in \mathbb{R} \,\} = P_2. \end{aligned}$$

The rank of T is the dimension of $R(T)$ which is 3.

(39.2) Find the real valued eigenvalues and corresponding eigenspaces of T.

The eigenvalue equation for T is

$$T(a+bx+cx^2) = \lambda(a+bx+cx^2) \quad \Rightarrow \quad b+cx+ax^2 = (\lambda a) + (\lambda b)x + (\lambda c)x^2$$

where $a,b,c,\lambda \in \mathbb{R}$, $(a,b,c) \neq (0,0,0)$. Comparing coefficients of elements of the standard basis in P_2 yields

$$b = \lambda a, \qquad c = \lambda b = \lambda^2 a, \qquad a = \lambda c = \lambda^3 a.$$

If $a = 0$, then $b = 0$ and $c = 0$ which contradicts $(a,b,c) \neq (0,0,0)$. So $a \neq 0$. Consequently $\lambda^3 = 1$ which has the only real solution $\lambda = 1$. From $\lambda = 1$ we find $b = \lambda a = a$ and $c = \lambda b = b = a$. Thus 1 is the only real eigenvalue of T and the corresponding eigenspace is

$$\{\, a+ax+ax^2 \,:\, a \in \mathbb{R} \,\} = \{\, a(1+x+x^2) \,:\, a \in \mathbb{R} \,\}.$$

Alternative:

We find the matrix representation of T with respect to the standard basis $B = \{1, x, x^2\}$ in P_2, with the usual left to right ordering. We have

$$\begin{aligned} T(1) &= T(1+0x+0x^2) = 0+0x+1x^2 \\ T(x) &= T(0+1x+0x^2) = 1+0x+0x^2 \\ T(x^2) &= T(0+0x+1x^2) = 0+1x+0x^2 \end{aligned}$$

so that

$$[T]_B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

The characteristic equation in λ for $[T]_B$ is

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right) = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{vmatrix} = \lambda^3 - 1 = 0.$$

The only real solution is $\lambda = 1$. Solving

$$\left(1 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \right) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$ yields $a = b$, $b = c$ and $a = c$ so that $a = b = c$. The eigenspace of T corresponding to the eigenvalue 1 is given by

$$\left\{ p(x) \in P_2 \,:\, [p(x)]_B = \begin{bmatrix} a \\ a \\ a \end{bmatrix}, \, a \in \mathbb{R} \right\} = \left\{ p(x) \in P_2 \,:\, p(x) = a \cdot 1 + a \cdot x + a \cdot x^2, \, a \in \mathbb{R} \right\}$$
$$= \left\{ a(1 + x + x^2) \,:\, a \in \mathbb{R} \right\}.$$

(39.3) Find $T^3 := T \circ T \circ T$.

We have

$$\begin{aligned} T^3(a + bx + cx^2) &= T(T(T(a + bx + cx^2))) \\ &= T(T(b + cx + ax^2)) \\ &= T(c + ax + bx^2) \\ &= a + bx + cx^2 \end{aligned}$$

for all $a, b, c \in \mathbb{R}$, i.e. T^3 is the identity on P_2 and $T^2 = T^{-1}$.

F.2 Previous multiple choice questions

Questions

Question 1

Consider the set

$$X := \{\, (x, y) \;:\; x, y \in \mathbb{R} \,\}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\begin{aligned} \cdot : \mathbb{R} \times X \to X, \qquad & k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx - k + 1, ky), \\ + : X \times X \to X, \qquad & \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha - 1, y + \beta). \end{aligned}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. The zero vector for X is

1. $(1, 0)$
2. $(1, 1)$
3. $(0, 1)$
4. $(0, 0)$
5. None of the above.

Question 2

Consider the set

$$X := \{\, (x, y) \;:\; x, y \in \mathbb{R} \,\}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\begin{aligned} \cdot : \mathbb{R} \times X \to X, \qquad & k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx - k + 1, ky), \\ + : X \times X \to X, \qquad & \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha - 1, y + \beta). \end{aligned}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements are true in this vector space?

1. $-(1, 0) = (-1, 0)$
2. $-(1, 0) = (1, 1)$
3. $-(1, 0) = (1, 0)$
4. $-(1, 0) = (0, 0)$
5. None of the above.

Question 3

Which of the following are subspaces of $\mathbb{R}^2$ with the usual operations ?

A. $\operatorname{span}\{(2,3)\}$

B. $\{(1,x) : x \in \mathbb{R}\}$

C. $\{(0,x) : x \in \mathbb{R},\, x \geq 0\}$

D. $\{(x,x) : x \in \mathbb{R}\}$

Select from the following:

1. Only A.
2. Only A and D.
3. Only C.
4. Only C and D.
5. None of the above.

Question 4

Which of the following are subspaces of $\mathbb{R}^2$ with the usual operations ?

A. $\operatorname{span}\{(\pi,0)\}$

B. $\{(2,x) : x \in \mathbb{R}\}$

C. $\{(x,y) : x,y \in \mathbb{N}\}$

D. $\{(x,-x) : x \in \mathbb{R}\}$

Select from the following:

1. Only A and D.
2. Only A, B and D.
3. Only C.
4. Only D.
5. None of the above.

Question 5

Which of the following sets are linearly independent?

A. $\operatorname{span}\{(2,3)\}$ in $\mathbb{R}^2$

B. $\{(1,1),\ (1,-1)\}$ in $\mathbb{R}^2$

C. $\{(1,1),\ (1,-1),\ (0,1)\}$ in $\mathbb{R}^2$

D. $\{1+x,\ 1-x\}$ in P_1

Select from the following:

1. Only A.
2. Only B.
3. Only B and C.
4. Only B and D.
5. None of the above.

Question 6

Which of the following sets are linearly independent?

A. $\operatorname{span}\{(\pi,0)\}$ in $\mathbb{R}^2$

B. $\{(1,2),\ (2,1)\}$ in $\mathbb{R}^2$

C. $\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$ in M_{22}

D. $\{1+2x,\ 2-x\}$ in P_1

Select from the following:

1. Only B and D.
2. Only B.
3. Only C and D.
4. Only D.
5. None of the above.

Question 7

Which of the following sets are identical?

A. $\text{span}\{(1,0,1),(1,0,-1)\}$ in $\mathbb{R}^3$

B. $\text{span}\{(0,0,1),(5,0,0)\}$ in $\mathbb{R}^3$

C. $\text{span}\{(3,0,7),(0,0,1),(5,0,0)\}$ in $\mathbb{R}^3$

D. $\{(1,0,1),(1,0,-1)\}$

E. $\text{span}\{(1,1,1),(1,-1,-1)\}$ in $\mathbb{R}^3$

Select from the following:

1. Only A and D.
2. Only A and E.
3. Only A, B and C.
4. Only B and C.
5. None of the above.

Question 8

Which of the following sets are identical?

A. $\text{span}\{(1,0,1),(2,0,2)\}$ in $\mathbb{R}^3$

B. $\text{span}\{(1,0,1),(1,0,-1)\}$ in $\mathbb{R}^3$

C. $\text{span}\{(1,0,1),(1,1,1)\}$ in $\mathbb{R}^3$

D. $\text{span}\{(3,0,3)\}$ in $\mathbb{R}^3$

E. $\{(1,0,1),(2,0,2)\}$

Select from the following:

1. Only A and B.
2. Only A and D.
3. Only A and E.
4. Only B and C.
5. None of the above.

Question 9

Which of the following sets are a basis for the following vector subspace of P_2:

$$X = \{ p(x) \in P_2 \, : \, p(3) = 0 \} .$$

A. $\{ 1, x, x^2 \}$

B. $\{ x - 3, x^2 - 9 \}$

C. $\{ x^2 + 2x - 15, x^2 - 2x - 3 \}$

D. $\{ x - 3, x^3 - 27 \}$

Select from the following:

1. Only B and C.
2. Only B.
3. Only A.
4. Only B and D.
5. None of the above.

Question 10

Which of the following sets are a basis for the following vector subspace of M_{22}:

$$X = \left\{ A \in M_{22} \, : \, A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} .$$

A. $\left\{ \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only C and D.
5. None of the above.

Question 11

Which of the following statements are true:

A. $\dim(\text{span}\{\,(1,0,1),(1,0,-1)\,\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\text{span}\{\,(0,0,0),(5,0,0)\,\}) = 2$ in $\mathbb{R}^3$

C. $\dim(\text{span}\{\,(3,0,7),(0,0,1),(5,0,0)\,\}) = 2$ in $\mathbb{R}^3$

D. $\dim(\text{span}\{\,(3,0,7),(0,0,1),(5,0,0)\,\}) = 3$ in $\mathbb{R}^3$

Select from the following:

1. All of A, B, C and D.
2. Only A and B.
3. Only A and C.
4. Only A and D.
5. None of the above.

Question 12

Which of the following statements are true:

A. $\dim(\text{span}\{\,(1,0,1),(1,0,-1)\,\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\text{span}\{\,(0,0,0),(5,0,0)\,\}) = 2$ in $\mathbb{R}^3$

C. $\dim(\text{span}\{\,(1,0,1),(1,0,-1),(5,0,0)\,\}) = 2$ in $\mathbb{R}^3$

D. $\dim(\text{span}\{\,(1,0,1),(1,0,-1),(5,0,0)\,\}) = 3$ in $\mathbb{R}^3$

Select from the following:

1. All of A, B, C and D.
2. Only A, B and D.
3. Only A and C.
4. Only A and D.
5. None of the above.

Question 13

Which of the following sets are a basis for the row space of $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 0 & -3 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only C.
3. Only A and C.
4. Only B.
5. None of the above.

Question 14

Which of the following sets are a basis for the row space of $\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 2 & -4 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T, \begin{bmatrix} -4 & 2 & -4 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} 2 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -4 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 2 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 1 & 2 \end{bmatrix} \right\}$

E. $\left\{ \begin{bmatrix} 2 & -1 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B, C, and E.
3. Only E.
4. Only D.
5. None of the above.

Question 15

Which of the following sets are contained in (i.e. subset of) the column space of $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \begin{bmatrix} -1 & 2 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} -1 & 2 \end{bmatrix}^T, \begin{bmatrix} 2 & 1 \end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 \end{bmatrix}^T \right\}$

D. $\left\{ \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \right\}$

Select from the following:

1. Only D.
2. Only A, B and C.
3. Only A and B.
4. Only A and C.
5. None of the above.

Question 16

Which of the following sets are contained in (i.e. subset of) the column space of $\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 2 & -4 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T, \begin{bmatrix} -4 & 2 & -4 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} 2 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -4 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 2 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T \right\}$

E. $\left\{ \begin{bmatrix} 1 & 2 \end{bmatrix} \right\}$

Select from the following:

1. Only E.
2. Only B, C and D.
3. Only B and C.
4. Only A and D.
5. None of the above.

Question 17

Which of the following sets are a basis for the null space of $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} 7 & -7 & -7 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \right\}$

Select from the following:

1. Only B and C.
2. Only B.
3. Only C.
4. Only A.
5. None of the above.

Question 18

Which of the following sets are a basis for the null space of $\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 2 & -4 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T, \begin{bmatrix} -4 & 2 & -4 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} 2 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 2 \end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix} 2 & -1 \end{bmatrix} \right\}$

Select from the following:

1. Only C and D.
2. Only D.
3. Only B.
4. Only A.
5. None of the above.

Question 19

Which of the following statements are always true for for all $m, n \in \mathbb{N}$ and $m \times n$ matrix A ?

A. $\text{nullity}(A) = \text{nullity}(A^T)$

B. $\text{rank}(A^T) + \text{nullity}(A) = m$

C. $\text{rank}(A^T) + \text{nullity}(A) = n$

D. row space(A) = column space(A)

Select from the following:

1. Only A.
2. Only B.
3. Only C and D.
4. Only C.
5. None of the above.

Question 20

Which of the following statements are always true for for all $m, n \in \mathbb{N}$ and $m \times n$ matrix A ?

A. $\text{nullity}(A) = \text{nullity}(A^T)$

B. $\text{rank}(A^T) + \text{nullity}(A) = m$

C. $\text{rank}(A^T) + \text{nullity}(A) = n$

D. row space(A) = column space(A)

Select from the following:

1. Only A.
2. Only C and D.
3. Only C.
4. Only B.
5. None of the above.

Question 21

Let A be an $n \times n$ matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The equation $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$ has the *unique* solution $\mathbf{x} = \mathbf{0}$ if and only if

1. $\lambda = 0$.
2. $\lambda = 0$ and 0 is an eigenvalue of A.
3. λ is not an eigenvalue of A.
4. A is invertible.
5. None of the above.

Question 22

Let A be an $n \times n$ matrix, $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The equation $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$ has the *unique* solution $\mathbf{x} = \mathbf{0}$ if and only if

1. λ is not an eigenvalue of A.
2. $\lambda = 0$.
3. A is invertible.
4. $\lambda = 0$ and 0 is an eigenvalue of A.
5. None of the above.

Question 23

Let A be an $n \times n$ matrix with eigenvalue 2 and let I be the $n \times n$ identity matrix. Which of the following are true?

A. -1 is an eigenvalue of $A - 3I$.

B. $\text{rank}(A + 3I) = n$.

C. 8 is an eigenvalue of A^3.

D. 6 is an eigenvalue of $3A$.

Select from the following:

1. Only B, C and D.
2. Only B.
3. Only A, C and D.
4. Only C.
5. None of the above.

Question 24

Let A be an $n \times n$ matrix with eigenvalue 3 and let I be the $n \times n$ identity matrix. Which of the following are true?

A. 4 is an eigenvalue of $A + I$.

B. $A + 3I$ is invertible.

C. 9 is an eigenvalue of A^2.

D. 6 is an eigenvalue of $2A$.

Select from the following:

1. Only A, C and D.
2. Only B.
3. Only B, C and D.
4. Only C and D.
5. None of the above.

Question 25

Which of the following matrices are diagonalizable?

A. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. B. $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. D. $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only C and D.
2. Only C.
3. Only B.
4. Only A and C.
5. None of the above.

Question 26

Which of the following matrices are diagonalizable?

A. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. B. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. C. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only A and C.
2. Only B.
3. Only B and D.
4. Only D.
5. None of the above.

Question 27

Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Then

1. If A is diagonalizable then A is invertible.
2. If $\lambda = 0$ is and eigenvalue of A, then A is not diagonalizable.
3. If A is invertible then A is diagonalizable.
4. If A is diagonalizable then $A + xI$ is diagonalizable for all $x \in \mathbb{R}$.
5. None of the above.

Question 28

Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Then

1. If A is diagonalizable then $A + xI$ is diagonalizable for all $x \in \mathbb{R}$.
2. If A is diagonalizable then A is invertible.
3. If A is invertible then A is diagonalizable.
4. If $\lambda = 0$ is and eigenvalue of A, then A is not diagonalizable.
5. None of the above.

Question 29

Which one of the following defines an inner product?

1. $\langle A, B\rangle = \operatorname{tr}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} AB^T\right)$ in M_{22}.
2. $\langle (x_1, x_2), (y_1, y_2)\rangle = 2x_1y_1 + 3x_2y_2$ in $\mathbb{R}_2$.
3. $\langle (x_1, x_2), (y_1, y_2)\rangle = (x_1 + x_2)(y_1 + y_2)$ in $\mathbb{R}_2$.
4. $\langle (x_1, x_2), (y_1, y_2)\rangle = x_1y_1 + x_2y_2 + 1$ in $\mathbb{R}_2$.
5. None of the above.

Question 30

Which one of the following defines an inner product?

1. $\langle(x_1, x_2), (y_1, y_2)\rangle = (x_1 + x_2)(y_1 + y_2)$ in $\mathbb{R}_2$.
2. $\langle(x_1, x_2), (y_1, y_2)\rangle = x_1y_1 - x_2y_2$ in $\mathbb{R}_2$.
3. $\langle A, B\rangle = \operatorname{tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} AB^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ in M_{22}.
4. $\langle A, B\rangle = \operatorname{tr}(AB)$ in M_{22}.
5. None of the above.

Question 31

Which of the following vectors are unit vectors with respect to the inner product $\langle(x_1, x_2, x_3), (y_1, y_2.y_3)\rangle = 2x_1y_1 + 2x_2y_2 + 2x_3y_3$ in $\mathbb{R}^3$?

A. $(1, 0, 0)$ B. $(0, 1, 0)$ C. $(1, 0, 0)/\sqrt{2}$ D. $(1, 1, 0)/2$

Select from the following:

1. Only A and B.
2. Only A and C.
3. Only C and D.
4. Only A, B and D.
5. None of the above.

Question 32

Which of the following vectors are unit vectors with respect to the inner product $\langle(x_1, x_2, x_3), (y_1, y_2.y_3)\rangle = x_1y_1 + x_2y_2 + 7x_3y_3$ in $\mathbb{R}^3$?

A. $(1, 0, 0)$ B. $(0, 0, 1)$ C. $(1, 1, 1)/\sqrt{3}$ D. $(1, 1, 1)/3$

Select from the following:

1. Only A and B.
2. Only A and C.
3. Only A and D.
4. Only A, B and C.
5. None of the above.

Question 33

Which of the following vectors are orthogonal to each other with respect to the inner product $\langle A, B\rangle = \mathrm{tr}(A^T B)$ in M_{22} ?

A. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. B. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. C. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only A and B are orthogonal.
2. Only A and C are orthogonal, C and D are orthogonal.
3. Only B and C are orthogonal, B and D are orthogonal.
4. Only A and D are orthogonal.
5. None of the above.

Question 34

Which of the following vectors are *not* orthogonal to each other with respect to the inner product $\langle A, B\rangle = \mathrm{tr}(A^T B)$ in M_{22} ?

A. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. B. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. C. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only A and B are not orthogonal, B and D are not orthogonal.
2. Only A and C are not orthogonal, B and D are not orthogonal.
3. Only A and C are not orthogonal.
4. Only A and D are not orthogonal.
5. None of the above.

Question 35

Consider the vector subspace $W = \mathrm{span}\{\, 1 + x,\ 1 + x^2 \,\}$ of P_2 with the *evaluation inner product* at 0, 1 and -1 (sample points). Which of the following vectors in P_2 lie in the subspace $W^{\perp}$?

1. $5x^2 + x - 4$.
2. $x^2 + x - 1$.
3. $-x^2 - x + 1$.
4. $x - 1$.
5. None of the above.

Question 36

Consider the vector subspace $W = \text{span}\{\,1+x,\ 1+x^2\,\}$ of P_2 with the *standard inner product.* Which of the following vectors in P_2 lie in the subspace $W^{\perp}$?

1. $x^2 + x - 1$.
2. $5x^2 + x - 4$.
3. $x^2 - 2x + 1$.
4. $x^2 - x$.
5. None of the above.

Solutions

Question 1

Consider the set

$$X := \{\, (x, y) \,:\, x, y \in \mathbb{R} \,\}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\begin{aligned} \cdot : \mathbb{R} \times X \to X, \qquad & k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx - k + 1, ky), \\ + : X \times X \to X, \qquad & \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha - 1, y + \beta). \end{aligned}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. The zero vector for X is

1. $(1, 0)$
2. $(1, 1)$
3. $(0, 1)$
4. $(0, 0)$
5. None of the above.

Answer: 1

For all $\mathbf{a} = (x, y) \in X$ *we have*

$$\mathbf{a} + (1, 0) = (x, y) + (1, 0) = (x + 1 - 1, y + 0) = (x, y) = \mathbf{a}$$

i.e. $\mathbf{0} = (1, 0)$ *in* X*. The zero vector is unique (exercise: prove uniqueness of the zero vector using the existence of the negative of a vector).*

Alternative: $\mathbf{0} = 0 \cdot \mathbf{a} = 0 \cdot (x, y) = (0x - 0 + 1, 0y) = (1, 0)$.

Question 2

Consider the set

$$X := \{\, (x, y) \,:\, x, y \in \mathbb{R} \,\}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\begin{aligned} \cdot : \mathbb{R} \times X \to X, \qquad & k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx - k + 1, ky), \\ + : X \times X \to X, \qquad & \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha - 1, y + \beta). \end{aligned}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements are true in this vector space?

1. $-(1, 0) = (-1, 0)$

2. $-(1,0) = (1,1)$
3. $-(1,0) = (1,0)$
4. $-(1,0) = (0,0)$
5. None of the above.

Answer: 3

Since $-(1,0) = (-1)\cdot(1,0)$ *we find* $-(1,0) = ((-1)\cdot 1 - (-1) + 1, (-1)\cdot 0) = (1,0)$*.*
(This is because $(1,0)$ *is the zero vector in* X*).*

Question 3

Which of the following are subspaces of $\mathbb{R}^2$ with the usual operations ?

A. $\operatorname{span}\{\,(2,3)\,\}$

B. $\{\,(1,x)\;:\;x\in\mathbb{R}\,\}$

C. $\{\,(0,x)\;:\;x\in\mathbb{R},\,x\geq 0\,\}$

D. $\{\,(x,x)\;:\;x\in\mathbb{R}\,\}$

Select from the following:

1. Only A.
2. Only A and D.
3. Only C.
4. Only C and D.
5. None of the above.

Answer: 2

The span of a set of element of a vector space is a subspace by definition, so A gives a subspace. A subspace always includes the zero vector, in this case the zero vector is $(0,0)$ *in* $\mathbb{R}^2$*, however B only contains elements of the form* $(1,x)$*, i.e. the first component of the pair is 1. Thus the set given in B does not contain the zero vector and is not a subspace of* $\mathbb{R}^2$*. A subspace must be closed under scalar multiplication. Note that* $(0,1)\in\{\,(0,x)\;:\;x\in\mathbb{R},\,x\geq 0\,\}$*, but* $(-1)\cdot(0,1) = (0,-1)\notin\{\,(0,x)\;:\;x\in\mathbb{R},\,x\geq 0\,\}$ *in* $\mathbb{R}^2$*. So C does not provide a subspace. Now let us consider D.*

1. $(0,0)\in\{\,(x,x)\;:\;x\in\mathbb{R}\,\}$
2. *For all* $a,b\in\mathbb{R}$*:* $(a,a),(b,b)\in\{\,(x,x)\;:\;x\in\mathbb{R}\,\}\Rightarrow(a,a)+(b,b) = (a+b,a+b)\in\{\,(x,x)\;:\;x\in\mathbb{R}\,\}$ *since* $a+b\in\mathbb{R}$ *and the first an second component of the pair are identical.*
3. *For all* $k,a\in\mathbb{R}$*:* $(a,a)\in\{\,(x,x)\;:\;x\in\mathbb{R}\,\}\Rightarrow k\cdot(a,a) = (ka,ka)\in\{\,(x,x)\;:\;x\in\mathbb{R}\,\}$ *since* $ka\in\mathbb{R}$ *and the first an second component of the pair are identical.*

Thus D provides a subspace of $\mathbb{R}^2$*. (This can also be seen from* $\{\,(x,x)\;:\;x\in\mathbb{R}\,\} = \operatorname{span}\{(1,1)\}$*.)*

Question 4

Which of the following are subspaces of $\mathbb{R}^2$ with the usual operations ?

A. $\text{span}\{(\pi, 0)\}$

B. $\{(2, x) : x \in \mathbb{R}\}$

C. $\{(x, y) : x, y \in \mathbb{N}\}$

D. $\{(x, -x) : x \in \mathbb{R}\}$

Select from the following:

1. Only A and D.
2. Only A, B and D.
3. Only C.
4. Only D.
5. None of the above.

Answer: 1

The span of a set of element of a vector space is a subspace by definition, so A gives a subspace. A subspace always includes the zero vector, in this case the zero vector is $(0, 0)$ in $\mathbb{R}^2$, however B only contains elements of the form $(2, x)$, i.e. the first component of the pair is 2. Thus the set given in B does not contain the zero vector and is not a subspace of $\mathbb{R}^2$. A subspace must be closed under scalar multiplication. Note that $(1, 0) \in \{(x, y) : x, y \in \mathbb{N}\}$, but $0.5 \cdot (1, 0) = (1, 0) \notin \{(x, y) : x, y \in \mathbb{N}\}$ in $\mathbb{R}^2$. So C does not provide a subspace. Now let us consider D.

1. $(0, 0) \in \{(x, -x) : x \in \mathbb{R}\}$
2. *For all $a, b \in \mathbb{R}$: $(a, -a), (b, -b) \in \{(x, -x) : x \in \mathbb{R}\} \Rightarrow (a, -a) + (b, -b) = (a + b, -(a + bi)) \in \{(x, -x) : x \in \mathbb{R}\}$ since $a + b \in \mathbb{R}$ and the first an second component of the pair are negatives of each other.*
3. *For all $k, a \in \mathbb{R}$: $(a, -a) \in \{(x, -x) : x \in \mathbb{R}\} \Rightarrow k \cdot (a, -a) = (ka, -ka) \in \{(x, -x) : x \in \mathbb{R}\}$ since $ka \in \mathbb{R}$ and the first an second component of the pair are negatives of each other.*

Thus D provides a subspace of $\mathbb{R}^2$. (This can also be seen from $\{(x, -x) : x \in \mathbb{R}\} = \text{span}\{(1, -1)\}$.)

Question 5

Which of the following sets are linearly independent?

A. $\text{span}\{(2, 3)\}$ in $\mathbb{R}^2$

B. $\{(1, 1), (1, -1)\}$ in $\mathbb{R}^2$

C. $\{\,(1,1),\,(1,-1),\,(0,1)\,\}$ in $\mathbb{R}^2$

D. $\{\,1+x,\,1-x\,\}$ in P_1

Select from the following:

1. Only A.
2. Only B.
3. Only B and C.
4. Only B and D.
5. None of the above.

Answer: 4

By definition, the span of a set (given in A) is linearly dependent. Consider B, solving

$$c_1(1,1)+c_2(1,-1)=(c_1+c_2,c_1-c_2)=(0,0)$$

yields $c_1=-c_2$ *and* $c_1=c_2=-c_1$, *i.e.* $2c_1=0$ *so that* $c_1=c_2=0$ *which is the only solution. Thus the set given by B is linearly independent. Consider C, solving*

$$c_1(1,1)+c_2(1,-1)+c_3(0,1)=(c_1+c_2,c_1-c_2+c_3)=(0,0)$$

yields $c_1=-c_2$ *and* $c_1=c_2-c_3=-c_1-c_3$, *i.e.* $2c_1=-c_3$ *so that* $c_1=-c_2=-c_3/2$ *which yields infinitely many solutions, for example take* $c_1=1$, $c_2=-1$ *and* $c_3=-2$. *This is a non-zero solution. Thus the set given by C is linearly dependent. Consider D, solving*

$$c_1(1+x)+c_2(1-x)=(c_1+c_2)\cdot 1+(c_1-c_2)x=0\cdot 1+0x$$

and comparing coefficients in the standard basis $\{1,x\}$ *in* P_1 *yields* $c_1=-c_2$ *and* $c_1=c_2=-c_1$, *i.e.* $2c_1=0$ *so that* $c_1=c_2=0$ *which is the only solution. Thus the set given by D is linearly independent.*

Question 6

Which of the following sets are linearly independent?

A. $\operatorname{span}\{\,(\pi,0)\,\}$ in $\mathbb{R}^2$

B. $\{\,(1,2),\,(2,1)\,\}$ in $\mathbb{R}^2$

C. $\left\{\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}0&1\\-1&0\end{bmatrix},\begin{bmatrix}0&0\\1&0\end{bmatrix}\right\}$ in M_{22}

D. $\{\,1+2x,\,2-x\,\}$ in P_1

Select from the following:

1. Only B and D.

2. Only B.

3. Only C and D.

4. Only D.

5. None of the above.

Answer: 1

By definition, the span of a set (given in A) is linearly dependent. Consider B, solving

$$c_1(1,2) + c_2(2,1) = (c_1 + 2c_2, 2c_1 + c_2) = (0,0)$$

yields $c_1 = c_2$*, i.e.* $3c_1 = 0$ *so that* $c_1 = c_2 = 0$ *which is the only solution. Thus the set given by B is linearly independent. Consider C, solving*

$$c_1 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & c_1 + c_2 \\ c_1 - c_2 + c_3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

yields $c_1 = -c_2$ *and* $c_3 = 2c_2$ *which yields infinitely many solutions, for example take* $c_1 = 1$*,* $c_2 = -1$ *and* $c_3 = -2$*. This is a non-zero solution. Thus the set given by C is linearly dependent. Consider D, solving*

$$c_1(1 + 2x) + c_2(2 - x) = (c_1 + 2c_2) \cdot 1 + (2c_1 - c_2)x = 0 \cdot 1 + 0x$$

and comparing coefficients in the standard basis $\{1, x\}$ *in* P_1 *yields* $c_1 = -2c_2$ *and* $c_2 = 2c_1 = -4c_2$ *so that* $c_1 = c_2 = 0$ *which is the only solution. Thus the set given by D is linearly independent.*

Question 7

Which of the following sets are identical?

A. span $\{ (1,0,1), (1,0,-1) \}$ in $\mathbb{R}^3$

B. span $\{ (0,0,1), (5,0,0) \}$ in $\mathbb{R}^3$

C. span $\{ (3,0,7), (0,0,1), (5,0,0) \}$ in $\mathbb{R}^3$

D. $\{ (1,0,1), (1,0,-1) \}$

E. span $\{ (1,1,1), (1,-1,-1) \}$ in $\mathbb{R}^3$

Select from the following:

1. Only A and D.

2. Only A and E.

3. Only A, B and C.

4. Only B and C.

5. None of the above.

Answer: 3

The sets in A, B, C and E have infinitely many elements (due to the span) while the set in D has only two elements and so cannot be equal to any of the other sets. Now we compare A and B. If each element of the set in A (respectively B) can be expressed as a linear combination of elements in B (respectively A) then the two sets are equal:

$$\begin{aligned}(1,0,1) &= a_1(0,0,1) + b_1(5,0,0) && \Rightarrow \quad a_1 = 1,\ b_1 = 1/5\\ (1,0,-1) &= a_2(0,0,1) + b_2(5,0,0) && \Rightarrow \quad a_2 = -1,\ b_2 = 1/5\\ (0,0,1) &= a_3(1,0,1) + b_3(1,0,-1) && \Rightarrow \quad a_3 = 1/2,\ b_3 = -1/2\\ (5,0,0) &= a_4(1,0,1) + b_4(1,0,-1) && \Rightarrow \quad a_4 = 1/2,\ b_4 = -1/2\end{aligned}$$

Since we found a solution in each case, the two sets are equal. Now consider, B and C:

$$\begin{aligned}(0,0,1) &= a_1(3,0,7) + b_1(0,0,1) + c_1(5,0,0) && \Rightarrow \quad b_1 = 1-7a_1,\ c_1 = -3a_1/5\\ (5,0,0) &= a_2(3,0,7) + b_2(0,0,1) + c_2(5,0,0) && \Rightarrow \quad b_2 = -7a_2,\ c_2 = 1-3/5a_2\\ (3,0,7) &= a_3(0,0,1) + b_3(5,0,0) && \Rightarrow \quad a_3 = 7,\ b_3 = 5/3\\ (0,0,1) &= a_4(0,0,1) + b_4(5,0,0) && \Rightarrow \quad a_4 = 1,\ b_4 = 0\\ (5,0,0) &= a_5(0,0,1) + b_5(5,0,0) && \Rightarrow \quad a_5 = 0,\ b_5 = 1\end{aligned}$$

It is sufficient to note that $a_1 = 0$, $b_1 = 1$, $c_1 = 0$, $a_2 = 0$, $b_2 = 0$ and $c_2 = 1$ satisfy the first two equations (instead of providing all solutions as above). Since we found a solution in each case, the two sets are equal. Hence, the sets given in A, B and C are all equal. Now consider A and E:

$$(1,0,1) = a(1,1,1) + b(1,-1,-1) \qquad \Rightarrow \qquad \textit{no solution for } a,b \in \mathbb{R}$$

so that the set in E is not equal to any of the other sets.

Question 8

Which of the following sets are identical?

A. $\operatorname{span}\{\,(1,0,1),(2,0,2)\,\}$ in $\mathbb{R}^3$

B. $\operatorname{span}\{\,(1,0,1),(1,0,-1)\,\}$ in $\mathbb{R}^3$

C. $\operatorname{span}\{\,(1,0,1),(1,1,1)\,\}$ in $\mathbb{R}^3$

D. $\operatorname{span}\{\,(3,0,3)\,\}$ in $\mathbb{R}^3$

E. $\{\,(1,0,1),(2,0,2)\,\}$

Select from the following:

1. Only A and B.
2. Only A and D.

3. Only A and E.
4. Only B and C.
5. None of the above.

Answer: 2

The sets in A, B, C and D have infinitely many elements (due to the span) while the set in E has only two elements and so cannot be equal to any of the other sets. Now we compare A and B. If each element of the set in A (respectively B) can be expressed as a linear combination of elements in B (respectively A) then the two sets are equal:

$$\begin{aligned}
(1,0,1) &= a_1(1,0,1) + b_1(1,0,-1) && \Rightarrow \quad a_1 = 1,\ b_1 = 0 \\
(2,0,2) &= a_2(1,0,1) + b_2(1,0,-1) && \Rightarrow \quad a_2 = 2,\ b_2 = 0 \\
(1,0,1) &= a_3(1,0,1) + b_3(2,0,2) && \Rightarrow \quad a_3 = 1,\ b_3 = 0 \text{ (amongst others)} \\
(1,0,-1) &= a_4(1,0,1) + b_4(2,0,2) && \Rightarrow \quad \text{no solution}
\end{aligned}$$

The two sets are not equal. Now consider, A and C (we show only one equation which provides no solution):

$$(1,1,1) = a_5(1,0,1) + b_5(2,0,2) \qquad \Rightarrow \qquad \text{no solution}$$

The two sets are not equal. Now consider, B and C (we show only one equation which provides no solution):

$$(1,1,1) = a_6(1,0,1) + b_6(1,0,-1) \qquad \Rightarrow \qquad \text{no solution}$$

Thus A, B and C are all pair wise unequal. Similarly B, C and D are all pair wise unequal. Finally, consider A and D

$$\begin{aligned}
(1,0,1) &= a_7(3,0,3) && \Rightarrow \quad a_7 = 1/3 \\
(2,0,2) &= a_8(3,0,3) && \Rightarrow \quad a_8 = 2/3 \\
(3,0,3) &= a_9(1,0,1) + b_9(2,0,2) && \Rightarrow \quad a_9 = 3,\ b_9 = 0 \text{ (amongst others)}
\end{aligned}$$

so that A and D are equal.

Question 9

Which of the following sets are a basis for the following vector subspace of P_2:

$$X = \{\, p(x) \in P_2 \,:\, p(3) = 0 \,\}.$$

A. $\{\, 1,\ x,\ x^2 \,\}$

B. $\{\, x-3,\ x^2-9 \,\}$

C. $\{\, x^2+2x-15,\ x^2-2x-3 \,\}$

D. $\{\, x-3,\ x^3-27 \,\}$

Select from the following:

1. Only B and C.
2. Only B.
3. Only A.
4. Only B and D.
5. None of the above.

Answer: 1

Since $1 \neq 0$, *we have* $1|_{x=3} = 1 \notin X$ *(also* $x|_{x=3} = 3 \neq 0$, $x \notin X$, $x^2|_{x=3} = 9 \neq 0$, $x^2 \notin X$*). Thus A does not provide a basis for* X*. Since* $X \subseteq P_2$, X *does not include any cubic polynomials and D also does not provide a basis for* X*. Since all the polynomials which appear in B and C are either quadratic or linear, they are all in* P_2*. We also have*

$$(x-3)|_{x=3} = (x^2-9)|_{x=3} = (x^2+2x-15)|_{x=3} = (x^2-2x-3)|_{x=3} = 0$$

so that $\{\, x-3,\ x^2-9 \,\} \subseteq X$ *and* $\{\, x^2+2x-15,\ x^2-2x-3 \,\} \subseteq X$*. We describe* X *more explicitly:*

$$\begin{aligned} X &= \{\, p(x) \in P_2 \,:\, p(3) = 0 \,\} \\ &= \left\{\, ax^2+bx+c \,:\, a,b,c \in \mathbb{R},\ (ax^2+bx+c)|_{x=3} = 0 \,\right\} \\ &= \left\{\, ax^2+bx+c \,:\, a,b,c \in \mathbb{R},\ 9a+3b+c = 0 \,\right\} \\ &= \left\{\, ax^2+bx-9a-3b \,:\, a,b \in \mathbb{R} \,\right\}. \end{aligned}$$

Now we need to determine, for B and C, whether each set is linearly dependent and spans X*. For B we have*

$$\begin{aligned} c_1(x-3)+c_2(x^2-9) = 0+0\cdot x+0\cdot x^2 \quad &\Leftrightarrow \quad (-3c_1-9c_2)+c_1x+c_2x^2 = 0+0\cdot x+0\cdot x^2 \\ &\Leftrightarrow \quad c_1 = c_2 = 0 \end{aligned}$$

so that the set in B is linearly independent (here we used the fact that $\{1, x, x^2\}$ *is a basis for* P_2 *and compared coefficients). We also have*

$$\begin{aligned} \operatorname{span}\left\{\, x-3,\ x^2-9 \,\right\} &= \left\{\, a(x-3)+b(x^2-9) \,:\, a,b \in \mathbb{R} \,\right\} \\ &= \left\{\, bx^2+ax-3a-9b \,:\, a,b \in \mathbb{R} \,\right\} \\ &= \left\{\, ax^2+bx-9a-3b \,:\, a,b \in \mathbb{R} \,\right\} \\ &= X \end{aligned}$$

so that $\{\, x-3,\ x^2-9 \,\}$ *spans* X*. Thus B provides a basis for* X*. For C we have*

$$\begin{aligned} & c_1(x^2+2x-15)+c_2(x^2-2x-3) = 0+0\cdot x+0\cdot x^2 \\ \Leftrightarrow \quad & (-15c_1-3c_2)+(2c_1-2c_2)x+(c_1+c_2)x^2 = 0+0\cdot x+0\cdot x^2 \\ \Leftrightarrow \quad & 5c_1+3c_2 = c_1+c_2 = c_1-c_2 = 0 \\ \Leftrightarrow \quad & c_1 = c_2 = 0 \end{aligned}$$

so that the set in C is linearly independent (here we used the fact that $\{1, x, x^2\}$ is a basis for P_2 and compared coefficients). We also have

$$\begin{aligned}\operatorname{span}\left\{ x^2+2x-15,\, x^2-2x-3 \right\} &= \left\{ a(x^2+2x-15)+b(x^2-2x-3) \,:\, a,b\in\mathbb{R} \right\} \\ &= \left\{ (a+b)x^2+2(a-b)x-15a-3b \,:\, a,b\in\mathbb{R} \right\} \\ &= \left\{ a'x^2+b'x-9a'-3b' \,:\, a',b'\in\mathbb{R} \right\} \\ &= X\end{aligned}$$

where we set $a' = a + b$ and $b' = 2(a - b)$ so that $a = a'/2 + b'/4$ and $b = a'/2 - b'/4$. Note that $\{a + b \,:\, a, b \in \mathbb{R}\} = \mathbb{R}$ and $\{2(a - b) \,:\, a, b \in \mathbb{R}\} = \mathbb{R}$. Thus $\{ x^2 + 2x - 15,\, x^2 - 2x - 3 \}$ spans X and C also provides a basis for X.

Question 10

Which of the following sets are a basis for the following vector subspace of M_{22}:

$$X = \left\{ A \in M_{22} \,:\, A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \right\}.$$

A. $\left\{ \begin{bmatrix}2&0\\0&0\end{bmatrix}, \begin{bmatrix}0&-1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\2&0\end{bmatrix}, \begin{bmatrix}0&0\\0&-1\end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix}2&-1\\2&-1\end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix}2&-1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\2&-1\end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix}2&-1\\2&-1\end{bmatrix}, \begin{bmatrix}2&-1\\-2&1\end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only C and D.
5. None of the above.

Answer: 4

A basis consists of elements of the vector space for which it is a basis. Since $\left[\begin{smallmatrix}2&0\\0&0\end{smallmatrix}\right] \notin X$, A cannot describe a basis for X. The remaining sets are all subsets of X (check this yourself). Since $\left[\begin{smallmatrix}2&-1\\4&-2\end{smallmatrix}\right] \in X$ but $\left[\begin{smallmatrix}2&-1\\4&-2\end{smallmatrix}\right] \notin \operatorname{span}\{\begin{bmatrix}2&-1\\2&-1\end{bmatrix}\}$, B does not provide a basis for X. We notice that

$$\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}a+2b\\c+2d\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \quad\Leftrightarrow\quad a=-2b,\ c=-2d$$

so that

$$X = \left\{ \begin{bmatrix} -2b & b \\ -2d & d \end{bmatrix} : b, d \in \mathbb{R} \right\} = \left\{ b \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} : b, d \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} \right\}.$$

Since the matrices in C span X and are linearly independent (check this yourself) this set forms a basis for X. Thus X is two dimensional. Since the set given in D is linearly independent (check this yourself), and consists of 2 elements of X, it must be a basis for X (since every linearly independent 2-element subset of X is a basis for X). Alternatively,

$$\begin{aligned}
&\operatorname{span} \left\{ \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} \right\} \\
&= \left\{ a \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} + b \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix} : a, b \in \mathbb{R} \right\} \\
&= \left\{ (a+b) \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + (a-b) \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} : a, b \in \mathbb{R} \right\} \\
&= \left\{ a' \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} + b' \begin{bmatrix} 0 & 0 \\ 2 & -1 \end{bmatrix} : a', b' \in \mathbb{R} \right\} \qquad \textit{(set } a' = a + b \textit{ and } b' = a - b\textit{)} \\
&= \operatorname{span} \left\{ \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} \right\} = X.
\end{aligned}$$

The second last equivalence follow by noting that if given two values $\alpha, \beta \in \mathbb{R}$, $a = (\alpha + \beta)/2$ and $b = (\alpha - \beta)/2$ provides $a' = \alpha$ and $b' = \beta$.

Question 11

Which of the following statements are true:

A. $\dim(\operatorname{span}\{ (1,0,1), (1,0,-1) \}) = 2$ in $\mathbb{R}^3$

B. $\dim(\operatorname{span}\{ (0,0,0), (5,0,0) \}) = 2$ in $\mathbb{R}^3$

C. $\dim(\operatorname{span}\{ (3,0,7), (0,0,1), (5,0,0) \}) = 2$ in $\mathbb{R}^3$

D. $\dim(\operatorname{span}\{ (3,0,7), (0,0,1), (5,0,0) \}) = 3$ in $\mathbb{R}^3$

Select from the following:

1. All of A, B, C and D.
2. Only A and B.
3. Only A and C.
4. Only A and D.
5. None of the above.

Answer: 3

Since $\{ (1,0,1), (1,0,-1) \}$ *consists of two linearly independent vectors (prove this!) A is true. Since* $\text{span}\{ (0,0,0), (5,0,0) \} = \text{span}\{ (5,0,0) \}$ *which has dimension 1, B is false. Now consider C:*

$$c_1(3,0,7) + c_2(0,0,1) + c_3(5,0,0) = (0,0,0)$$

yields $3c_1 + 5c_3 = 0$ *and* $7c_1 + c_2 = 0$. *We apply row reduction to find all solutions:*

$$\begin{bmatrix} 3 & 0 & 5 & : & 0 \\ 7 & 1 & 0 & : & 0 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 5 & : & 0 \\ 0 & 3 & -35 & : & 0 \end{bmatrix} \qquad (R_2 \leftarrow 3R_2 - 7R_1)$$

so that $c_1 = -5c_3/3$, $c_2 = 35c_3/3$ *and* c_3 *is arbitrary (free). Thus* $\{ (3,0,7), (0,0,1), (5,0,0) \}$ *is not a linearly independent set. However* $\{ (0,0,1), (5,0,0) \}$ *is linearly independent, so that*

$$\text{span}\{ (3,0,7), (0,0,1), (5,0,0) \} = \text{span}\{ (0,0,1), (5,0,0) \}$$

and C is true. Consequently, D is false.

Question 12

Which of the following statements are true:

A. $\dim(\text{span}\{ (1,0,1), (1,0,-1) \}) = 2$ in $\mathbb{R}^3$

B. $\dim(\text{span}\{ (0,0,0), (5,0,0) \}) = 2$ in $\mathbb{R}^3$

C. $\dim(\text{span}\{ (1,0,1), (1,0,-1), (5,0,0) \}) = 2$ in $\mathbb{R}^3$

D. $\dim(\text{span}\{ (1,0,1), (1,0,-1), (5,0,0) \}) = 3$ in $\mathbb{R}^3$

Select from the following:

1. All of A, B, C and D.
2. Only A, B and D.
3. Only A and C.
4. Only A and D.
5. None of the above.

Answer: 3

Since $\{ (1,0,1), (1,0,-1) \}$ *consists of two linearly independent vectors (prove this!) A is true. Since* $\text{span}\{ (0,0,0), (5,0,0) \} = \text{span}\{ (5,0,0) \}$ *which has dimension 1, B is false. Now consider C:*

$$c_1(1,0,1) + c_2(1,0,-1) + c_3(5,0,0) = (0,0,0)$$

yields $c_1 + c_2 + 5c_3 = 0$ *and* $c_1 - c_2 = 0$. *Since* $c_1 = c_2 = 5$ *and* $c_3 = -2$ *provides a non-zero solution. Thus the dimension is less than 3 (and D is false).*

$$c_1(1,0,1) + c_2(1,0,-1) = (0,0,0)$$

however does yield only the trivial solution, so the dimension is at least 2. Thus the dimension must be 2.

Question 13

Which of the following sets are a basis for the row space of $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$?

A. $\{ \begin{bmatrix} 0 & -3 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \}$

B. $\{ \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \}$

C. $\{ \begin{bmatrix} 1 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \}$

Select from the following:

1. Only A.
2. Only C.
3. Only A and C.
4. Only B.
5. None of the above.

Answer: 3

In order for each set to be a basis for the row space, it must be linearly independent and each element must be expressible as a linear combination of the rows. For A we have

$$c_1 \begin{bmatrix} 0 & -3 & 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} >$$

Thus $c_2 = 0$ *and* $-3c_1 + 2c_2 = 0$ *so that* $c_1 = 0$*. Since* $c_1 = c_2 = 0$ *is the only solution the set in A is linearly independent. Now we solve*

$$\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} = a_1 \begin{bmatrix} 0 & -3 & 3 \end{bmatrix} + b_1 \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} = a_2 \begin{bmatrix} 0 & -3 & 3 \end{bmatrix} + b_2 \begin{bmatrix} 1 & 2 & -1 \end{bmatrix}$$

which yields $b_1 = 1$, $a_1 = 1$, $b_2 = 1$ *and* $a_2 = 0$*. Thus A provides a basis for the row space. It is straightforward to verify that the set in B is linearly independent, and solving*

$$\begin{bmatrix} 1 & -1 & 2 \end{bmatrix} = a_1 \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} + b_1 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} = a_2 \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} + b_2 \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

yields $a_1 = -1$, $b_1 = 3$*, but no solution for* a_2*. Thus B does not provide a basis. The set given in C is exactly the rows of the matrix. It suffices to check for linear independence:*

$$c_1 \begin{bmatrix} 1 & -1 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

yields $c_1 + c_2 = 0$, $-c_1 + 2c_2 = 0$ *and* $2c_1 - c_2 = 0$. *We apply row reduction to find all solutions:*

$$\begin{bmatrix} 1 & 1 & : & 0 \\ -1 & 2 & : & 0 \\ 2 & -1 & : & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & : & 0 \\ 0 & 2 & : & 0 \\ 2 & -1 & : & 0 \end{bmatrix} \qquad (R_2 \leftarrow R_2 + R_1)$$

$$\to \begin{bmatrix} 1 & 1 & : & 0 \\ 0 & 2 & : & 0 \\ 0 & -3 & : & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - 2R_1)$$

$$\to \begin{bmatrix} 1 & 1 & : & 0 \\ 0 & 2 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \qquad (R_3 \leftarrow 2R_3 + 3R_1)$$

$$\to \begin{bmatrix} 1 & 1 & : & 0 \\ 0 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \qquad (R_2 \leftarrow R_2/2)$$

$$\to \begin{bmatrix} 1 & 0 & : & 0 \\ 0 & 1 & : & 0 \\ 0 & 0 & : & 0 \end{bmatrix} \qquad (R_1 \leftarrow R_1 - R_2)$$

so that $c_1 = 0$ *and* $c_2 = 0$ *is the only solution. Thus* C *provides a basis for the row space.*

Question 14

Which of the following sets are a basis for the row space of $\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 2 & -4 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T, \begin{bmatrix} -4 & 2 & -4 \end{bmatrix}^T \right\}$

B. $\{ \begin{bmatrix} 2 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -4 \end{bmatrix} \}$

C. $\{ \begin{bmatrix} 2 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix} \}$

D. $\{ \begin{bmatrix} 1 & 2 \end{bmatrix} \}$

E. $\{ \begin{bmatrix} 2 & -1 \end{bmatrix} \}$

Select from the following:

1. Only A.
2. Only B, C, and E.
3. Only E.
4. Only D.
5. None of the above.

Answer: 5

The row space consists of 2 coordinate row vectors, so A and D make no sense. B is not a basis since the third vector is a (trivial) linear combination of the first two. C is not a basis since the first vector is a scalar multiple (i.e. -2*) the second vector. The set E lists a vector which cannot be expressed as a linear combination of the rows:*

$$\begin{bmatrix}1 & 2\end{bmatrix} = a\begin{bmatrix}2 & -4\end{bmatrix} + b\begin{bmatrix}-1 & 2\end{bmatrix} + c\begin{bmatrix}2 & -4\end{bmatrix} = \begin{bmatrix}2a-b+2c & -4a+2b-4c\end{bmatrix}$$

The equations

$$\begin{aligned} 2a - b + 2c &= 1 \\ -4a + 2b - 4c &= 2 \end{aligned}$$

Adding twice the first equation to the second yield $0 = 3$*, obviously a contradiction.*

Question 15

Which of the following sets are contained in (i.e. subset of) the column space of $\begin{bmatrix}1 & -1 & 2 \\ 1 & 2 & -1\end{bmatrix}$?

A. $\left\{ \begin{bmatrix}1 & 1\end{bmatrix}^T, \begin{bmatrix}-1 & 2\end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix}-1 & 2\end{bmatrix}^T, \begin{bmatrix}2 & 1\end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix}1 & 0\end{bmatrix}^T, \begin{bmatrix}0 & 1\end{bmatrix}^T, \begin{bmatrix}1 & 1\end{bmatrix}^T \right\}$

D. $\left\{ \begin{bmatrix}-1 & 1 & 1\end{bmatrix} \right\}$

Select from the following:

1. Only D.
2. Only A, B and C.
3. Only A and B.
4. Only A and C.
5. None of the above.

Answer: 2

There are 2 rows, so D is not relevant. The set in A consists of columns of the matrix and is by definition in the column space of the matrix. Similarly the first element $\begin{bmatrix}-1 & 2\end{bmatrix}^T$ *in the set given in B is in the column space of the matrix, but we must still check the second element:*

$$\begin{bmatrix}2 \\ 1\end{bmatrix} = a\begin{bmatrix}1 \\ 1\end{bmatrix} + b\begin{bmatrix}-1 \\ 2\end{bmatrix} + c\begin{bmatrix}2 \\ -1\end{bmatrix}$$

yields $a - b + 2c = 2$ and $a + 2b - c = 1$ and using row reduction we find

$$\begin{bmatrix} 1 & -1 & 2 & : & 2 \\ 1 & 2 & -1 & : & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & : & 2 \\ 0 & 3 & -3 & : & -1 \end{bmatrix} \qquad (R_2 \leftarrow R_2 - R_1)$$

$$\to \begin{bmatrix} 1 & -1 & 2 & : & 2 \\ 0 & 1 & -1 & : & -1/3 \end{bmatrix} \qquad (R_2 \leftarrow R_2/3)$$

$$\to \begin{bmatrix} 1 & 0 & 1 & : & 5/3 \\ 0 & 1 & -1 & : & -1/3 \end{bmatrix} \qquad (R_1 \leftarrow R_1 + R_2)$$

so that we have a solution ($a = 5/3 - c$, $b = c - 1/3$ and c is free). Thus $\begin{bmatrix} 2 & 1 \end{bmatrix}^T$ is in the column space and the set given in B is a subset of the column space. Similarly, for C we need only check the first two elements

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = a_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

with solution $a_1 = 2/3 - c_1$, $b_1 = c_1 - 1/3$, $a_2 = 1/3 - c_2$ and $b_2 = 1/3 + c_2$. It follows that the set given in C is a subset of the column space.

A much simpler solution is found by noting that the column space is the vector space of column vectors $\mathbb{R}^2$ (since two of the columns are linearly independent). Then the sets in A, B and C are obviously subsets of the column space.

Question 16

Which of the following sets are contained in (i.e. subset of) the column space of $\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 2 & -4 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T, \begin{bmatrix} -4 & 2 & -4 \end{bmatrix}^T \right\}$

B. $\{ \begin{bmatrix} 2 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -4 \end{bmatrix} \}$

C. $\{ \begin{bmatrix} 2 & -4 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix} \}$

D. $\left\{ \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}^T \right\}$

E. $\{ \begin{bmatrix} 1 & 2 \end{bmatrix} \}$

Select from the following:

1. Only E.
2. Only B, C and D.
3. Only B and C.

4. Only A and D.

5. None of the above.

Answer: 4

There are 3 rows, so B, C and E is not relevant. The set in A consists of columns of the matrix and is by definition in the column space of the matrix. Similarly the only element in the set given in D is in the column space of the matrix.

Question 17

Which of the following sets are a basis for the null space of $\begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 2 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} 7 & -7 & -7 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} -1 & 1 & 1 \end{bmatrix} \right\}$

Select from the following:

1. Only B and C.

2. Only B.

3. Only C.

4. Only A.

5. None of the above.

Answer: 1

The null space is given by

$$\left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R}, \begin{bmatrix} 1 & -1 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : a, b, c \in \mathbb{R}, \begin{bmatrix} a - b + 2c \\ a + 2b - c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}$$

$$= \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : c \in \mathbb{R}, a = -c, b = c, \right\}$$

$$= \left\{ \begin{bmatrix} -c \\ c \\ c \end{bmatrix} : c \in \mathbb{R} \right\}$$

$$= \operatorname{span} \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Since $\begin{bmatrix}-1 & 1 & 1\end{bmatrix}^T, \begin{bmatrix}7 & -7 & -7\end{bmatrix}^T \in \text{span}\left\{ \begin{bmatrix}-1 & 1 & 1\end{bmatrix}^T \right\}$ *are both non-zero vectors in a one dimensional vector space (the null space) both B and C provide a basis for the null space.*

Question 18

Which of the following sets are a basis for the null space of $\begin{bmatrix} 2 & -4 \\ -1 & 2 \\ 2 & -4 \end{bmatrix}$?

A. $\left\{ \begin{bmatrix}2 & -1 & 2\end{bmatrix}^T, \begin{bmatrix}-4 & 2 & -4\end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix}2 & -4\end{bmatrix}, \begin{bmatrix}-1 & 2\end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix}1 & 2\end{bmatrix} \right\}$

D. $\left\{ \begin{bmatrix}2 & -1\end{bmatrix} \right\}$

Select from the following:

1. Only C and D.
2. Only D.
3. Only B.
4. Only A.
5. None of the above.

Answer: 5

The null space consists of matrix multiplication compatible vectors, i.e. since the matrix is 3×2 *- which means that the null space consists of* 2×1 *column vectors. None of these sets consist of* 2×1 *matrices.*

Question 19

Which of the following statements are always true for for all $m, n \in \mathbb{N}$ and $m \times n$ matrix A ?

A. $\text{nullity}(A) = \text{nullity}(A^T)$

B. $\text{rank}(A^T) + \text{nullity}(A) = m$

C. $\text{rank}(A^T) + \text{nullity}(A) = n$

D. $\text{row space}(A) = \text{column space}(A)$

Select from the following:

1. Only A.

2. Only B.
3. Only C and D.
4. Only C.
5. None of the above.

Answer: 4

Statement A is false (for some matrices). For example, consider $A = \left[\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix}\right]$ *so that* $\text{nullity}(A) = 2$ *and* $\text{nullity}(A^T) = 1$. *Statement B is false (for some matrices) since for the previous example* $\text{rank}(A^T) + \text{nullity}(A) = 1 + 2 = 3 \neq 2$ *which is the number of rows (m). We know that* $\text{rank}(A) + \text{nullity}(A) = n$ *and that* $\text{rank}(A) = \text{rank}(A^T)$. *Thus* $\text{rank}(A^T) + \text{nullity}(A) = n$ *is true for all* $m \times n$ *matrices A. Since the number of rows and columns in A may be different, statement D makes no sense in general, and is false for some matrices.*

Challenge: for each of A, B and D, characterize all the matrices for which the statement is true.

Question 20

Which of the following statements are always true for for all $m, n \in \mathbb{N}$ and $m \times n$ matrix A ?

A. $\text{nullity}(A) = \text{nullity}(A^T)$

B. $\text{rank}(A^T) + \text{nullity}(A) = m$

C. $\text{rank}(A^T) + \text{nullity}(A) = n$

D. $\text{row space}(A) = \text{column space}(A)$

Select from the following:

1. Only A.
2. Only C and D.
3. Only C.
4. Only B.
5. None of the above.

Answer: 3

Statement A is false (for some matrices). For example, consider $A = \left[\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix}\right]$ *so that* $\text{nullity}(A) = 2$ *and* $\text{nullity}(A^T) = 1$. *Statement B is false (for some matrices) since for the previous example* $\text{rank}(A^T) + \text{nullity}(A) = 1 + 2 = 3 \neq 2$ *which is the number of rows (m). We know that* $\text{rank}(A) + \text{nullity}(A) = n$ *and that* $\text{rank}(A) = \text{rank}(A^T)$. *Thus* $\text{rank}(A^T) + \text{nullity}(A) = n$ *is true for all* $m \times n$ *matrices A. Since the number of rows and columns in A may be different, statement D makes no sense in general, and is false for some matrices.*

Challenge: for each of A, B and D, characterize all the matrices for which the statement is true.

Question 21

Let A be an $n \times n$ matrix, $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The equation $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$ has the *unique* solution $\mathbf{x} = \mathbf{0}$ if and only if

1. $\lambda = 0$.
2. $\lambda = 0$ and 0 is an eigenvalue of A.
3. λ is not an eigenvalue of A.
4. A is invertible.
5. None of the above.

Answer: 3

First note that $\mathbf{x} = \mathbf{0}$ always satisfies the equation. If $\lambda = 0$, and the solution $\mathbf{x} = \mathbf{0}$ is unique, then A must be invertible – but this was not given. If $\lambda = 0$ is an eigenvalue of A, then there exists a non-zero vector $\mathbf{x} \neq 0$ such that $A\mathbf{x} = \mathbf{0}$ which contradicts the uniqueness of the solution $\mathbf{x} = \mathbf{0}$. If λ is and eigenvalue of A then $A\mathbf{x} = \lambda\mathbf{x}$ has a non-zero solution for $\mathbf{x}$ (contradicting the uniqueness of the solution $\mathbf{x} = \mathbf{0}$), thus λ is not an eigenvalue of A. Conversely, if λ is not an eigenvalue of A, then no non-zero vector $\mathbf{x}$ exists which satisfies $A\mathbf{x} = \lambda\mathbf{x}$ (ensuring the uniqueness of the solution $\mathbf{x} = \mathbf{0}$). Thus 3 is valid. Consider $A = I$ and $\lambda = 1$, then A is invertible but any non-zero vector $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$. Thus invertibility is not necessary for the uniqueness of the solution $\mathbf{x} = \mathbf{0}$.

This question emphasizes the fact that no eigenvalue of A has an eigenspace consisting of only the zero vector. If a calculation to find an eigenspace for the "eigenvalue" λ yields only the zero vector, either the calculation is incorrect or λ is not an eigenvalue.

Question 22

Let A be an $n \times n$ matrix, $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. The equation $A\mathbf{x} = \lambda\mathbf{x}$ for $\mathbf{x}$ has the *unique* solution $\mathbf{x} = \mathbf{0}$ if and only if

1. λ is not an eigenvalue of A.
2. $\lambda = 0$.
3. A is invertible.
4. $\lambda = 0$ and 0 is an eigenvalue of A.
5. None of the above.

Answer: 1

If λ is and eigenvalue of A then $A\mathbf{x} = \lambda\mathbf{x}$ has a non-zero solution for $\mathbf{x}$ (contradicting the uniqueness of the solution $\mathbf{x} = \mathbf{0}$), thus λ is not an eigenvalue of A. Conversely, if λ is not an eigenvalue of A, *then no non-zero vector* $\mathbf{x}$ *exists which satisfies* $A\mathbf{x} = \lambda\mathbf{x}$ *(ensuring the uniqueness of the solution* $\mathbf{x} = \mathbf{0}$*). Thus 1 is valid. If* $\lambda = 0$*, and the solution* $\mathbf{x} = \mathbf{0}$ *is unique, then* A *must be invertible – but this was not given. Consider* $A = I$ *and* $\lambda = 1$*, then* A *is invertible but any non-zero vector* $\mathbf{x}$ *satisfies* $A\mathbf{x} = \lambda\mathbf{x}$*. Thus invertibility is not necessary for the uniqueness of the solution* $\mathbf{x} = \mathbf{0}$*. If* $\lambda = 0$ *is an eigenvalue of* A*, then there exists a non-zero vector* $\mathbf{x} \neq 0$ *such that* $A\mathbf{x} = \mathbf{0}$ *which contradicts the uniqueness of the solution* $\mathbf{x} = \mathbf{0}$*.*

This question emphasizes the fact that no eigenvalue of A *has an eigenspace consisting of only the zero vector. If a calculation to find an eigenspace for the "eigenvalue"* λ *yields only the zero vector, either the calculation is incorrect or* λ *is not an eigenvalue.*

Question 23

Let A be an $n \times n$ matrix with eigenvalue 2 and let I be the $n \times n$ identity matrix. Which of the following are true?

A. -1 is an eigenvalue of $A - 3I$.

B. $\text{rank}(A + 3I) = n$.

C. 8 is an eigenvalue of A^3.

D. 6 is an eigenvalue of $3A$.

Select from the following:

1. Only B, C and D.
2. Only B.
3. Only A, C and D.
4. Only C.
5. None of the above.

Answer: 3

Since 2 is an eigenvalue of A*, there exists a non-zero vector* $\mathbf{x} \in \mathbb{R}^n$ *such that* $A\mathbf{x} = 2\mathbf{x}$*. Now* $(A - 3I)\mathbf{x} = A\mathbf{x} - 3\mathbf{x} = 2\mathbf{x} - 3\mathbf{x} = -\mathbf{x}$*. Since* $\mathbf{x}$ *is non-zero, it is clear that -1 is an eigenvalue of* $A - 3I$ *with corresponding eigenvector* $\mathbf{x}$*. Thus A is true. Let*

$$A = \begin{bmatrix} 2 & 0 & 0 & 0 & \cdots \\ 0 & -3 & 0 & 0 & \cdots \\ 0 & 0 & -3 & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Then 2 is an eigenvalue of A*, but* $\text{rank}(A + 3I) = 1$ *and B is false in general. We have* $A^3\mathbf{x} = A(A(A\mathbf{x})) = A(A(2\mathbf{x})) = 2(A(A\mathbf{x})) = 4(A\mathbf{x}) = 8\mathbf{x}$ *and since* $\mathbf{x} \neq \mathbf{0}$*, C is true. Also,* $3A\mathbf{x} = 3(A\mathbf{x}) = 6\mathbf{x}$ *so that D is true.*

Question 24

Let A be an $n \times n$ matrix with eigenvalue 3 and let I be the $n \times n$ identity matrix. Which of the following are true?

A. 4 is an eigenvalue of $A + I$.

B. $A + 3I$ is invertible.

C. 9 is an eigenvalue of A^2.

D. 6 is an eigenvalue of $2A$.

Select from the following:

1. Only A, C and D.
2. Only B.
3. Only B, C and D.
4. Only C and D.
5. None of the above.

Answer: 1

Since 3 is an eigenvalue of A, there exists a non-zero vector $\mathbf{x} \in \mathbb{R}^n$ *such that* $A\mathbf{x} = 3\mathbf{x}$. *Now* $(A + I)\mathbf{x} = A\mathbf{x} + \mathbf{x} = 3\mathbf{x} + \mathbf{x} = 4\mathbf{x}$. *Since* $\mathbf{x}$ *is non-zero, it is clear that 4 is an eigenvalue of* $A + I$ *with corresponding eigenvector* $\mathbf{x}$. *Thus A is true. If* $A = -3I$ *then* $A + 3I$ *is obviously not invertible and B is false in general. We have* $A^2\mathbf{x} = A(A\mathbf{x}) = A(3\mathbf{x}) = 3(A\mathbf{x}) = 9\mathbf{x}$ *and since* $\mathbf{x} \neq \mathbf{0}$, *C is true. Also,* $2A\mathbf{x} = 2(A\mathbf{x}) = 6\mathbf{x}$ *so that D is true.*

Question 25

Which of the following matrices are diagonalizable?

A. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. B. $\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$. C. $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$. D. $\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only C and D.
2. Only C.
3. Only B.
4. Only A and C.
5. None of the above.

Answer: 1

The matrix D is lower triangular, the eigenvalues lie on the diagonal i.e. 0 and 1. Since all of the eigenvalues are distinct, D is diagonal. We consider the eigenvalues and eigenspaces for the matrices in A and B. These matrices are all upper or lower triangular, which means the eigenvalues are given by the diagonal entries of the matrices. Since the matrices A and B have only one eigenvalue with algebraic multiplicity 2, we need only check whether the geometric multiplicity is also 2 for each case.

Matrix	*Eigenvalue*	*Algebraic multiplicity*	*Eigenspace*	*Geometric multiplicity*	*Diagonalizable*
$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	*0*	*2*	$\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$	*1*	*No*
$\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$	-1	*2*	$\text{span}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$	*1*	*No*

The last matrix (C) has characteristic equation

$$\lambda^2 - 3\lambda = \lambda(\lambda - 3) = 0$$

so that C has two distinct eigenvalues and is diagonalizable.

Question 26

Which of the following matrices are diagonalizable?

A. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. B. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$. C. $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only A and C.
2. Only B.
3. Only B and D.
4. Only D.
5. None of the above.

Answer: 4

The matrix in D is symmetric and therefore diagonalizable. We consider the eigenvalues and eigenspaces for the matrices in A, B and C. These matrices are all upper or lower triangular, which means the eigenvalues are given by the diagonal entries of the matrices. Since each matrix has only one eigenvalue with algebraic multiplicity 2, we need only check whether the geometric multiplicity is also 2 for each case.

Matrix	*Eigenvalue*	*Algebraic multiplicity*	*Eigenspace*	*Geometric multiplicity*	*Diagonalizable*
$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$	*0*	*2*	$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$	*1*	*No*
$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$	*2*	*2*	$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$	*1*	*No*
$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$	*0*	*2*	$\text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$	*1*	*No*

Question 27

Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Then

1. If A is diagonalizable then A is invertible.
2. If $\lambda = 0$ is and eigenvalue of A, then A is not diagonalizable.
3. If A is invertible then A is diagonalizable.
4. If A is diagonalizable then $A + xI$ is diagonalizable for all $x \in \mathbb{R}$.
5. None of the above.

Answer: 4

The $n \times n$ zero matrix is trivially diagonalizable (it is already diagonal) but is not invertible. Thus 1 is false. The $n \times n$ zero matrix is trivially diagonalizable (it is already diagonal) with all eigenvalues equal to zero. Thus 2 is false. The matrix $\left[\begin{smallmatrix} -1 & 0 \\ 1 & -1 \end{smallmatrix}\right]$ (from above) is invertible but not diagonalizable, hence 3 is false (can you think of a counter example for 3×3 matrices? $n \times n$ matrices?). If A is diagonalizable, then there exists an invertible $n \times n$ matrix P such that $P^{-1}AP$ is diagonal. Now, $P^1(A + xI)P = P^{-1}AP + xPP^{-1} = P^{-1}AP + xI$ is the sum of two diagonal matrices which is also diagonal. Thus $A + xI$ is diagonalizable (diagonalized by P). Thus 4 is true.

Question 28

Let A be an $n \times n$ matrix and let I be the $n \times n$ identity matrix. Then

1. If A is diagonalizable then $A + xI$ is diagonalizable for all $x \in \mathbb{R}$.
2. If A is diagonalizable then A is invertible.
3. If A is invertible then A is diagonalizable.
4. If $\lambda = 0$ is and eigenvalue of A, then A is not diagonalizable.

5. None of the above.

Answer: 1

If A is diagonalizable, then there exists an invertible $n \times n$ matrix P such that $P^{-1}AP$ is diagonal. Now, $P^{1}(A + xI)P = P^{-1}AP + xPP^{-1} = P^{-1}AP + xI$ is the sum of two diagonal matrices which is also diagonal. Thus $A + xI$ is diagonalizable (diagonalized by P). Thus 1 is true. The $n \times n$ zero matrix is trivially diagonalizable (it is already diagonal) but is not invertible. Thus 2 is false. The matrix $\left[\begin{smallmatrix} 2 & 1 \\ 0 & 2 \end{smallmatrix}\right]$ (from above) is invertible but not diagonalizable, hence 3 is false (can you think of a counter example for 3×3 matrices? $n \times n$ matrices?). The $n \times n$ zero matrix is trivially and has the eigenvalue 0. Thus 5 is false.

Question 29

Which one of the following defines an inner product?

1. $\langle A, B \rangle = \operatorname{tr}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} AB^T\right)$ in M_{22}.
2. $\langle (x_1, x_2), (y_1, y_2) \rangle = 2x_1y_1 + 3x_2y_2$ in $\mathbb{R}_2$.
3. $\langle (x_1, x_2), (y_1, y_2) \rangle = (x_1 + x_2)(y_1 + y_2)$ in $\mathbb{R}_2$.
4. $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 + x_2y_2 + 1$ in $\mathbb{R}_2$.
5. None of the above.

Answer: 2

1. *Consider $A = \left[\begin{smallmatrix} 1 & 1 \\ -1 & -1 \end{smallmatrix}\right]$, then $\langle A, A \rangle = 0$ which violates positivity.*
2. *This is an example of a weighted Euclidean inner product (see the text book).*
3. *Consider $\langle (1, -1), (1, -1) \rangle = 0$ which violates positivity.*
4. *Here $\langle (0, 0), (0, 0) \rangle = 1 \neq 0$.*

Question 30

Which one of the following defines an inner product?

1. $\langle (x_1, x_2), (y_1, y_2) \rangle = (x_1 + x_2)(y_1 + y_2)$ in $\mathbb{R}_2$.
2. $\langle (x_1, x_2), (y_1, y_2) \rangle = x_1y_1 - x_2y_2$ in $\mathbb{R}_2$.
3. $\langle A, B \rangle = \operatorname{tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} AB^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$ in M_{22}.

4. $\langle A, B\rangle = \mathrm{tr}\,(AB)$ in M_{22}.

5. None of the above.

Answer: 3

Consider 3 first. Since

$$\left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle = \mathrm{tr}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)$$
$$= \mathrm{tr} \begin{bmatrix} a_3b_3 + a_4b_4 & a_3b_1 + a_4b_2 \\ a_1b_3 + a_2b_4 & a_1b_1 + a_2b_2 \end{bmatrix}$$
$$= a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4$$

which is the inner product on M_{22} given as an example in the text book. (This follows trivially from $\mathrm{tr}(ABC) = \mathrm{tr}(CAB)$, and the consequence $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for product compatible square matrices A, B and C.) It is left as an exercise to show that each of the axioms hold (note that this inner product is essentially identical to the standard inner product on $\mathbb{R}^4$. For the remaining cases we provide examples of axioms which do not hold (other examples also exist).

1. $\langle(1,-1),(1,-1)\rangle = 0$ *which violates positivity (since* $(1,-1) \neq (0,0)$*).*

2. $\langle(1,1),(1,1)\rangle = 0$ *which violates positivity (since* $(1,1) \neq (0,0)$*).*

4. $\left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\rangle = 0$ *which violates positivity.*

Question 31

Which of the following vectors are unit vectors with respect to the inner product $\langle(x_1, x_2, x_3),(y_1, y_2.y_3)\rangle = 2x_1y_1 + 2x_2y_2 + 2x_3y_3$ in $\mathbb{R}^3$?

A. $(1,0,0)$ B. $(0,1,0)$ C. $(1,0,0)/\sqrt{2}$ D. $(1,1,0)/2$

Select from the following:

1. Only A and B.

2. Only A and C.

3. Only C and D.

4. Only A, B and D.

5. None of the above.

Answer: 3

A. $\|(1,0,0)\| = \sqrt{2 \cdot 1^2 + 2 \cdot 0^2 + 2 \cdot 0^2} = \sqrt{2}$

B. $\|(0,1,0)\| = \sqrt{2 \cdot 0^2 + 2 \cdot 1^2 + 2 \cdot 0^2} = \sqrt{2}$

C. $\|(1,0,0)/\sqrt{2}\| = \sqrt{2\left(\frac{1}{\sqrt{2}}\right)^2 + 2 \cdot 0^2 + 2 \cdot 0^2} = 1$

D. $\|(1,1,0)/2\| = \sqrt{2\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 + 2 \cdot 0^2} = 1$

Question 32

Which of the following vectors are unit vectors with respect to the inner product $\langle (x_1, x_2, x_3), (y_1, y_2.y_3) \rangle = x_1y_1 + x_2y_2 + 7x_3y_3$ in $\mathbb{R}^3$?

A. $(1,0,0)$ B. $(0,0,1)$ C. $(1,1,1)/\sqrt{3}$ D. $(1,1,1)/3$

Select from the following:

1. Only A and B.
2. Only A and C.
3. Only A and D.
4. Only A, B and C.
5. None of the above.

Answer: 3

A. $\|(1,0,0)\| = \sqrt{1^2 + 0^2 + 7 \cdot 0^2} = 1$

B. $\|(0,0,1)\| = \sqrt{0^2 + 0^2 + 7 \cdot 1^2} = \sqrt{7}$

C. $\|(1,1,1)/\sqrt{3}\| = \sqrt{\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + 7\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{3}{\sqrt{3}}$

D. $\|(1,1,1)/3\| = \sqrt{\left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + 7\left(\frac{1}{3}\right)^2} = 1$

Question 33

Which of the following vectors are orthogonal to each other with respect to the inner product $\langle A, B \rangle = \mathrm{tr}(A^T B)$ in M_{22} ?

A. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. B. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. C. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only A and B are orthogonal.
2. Only A and C are orthogonal, C and D are orthogonal.
3. Only B and C are orthogonal, B and D are orthogonal.
4. Only A and D are orthogonal.
5. None of the above.

Answer: 2

since

$$\left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\rangle = 1, \quad \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle = 0, \quad \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = 2,$$

$$\left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle = -1, \quad \left\langle \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = 1,$$

$$\left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = 0.$$

Question 34

Which of the following vectors are *not* orthogonal to each other with respect to the inner product $\langle A, B\rangle = \text{tr}(A^T B)$ in M_{22} ?

A. $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. B. $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. C. $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. D. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

Select from the following:

1. Only A and B are not orthogonal, B and D are not orthogonal.
2. Only A and C are not orthogonal, B and D are not orthogonal.
3. Only A and C are not orthogonal.
4. Only A and D are not orthogonal.
5. None of the above.

Answer: 1

since

$$\left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = -1, \quad \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle = 0, \quad \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = 0,$$

$$\left\langle \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\rangle = 0, \quad \left\langle \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = 3,$$

$$\left\langle \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\rangle = 0.$$

Question 35

Consider the vector subspace $W = \text{span}\{\,1+x,\ 1+x^2\,\}$ of P_2 with the *evaluation inner product* at 0, 1 and -1 (sample points). Which of the following vectors in P_2 lie in the subspace $W^\perp$?

1. $5x^2 + x - 4$.
2. $x^2 + x - 1$.
3. $-x^2 - x + 1$.
4. $x - 1$.
5. None of the above.

Answer: 1

If $p(x) \in W^\perp$ then for all $a, b \in \mathbb{R}$

$$\langle p(x), a(1+x) + b(1+x^2)\rangle = a\langle p(x), 1+x\rangle + b\langle p(x), 1+x^2\rangle = 0.$$

By choosing $a = 1$ and $b = 0$ we must have $\langle p(x), 1+x\rangle = 0$. By choosing $a = 0$ and $b = 1$ we must have $\langle p(x), 1+x^2\rangle = 0$. Imposing these two conditions yields $\langle p(x), a(1+x) + b(1+x^2)\rangle = 0$. Thus it is sufficient to determine whether $\langle p(x), 1+x\rangle = 0$ and $\langle p(x), 1+x^2\rangle = 0$. The evaluation inner products (using the sample points 0, 1 and -1) are given by

$$\begin{aligned}
\langle p(x), 1+x\rangle &= p(0)\cdot[1+x]_{x\to 0} + p(1)\cdot[1+x]_{x\to 1} + p(-1)\cdot[1+x]_{x\to -1}\\
&= p(0)\cdot 1 + p(1)\cdot 2 + p(-1)\cdot 0,\\
\langle p(x), 1+x^2\rangle &= p(0)\cdot[1+x^2]_{x\to 0} + p(1)\cdot[1+x^2]_{x\to 1} + p(-1)\cdot[1+x^2]_{x\to -1}\\
&= p(0)\cdot 1 + p(1)\cdot 2 + p(-1)\cdot 2.
\end{aligned}$$

Thus we find

1. $\langle 5x^2 + x - 4, 1+x\rangle = (-4)\cdot 1 + 2\cdot 2 + 0\cdot 0 = 0,$
   $\langle 5x^2 + x - 4, 1+x^2\rangle = (-4)\cdot 1 + 2\cdot 2 + 0\cdot 2 = 0.$
2. $\langle x^2 + x - 1, 1+x\rangle = (-1)\cdot 1 + 1\cdot 2 + (-1)\cdot 0 \neq 0.$
3. $\langle -x^2 - x - 1, 1+x\rangle = (-1)\cdot 1 + (-3)\cdot 2 + (-1)\cdot 0 \neq 0.$
4. $\langle x - 1, 1+x\rangle = (-1)\cdot 1 + 0\cdot 2 + (-2)\cdot 0 \neq 0.$

Question 36

Consider the vector subspace $W = \text{span}\{\,1+x,\ 1+x^2\,\}$ of P_2 with the *standard inner product.* Which of the following vectors in P_2 lie in the subspace $W^\perp$?

1. $x^2 + x - 1$.

2. $5x^2 + x - 4$.

3. $x^2 - 2x + 1$.

4. $x^2 - x$.

5. None of the above.

Answer: 1

If $p(x) \in W^\perp$ *then for all* $a, b \in \mathbb{R}$

$$\langle p(x), a(1+x) + b(1+x^2)\rangle = a\langle p(x), 1+x\rangle + b\langle p(x), 1+x^2\rangle = 0.$$

By choosing $a = 1$ *and* $b = 0$ *we must have* $\langle p(x), 1+x\rangle = 0$*. By choosing* $a = 0$ *and* $b = 1$ *we must have* $\langle p(x), 1+x^2\rangle = 0$*. Imposing these two conditions yields* $\langle p(x), a(1+x) + b(1+x^2)\rangle = 0$*. Thus it is sufficient to determine whether* $\langle p(x), 1+x\rangle = 0$ *and* $\langle p(x), 1+x^2\rangle = 0$*. The standard inner products are given by*

1. $\langle -1 + x + x^2, 1 + x + 0x^2\rangle = -1\cdot 1 + 1\cdot 1 + 1\cdot 0 = 0,$
   $\langle -1 + x + x^2, 1 + 0x + x^2\rangle = -1\cdot 1 + 1\cdot 0 + 1\cdot 1 = 0.$

2. $\langle -4 + x + 5x^2, 1 + x + 0x^2\rangle = -4\cdot 1 + 1\cdot 1 + 5\cdot 0 = -3 \neq 0.$

3. $\langle 1 - 2x + x^2, 1 + x + 0x^2\rangle = 1\cdot 1 + (-2)\cdot 1 + 1\cdot 0 = -1 \neq 0.$

4. $\langle 0 - x + x^2, 1 + x + 0x^2\rangle = 0\cdot 1 + (-1)\cdot 1 + 1\cdot 0 = -1 \neq 0.$

F.3 2016 Semester 1: Exam

Question paper

Question 1: 16 Marks

This question is a **multiple choice** question and should be answered in the **answer book**. Any rough work should be clearly marked and appear on the last pages of the answer book. Write only the *number* for your answer.

(1.1) Consider the set (2)

$$X := \{\spadesuit\}$$

and the operations (for all $k \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} := \spadesuit,$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} := \spadesuit.$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which of the following statements are true in X ?

A. for all $\mathbf{x} \in X$: $-\mathbf{x} = \spadesuit$

B. for all $\mathbf{x} \in X$: $-\mathbf{x} = \mathbf{x}$

C. $\mathbf{0} = 0$

D. $\mathbf{0} = (0, 0)$

Choose from the following:

1. A
2. B
3. A and B
4. C or D
5. None of the above.

(1.2) Which of the following are subspaces of M_{22} with the usual operations ? (2)

A. $\text{span}\left\{ \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} : a \geq 0 \right\}$

C. $\left\{ \begin{bmatrix} a & -1 \\ 0 & a \end{bmatrix} : a \in \mathbb{R} \right\}$

Select from the following:

1. Only A.
2. Only A and B.
3. Only B and C.
4. All of A, B and C.
5. None of the above.

(1.3) Which of the following sets are linearly independent? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$ in M_{22}

B. $\{ (1,0,1), (0,1,0), (1,1,-1) \}$ in $\mathbb{R}^3$

C. $\{ 1-x, 1-x^2, 1-x+x^2 \}$ in P_2

Select from the following:

1. Only A and C.
2. Only B and C.
3. Only B.
4. Only C.
5. None of the above.

(1.4) Which of the following sets are a basis for the following vector subspace of M_{22}: (2)

$$X = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

A. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right\}$

Select from the following:

1. Both A and B.
2. Only A.
3. Only B.
4. None of the above.

(1.5) Which of the following statements are true: (2)

A. $\dim \left(\operatorname{span} \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\} \right) = 2$ in M_{22}

B. $\dim (\operatorname{span} \{ (1,0,1),\ (0,1,0),\ (1,1,-1) \}) = 3$ in $\mathbb{R}^3$

C. $\dim (\operatorname{span} \{ 1-x,\ 1-x^2,\ 1-x+x^2 \}) = 2$ in P_2

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only A and B.
5. None of the above.

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -2 & 2 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both A and B.
4. Only C.
5. None of the above.

(1.7) Which of the following sets are a basis for the null space of $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both B and C.
4. All of A, B and C.
5. None of the above.

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$? (2)

1. $\text{rank}(A) = 3$, $\text{nullity}(A) = 0$.
2. $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.
3. $\text{rank}(A) = 2$, $\text{nullity}(A) = 2$.
4. $\text{rank}(A) = 2$, $\text{nullity}(A) = 1$.
5. None of the above.

Question 2: 34 Marks

Consider the vector space M_{22}.

(2.1) Show that (12)

$$\langle A, B \rangle = \text{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T \right)$$

is an inner product on M_{22}.

(2.2) Prove that if $A, B \in M_{22}$, where $A, B \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, are orthogonal to each other with respect to the inner product **defined in 2.1** above, then $\{ A, B \}$ is a linearly independent set. (6)

(2.3) Apply the Gram-Schmidt process to the following subset of M_{22}: (12)

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** above for the span of this subset.

(2.4) Let V be a vector space with zero vector $\mathbf{0}$ and let $\langle \cdot, \cdot \rangle$ denote an inner product on V. Prove that $\langle \mathbf{0}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$. (4)

Question 3: 28 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

(3.1) Determine the nullity of A. (2)

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (3)

$$\lambda(\lambda-1)^2 = 0.$$

(3.3) Find bases for the eigenspaces of A. (14)

(3.4) For each eigenvalue, determine the algebraic and geometric multiplicity. Is A diagonalizable? (5)

(3.5) Prove or disprove: (2)

If B is a 2×2 matrix, then B is diagonalizable if and only if B^2 is diagonalizable.

(3.6) Let B be an $n \times n$ matrix. Prove that $B + B^T$ is diagonalizable. (2)

Question 4: 22 Marks

Let $T : \mathbb{R}^3 \to M_{22}$ be defined by $T(x, y, z) = \begin{bmatrix} x & y \\ z & x \end{bmatrix}$.

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation $[T]_{B_2,B_1}$ of T relative to the basis (8)

$$B_1 = \{\,(1,0,1),\ (0,1,0),\ (1,0,-1)\,\}$$

in $\mathbb{R}^3$ and the basis

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

in M_{22}, ordered from left to right.

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = M_{22}$? (4)

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

(4.5) Is T one-to-one? Motivate your answer. (2)

Solution

Question 1: 16 Marks

This question is a **multiple choice** question and should be answered in the **answer book**. Any rough work should be clearly marked and appear on the last pages of the answer book. Write only the *number* for your answer.

(1.1) Consider the set (2)

$$X := \{\spadesuit\}$$

and the operations (for all $k \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} := \spadesuit,$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} := \spadesuit.$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which of the following statements are true in X ?

A. for all $\mathbf{x} \in X$: $-\mathbf{x} = \spadesuit$

B. for all $\mathbf{x} \in X$: $-\mathbf{x} = \mathbf{x}$

C. $\mathbf{0} = 0$

D. $\mathbf{0} = (0, 0)$

Choose from the following:

1. A
2. B
3. A and B
4. C or D
5. None of the above.

Answer: 3

(1.2) Which of the following are subspaces of M_{22} with the usual operations ? (2)

A. $\text{span}\left\{\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & -1\end{bmatrix}\right\}$

B. $\left\{\begin{bmatrix}0 & a\\-a & 0\end{bmatrix} : a \geq 0\right\}$

C. $\left\{\begin{bmatrix}a & -1\\0 & a\end{bmatrix} : a \in \mathbb{R}\right\}$

Select from the following:

1. Only A.
2. Only A and B.
3. Only B and C.
4. All of A, B and C.
5. None of the above.

Answer: 1

(1.3) Which of the following sets are linearly independent? (2)

A. $\left\{\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}2 & 1\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}\right\}$ in M_{22}

B. $\{(1,0,1), (0,1,0), (1,1,-1)\}$ in $\mathbb{R}^3$

C. $\{1-x, 1-x^2, 1-x+x^2\}$ in P_2

Select from the following:

1. Only A and C.
2. Only B and C.
3. Only B.
4. Only C.
5. None of the above.

Answer: 2

(1.4) Which of the following sets are a basis for the following vector subspace of M_{22}: (2)

$$X = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

A. $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \right\}$

Select from the following:

1. Both A and B.
2. Only A.
3. Only B.
4. None of the above.

Answer: 1

(1.5) Which of the following statements are true: (2)

A. $\dim\left(\text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}\right) = 2$ in M_{22}

B. $\dim(\text{span}\{ (1,0,1), (0,1,0), (1,1,-1) \}) = 3$ in $\mathbb{R}^3$

C. $\dim(\text{span}\{ 1-x, 1-x^2, 1-x+x^2 \}) = 2$ in P_2

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only A and B.
5. None of the above.

Answer: 4

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix}1 & 1 & -1\\0 & 1 & -1\\1 & 0 & 0\\1 & -2 & 2\end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix}1 & 1 & -1\end{bmatrix}, \begin{bmatrix}0 & 1 & -1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix}1 & 1 & -1\end{bmatrix}, \begin{bmatrix}0 & 1 & -1\end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix}1 & 1 & -1\end{bmatrix}, \begin{bmatrix}0 & 1 & -1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & -2 & 2\end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both A and B.
4. Only C.
5. None of the above.

Answer: 2

(1.7) Which of the following sets are a basis for the null space of $\begin{bmatrix}1 & 1 & -1\\0 & 1 & -1\\1 & 0 & 0\\1 & -2 & 2\end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix}0\\1\\1\end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix}0\\2\\2\end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Both B and C.
4. All of A, B and C.
5. None of the above.

Answer: 3

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 1 & 0 & 0 \\ 1 & -2 & 2 \end{bmatrix}$? (2)

1. $\operatorname{rank}(A) = 3$, $\operatorname{nullity}(A) = 0$.
2. $\operatorname{rank}(A) = 3$, $\operatorname{nullity}(A) = 1$.
3. $\operatorname{rank}(A) = 2$, $\operatorname{nullity}(A) = 2$.
4. $\operatorname{rank}(A) = 2$, $\operatorname{nullity}(A) = 1$.
5. None of the above.

Answer: 4

Question 2: 34 Marks

Consider the vector space M_{22}.

(2.1) Show that (12)

$$\langle A, B\rangle = \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T\right)$$

is an inner product on M_{22}.

- *For all* $A, B \in M_{22}$

  $$\langle A, B\rangle = \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T\right) = \operatorname{tr}\left(BA^T \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right) = \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} BA^T\right) = \langle B, A\rangle \checkmark^{2}$$

  where we used Theorem TI and Theorem CT.

- *For all* $k \in \mathbb{R}$ *and* $A \in M_{22}$ *we have*

  $$\langle kA, B\rangle = \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} (kA)B^T\right) = k\operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T\right) = k\langle A, B\rangle \checkmark^{2}$$

  since $\operatorname{tr}(kA) = k\operatorname{tr}(A)$.

- *For all* $A, B, C \in M_{22}$

  $$\begin{aligned} \langle A, B + C\rangle &= \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A(B + C)^T\right) = \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A(B^T + C^T)\right) \\ &= \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T + \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AC^T\right) \\ &= \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AB^T\right) + \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AC^T\right) = \langle A, B\rangle + \langle A, C\rangle \checkmark^{4} \end{aligned}$$

  since $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$.

- Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

  $$\langle A, A\rangle = \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} AA^T\right) = \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix}\right) = 2a^2+2b^2+c^2+d^2 \geq 0 \checkmark^2$$

  and $\langle A, A\rangle = 0$ if and only if $a = b = c = d = 0$ (since $a^2, b^2, c^2, d^2 \geq 0$). $\checkmark^2$

Alternative:

Note that

$$\left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle = \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} b_1 & b_3 \\ b_2 & b_4 \end{bmatrix}\right) = 2a_1b_1 + 2a_2b_2 + a_3b_3 + a_4b_4.$$

- For all $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \in M_{22}$

  $$\begin{aligned}\left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle &= 2a_1b_1 + 2a_2b_2 + a_3b_3 + a_4b_4 \\ &= 2b_1a_1 + 2b_2a_2 + b_3a_3 + b_4a_4 = \left\langle \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \right\rangle.\end{aligned}$$

- For all $k \in \mathbb{R}$ and $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \in M_{22}$

  $$\begin{aligned}\left\langle k\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} ka_1 & ka_2 \\ ka_3 & ka_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle \\ &= 2(ka_1)b_1 + 2(ka_2)b_2 + (ka_3)b_3 + (ka_4)b_4 \\ &= k(2a_1b_1 + 2a_2b_2 + a_3b_3 + a_4b_4) = k\left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle.\end{aligned}$$

- For all $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \in M_{22}$

  $$\begin{aligned}\left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} + \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \right\rangle &= \left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1+c_1 & b_2+c_2 \\ b_3+c_3 & b_4+c_4 \end{bmatrix} \right\rangle \\ &= 2a_1(b_1+c_1) + 2a_2(b_2+c_2) + a_3(b_3+c_3) + a_4(b_4+c_4) \\ &= (2a_1b_1 + 2a_2b_2 + a_3b_3 + a_4b_4) \\ &+ (2a_1c_1 + 2a_2c_2 + a_3c_3 + a_4c_4) \\ &= \left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \right\rangle + \left\langle \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \right\rangle.\end{aligned}$$

- Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$. Then

  $$\langle A, A\rangle = 2a_1^2 + 2a_2^2 + a_3^2 + a_4^2 \geq 0$$

  and $\langle A, A\rangle = 0$ if and only if $a_1 = a_2 = a_3 = a_4 = 0$ (since $a_1^2, a_2^2, a_3^2, a_4^2 \geq 0$).

(2.2) Prove that if $A, B \in M_{22}$, where $A, B \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, are orthogonal to each other with respect to the inner product **defined in 2.1** above, then $\{ A, B \}$ is a linearly independent set. (6)

Suppose $c_1A + c_2B = 0$ *where* $c_1, c_2 \in \mathbb{R}$. *Since* A *and* B *are orthogonal to each other we have* $\langle A, B \rangle = \langle B, A \rangle = 0$.✓

$$\begin{aligned} c_1A + c_2B = 0 &\Rightarrow \langle A, c_1A + c_2B \rangle = \langle A, 0 \rangle ✓ \\ &\Rightarrow c_1\langle A, A \rangle + c_2\langle A, B \rangle = 0 ✓ \\ &\Rightarrow c_1\langle A, A \rangle = 0 ✓ \\ &\Rightarrow c_1 = 0 ✓ \end{aligned}$$

since $\langle A, A \rangle \neq 0$. *Similarly*

$$\begin{aligned} c_1A + c_2B = 0 &\Rightarrow \langle B, c_1A + c_2B \rangle = \langle B, 0 \rangle \\ &\Rightarrow c_2 = 0. ✓ \end{aligned}$$

Thus $\{ A, B \}$ *is a linearly independent set.*

(2.3) Apply the Gram-Schmidt process to the following subset of M_{22}: (12)

$$\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** above for the span of this subset.

Let

$$u_1 := \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \quad u_2 := \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad u_3 := \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

Then the Gram-Schmidt process provides

$$
\begin{aligned}
v_1 &:= u_1 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \checkmark \\
\langle v_1, v_1 \rangle &= \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^T \right) = 8 \checkmark \\
\langle u_2, v_1 \rangle &= \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^T \right) = 4 \checkmark \\
v_2 &:= u_2 - \frac{\langle u_2, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1(x) \checkmark \\
&= \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} - \frac{4}{8} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \checkmark \\
\langle v_2, v_2 \rangle &= \frac{1}{4} \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}^T \right) = 2 \checkmark \\
\langle u_3, v_1 \rangle &= \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}^T \right) = 4 \checkmark \\
\langle u_3, v_2 \rangle &= \frac{1}{2} \operatorname{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}^T \right) = 0 \checkmark \\
v_3 &:= u_3 - \frac{\langle u_3, v_1 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle u_3, v_2 \rangle}{\langle v_2, v_2 \rangle} v_2 \checkmark \\
&= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} - \frac{4}{8} \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} - \frac{0}{2} \cdot \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} . \checkmark
\end{aligned}
$$

Thus we have the orthogonal basis

$$
\left\{ \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \right\} . \checkmark^2
$$

(2.4) Let V be a vector space with zero vector $\mathbf{0}$ and let $\langle \cdot, \cdot \rangle$ denote an inner product on V. Prove that $\langle \mathbf{0}, \mathbf{v} \rangle = 0$ for all $\mathbf{v} \in V$. (4)

Since $\mathbf{0} = 0 \cdot \mathbf{0} \checkmark^2$ *(Theorem VZ) we have* $\langle \mathbf{0}, \mathbf{v} \rangle = \langle 0 \cdot \mathbf{0}, \mathbf{v} \rangle = 0 \langle \mathbf{0}, \mathbf{v} \rangle = 0 \checkmark^2$ *by IP2.*
Alternative:

Since $\mathbf{0} = \mathbf{0} + \mathbf{0}$ *(VS4) we have* $\langle \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{0} + \mathbf{0}, \mathbf{v} \rangle = \langle \mathbf{0}, \mathbf{v} \rangle + \langle \mathbf{0}, \mathbf{v} \rangle$ *by IP3 and IP1, so that* $\langle \mathbf{0}, \mathbf{v} \rangle = 0$. ***Alternative:***

Since the Cauchy-Schwarz inequality yields $|\langle \mathbf{0}, \mathbf{v} \rangle| \leq \sqrt{\langle \mathbf{0}, \mathbf{0} \rangle \langle \mathbf{v}, \mathbf{v} \rangle} = 0$ *by IP4b, it follows that* $|\langle \mathbf{0}, \mathbf{v} \rangle| = 0$. *Thus* $\langle \mathbf{0}, \mathbf{v} \rangle = 0$.

Question 3: 28 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

(3.1) Determine the nullity of A. (2)

Row reduction of A yields

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_1)$$

which is in upper triangular form, with two nonzero rows. Hence the rank is 2, and the nullity is $3 - 2 = 1.\checkmark^2$

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (3)

$$\lambda(\lambda - 1)^2 = 0.$$

The characteristic equation is

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}\right) \checkmark = \begin{vmatrix} \lambda - 1 & -1 & 0 \\ 0 & \lambda - 1 & 0 \\ -1 & -1 & \lambda \end{vmatrix}$$
$$= (\lambda - 1)^2 \lambda = 0 \checkmark^2.$$

(3.3) Find bases for the eigenspaces of A. (14)

From the characteristic equation we obtain the eigenvalues 0, and 1 (twice)$\checkmark^2$. *For the eigenspace corresponding to the eigenvalue 0 we solve*

$$\begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 0 \\ -1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark^2$$

for $x, y, z \in \mathbb{R}$. *Clearly* $x = -y = 0$. *We find the 1-dimensional eigenspace*

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ z \end{bmatrix} : z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}.\checkmark^2$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.\checkmark^2$$

For the eigenspace corresponding to the eigenvalue 1 we solve

$$\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark^2$$

for $x, y, z \in \mathbb{R}$. Obviously $y = 0$ and $x = z$. The corresponding eigenspace is

$$\left\{ \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} : x \in \mathbb{R} \right\} = \left\{ x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} : x \in \mathbb{R} \right\}. \checkmark^2$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}. \checkmark^2$$

(3.4) For each eigenvalue, determine the algebraic and geometric multiplicity. Is A diagonalizable? (5)

The algebraic multiplicity of $\lambda = 1$ is 2,✓ and the geometric multiplicity is 1.✓ Thus A is not diagonalizable (Theorem DM).✓ The algebraic multiplicity of $\lambda = 0$ is 1,✓ and the geometric multiplicity is 1.✓

(3.5) Prove or disprove: (2)

If B is a 2×2 matrix, then B is diagonalizable if and only if B^2 is diagonalizable.

The statement is false, for example the matrix $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable, but $B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is diagonalizable (and diagonal).$\checkmark^2$

(3.6) Let B be an $n \times n$ matrix. Prove that $B + B^T$ is diagonalizable. (2)

Since $(B + B^T)^T = B^T + (B^T)^T = B^T + B = B + B^T$, $B + B^T$ is symmetric✓ and consequently diagonalizable (Theorem DS).✓

Question 4: 22 Marks

Let $T : \mathbb{R}^3 \to M_{22}$ be defined by $T(x, y, z) = \begin{bmatrix} x & y \\ z & x \end{bmatrix}$.

(4.1) Show that T is a linear transformation. (4)

Let $k \in \mathbb{R}$ and $(x_1, y_1, z_1), (x_2, y_2, z_2) \in \mathbb{R}^3$.

- $T((x_1, y_1, z_1) + (x_2, y_2, z_2)) = T(x_1 + x_2, y_1 + y_2, z_1 + z_2) = \begin{bmatrix} x_1 + x_2 & y_1 + y_2 \\ z_1 + z_2 & x_1 + x_2 \end{bmatrix}$
  $= \begin{bmatrix} x_1 & y_1 \\ z_1 & x_1 \end{bmatrix} + \begin{bmatrix} x_2 & y_2 \\ z_2 & x_2 \end{bmatrix} = T(x_1, y_1, z_1) + T(x_2, y_2, z_2). \checkmark^2$
- $T(k \cdot (x_1, y_1, z_1)) = T(kx_1, ky_1, kz_1) = \begin{bmatrix} kx_1 & ky_1 \\ kz_1 & kx_1 \end{bmatrix} = k \begin{bmatrix} x_1 & y_1 \\ z_1 & x_1 \end{bmatrix} = kT(x_1, y_1, z_1). \checkmark^2$

(4.2) Find the matrix representation $[T]_{B_2,B_1}$ of T relative to the basis (8)

$$B_1 = \{ (1,0,1), (0,1,0), (1,0,-1) \}$$

in $\mathbb{R}^3$ and the basis

$$B_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

in M_{22}, ordered from left to right.

From

$$T(1,0,1) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \checkmark^2$$

$$T(0,1,0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \checkmark^2$$

$$T(1,0,-1) = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 0\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \checkmark^2$$

the coefficients of the basis elements in each equation provide the columns of the matrix representation:

$$\frac{1}{2}\begin{bmatrix} 2 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}. \checkmark^2$$

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = M_{22}$? (4)

The range of T is

$$\begin{aligned} R(T) &= \{ T(x,y,z) \ : \ x,y,z \in \mathbb{R} \} \\ &= \left\{ \begin{bmatrix} x & y \\ z & x \end{bmatrix} \ : \ x,y,z \in \mathbb{R} \right\}. \checkmark^2 \end{aligned}$$

Since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in M_{22}$ *but* $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \notin R(T)$, *$T$ is not onto.* $\checkmark^2$

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

$$\begin{aligned} \ker(T) &= \left\{ (x,y,z) \in \mathbb{R}^3 \ : \ T(x,y,z) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \left\{ (x,y,z) \in \mathbb{R}^3 \ : \ \begin{bmatrix} x & y \\ z & x \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \\ &= \{ (0,0,0) \}. \checkmark^2 \end{aligned}$$

We have a zero-dimensional space and the nullity of T is 0. $\checkmark^2$

(4.5) Is T one-to-one? Motivate your answer. (2)

Yes, since $\text{nullity}(T) = 0$ *(or equivalently* $\ker(T) = \{ (0,0,0) \}$, *Theorem TO).* $\checkmark^2$

F.4 2015 Semester 1: Exam

Question paper

Question 1: 16 Marks

This question is a **multiple choice** question and should be answered in the **green answer book**. Any rough work should be clearly marked and appear on the last pages of the answer book.

(1.1) Consider the set (2)

$$X := \{\, (x, y) \ :\ x, y \in \mathbb{R} \,\}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx + k - 1, ky),$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha + 1, y + \beta).$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements is true in X ?

1. $-(1, 1) = (-3, -1)$
2. $-(1, 1) = (-2, -1)$
3. $-(1, 1) = (-1, -1)$
4. $-(1, 1) = (0, -1)$
5. None of the above.

(1.2) Which of the following are subspaces of P_1 with the usual operations ? (2)

A. $\text{span}\,\{\, 1 + x \,\}$

B. $\{\, ax \ :\ a \in \mathbb{R} \,\}$

C. $\{\, 1 + ax \ :\ a \in \mathbb{R} \,\}$

D. $\{\, (1 + a)x \ :\ a \in \mathbb{R} \,\}$

Select from the following:

1. All of A, B, C and D.
2. Only A, B and D.
3. Only A, B and C.
4. Only B, C and D.
5. None of the above.

(1.3) Which of the following sets are linearly independent? (2)

A. span$\{(1,0,1),\ (1,0,2)\}$ in $\mathbb{R}^3$

B. $\{(1,0,1),\ (1,0,2)\}$ in $\mathbb{R}^3$

C. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right\}$ in M_{22}

Select from the following:

1. Only A.
2. Only A and B.
3. Only B.
4. Only B and C.
5. None of the above.

(1.4) Which of the following sets are a basis for the following vector subspace of P_2: (2)

$$X = \{ p(x) \in P_2 \, : \, p(1) = 0 \} .$$

A. $\{ 1 - 2x + x^2,\ 2 - 3x + x^2 \}$

B. $\{ 1 - x \}$

C. $\{ 1 - 2x + x^2 \}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only B and C.
5. None of the above.

(1.5) Which of the following statements are true: (2)

A. $\dim(\text{span}\{(1,1,1),\ (1,1,0)\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\text{span}\{(0,0,0),\ (1,1,1),\ (1,1,0)\}) = 3$ in $\mathbb{R}^3$

C. $\dim\left(\text{span}\left\{\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right\}\right) = 2$ in M_{22}

Select from the following:

1. All of A, B and C.
2. Only A.
3. Only A and B.
4. Only A and C.
5. None of the above.

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$? (2)

A. $\{\begin{bmatrix}-1 & 1 & 1 & -1\end{bmatrix}\}$

B. $\{\begin{bmatrix}1 & -1\end{bmatrix}\}$

C. $\{\begin{bmatrix}1 & -1\end{bmatrix},\begin{bmatrix}-1 & 1\end{bmatrix}\}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only A and C.
5. None of the above.

(1.7) Which of the following sets are a basis for the column space of $\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$? (2)

A. $\{\begin{bmatrix}1\end{bmatrix},\begin{bmatrix}-1\end{bmatrix}\}$

B. $\{\begin{bmatrix}-1\end{bmatrix}\}$

C. $\{\begin{bmatrix}1 & -1\end{bmatrix}\}$

D. $\{\begin{bmatrix}1 & -1\end{bmatrix},\begin{bmatrix}-1 & 1\end{bmatrix}\}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only D.
5. None of the above.

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix}$? (2)

1. $\text{rank}(A) = 0$, $\text{nullity}(A) = 4$.
2. $\text{rank}(A) = 1$, $\text{nullity}(A) = 3$.
3. $\text{rank}(A) = 2$, $\text{nullity}(A) = 2$.
4. $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.
5. None of the above.

Question 2: **30 Marks**

Consider the vector space P_3.

(2.1) Show that (12)

$$\langle p(x), q(x) \rangle := p_0 q_0 + 2p_1 q_1 + 2p_2 q_2 + p_3 q_3,$$

where

$$p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3 \quad \text{and} \quad q(x) = q_0 + q_1 x + q_2 x^2 + q_3 x^3,$$

is an inner product on P_3.

(2.2) Are the vectors (6)

$$1 + x^2 + x^3, \ -1 - x^2 + x^3, \ -1 + x - x^2 + x^3$$

linearly independent?

(2.3) Apply the Gram-Schmidt process to the following subset of P_3: (12)

$$\left\{ 1 + x^2 + x^3, \ -1 - x^2 + x^3, \ -1 + x - x^2 + x^3 \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** above for the span of this subset.

Question 3: **30 Marks**

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ -1 & 2 & 2 \end{bmatrix}.$$

(3.1) Determine the nullity of A. (2)

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (3)

$$\lambda^2(\lambda - 3) = 0.$$

(3.3) Find bases for the eigenspaces of A. (18)

(3.4) Is A diagonalizable? Motivate your answer. (2)

(3.5) Let B be an $n \times n$ matrix and B^T be the transpose of B. Prove that: (5)

$$B \text{ is diagonalizable if and only if } B^T \text{ is diagonalizable.}$$

Hint: recall that $(P^{-1})^T = (P^T)^{-1}$ for any invertible matrix P.

Question 4: 24 Marks

Let $T : M_{22} \to \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a, b) + (c, d)$$

where $a, b, c, d \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation $[T]_{B',B}$ of T relative to the basis (10)

$$B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

in M_{22}, and the basis

$$B' = \{ (1, 1), (1, -1) \}$$

in $\mathbb{R}^2$, ordered from left to right.

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = \mathbb{R}^2$? (4)

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

(4.5) Is T one-to-one? Motivate your answer. (2)

Solution

Question 1: **16 Marks**

This question is a **multiple choice** question and should be answered in the **green answer book**. Any rough work should be clearly marked and appear on the last pages of the answer book.

(1.1) Consider the set (2)

$$X := \{\, (x, y) \;:\; x, y \in \mathbb{R} \,\}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\begin{aligned} \cdot : \mathbb{R} \times X \to X, \qquad & k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx + k - 1, ky), \\ + : X \times X \to X, \qquad & \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha + 1, y + \beta). \end{aligned}$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements is true in X ?

1. $-(1, 1) = (-3, -1)$
2. $-(1, 1) = (-2, -1)$
3. $-(1, 1) = (-1, -1)$
4. $-(1, 1) = (0, -1)$
5. None of the above.

Answer: 1

(1.2) Which of the following are subspaces of P_1 with the usual operations ? (2)

A. $\operatorname{span}\{\, 1 + x \,\}$

B. $\{\, ax \;:\; a \in \mathbb{R} \,\}$

C. $\{\, 1 + ax \;:\; a \in \mathbb{R} \,\}$

D. $\{\, (1 + a)x \;:\; a \in \mathbb{R} \,\}$

Select from the following:

1. All of A, B, C and D.
2. Only A, B and D.
3. Only A, B and C.
4. Only B, C and D.
5. None of the above.

Answer: 2

(1.3) Which of the following sets are linearly independent? (2)

A. $\operatorname{span}\{(1,0,1), (1,0,2)\}$ in $\mathbb{R}^3$

B. $\{(1,0,1), (1,0,2)\}$ in $\mathbb{R}^3$

C. $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \right\}$ in M_{22}

Select from the following:

1. Only A.
2. Only A and B.
3. Only B.
4. Only B and C.
5. None of the above.

Answer: 4

(1.4) Which of the following sets are a basis for the following vector subspace of P_2: (2)

$$X = \{ p(x) \in P_2 \,:\, p(1) = 0 \}.$$

A. $\{1 - 2x + x^2, 2 - 3x + x^2\}$

B. $\{1 - x\}$

C. $\{1 - 2x + x^2\}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only B and C.
5. None of the above.

Answer: 1

(1.5) Which of the following statements are true: (2)

A. $\dim(\text{span}\{(1,1,1),\ (1,1,0)\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\text{span}\{(0,0,0),\ (1,1,1),\ (1,1,0)\}) = 3$ in $\mathbb{R}^3$

C. $\dim\left(\text{span}\left\{\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\right\}\right) = 2$ in M_{22}

Select from the following:

1. All of A, B and C.
2. Only A.
3. Only A and B.
4. Only A and C.
5. None of the above.

Answer: 2

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$? (2)

A. $\left\{\begin{bmatrix}-1 & 1 & 1 & -1\end{bmatrix}\right\}$

B. $\left\{\begin{bmatrix}1 & -1\end{bmatrix}\right\}$

C. $\left\{\begin{bmatrix}1 & -1\end{bmatrix}, \begin{bmatrix}-1 & 1\end{bmatrix}\right\}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only A and C.
5. None of the above.

Answer: 1

(1.7) Which of the following sets are a basis for the column space of $\begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$? (2)

A. $\{ \begin{bmatrix}1\end{bmatrix}, \begin{bmatrix}-1\end{bmatrix} \}$

B. $\{ \begin{bmatrix}-1\end{bmatrix} \}$

C. $\{ \begin{bmatrix}1 & -1\end{bmatrix} \}$

D. $\{ \begin{bmatrix}1 & -1\end{bmatrix}, \begin{bmatrix}-1 & 1\end{bmatrix} \}$

Select from the following:

1. Only A.
2. Only B.
3. Only C.
4. Only D.
5. None of the above.

Answer: 2

(1.8) Which one of the following statements is true for the matrix $A = \begin{bmatrix}1 & -1 & -1 & 1\end{bmatrix}$? (2)

1. $\text{rank}(A) = 0$, $\text{nullity}(A) = 4$.
2. $\text{rank}(A) = 1$, $\text{nullity}(A) = 3$.
3. $\text{rank}(A) = 2$, $\text{nullity}(A) = 2$.
4. $\text{rank}(A) = 3$, $\text{nullity}(A) = 1$.
5. None of the above.

Answer: 2

Question 2: 30 Marks

Consider the vector space P_3.

(2.1) Show that (12)

$$\langle p(x), q(x)\rangle := p_0q_0 + 2p_1q_1 + 2p_2q_2 + p_3q_3,$$

where

$$p(x) = p_0 + p_1x + p_2x^2 + p_3x^3 \text{ and } q(x) = q_0 + q_1x + q_2x^2 + q_3x^3,$$

is an inner product on P_3.

We have for $k \in \mathbb{R}$ and

$$p(x) = p_0 + p_1x + p_2x^2 + p_3x^3, q(x) = q_0 + q_1x + q_2x^2 + q_3x^3, r(x) = r_0 + r_1x + r_2x^2 + r_3x^3$$

1. $\langle p(x), q(x)\rangle = p_0q_0+2p_1q_1+2p_2q_2+p_3q_3 = q_0p_0+2q_1p_1+2q_2p_2+q_3p_3 = \langle q(x), p(x)\rangle$ ✓^{2}

2. $$\begin{aligned}\langle p(x)+r(x), q(x)\rangle &= (p_0+r_0)q_0 + 2(p_1+r_1)q_1 + 2(p_2+r_2)q_2 + (p_3+r_3)q_3\\ &= p_0q_0 + r_0q_0 + 2p_1q_1 + 2r_1q_1 + 2p_2q_2 + 2r_2q_2 + p_3q_3 + r_3q_3\\ &= p_0q_0 + 2p_1q_1 + 2p_2q_2 + p_3q_3 + r_0q_0 + 2r_1q_1 + 2r_2q_2 + r_3q_3\\ &= \langle p(x), q(x)\rangle + \langle r(x), q(x)\rangle \checkmark^4\end{aligned}$$

3. $\langle kp(x), q(x)\rangle = (kp_0)q_0 + 2(kp_1)q_1 + 2(kp_2)q_2 + (kp_3)q_3 = k(p_0q_0 + 2p_1q_1 + 2p_2q_2 + p_3q_3) = k\langle p(x), q(x)\rangle$ ✓^{2}

4. $\langle p(x), p(x)\rangle = p_0^2+2p_1^2+2p_2^2+p_3^2 \geq 0$ ✓^{2} *so that* $\langle p(x), p(x)\rangle \geq 0$ *and* $\langle p(x), p(x)\rangle = 0$ *if and only if* $p_0 = p_1 = p_2 = p_3 = 0$ *(since* $p_0^2, p_1^2, p_2^2, p_3^2 \geq 0$*)*✓^{2}, *i.e.* $p(x) = 0$.

(2.2) Are the vectors (6)

$$1 + x^2 + x^3,\ -1 - x^2 + x^3,\ -1 + x - x^2 + x^3$$

linearly independent?

Consider the equation

$$a(1 + x^2 + x^3) + b(-1 - x^2 + x^3) + c(-1 + x - x^2 + x^3) = 0 \checkmark^2$$

where $a, b, c \in \mathbb{R}$. Thus we have the equations

$$\begin{aligned} a - b - c &= 0\\ c &= 0\\ a - b - c &= 0\\ a + b + c &= 0\end{aligned}$$

Adding the fourth equation to the third provides $a = 0$. The second equation provides $c = 0$. Inserting the solutions for a and c into the first equation yields $b = 0$. This is the only solution. Thus these vectors are linearly independent.✓^{4}

(2.3) Apply the Gram-Schmidt process to the following subset of P_3: (12)

$$\{\,1+x^2+x^3,\ -1-x^2+x^3,\ -1+x-x^2+x^3\,\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** above for the span of this subset.

Let

$$u_1(x) := 1+x^2+x^3, \quad u_2(x) := -1-x^2+x^3, \quad u_3(x) := -1+x-x^2+x^3.$$

Then the Gram-Schmidt process provides

$$\begin{aligned}
v_1(x) &:= u_1 = 1+x^2+x^3 \checkmark \\
\langle v_1(x), v_1(x)\rangle &= 1^2 + 2\cdot 0^2 + 2\cdot 1^2 + 1^2 = 4 \checkmark \\
\langle u_2(x), v_1(x)\rangle &= (-1)\cdot 1 + 2\cdot 0\cdot 0 + 2\cdot(-1)\cdot 1 + 1\cdot 1 = -2 \checkmark \\
v_2(x) &:= u_2(x) - \frac{\langle u_2(x), v_1(x)\rangle}{\langle v_1(x), v_1(x)\rangle} v_1(x) \checkmark \\
&= (-1-x^2+x^3) - \frac{-2}{4}(1+x^2+x^3) = \frac{1}{2}(-1-x^2+3x^3) \checkmark \\
\langle v_2(x), v_2(x)\rangle &= \frac{1}{4}((-1)^2 + 2\cdot 0^2 + 2(-1)^2 + 3^2) = 3 \checkmark \\
\langle u_3(x), v_1(x)\rangle &= (-1)\cdot 1 + 2\cdot 1\cdot 0 + 2\cdot(-1)\cdot 1 + 1\cdot 1 = -2 \checkmark \\
\langle u_3(x), v_2(x)\rangle &= (-1)\cdot\left(-\frac{1}{2}\right) + 2\cdot 1\cdot 0 + 2\cdot(-1)\cdot\left(-\frac{1}{2}\right) + 1\cdot\frac{3}{2} = 3 \checkmark \\
v_3(x) &:= u_3(x) - \frac{\langle u_3(x), v_1(x)\rangle}{\langle v_1(x), v_1(x)\rangle} v_1 - \frac{\langle u_3(x), v_2(x)\rangle}{\langle v_2(x), v_2(x)\rangle} v_2 \checkmark \\
&= (-1+x-x^2+x^3) - \frac{-2}{4}(1+x^2+x^3) - \frac{3}{3}\cdot\frac{1}{2}(-1-x^2+3x^3) \\
&= x. \checkmark
\end{aligned}$$

Thus we have the orthogonal basis

$$\left\{\,1+x^2+x^3,\ \frac{1}{2}(-1-x^2+3x^3),\ x\,\right\}. \checkmark^{2}$$

Question 3: 30 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ -1 & 2 & 2 \end{bmatrix}.$$

(3.1) Determine the nullity of A. (2)

Row reduction of A yields

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ -1 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ -1 & 2 & 2 \end{bmatrix} \qquad (R_2 \leftarrow R_2 - 3R_1)$$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 3 & 3 \end{bmatrix} \qquad (R_3 \leftarrow R_3 + R_1)$$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_2 + R_3)$$

which is in upper triangular form, with two nonzero rows. Hence the rank is 2, and the nullity is $3 - 2 = 1$.✓ 2

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (3)

$$\lambda^2(\lambda - 3) = 0.$$

The characteristic equation is

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 3 & 0 & 0 \\ -1 & 2 & 2 \end{bmatrix}\right) \checkmark = \begin{vmatrix} \lambda - 1 & -1 & -1 \\ -3 & \lambda & 0 \\ 1 & -2 & \lambda - 2 \end{vmatrix}$$

$$= \lambda(\lambda - 1)(\lambda - 2) - 6 - 3(\lambda - 2) + \lambda \qquad = \lambda^3 - 3\lambda^2$$

$$= \lambda^2(\lambda - 3) = 0 \checkmark^2.$$

(3.3) Find bases for the eigenspaces of A. (18)

From the characteristic equation we obtain the eigenvalues 0 (twice), and 3✓ 2*. For the eigenspace corresponding to the eigenvalue 0 we solve*

$$\begin{bmatrix} -1 & -1 & -1 \\ -3 & 0 & 0 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark^2$$

for $x, y, z \in \mathbb{R}$. *Row reduction yields*

$$\begin{bmatrix} -1 & -1 & -1 \\ -3 & 0 & 0 \\ 1 & -2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ -3 & 0 & 0 \\ 1 & -2 & -2 \end{bmatrix} \qquad (R_1 \leftarrow -R_1)$$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 1 & -2 & -2 \end{bmatrix} \qquad (R_2 \leftarrow R_2 + 3R_1)$$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & -3 & -3 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_1)$$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_2 + R_3)$$

$$\to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_2 \leftarrow R_2/3)$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_1 \leftarrow R_1 - R_2)$$

so that $x = 0$ and $y = -z$. *We find the 1-dimensional eigenspace*

$$\left\{ \begin{bmatrix} 0 \\ z \\ -z \end{bmatrix} : z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} : z \in \mathbb{R} \right\}. \checkmark^{4}$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}. \checkmark^{2}$$

For the eigenspace corresponding to the eigenvalue 3 we solve

$$\begin{bmatrix} 2 & -1 & -1 \\ -3 & 3 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark^{2}$$

for $x, y, z \in \mathbb{R}$. *Row reduction yields*

$$\begin{bmatrix} 2 & -1 & -1 \\ -3 & 3 & 0 \\ 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ -3 & 3 & 0 \\ 2 & -1 & -1 \end{bmatrix} \qquad (R_1 \leftrightarrow R_3)$$

$$\to \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 2 & -1 & -1 \end{bmatrix} \qquad (R_2 \leftarrow R_2 + 3R_1)$$

$$\to \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 3 & -3 \end{bmatrix} \qquad (R_3 \leftarrow R_2 - 2R_1)$$

$$\to \begin{bmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 + R_2)$$

$$\to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_2 \leftarrow -R_2/3)$$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_1 \leftarrow R_1 + 2R_2)$$

so that $x = y = z$. *The corresponding eigenspace is*

$$\left\{ \begin{bmatrix} z \\ z \\ z \end{bmatrix} : z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\}. \checkmark^4$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \checkmark^2$$

(3.4) Is A diagonalizable? Motivate your answer. (2)

No, because the geometric and algebraic multiplicities are not equal for the eigenvalue 0 (i.e. the algebraic multiplicity is 2 while the geometric multiplicity is 1). $\checkmark^2$

(3.5) Let B be an $n \times n$ matrix and B^T be the transpose of B. Prove that: (5)

B is diagonalizable if and only if B^T is diagonalizable.

Hint: recall that $(P^{-1})^T = (P^T)^{-1}$ for any invertible matrix P.

Assume B *is diagonalizable, then there exists an invertible* $n \times n$ *matrix* P *such that* $P^{-1}BP$ *is diagonal.* $\checkmark$ *It follows that*

$$(P^{-1}BP)^T = P^T B^T (P^{-1})^T$$

is diagonal (since the transpose of a diagonal matrix is diagonal). $\checkmark$ *Let* $Q = (P^{-1})^T$ *so that* $Q^{-1} = P^T$. *Obviously* Q *is invertible and* $Q^{-1}B^TQ$ *is diagonal.* $\checkmark$ *Hence* B^T *is* *diagonalizable.*✓

Similarly, if B^T is diagonalizable then there exists invertible Q such that $Q^{-1}BQ$ is diagonal. Let $P = (Q^{-1})^T$. Then $P^{-1}BP$ is diagonal.✓

Question 4: 24 Marks

Let $T : M_{22} \to \mathbb{R}^2$ be defined by

$$T\begin{bmatrix} a & b \\ c & d \end{bmatrix} = (a, b) + (c, d)$$

where $a, b, c, d \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

Let $k, a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$. Using the definition of $\mathbb{R}^2$, M_{22} and T we find

-
$$T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} = (ka, kb) + (kc, kd) = k(a, b) + k(c, d)$$
$$= k((a, b) + (c, d)) = kT\begin{bmatrix} a & b \\ c & d \end{bmatrix}. \checkmark^2$$

-
$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) = T\begin{bmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{bmatrix} = (a+\alpha, b+\beta) + (c+\gamma, d+\delta)$$
$$= ((a, b) + (c, d)) + ((\alpha, \beta) + (\gamma, \delta)) = T\begin{bmatrix} a & b \\ c & d \end{bmatrix} T + T\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}. \checkmark^2$$

(4.2) Find the matrix representation $[T]_{B',B}$ of T relative to the basis (10)

$$B = \left\{ \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} \right\}$$

in M_{22}, and the basis

$$B' = \{ (1, 1), (1, -1) \}$$

in $\mathbb{R}^2$, ordered from left to right.

From

$$T\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = (1, 1) + (0, 0) = (1, 1) \qquad = 1 \cdot (1, 1) + 0 \cdot (1, -1)\checkmark^2$$
$$T\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = (0, 0) + (1, 1) = (1, 1) \qquad = 1 \cdot (1, 1) + 0 \cdot (1, -1)\checkmark^2$$
$$T\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = (1, -1) + (0, 0) = (1, -1) \qquad = 0 \cdot (1, 1) + 1 \cdot (1, -1)\checkmark^2$$
$$T\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix} = (0, 0) + (1, -1) = (1, -1) \qquad = 0 \cdot (1, 1) + 1 \cdot (1, -1)\checkmark^2$$

the coefficients of the basis elements in each equation provide the columns of the matrix representation:

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}. \checkmark^2$$

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = \mathbb{R}^2$? (4)

T is onto since

$$\begin{aligned} R(T) &= \left\{ T \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} \checkmark^2 \\ &= \{ (a,b) + (c,d) : a, b, c, d \in \mathbb{R} \} \\ &= \{ (\alpha, \beta) : \alpha, \beta \in \mathbb{R} \} \qquad \textit{(setting } \alpha = a + c \textit{ and } \beta = b + d) \\ &= \mathbb{R}^2. \checkmark^2 \end{aligned}$$

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

$$\begin{aligned} \ker(T) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = (0,0) \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R},\ (a,b) + (c,d) = (a+c, b+d) = 0 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b \in \mathbb{R},\ c = -a,\ d = -b \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ -a & -b \end{bmatrix} : a, b \in \mathbb{R} \right\} \\ &= \left\{ a \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} : a, b \in \mathbb{R} \right\}. \checkmark^2 \end{aligned}$$

Since

$$\left\{ \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \right\}$$

is a linearly independent set, we have a two-dimensional space and the nullity of T is 2. $\checkmark^2$

(4.5) Is T one-to-one? Motivate your answer. (2)

No, since (for example)

$$T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = T \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = (1,0). \checkmark^2$$

F.5 2014 Semester 1: Exam

Question paper

Question 1: **16 Marks**

This question is a **multiple choice** question and should be filled in on the multiple choice **answer sheet** (mark reading sheet).

(1.1) Consider the set (2)

$$X := \{ (x, y) \,:\, x, y \in \mathbb{R} \}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx - k + 1, ky),$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha - 1, y + \beta).$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements is true in X ?

1. $-(0, 0) = (1, 0)$
2. $-(0, 0) = (1, 1)$
3. $-(0, 0) = (0, 1)$
4. $-(0, 0) = (2, 0)$
5. None of the above.

(1.2) Which of the following are subspaces of $\mathbb{R}^2$ with the usual operations ? (2)

A. $\operatorname{span}\{ (2, 3) \}$

B. $\{ (x, 1) \,:\, x \in \mathbb{R} \}$

C. $\{ (0, x) \,:\, x \in \mathbb{R},\, x \geq 0 \}$

D. $\{ (0, x - 1) \,:\, x \in \mathbb{R} \}$

Select from the following:

1. Only A.
2. Only A and D.
3. Only C.
4. Only C and D.
5. None of the above.

(1.3) Which of the following sets are linearly independent? (2)

A. $\operatorname{span}\{(2,3)\}$ in $\mathbb{R}^2$

B. $\{(1,1), (-1,1)\}$ in $\mathbb{R}^2$

C. $\{(2,4), (1,-1), (1,1)\}$ in $\mathbb{R}^2$

D. $\{1+x, 1-x\}$ in P_1

Select from the following:

1. Only A.
2. Only B.
3. Only B and C.
4. Only B and D.
5. None of the above.

(1.4) Which of the following sets are a basis for the following vector subspace of P_2: (2)

$$X = \{ p(x) \in P_2 : p(1) = 0 \}.$$

A. $\{1, x, x^2\}$

B. $\{1-x, 1-x^2\}$

C. $\{1, 1-x, 1-x^2\}$

D. $\{1-x, 1-x^2, 3-2x-x^2\}$

Select from the following:

1. Only A.
2. Only B.
3. Only A and C.
4. A, B, C and D.
5. None of the above.

(1.5) Which of the following statements are true: (2)

A. $\dim(\text{span}\{(1,1,1),(1,1,-1)\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\text{span}\{(0,0,0),(1,1,1)\}) = 2$ in $\mathbb{R}^3$

C. $\dim(\text{span}\{(1,1,1),(1,-1,1),(1,1,-1)\}) = 2$ in $\mathbb{R}^3$

D. $\dim(\text{span}\{(1,1,1),(1,-1,1),(1,1,-1)\}) = 3$ in $\mathbb{R}^3$

Select from the following:

1. All of A, B, C and D.
2. Only A and B.
3. Only A and C.
4. Only A and D.
5. None of the above.

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$? (2)

A. $\{ \begin{bmatrix} 0 & -3 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 2 \end{bmatrix} \}$

B. $\{ \begin{bmatrix} 0 & -3 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \}$

C. $\{ \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 & -1 \end{bmatrix} \}$

Select from the following:

1. Only A.
2. Only B.
3. Only A and B.
4. A, B and C.
5. None of the above.

(1.7) Which of the following sets are contained in (i.e. subset of) the column space of $\begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \begin{bmatrix} -1 & 2 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} -1 & 2 \end{bmatrix}^T, \begin{bmatrix} 2 & 1 \end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 \end{bmatrix}^T \right\}$

D. $\left\{ \begin{bmatrix} 0 & 3 & -3 \end{bmatrix} \right\}$

Select from the following:

1. Only D.
2. Only A, B and C.
3. Only A and B.
4. Only A and C.
5. None of the above.

(1.8) Which of the following sets are a basis for the null space of $\begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \begin{bmatrix} -1 & 2 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix} -1 & 3 & 3 \end{bmatrix}^T \right\}$

Select from the following:

1. Only B and C.
2. Only B.
3. Only C.
4. Only A.
5. None of the above.

Question 2: 30 Marks

Consider the vector space $\mathbb{R}^4$.

(2.1) Show that (12)

$$\langle \mathbf{x}, \mathbf{y} \rangle := 2x_1y_1 + 2x_2y_2 + x_3y_3 + x_4y_4, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^4$$

is an inner product on $\mathbb{R}^4$.

(2.2) Are the vectors (6)

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

linearly independent?

(2.3) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^4$: (12)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** for the span of this subset.

Question 3: 30 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(3.1) Determine the rank of A. (2)

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (6)

$$\lambda(\lambda - 1)^2 = 0.$$

(3.3) Find bases for the eigenspaces of A. (18)

(3.4) Is A diagonalizable? Motivate your answer. (2)

(3.5) Is the matrix $A - I_3$ diagonalizable? Motivate your answer. (Here I_3 is the 3×3 identity matrix). (2)

Question 4: 24 Marks

Let $T : M_{22} \to P_2$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + \frac{b-c}{2}x + dx^2$$

where $a, b, c \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation $[T]_{B',B}$ of T relative to the basis (12)

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

in M_{22}, and the basis

$$B' = \{\, 1 + x,\, 1 - x,\, x^2 \,\}$$

in P_2, ordered from left to right.

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = P_2$? (4)

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

Solution

Question 1: **16 Marks**

This question is a **multiple choice** question and should be filled in on the multiple choice **answer sheet** (mark reading sheet).

(1.1) Consider the set (2)

$$X := \{\, (x, y) \,:\, x, y \in \mathbb{R} \,\}$$

and the operations (for all $k, x, y, \alpha, \beta \in \mathbb{R}$, $\mathbf{a} = (x, y) \in X$ and $\mathbf{b} = (\alpha, \beta) \in X$)

$$\cdot : \mathbb{R} \times X \to X, \qquad k \cdot \mathbf{a} \equiv k \cdot (x, y) := (kx - k + 1, ky),$$
$$+ : X \times X \to X, \qquad \mathbf{a} + \mathbf{b} \equiv (x, y) + (\alpha, \beta) := (x + \alpha - 1, y + \beta).$$

The set X with these definitions of $\cdot$ and $+$ forms a vector space. Which one of the following statements is true in X ?

1. $-(0, 0) = (1, 0)$
2. $-(0, 0) = (1, 1)$
3. $-(0, 0) = (0, 1)$
4. $-(0, 0) = (2, 0)$
5. None of the above.

Answer: 4

(1.2) Which of the following are subspaces of $\mathbb{R}^2$ with the usual operations ? (2)

A. $\operatorname{span}\{\, (2, 3) \,\}$

B. $\{\, (x, 1) \,:\, x \in \mathbb{R} \,\}$

C. $\{\, (0, x) \,:\, x \in \mathbb{R},\ x \geq 0 \,\}$

D. $\{\, (0, x - 1) \,:\, x \in \mathbb{R} \,\}$

Select from the following:

1. Only A.
2. Only A and D.
3. Only C.
4. Only C and D.
5. None of the above.

Answer: 2

(1.3) Which of the following sets are linearly independent? (2)

A. $\operatorname{span}\{(2,3)\}$ in $\mathbb{R}^2$

B. $\{(1,1), (-1,1)\}$ in $\mathbb{R}^2$

C. $\{(2,4), (1,-1), (1,1)\}$ in $\mathbb{R}^2$

D. $\{1+x, 1-x\}$ in P_1

Select from the following:

1. Only A.
2. Only B.
3. Only B and C.
4. Only B and D.
5. None of the above.

Answer: 4

(1.4) Which of the following sets are a basis for the following vector subspace of P_2: (2)

$$X = \{ p(x) \in P_2 : p(1) = 0 \}.$$

A. $\{1, x, x^2\}$

B. $\{1-x, 1-x^2\}$

C. $\{1, 1-x, 1-x^2\}$

D. $\{1-x, 1-x^2, 3-2x-x^2\}$

Select from the following:

1. Only A.
2. Only B.
3. Only A and C.
4. A, B, C and D.
5. None of the above.

Answer: 2

(1.5) Which of the following statements are true: (2)

A. $\dim(\text{span}\{(1,1,1),(1,1,-1)\}) = 2$ in $\mathbb{R}^3$

B. $\dim(\text{span}\{(0,0,0),(1,1,1)\}) = 2$ in $\mathbb{R}^3$

C. $\dim(\text{span}\{(1,1,1),(1,-1,1),(1,1,-1)\}) = 2$ in $\mathbb{R}^3$

D. $\dim(\text{span}\{(1,1,1),(1,-1,1),(1,1,-1)\}) = 3$ in $\mathbb{R}^3$

Select from the following:

1. All of A, B, C and D.
2. Only A and B.
3. Only A and C.
4. Only A and D.
5. None of the above.

Answer: 4

(1.6) Which of the following sets are a basis for the row space of $\begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 0 & -3 & 3 \end{bmatrix}, \begin{bmatrix} 3 & -1 & 2 \end{bmatrix} \right\}$

B. $\left\{ \begin{bmatrix} 0 & -3 & 3 \end{bmatrix}, \begin{bmatrix} 3 & 0 & 1 \end{bmatrix} \right\}$

C. $\left\{ \begin{bmatrix} 3 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 2 & -1 \end{bmatrix} \right\}$

Select from the following:

1. Only A.
2. Only B.
3. Only A and B.
4. A, B and C.
5. None of the above.

Answer: 4

(1.7) Which of the following sets are contained in (i.e. subset of) the column space of $\begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \begin{bmatrix} -1 & 2 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} -1 & 2 \end{bmatrix}^T, \begin{bmatrix} 2 & 1 \end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix} 1 & 0 \end{bmatrix}^T, \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \begin{bmatrix} 1 & 1 \end{bmatrix}^T \right\}$

D. $\{ \begin{bmatrix} 0 & 3 & -3 \end{bmatrix} \}$

Select from the following:

1. Only D.
2. Only A, B and C.
3. Only A and B.
4. Only A and C.
5. None of the above.

Answer: 2

(1.8) Which of the following sets are a basis for the null space of $\begin{bmatrix} 3 & -1 & 2 \\ 3 & 2 & -1 \end{bmatrix}$? (2)

A. $\left\{ \begin{bmatrix} 1 & 1 \end{bmatrix}^T, \begin{bmatrix} -1 & 2 \end{bmatrix}^T \right\}$

B. $\left\{ \begin{bmatrix} -1 & 1 & 1 \end{bmatrix}^T \right\}$

C. $\left\{ \begin{bmatrix} -1 & 3 & 3 \end{bmatrix}^T \right\}$

Select from the following:

1. Only B and C.
2. Only B.
3. Only C.
4. Only A.
5. None of the above.

Answer: 3

Question 2: 30 Marks

Consider the vector space $\mathbb{R}^4$.

(2.1) Show that (12)

$$\langle \mathbf{x}, \mathbf{y} \rangle := 2x_1y_1 + 2x_2y_2 + x_3y_3 + x_4y_4, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^4$$

is an inner product on $\mathbb{R}^4$.

We have for $k \in \mathbb{R}$ and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \in \mathbb{R}^4$$

1. $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1y_1 + 2x_2y_2 + x_3y_3 + x_4y_4 = 2y_1x_1 + 2y_2x_2 + y_3x_3 + y_4x_4 = \langle \mathbf{y}, \mathbf{x} \rangle$ ✓^{2}
2. $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = 2(x_1 + z_1)y_1 + 2(x_2 + z_2)y_2 + (x_3 + z_3)y_3 + (x_4 + z_4)y_4$
   $= 2x_1y_1 + 2z_1y_1 + 2x_2y_2 + 2z_2y_2 + x_3y_3 + z_3y_3 + x_4y_4 + z_4y_4$
   $= 2x_1y_1 + 2x_2y_2 + x_3y_3 + x_4y_4 + 2z_1y_1 + 2z_2y_2 + z_3y_3 + z_4y_4 = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$
   ✓^{4}
3. $\langle k\mathbf{x}, \mathbf{y} \rangle = 2(kx_1)y_1 + 2(kx_2)y_2 + (kx_3)y_3 + (kx_4)y_4 = k(2x_1y_1 + 2x_2y_2 + x_3y_3 + x_4y_4) = k\langle \mathbf{x}, \mathbf{y} \rangle$ ✓^{2}
4. $\langle \mathbf{x}, \mathbf{x} \rangle = 2x_1^2 + 2x_2^2 + x_3^2 + x_4^2 \geq 0$ ✓^{2} *so that* $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *if and only if* $x_1 = x_2 = x_3 = x_4 = 0$ *(since* $x_1^2, x_2^2, x_3^2, x_4^2 \geq 0$*)* ✓^{2}, *i.e.* $\mathbf{x} = \mathbf{0}$.

(2.2) Are the vectors (6)

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

linearly independent?

Consider the equation

$$a \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} = 0 \checkmark^2$$

where $a, b, c \in \mathbb{R}$. Thus we have the equations

$$\begin{aligned} a + b + c &= 0 \\ c &= 0 \\ a - b - c &= 0 \\ a + b + c &= 0 \end{aligned}$$

Adding the fourth equation to the third provides $a = 0$. The second equation provides $c = 0$. Inserting the solutions for a and c into the first equation yields $b = 0$. This is the only solution. Thus these vectors are linearly independent.✓[4]

(2.3) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^4$: (12)

$$\left\{ \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** for the span of this subset.

Let

$$\mathbf{u}_1 := \begin{bmatrix}1\\0\\1\\1\end{bmatrix}, \quad \mathbf{u}_2 := \begin{bmatrix}1\\0\\-1\\1\end{bmatrix}, \quad \mathbf{u}_3 := \begin{bmatrix}1\\1\\-1\\1\end{bmatrix}.$$

Then the Gram-Schmidt process provides

$$\mathbf{v}_1 := \mathbf{u}_1 = \begin{bmatrix}1\\0\\1\\1\end{bmatrix} ✓$$

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 2 \cdot 1^2 + 2 \cdot 0^2 + 1^2 + 1^2 = 4 ✓$$

$$\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = 2 \cdot 1 \cdot 1 + 2 \cdot 0 \cdot 0 + 1 \cdot (-1) + 1 \cdot 1 = 2 ✓$$

$$\mathbf{v}_2 := \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 ✓ = \begin{bmatrix}1\\0\\-1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1\\0\\-3\\1\end{bmatrix} ✓$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \frac{1}{4}(2 \cdot 1^2 + 2 \cdot 0^2 + (-3)^2 + 1^2) = 3 ✓$$

$$\langle \mathbf{u}_3, \mathbf{v}_1 \rangle = 2 \cdot 1 \cdot 1 + 2 \cdot 1 \cdot 0 + (-1) \cdot 1 + 1 \cdot 1 = 2 ✓$$

$$\langle \mathbf{u}_3, \mathbf{v}_2 \rangle = 2 \cdot 1 \cdot \frac{1}{2} + 2 \cdot 1 \cdot 0 + (-1) \cdot \left(-\frac{3}{2}\right) + 1 \cdot \frac{1}{2} = 3 ✓$$

$$\mathbf{v}_3 := \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 ✓$$

$$= \begin{bmatrix}1\\1\\-1\\1\end{bmatrix} - \frac{2}{4}\begin{bmatrix}1\\0\\1\\1\end{bmatrix} - \frac{3}{3} \cdot \frac{1}{2}\begin{bmatrix}1\\0\\-3\\1\end{bmatrix}$$

$$= \begin{bmatrix}0\\1\\0\\0\end{bmatrix}. ✓$$

Thus we have the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}. \checkmark^{2}$$

Question 3: 30 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

(3.1) Determine the rank of A. (2)

Row reduction of A yields

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad (R_2 \leftarrow R_2 - R_1/2,\ R_3 \leftarrow R_3 + R_1/2)$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_2)$$

which is in upper triangular form, with two nonzero rows. Hence the rank is 2. $\checkmark^{2}$

(3.2) Show that the characteristic equation for the eigenvalues λ of A is given by (6)

$$\lambda(\lambda - 1)^2 = 0.$$

The characteristic equation is

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) \checkmark^{2} = \begin{vmatrix} \lambda - 1 & 0 & 0 \\ -\frac{1}{2} & \lambda - \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \lambda - \frac{1}{2} \end{vmatrix}$$

$$= (\lambda - 1)\begin{vmatrix} \lambda - \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \lambda - \frac{1}{2} \end{vmatrix}$$

$$= (\lambda - 1)\left(\left(\lambda - \frac{1}{2}\right)^2 - \frac{1}{4}\right) = (\lambda - 1)(\lambda - 1)\lambda = 0 \checkmark^{4}.$$

(3.3) Find bases for the eigenspaces of A. (18)

From the characteristic equation we obtain the eigenvalues 0, and 1✓[2] *(twice). For the eigenspace corresponding to the eigenvalue 0 we solve*

$$\begin{bmatrix} -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark^2$$

for $x, y, z \in \mathbb{R}$. *Row reduction yields*

$$\begin{bmatrix} -1 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \qquad (R_2 \leftarrow R_2 - R_1/2,\ R_3 \leftarrow R_3 + R_1/2)$$

$$\to \begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_3 \leftarrow R_3 - R_2)$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_1 \leftarrow -R_1,\ R_2 \leftarrow -2R_2)$$

so that $x = 0$ *and* $y = -z$. *We find the 1-dimensional eigenspace*

$$\left\{ \begin{bmatrix} 0 \\ -z \\ z \end{bmatrix} : z \in \mathbb{R} \right\} = \left\{ z \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} : z \in \mathbb{R} \right\} . \checkmark^4$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} . \checkmark^2$$

For the eigenspace corresponding to the eigenvalue 1 we solve

$$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark^2$$

for $x, y, z \in \mathbb{R}$. *Row reduction yields*

$$\begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_1 \leftrightarrow R_3)$$

$$\to \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_2 \leftarrow R_2 + R_1)$$

$$\to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (R_1 \leftarrow 2R_1)$$

Thus we find the 2-dimensional eigenspace

$$\left\{\begin{bmatrix} y-z \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R}\right\} = \left\{ y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : x, y \in \mathbb{R}\right\}. \checkmark^4$$

Thus a basis is given by

$$\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}. \checkmark^2$$

(3.4) Is A diagonalizable? Motivate your answer. (2)

Yes, because the geometric and algebraic multiplicities are equal for each eigenvalue $\checkmark^2$ *(i.e. 1 for* $\lambda = 0$ *and 2 for* $\lambda = 1$*).*

(3.5) Is the matrix $A - I_3$ diagonalizable? Motivate your answer. (Here I_3 is the 3×3 identity matrix). (2)

Yes. Since the eigenvalues of $A - I_3$ *are* $0 - 1 = -1$ *and* $1 - 1 = 0$ *with algebraic multiplicities 1 and 2 respectively. The eigenspaces of* A *are also eigenspaces of* $A - I_3$*, i.e. the eigenspace* E_0 *of* A *(with dimension 1) is the eigenspace of* $A - I_3$ *corresponding to the eigenvalue -1, and similarly* E_1 *(with dimension 2) to the eigenvalue 0. Thus the geometric and algebraic multiplicities are equal.* $\checkmark^2$

Question 4: 24 Marks

Let $T : M_{22} \to P_2$ be defined by

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = a + \frac{b-c}{2}x + dx^2$$

where $a, b, c \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

Let $k, a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$*. Using the definition of* M_{22} *and* T *we find*

- $$\begin{aligned} T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) &= T\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) \\ &= ka + \frac{kb-kc}{2}x + kdx^2 \\ &= k\left(a + \frac{b-c}{2}x + dx^2\right) \\ &= kT\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right). \checkmark^2 \end{aligned}$$

- $$\begin{aligned} T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) &= T\left(\begin{bmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{bmatrix}\right) \\ &= (a+\alpha) + \frac{b+\beta-(c+\gamma)}{2}x + (d+\delta)x^2 \\ &= (a+\alpha) + \frac{b-c+\beta-\gamma}{2}x + (d+\delta)x^2 \\ &= (a + \frac{b-c}{2}x + dx^2) + (\alpha + \frac{\beta-\gamma}{2}x + \delta x^2) \\ &= T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right). \checkmark^2 \end{aligned}$$

(4.2) Find the matrix representation $[T]_{B',B}$ of T relative to the basis (12)

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

in M_{22}, and the basis

$$B' = \{ 1+x,\ 1-x,\ x^2 \}$$

in P_2, ordered from left to right.

From

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = 1 + \frac{0-0}{2}x + 0x^2 \qquad = \frac{1}{2}\cdot(1+x) + \frac{1}{2}\cdot(1-x) + 0\cdot x^2 \checkmark^2 \qquad (\dagger)$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0 + \frac{0-0}{2}x + 1x^2 \qquad = 0\cdot(1+x) + 0\cdot(1-x) + 1\cdot x^2 \checkmark^2$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right) = 0 + \frac{1-1}{2}x + 0x^2 \qquad = 0\cdot(1+x) + 0\cdot(1-x) + 0\cdot x^2 \checkmark^2$$

$$T\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = 0 + \frac{1+1}{2}x + 0x^2 \qquad = \frac{1}{2}\cdot(1+x) - \frac{1}{2}\cdot(1-x) + 0\cdot x^2 \checkmark^2 \qquad (\ddagger)$$

the coefficients of the basis elements in each equation provide the columns of the matrix representation:

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & 0 \end{bmatrix}. \checkmark^4$$

The coefficients in $(\dagger)$ *are found as follows. Let* $u, v, w \in \mathbb{R}$ *such that*

$$u(1+x) + v(1-x) + wx^2 = 1$$

i.e. $u+v=1$, $u-v=0$ *and* $w=0$. *These linear equations are easily solved to yield* $u=1/2$, $v=1/2$ *and* $w=0$.

The coefficients in $(\ddagger)$ *are found as follows. Let* $u, v, w \in \mathbb{R}$ *such that*

$$u(1+x) + v(1-x) + wx^2 = x$$

i.e. $u+v=0$, $u-v=1$ *and* $w=0$. *These linear equations are easily solved to yield* $u=1/2$, $v=-1/2$ *and* $w=0$.

(4.3) Determine the range $R(T)$ of T. Is T onto? In other words, is it true that $R(T) = P_2$? (4)

$$\begin{aligned} R(T) &= \left\{ T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \,:\, a,b,c,d \in \mathbb{R} \right\} \checkmark^2 \\ &= \left\{ a + \frac{b-c}{2}x + dx^2 \,:\, a,b,c,d \in \mathbb{R} \right\} \\ &= \{ a + b'x + dx^2 \,:\, a,b',d \in \mathbb{R} \} \qquad \textit{(Setting } b' = (b-c)/2\textit{)} \\ &= P_2. \end{aligned}$$

Thus T is onto. $\checkmark^2$

(4.4) Determine $\ker(T)$ and the nullity of T. (4)

$$\begin{aligned} \ker(T) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,:\, a,b,c,d \in \mathbb{R},\ T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 0 + 0x + 0x^2 \right\} \checkmark^2 \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,:\, a,b,c,d \in \mathbb{R},\ a + \frac{b-c}{2}x + dx^2 = 0 + 0x + 0x^2 \right\} \\ &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,:\, a,b,c,d \in \mathbb{R},\ a = 0, b = c, d = 0 \right\} \\ &= \left\{ \begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} \,:\, b \in \mathbb{R}, \right\} \\ &= \left\{ b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \,:\, b \in \mathbb{R}, \right\}. \end{aligned}$$

Thus we have a one-dimensional space and the nullity of T is 1. $\checkmark^2$

F.6 2013 Semester 1: Exam

Question paper

Question 1: 17 Marks

Consider the vector space M_{22}.

(1.1) Find (2)

$$S := \text{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

(1.2) Is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in S$? Explain. (3)

(1.3) Let (8)

$$X := \left\{ A \in M_{22} \;:\; A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \subset M_{22}.$$

Show that X is a vector subspace of M_{22}.

(1.4) What is the dimension of the vector subspace X? (4)

Question 2: 32 Marks

Consider the vector space $\mathbb{R}^4$.

(2.1) Show that (12)

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1y_1 + x_2y_2 + x_3y_3 + 3x_4y_4, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^4$$

is an inner product on $\mathbb{R}^4$.

(2.2) Are the vectors (6)

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

linearly independent?

(2.3) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^4$: (14)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** for the span of this subset.

Question 3: 31 Marks

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(3.1) Show that the characteristic equation for the eigenvalues λ of A is given by (6)

$$\lambda(\lambda - 3)^2 = 0.$$

(3.2) Find bases for the eigenspaces of A. (18)

(3.3) Is A diagonalizable? Motivate your answer. (2)

(3.4) Is the matrix (5)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

diagonalizable? Motivate your answer.

Question 4: 20 Marks

Let $T : P_2 \to P_2$ be defined by

$$T(a + bx + cx^2) = c + ax + bx^2$$

where $a, b, c \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation of T relative to the basis (11)

$$\{\, 1 + x + x^2,\ 1 - x,\ 1 + x - 2x^2 \,\}$$

ordered from left to right.

(4.3) The transform T has the eigenvalue 1. Find the corresponding eigenspace. (5)

Solution

Please note: any ***fundamental error*** *is grounds for* ***no marks*** *being awarded for an answer.*

Question 1: 17 Marks

Consider the vector space M_{22}.

(1.1) Find (2)

$$S := \operatorname{span}\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right\}.$$

We have

$$S = \left\{ a\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + b\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} : a, b \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} a+b & a \\ a & a-b \end{bmatrix} : a, b \in \mathbb{R} \right\}. \checkmark^{2}$$

(1.2) Is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in S$? Explain. (3)

Solving

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a+b & a \\ a & a-b \end{bmatrix}$$

yields $1 = a + b$, $a = 0$ and $1 = a - b$. Clearly $b = 1$ and $b = -1$ is impossible to satisfy. Hence $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin S$. $\checkmark^{3}$

(1.3) Let (8)

$$X := \left\{ A \in M_{22} : A\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\} \subset M_{22}.$$

Show that X is a vector subspace of M_{22}.

We have

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in X.$$

Thus X is non-empty. $\checkmark^{2}$ *Let $A, B \in X$ (i.e. $A, B \in M_{22}$ and $A\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = B\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$).*

1. *$A + B \in M_{22}$ (since M_{22} is a vector space)*

$$(A+B)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = A\begin{bmatrix} 1 \\ 1 \end{bmatrix} + B\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad A + B \in X. \checkmark^{3}$$

2. *Let $k \in \mathbb{R}$. $kA \in M_{22}$ (since M_{22} is a vector space)*

$$(kA)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = k\left(A\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = k\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad kA \in X. \checkmark^{3}$$

Thus this set forms a vector subspace.

(1.4) What is the dimension of the vector subspace X? (4)

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in X,$$

$a, b, c, d \in \mathbb{R}$. *It follows that*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so that $a = -b$, $c = -d$ *and*

$$X := \left\{ \begin{bmatrix} a & -a \\ c & -c \end{bmatrix} \,:\, a, c \in \mathbb{R} \right\}.$$

There are 2 free parameters and the dimension is 2.✓4

Question 2: 32 Marks

Consider the vector space $\mathbb{R}^4$.

(2.1) Show that (12)

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1y_1 + x_2y_2 + x_3y_3 + 3x_4y_4, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \ \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \in \mathbb{R}^4$$

is an inner product on $\mathbb{R}^4$.

We have for $k \in \mathbb{R}$ and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \in \mathbb{R}^4$$

1. $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + x_3y_3 + 3x_4y_4 = y_1x_1 + y_2x_2 + y_3x_3 + 3y_4x_4 = \langle \mathbf{y}, \mathbf{x} \rangle$ ✓^{2}
2. $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = (x_1 + z_1)y_1 + (x_2 + z_2)y_2 + (x_3 + z_3)y_3 + 3(x_4 + z_4)y_4$
   $= x_1y_1 + z_1y_1 + x_2y_2 + z_2y_2 + x_3y_3 + z_3y_3 + 3x_4y_4 + 3z_4y_4$
   $= x_1y_1 + x_2y_2 + x_3y_3 + 3x_4y_4 + z_1y_1 + z_2y_2 + z_3y_3 + 3z_4y_4 = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle$
   ✓^{4}
3. $\langle k\mathbf{x}, \mathbf{y} \rangle = (kx_1)y_1 + (kx_2)y_2 + (kx_3)y_3 + 3(kx_4)y_4 = k(x_1y_1 + x_2y_2 + x_3y_3 + 3x_4y_4) = k\langle \mathbf{x}, \mathbf{y} \rangle$ ✓^{2}
4. $\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2 + x_3^2 + 3x_4^2 \geq 0$ ✓^{2} *so that* $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *if and only if* $x_1 = x_2 = x_3 = x_4 = 0$ *(since* $x_1^2, x_2^2, x_3^2, x_4^2 \geq 0$*)* ✓^{2}*, i.e.* $\mathbf{x} = \mathbf{0}$.

(2.2) Are the vectors (6)

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

linearly independent?

Consider the equation

$$a \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = 0 \checkmark^2$$

where $a, b, c \in \mathbb{R}$. Thus we have the equations

$$\begin{aligned} a - b - c &= 0 \\ c &= 0 \\ 0 &= 0 \\ a + b + c &= 0 \end{aligned}$$

Adding the fourth equation to the first provides $a = 0$. The second equation provides $c = 0$. Inserting the solutions for a and c into the first equation yields $b = 0$. This is the only solution. Thus these vectors are linearly independent.$\checkmark^4$

(2.3) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^4$: (14)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** for the span of this subset.

Let

$$\mathbf{u}_1 := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 := \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{u}_3 := \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Then the Gram-Schmidt process provides

$$\mathbf{v}_1 := \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \checkmark$$

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 1^2 + 0^2 + 0^2 + 3 \cdot 1^2 = 4 \checkmark$$

$$\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = -1 \cdot 1 + 0 \cdot 0 + 0 \cdot 0 + 3 \cdot 1 \cdot 1 = 2 \checkmark$$

$$\mathbf{v}_2 := \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 \checkmark = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \checkmark^2$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \frac{1}{4}((-3)^2 + 0^2 + 0 \cdot 0 + 3 \cdot 1^2) = 3 \checkmark$$

$$\langle \mathbf{u}_3, \mathbf{v}_1 \rangle = -1 \cdot 1 + 1 \cdot 0 + 0 \cdot 0 + 3 \cdot 1 \cdot 1 = 2 \checkmark$$

$$\langle \mathbf{u}_3, \mathbf{v}_2 \rangle = -1 \cdot \left(-\frac{3}{2} \right) + 1 \cdot 0 + 0 \cdot 0 + 3 \cdot 1 \cdot \frac{1}{2} = 3 \checkmark$$

$$\begin{aligned} \mathbf{v}_3 &:= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle} \mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle} \mathbf{v}_2 \checkmark \\ &= \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{4} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3} \cdot \frac{1}{2} \begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} . \checkmark^2 \end{aligned}$$

Thus we have the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}. \checkmark^2$$

Question 3: 31 Marks

Consider the matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(3.1) Show that the characteristic equation for the eigenvalues λ of A is given by (6)

$$\lambda(\lambda - 3)^2 = 0.$$

The characteristic equation is

$$\begin{aligned} \det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \right) \checkmark^2 &= \begin{vmatrix} \lambda - 2 & 1 & 1 \\ 1 & \lambda - 2 & 1 \\ 1 & 1 & \lambda - 2 \end{vmatrix} \\ &= (\lambda-2)^3 + 1 + 1 - (\lambda-2) - (\lambda-2) - (\lambda-2) \\ &= \lambda^3 - 6\lambda^2 + 9\lambda = \lambda(\lambda-3)^2 = 0 \checkmark^4. \end{aligned}$$

(3.2) Find bases for the eigenspaces of A. (18)

From the characteristic equation we obtain the eigenvalues 0, and 3$\checkmark^2$ *(twice). For the eigenspace corresponding to the eigenvalue 0 we solve*

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark^2$$

for $x, y, z \in \mathbb{R}$. *The resulting equations are* $-2x + y + z = 0$, $x - 2y + z = 0$ *and* $x + y - 2z = 0$. *Obviously* $x = 2y - z$ *and* $y = 2x - z = 4y - 3z$. *Thus* $x = y = z$. *We find the 1-dimensional eigenspace*

$$\left\{ \begin{bmatrix} x \\ x \\ x \end{bmatrix} : x \in \mathbb{R} \right\} = \left\{ x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : x \in \mathbb{R} \right\}. \checkmark^4$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}. \checkmark^2$$

For the eigenspace corresponding to the eigenvalue 3 we solve

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \checkmark^2$$

for $x, y, z \in \mathbb{R}$. *The resulting equation is* $x + y + z = 0$, *i.e.* $x = -y - z$. *Thus we find the 2-dimensional eigenspace*

$$\left\{ \begin{bmatrix} -y - z \\ y \\ z \end{bmatrix} : y, z \in \mathbb{R} \right\} = \left\{ y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} : x, y \in \mathbb{R} \right\}. \checkmark^4$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}. \checkmark^2$$

(3.3) Is A diagonalizable? Motivate your answer. (2)

Yes, because the geometric and algebraic multiplicities are equal for each eigenvalue$\checkmark^2$ *(i.e. 1 for* $\lambda = 0$ *and 2 for* $\lambda = 3$*).*

Alternative:
Yes, since A *is symmetric.*

(3.4) Is the matrix (5)

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

diagonalizable? Motivate your answer.

Since the matrix is upper triangular, the eigenvalues are the diagonal entries, i.e. 1 (twice)$\checkmark^2$. *Solving the eigenvalue equation*

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

for $x, y \in \mathbb{R}$ *yields* $x + y = x$ *(and* $y = y$*), i.e.* $y = 0$. *Thus the 1-dimensional eigenspace is given by*

$$\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\} \checkmark^2$$

and the algebraic multiplicity (i.e. 2) is not equal to the geometric multiplicity (i.e 1). Thus the matrix is not diagonalizable.$\checkmark$

Question 4: 20 Marks

Let $T : P_2 \to P_2$ be defined by

$$T(a + bx + cx^2) = c + ax + bx^2$$

where $a, b, c \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

Let $k, a, b, c, \alpha, \beta, \gamma \in \mathbb{R}$. Using the definition of P_2 and T we find

- $$\begin{aligned} T\left(k(a + bx + cx^2)\right) &= T\left((ka) + (kb)x + (kc)x^2\right) \\ &= (kc) + (ka)x + (kb)x^2 \\ &= k(c + ax + bx^2) = kT\left(a + bx + cx^2\right). \checkmark^2 \end{aligned}$$

- $$\begin{aligned} T\left((a + bx + cx^2) + (\alpha + \beta x + \gamma x^2)\right) &= T\left((a + \alpha) + (b + \beta)x + (c + \gamma)x^2\right) \\ &= (c + \gamma) + (a + \alpha)x + (b + \beta)x^2 \\ &= (c + ax + bx^2) + (\gamma + \alpha x + \beta x^2) \\ &= T\left(a + bx + cx^2\right) + T\left(\alpha + \beta x + \gamma x^2\right). \checkmark^2 \end{aligned}$$

(4.2) Find the matrix representation of T relative to the basis (11)

$$\{\, 1 + x + x^2,\ 1 - x,\ 1 + x - 2x^2 \,\}$$

ordered from left to right.

From

$$\begin{aligned} T(1 + x + x^2) &= 1 + x + x^2 \\ &= 1 \cdot (1 + x + x^2) + 0 \cdot (1 - x) + 0 \cdot (1 + x - 2x^2) \checkmark^3 \\ T(1 - x) &= T(1 - x + 0 \cdot x^2) = x - x^2 \\ &= 0 \cdot (1 + x + x^2) + \left(-\frac{1}{2}\right) \cdot (1 - x) + \frac{1}{2} \cdot (1 + x - 2x^2) \checkmark^3 \qquad (\dagger) \\ T(1 + x - 2x^2) &= -2 + x + x^2 \\ &= 0 \cdot (1 + x + x^2) + \left(-\frac{3}{2}\right) \cdot (1 - x) + \left(-\frac{1}{2}\right) \cdot (1 + x - 2x^2) \checkmark^3 \qquad (\ddagger) \end{aligned}$$

the coefficients of the basis elements in each equation provide the columns of the matrix representation:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & -\frac{3}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}. \checkmark^2$$

The coefficients in (†) *are found as follows. Let* $u, v, w \in \mathbb{R}$ *such that*

$$u(1 + x + x^2) + v(1 - x) + w(1 + x - 2x^2) = x - x^2$$

i.e. $u + v + w = 0$, $u - v + w = 1$ *and* $u - 2w = -1$. *These linear equations are easily solved to yield* $u = 0$, $v = -1/2$ *and* $w = 1/2$.

The coefficients in (‡) *are found as follows. Let* $u, v, w \in \mathbb{R}$ *such that*

$$u(1 + x + x^2) + v(1 - x) + w(1 + x - 2x^2) = -2 + x + x^2$$

i.e. $u + v + w = -2$, $u - v + w = 1$ *and* $u - 2w = 1$. *These linear equations are easily solved to yield* $u = 0$, $v = -3/2$ *and* $w = -1/2$.

(4.3) The transform T has the eigenvalue 1. Find the corresponding eigenspace. (5)

The corresponding eigenvalue equation is

$$T(a + bx + cx^2) = c + ax + bx^2 = a + bx + cx^2 \checkmark^2$$

where $a, b, c \in \mathbb{R}$ *from which follows* $a = b = c$✓. *The eigenspace is*

$$\left\{ a + ax + ax^2 \,:\, a \in \mathbb{R} \right\}. \checkmark^2$$

F.7 2012 Semester 1: Exam

Question paper

Question 1: 16 Marks

Let $n \in \mathbb{N}$. Consider the set of $n \times n$ symmetric matrices over $\mathbb{R}$ with the usual addition and multiplication by a scalar.

(1.1) Show that this set with the given operations is a vector subspace of M_{nn}. (6)

(1.2) What is the dimension of this vector subspace? (4)

(1.3) Find a basis for the vector space of 2×2 symmetric matrices. (6)

Question 2: 36 Marks

Consider the vector space $\mathbb{R}^3$.

(2.1) Show that (12)

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1 y_1 + 2x_2 y_2 + x_3 y_3, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

is an inner product on $\mathbb{R}^3$.

(2.2) Are the vectors (10)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

linearly independent?

(2.3) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^3$: (14)

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product defined in 2.1 for the span of this subset.

Question 3: 28 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

(3.1) Show that the characteristic equation for the eigenvalues λ of A is given by (6)

$$(\lambda^2 - 1)(\lambda - 3) = 0.$$

(3.2) Find an orthogonal matrix P which diagonalizes A. (16)

(3.3) Find A^n (for $n \in \mathbb{N}$) as a matrix. (6)

Question 4: 20 Marks

Consider the vector space P_3.

(4.1) Is span$\{1+x,\ x+x^2,\ x^2+x^3,\ x^3+1\} = P_3$? Motivate your answer. (4)

(4.2) Let $D : P_3 \to P_3$ be the differentiation operator

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2.$$

(a) Find the matrix representation of D relative to the basis $\{1, x, x^2, x^3\}$ using the coefficient ordering (8)

$$a_0 + a_1x + a_2x^2 + a_3x^3 \to \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

(b) Find the kernel and range of D. (8)

Solution

Question 1: 16 Marks

Let $n \in \mathbb{N}$. Consider the set of $n \times n$ symmetric matrices over $\mathbb{R}$ with the usual addition and multiplication by a scalar.

(1.1) Show that this set with the given operations is a vector subspace of M_{nn}. (6)

Obviously the set of $n \times n$ symmetric matrices is a subset of M_{nn}. $\checkmark^2$
Let A and B be $n \times n$ symmetric matrices over $\mathbb{R}$, i.e. $A = A^T$ and $B = B^T$. Obviously the $n \times n$ zero matrix is symmetric, i.e. the set is not empty. The properties of the transpose provide

1. *$(A+B)^T = A^T + B^T = A + B$ i.e. $A + B$ is symmetric.* $\checkmark^2$
2. *Let $k \in \mathbb{R}$. Then $(kA)^T = k(A^T) = kA$ which is also symmetric.* $\checkmark^2$

Thus this set forms a vector subspace.

(1.2) What is the dimension of this vector subspace? (4)

Let

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \dots & a_{1,n} \\ a_{1,2} & a_{2,2} & a_{2,3} & \dots & a_{2,n} \\ a_{1,3} & a_{2,3} & a_{3,3} & \dots & a_{3,n} \\ \vdots & \vdots & \ddots & \vdots & \\ a_{1,n} & a_{2,n} & a_{3,n} & \dots & a_{n,n} \end{bmatrix}$$

be an element of this vector space, with the free parameters $a_{1,1}, \dots, a_{1,n} \in \mathbb{R}$, $a_{2,2}, \dots, a_{2,n} \in \mathbb{R}$, $a_{3,3}, \dots a_{3,n} \in \mathbb{R}, \dots \;\dots, a_{n,n} \in \mathbb{R}$. We sum the number of free parameters in each row to obtain

$$n + (n-1) + (n-2) + \dots + 1 = \sum_{j=1}^{n} j = \frac{n(n+1)}{2} \quad \checkmark^4$$

for the dimension.

(1.3) Find a basis for the vector space of 2×2 symmetric matrices. (6)

A typical 2×2 symmetric matrix over $\mathbb{R}$ has the form

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}$$

where $a, b, c \in \mathbb{R}$. Thus we have the vector space

$$\left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} = \left\{ a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} : a, b, c \in \mathbb{R} \right\}.$$

The dimension of the vector space is 3. An obvious choice for a basis is

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \checkmark^2, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \checkmark^2, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \checkmark^2 \right\}.$$

Other choices include

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} \quad \text{and} \quad \left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right\}.$$

Question 2: 36 Marks

Consider the vector space $\mathbb{R}^3$.

(2.1) Show that (12)

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1y_1 + 2x_2y_2 + x_3y_3, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

is an inner product on $\mathbb{R}^3$.
We have for $k \in \mathbb{R}$ and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{R}^3$$

1. $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + 2x_2y_2 + x_3y_3 = y_1x_1 + 2y_2x_2 + y_3x_3 = \langle \mathbf{y}, \mathbf{x} \rangle$ ✓^{2}
2. $$\begin{aligned} \langle \mathbf{x}+\mathbf{z}, \mathbf{y} \rangle &= (x_1+z_1)y_1 + 2(x_2+z_2)y_2 + (x_3+z_3)y_3 \\ &= x_1y_1 + z_1y_1 + 2x_2y_2 + 2z_2y_2 + x_3y_3 + z_3y_3 \\ &= x_1y_1 + 2x_2y_2 + x_3y_3 + z_1y_1 + 2z_2y_2 + z_3y_3 = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle \end{aligned}$$ ✓^{4}
3. $\langle k\mathbf{x}, \mathbf{y} \rangle = (kx_1)y_1 + 2(kx_2)y_2 + (kx_3)y_3 = k(x_1y_1 + 2x_2y_2 + x_3y_3) = k\langle \mathbf{x}, \mathbf{y} \rangle$✓^{2}
4. $\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + 2x_2^2 + x_3^2 \geq 0$✓^{2} *so that* $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *if and only if* $x_1 = x_2 = x_3 = 0$ *(since* $x_1^2, x_2^2, x_3^2 \geq 0$*)*✓^{2}*, i.e.* $\mathbf{x} = \mathbf{0}$.

(2.2) Are the vectors (10)

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

linearly independent?
Consider the equation

$$a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + c\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = 0$$ ✓^{2}

where $a, b, c \in \mathbb{R}$. Thus we have the equation

$$\begin{aligned} a + b + c &= 0 \\ a + b - c &= 0 \\ a - b - c &= 0 \end{aligned}$$

Adding the first and third equation provides $a = 0$✓^{2}*. Adding the first and second equation provides $b = -a = 0$*✓^{2}*. Adding the second and third equation provides*

$c = a = 0\checkmark^2$. *This is the only solution. Thus these vectors are linearly independent.*$\checkmark^2$

Alternative:
The coefficient matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

which has a determinant of -2, i.e. the determinant is non-zero from which follows that these vectors are linearly independent.

(2.3) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^3$: (14)

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product defined in 2.1 for the span of this subset.
Let

$$\mathbf{u}_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{u}_2 := \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{u}_3 := \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

Then the Gram-Schmidt process provides

$$\begin{aligned}
\mathbf{v}_1 &:= \mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \\
\langle \mathbf{v}_1, \mathbf{v}_1 \rangle &= 1^2 + 2 \cdot 1^2 + 1^2 = 4\checkmark^2 \\
\mathbf{v}_2 &:= \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \checkmark^4 \\
\langle \mathbf{v}_2, \mathbf{v}_2 \rangle &= \left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 + \left(-\frac{3}{2}\right)^2 = 3\checkmark^2 \\
\mathbf{v}_3 &:= \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 \\
&= \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \frac{-2}{4}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \checkmark^2 - \frac{1}{3} \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \checkmark^2 \\
&= \frac{1}{3}\begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} \checkmark^2.
\end{aligned}$$

Thus we have the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}, \frac{1}{3}\begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} \right\}.$$

Question 3: 28 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

(3.1) Show that the characteristic equation for the eigenvalues λ of A is given by (6)

$$(\lambda^2 - 1)(\lambda - 3) = 0.$$

The characteristic equation is

$$\det\left(\lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\right) \checkmark^2 = \begin{vmatrix} \lambda - 1 & 0 & -2 \\ 0 & \lambda - 1 & 0 \\ -2 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^3 - 4(\lambda - 1)\checkmark^2$$

$$= (\lambda - 1)((\lambda - 1)^2 - 4) = (\lambda - 1)(\lambda - 3)(\lambda + 1) = 0\checkmark^2.$$

Cofactor expansion along any row or column could also be used.

(3.2) Find an orthogonal matrix P which diagonalizes A. (16)

From the characteristic equation we obtain the eigenvalues -1$\checkmark^2$*, 1*$\checkmark^2$ *and 3*$\checkmark^2$*. For the eigenspace corresponding to the eigenvalue 1 we solve*

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for $x, y, z \in \mathbb{R}$*. The resulting equations are* $x + 2z = x$, $2x + z = z$ *and* $y = y$ *which is satisfied identically. Thus we find the 1-dimensional eigenspace*

$$\left\{ \begin{bmatrix} 0 \\ y \\ 0 \end{bmatrix} \checkmark^2 : y \in \mathbb{R} \right\}.$$

For the eigenspace corresponding to the eigenvalue -1 we solve

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = - \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for $x, y, z \in \mathbb{R}$*. The resulting equations are* $x + 2z = -x$, $2x + z = -z$ *and* $y = -y$ *i.e.* $z = -x$ *and* $y = 0$*. Thus we find the 1-dimensional eigenspace*

$$\left\{ \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} \checkmark^2 : x \in \mathbb{R} \right\}.$$

For the eigenspace corresponding to the eigenvalue 3 we solve

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for $x, y, z \in \mathbb{R}$. *The resulting equations are* $x + 2z = 3x$, $2x + z = 3z$ *and* $y = 3y$ *i.e.* $z = x$ *and* $y = 0$. *Thus we find the 1-dimensional eigenspace*

$$\left\{ \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} \checkmark^{2} : x \in \mathbb{R} \right\}.$$

Choosing orthonormal representative eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

from each eigenspace we find that

$$P = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \checkmark^{4}$$

diagonalizes A, *i.e.*

$$P^T A P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Of course, other P *could also be used (for example by rearranging columns or by multiplying a column by -1).*

(3.3) Find A^n (for $n \in \mathbb{N}$) as a matrix. (6)

From

$$(P^T A P)^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} \checkmark^{2}$$

we find

$$A^n = P \begin{bmatrix} 1 & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & 3^n \end{bmatrix} P^T = \begin{bmatrix} \frac{3^n + (-1)^n}{2} & 0 & \frac{3^n - (-1)^n}{2} \\ 0 & 1 & 0 \\ \frac{3^n - (-1)^n}{2} & 0 & \frac{3^n + (-1)^n}{2} \end{bmatrix}. \checkmark^{4}$$

Question 4: 20 Marks

Consider the vector space P_3.

(4.1) Is $\text{span}\{\,1+x,\ x+x^2,\ x^2+x^3,\ x^3+1\,\} = P_3$? Motivate your answer. (4)

No, since

$$a_0 + a_1x + a_2x^2 + a_3x^3 = b_0(1+x) + b_1(x+x^2) + b_2(x^2+x^3) + b_3(x^3+1)$$

gives $b_0+b_1=a_1$, $b_1+b_2=a_2$, $b_2+b_3=a_3$ and $b_3+b_0=a_0$ *so that*

$$b_0 = a_1 - b_1 = a_1 - a_2 + b_2 = a_1 - a_2 + a_3 - b_3 = a_1 - a_2 + a_3 - a_0 + b_0 \quad \Rightarrow \quad a_1 + a_3 = a_0 + a_2.$$

For example $x \notin \text{span}\{\,1+x,\ x+x^2,\ x^2+x^3\, x^3+1\,\}$. ✓4

Alternative:
The augmented coefficient matrix for the above equation is

$$\begin{bmatrix} 1 & 0 & 0 & 1 & : & a_0 \\ 1 & 1 & 0 & 0 & : & a_1 \\ 0 & 1 & 1 & 0 & : & a_2 \\ 0 & 0 & 1 & 1 & : & a_3 \end{bmatrix} \begin{matrix} \sim \\ R_2 - R_1 \\ \sim \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & : & a_0 \\ 0 & 1 & 0 & -1 & : & a_1 - a_0 \\ 0 & 1 & 1 & 0 & : & a_2 \\ 0 & 0 & 1 & 1 & : & a_3 \end{bmatrix}$$

$$\begin{matrix} \sim \\ \sim \\ R_3 - R_2 \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & : & a_0 \\ 0 & 1 & 0 & -1 & : & a_1 - a_0 \\ 0 & 0 & 1 & 1 & : & a_2 - a_1 + a_0 \\ 0 & 0 & 1 & 1 & : & a_3 \end{bmatrix}$$

$$\begin{matrix} \sim \\ \sim \\ \sim \\ R_4 - R_3 \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & : & a_0 \\ 0 & 1 & 0 & -1 & : & a_1 - a_0 \\ 0 & 0 & 1 & 1 & : & a_2 - a_1 + a_0 \\ 0 & 0 & 0 & 0 & : & a_3 - a_2 + a_1 - a_0 \end{bmatrix}$$

Clearly the last equation cannot be satisfied if $a_3 - a_2 + a_1 - a_0 \neq 0$.

Alternative:
Noting that the determinant of the coefficient matrix is zero, the dimension of $\text{span}\{\,1+x,\ x+x^2,\ x^2+x^3,\ x^3+1\,\}$ *is less than 4, while the dimension of* P_3 *is 4 : i.e. the two vector spaces cannot be the same.*

(4.2) Let $D : P_3 \to P_3$ be the differentiation operator

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2.$$

(a) Find the matrix representation of D relative to the basis $\{1, x, x^2, x^3\}$ using the coefficient ordering (8)

$$a_0 + a_1x + a_2x^2 + a_3x^3 \to \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

From

$$\begin{aligned}
D(1) &= D(1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3) = 0 = 0\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 \\
D(x) &= D(0\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3) = 1 = 1\cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 \\
D(x^2) &= D(0\cdot 1 + 0\cdot x + 1\cdot x^2 + 0\cdot x^3) = 2x = 0\cdot 1 + 2\cdot x + 0\cdot x^2 + 0\cdot x^3 \\
D(x^3) &= D(0\cdot 1 + 0\cdot x + 0\cdot x^2 + 1\cdot x^3) = 3x^2 = 0\cdot 1 + 0\cdot x + 3\cdot x^2 + 0\cdot x^3
\end{aligned}$$

the coefficients of the basis elements in each equation provide the columns of the matrix representation:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \checkmark^2 \\ \checkmark^2 \\ \checkmark^2 \\ \checkmark^2 \end{matrix}.$$

(b) Find the kernel and range of D. (8)

Solving

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2 = 0\checkmark^2$$

we find $a_1 = a_2 = a_3 = 0$ *so that*

$$\ker(D) = \{a_0 \cdot 1 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3 \ : \ a_0 \in \mathbb{R}\} \equiv \{a_0 \ : \ a_0 \in \mathbb{R}\} \equiv \mathbb{R}.\checkmark^2$$

We have

$$D\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = a_1 + 2a_2x + 3a_3x^2$$

where $a_0, a_1, a_2, a_3 \in \mathbb{R}$. *Thus*

$$\begin{aligned}
R(D) &= \{a_1 + 2a_2x + 3a_3x^2 + 0\cdot x^3 \ : \ a_1, a_2, a_3 \in \mathbb{R}\} \\
&\equiv \{a'_1 + a'_2x + a'_3x^2 \ : \ a'_1, a'_2, a'_3 \in \mathbb{R}\} \equiv P_2\checkmark^4
\end{aligned}$$

where $a'_1 = a_1$, $a'_2 = 2a_2$ *and* $a'_3 = 3a_3$.

F.8 2012 Semester 2: Exam

Question paper

Question 1: 18 Marks

Consider the vector space P_3.

(1.1) Is $\text{span}\{\,1,\ 1+x,\ x+x^2,\ x^2+x^3,\ x^3+1\,\} = P_3$? Motivate your answer. (7)

(1.2) Let $a \in \mathbb{R}$ and (8)

$$Z_a := \{\, p(x) \in P_3 \,:\, p(a) = 0 \,\} \subset P_3.$$

Show that Z_a is a vector subspace of P_3.

(1.3) What is the dimension of the vector subspace Z_a? (3)

Question 2: 36 Marks

Consider the vector space $\mathbb{R}^3$.

(2.1) Show that (12)

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1y_1 + x_2y_2 + 3x_3y_3, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

is an inner product on $\mathbb{R}^3$.

(2.2) Are the vectors (10)

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

linearly independent?

(2.3) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^3$: (14)

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** for the span of this subset.

Question 3: **23 Marks**

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(3.1) Show that the characteristic equation for the eigenvalues λ of A is given by (6)

$$\lambda(\lambda - 2)^2 = 0.$$

(3.2) Find bases for the eigenspaces of A. (10)

(3.3) Is A diagonalizable? Motivate your answer. (2)

(3.4) Is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ diagonalizable? Motivate your answer. (5)

Question 4: **23 Marks**

Let $T : M_{22} \to M_{22}$ be the transpose operation

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

where $a, b, c, d \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

(4.2) Find the matrix representation of T relative to the standard basis (8)

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

using the coefficient ordering

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

(4.3) Find the kernel of T. (4)

(4.4) Is T invertible? (2)
Explain using **only your answer for 4.3**.

(4.5) The transform T has the eigenvalue 1. Find the corresponding eigenspace. (5)

Solution

Question 1: 18 Marks

Consider the vector space P_3.

(1.1) Is $\operatorname{span}\{1,\,1+x,\,x+x^2,\,x^2+x^3,\,x^3+1\} = P_3$? Motivate your answer. (7)

Yes, since

$$a_0 + a_1x + a_2x^2 + a_3x^3 = b_0 + b_1(1+x) + b_2(x+x^2) + b_3(x^2+x^3) + b_4(x^3+1)\checkmark^2$$

gives $b_0 + b_1 + b_4 = a_0$, $b_1 + b_2 = a_1$, $b_2 + b_3 = a_2$, *and* $b_3 + b_4 = a_3$ $\checkmark^2$ *so that*

$$b_3 = a_3 - b_4, \quad b_2 = a_2 - b_3 = a_2 - a_3 + b_4, \quad b_1 = a_1 - b_2 = a_1 - a_2 + a_3 - b_4,$$

$$b_0 = a_0 - b_1 - b_4 = a_0 - a_1 + a_2 - a_3.\checkmark^2$$

where b_4 *is a free parameter. Thus we found a solution.*$\checkmark$

Alternative:
The augmented coefficient matrix for the above equation is

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & : & a_0 \\ 0 & 1 & 1 & 0 & 0 & : & a_1 \\ 0 & 0 & 1 & 1 & 0 & : & a_2 \\ 0 & 0 & 0 & 1 & 1 & : & a_3 \end{bmatrix} \begin{matrix} \sim \\ \sim \\ R_3 - R_4 \\ \sim \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & : & a_0 \\ 0 & 1 & 1 & 0 & 0 & : & a_1 \\ 0 & 0 & 1 & 0 & -1 & : & a_2 - a_3 \\ 0 & 0 & 0 & 1 & 1 & : & a_3 \end{bmatrix}$$

$$\begin{matrix} \sim \\ R_2 - R_3 \\ \sim \\ \sim \end{matrix} \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & : & a_0 \\ 0 & 1 & 0 & 0 & 1 & : & a_1 - a_2 + a_3 \\ 0 & 0 & 1 & 0 & -1 & : & a_2 - a_3 \\ 0 & 0 & 0 & 1 & 1 & : & a_3 \end{bmatrix}$$

$$\begin{matrix} R_1 - R_2 \\ \sim \\ \sim \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & : & a_0 - a_1 + a_2 - a_3 \\ 0 & 1 & 0 & 0 & 1 & : & a_1 - a_2 + a_3 \\ 0 & 0 & 1 & 0 & -1 & : & a_2 - a_3 \\ 0 & 0 & 0 & 1 & 1 & : & a_3 \end{bmatrix}$$

which yields the same solution as above.

(1.2) Let $a \in \mathbb{R}$ and (8)

$$Z_a := \{\, p(x) \in P_3 \;:\; p(a) = 0 \,\} \subset P_3.$$

Show that Z_a is a vector subspace of P_3.
We have $p_1(x) := -a + x \in Z_a$ *since* $p_1(x) = -a + x = -a\cdot 1 + 1\cdot x + 0\cdot x^2 + 0\cdot x^3 \in P_3$ *and* $p_1(a) = -a + a = 0$. *Thus* Z_a *is non-empty (we could also have used* $p_1(x) := 0 + 0\cdot x + 0\cdot x^2 + 0\cdot x^3$*).* $\checkmark^2$ *Let* $p(x), q(x) \in Z_a$ *(i.e.* $p(x), q(x) \in P_3$ *and* $p(a) = q(a) = 0$*).*

1. $p(x) + q(x) \in P_3$ $\checkmark$ *(since* P_3 *is a vector space)*
   $(p+q)(a) = p(a) + q(a) = 0 + 0 = 0\checkmark \Rightarrow p(x) + q(x) \in Z_a$ $\checkmark$

2. *Let* $k \in \mathbb{R}$. $kp(x) \in P_3$ $\checkmark$ *(since* P_3 *is a vector space)*
   $(kp)(a) = k\,p(a) = k \cdot 0 = 0\checkmark \Rightarrow kp(x) \in Z_a$ $\checkmark$

Thus this set forms a vector subspace.

(1.3) What is the dimension of the vector subspace Z_a? (3)

Let $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3 \in Z_a$, $p_0, p_1, p_2, p_3 \in \mathbb{R}$. It follows that

$$p(a) = p_0 + p_1a + p_2a^2 + p_3a^3 = 0 \quad \Rightarrow \quad p_0 = -p_1a - p_2a^2 - p_3a^3 \checkmark^2$$

so that

$$Z_a := \{\, (-p_1a - p_2a^2 - p_3a^3) + p_1x + p_2x^2 + p_3x^3 \;:\; p_1, p_2, p_3 \in \mathbb{R} \,\}.$$

There are 3 free parameters and the dimension is 3.✓

Question 2: 36 Marks

Consider the vector space $\mathbb{R}^3$.

(2.1) Show that (12)

$$\langle \mathbf{x}, \mathbf{y} \rangle := x_1y_1 + x_2y_2 + 3x_3y_3, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathbb{R}^3$$

is an inner product on $\mathbb{R}^3$.

We have for $k \in \mathbb{R}$ and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \in \mathbb{R}^3$$

1. $\langle \mathbf{x}, \mathbf{y} \rangle = x_1y_1 + x_2y_2 + 3x_3y_3 = y_1x_1 + y_2x_2 + 3y_3x_3 = \langle \mathbf{y}, \mathbf{x} \rangle \checkmark^2$
2. $\langle \mathbf{x} + \mathbf{z}, \mathbf{y} \rangle = (x_1 + z_1)y_1 + (x_2 + z_2)y_2 + 3(x_3 + z_3)y_3$
   $= x_1y_1 + z_1y_1 + x_2y_2 + z_2y_2 + 3x_3y_3 + 3z_3y_3$
   $= x_1y_1 + x_2y_2 + 3x_3y_3 + z_1y_1 + z_2y_2 + 3z_3y_3 = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{z}, \mathbf{y} \rangle \checkmark^4$
3. $\langle k\mathbf{x}, \mathbf{y} \rangle = (kx_1)y_1 + (kx_2)y_2 + 3(kx_3)y_3 = k(x_1y_1 + x_2y_2 + 3x_3y_3) = k\langle \mathbf{x}, \mathbf{y} \rangle \checkmark^2$
4. $\langle \mathbf{x}, \mathbf{x} \rangle = x_1^2 + x_2^2 + 3x_3^2 \geq 0 \checkmark^2$ *so that* $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$ *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *if and only if* $x_1 = x_2 = x_3 = 0$ *(since* $x_1^2, x_2^2, x_3^2 \geq 0$*)*$\checkmark^2$*, i.e.* $\mathbf{x} = \mathbf{0}$.

(2.2) Are the vectors (10)

$$\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix}$$

linearly independent?
Consider the equation

$$a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}-1\\0\\1\end{bmatrix} + c\begin{bmatrix}-1\\1\\1\end{bmatrix} = 0\checkmark^{2}$$

where $a, b, c \in \mathbb{R}$. Thus we have the equation

$$\begin{aligned} a - b - c &= 0 \\ c &= 0 \\ a + b + c &= 0 \end{aligned}$$

Adding the third equation to the first provides $a = 0\checkmark^{2}$. The second equation provides $c = 0.\checkmark^{2}$Inserting the solutions for a and c into the first equation yields $b = 0.\checkmark^{2}$This is the only solution. Thus these vectors are linearly independent.$\checkmark^{2}$

Alternative:
The coefficient matrix is

$$\begin{bmatrix}1 & -1 & -1\\0 & 0 & 1\\1 & 1 & 1\end{bmatrix}$$

which has a determinant of -2, i.e. the determinant is non-zero from which follows that these vectors are linearly independent.

(2.3) Apply the Gram-Schmidt process to the following subset of $\mathbb{R}^3$: (14)

$$\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\1\end{bmatrix} \right\}$$

to find an orthogonal basis with respect to the inner product **defined in 2.1** for the span of this subset.
Let

$$\mathbf{u}_1 := \begin{bmatrix}1\\0\\1\end{bmatrix}, \quad \mathbf{u}_2 := \begin{bmatrix}-1\\0\\1\end{bmatrix}, \quad \mathbf{u}_3 := \begin{bmatrix}-1\\1\\1\end{bmatrix}.$$

Then the Gram-Schmidt process provides

$$\mathbf{v}_1 := \mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \checkmark$$

$$\langle \mathbf{v}_1, \mathbf{v}_1 \rangle = 1^2 + 0^2 + 3 \cdot 1^2 = 4\checkmark$$

$$\langle \mathbf{u}_2, \mathbf{v}_1 \rangle = -1 \cdot 1 + 0 \cdot 0 + 3 \cdot 1 \cdot 1 = 2\checkmark$$

$$\mathbf{v}_2 := \mathbf{u}_2 - \frac{\langle \mathbf{u}_2, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1\checkmark = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \checkmark^2$$

$$\langle \mathbf{v}_2, \mathbf{v}_2 \rangle = \frac{1}{4}((-3)^2 + 0^2 + 3 \cdot 1^2) = 3\checkmark$$

$$\langle \mathbf{u}_3, \mathbf{v}_1 \rangle = -1 \cdot 1 + 1 \cdot 0 + 3 \cdot 1 \cdot 1 = 2\checkmark$$

$$\langle \mathbf{u}_3, \mathbf{v}_2 \rangle = -1 \cdot \left(-\frac{3}{2}\right) + 1 \cdot 0 + 3 \cdot 1 \cdot \frac{1}{2} = 3\checkmark$$

$$\mathbf{v}_3 := \mathbf{u}_3 - \frac{\langle \mathbf{u}_3, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 - \frac{\langle \mathbf{u}_3, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2\checkmark$$

$$= \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{4}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3} \cdot \frac{1}{2}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.\checkmark^2$$

Thus we have the orthogonal basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.\checkmark^2$$

Question 3: 23 Marks

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

(3.1) Show that the characteristic equation for the eigenvalues λ of A is given by (6)

$$\lambda(\lambda - 2)^2 = 0.$$

The characteristic equation is

$$\det\left(\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}\right)\checkmark^2 = \begin{vmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda - 2 & 0 \\ -1 & 0 & \lambda - 1 \end{vmatrix} = (\lambda - 1)^2(\lambda - 2) - (\lambda - 2)\checkmark^2$$

$$= (\lambda - 2)((\lambda - 1)^2 - 1) = (\lambda - 2)^2\lambda = 0\checkmark^2.$$

(3.2) Find bases for the eigenspaces of A. (10)

From the characteristic equation we obtain the eigenvalues 0, and 2$\checkmark^2$ *(twice). For the eigenspace corresponding to the eigenvalue 0 we solve*

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $x, y, z \in \mathbb{R}$. *The resulting equations are* $x + z = 0$, $2y = 0$ *and* $x + z = 0.\checkmark^2$ *Obviously* $y = 0$ *and* $z = -x$. *Thus we find the 1-dimensional eigenspace*

$$\left\{ \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} : x \in \mathbb{R} \right\} = \left\{ x \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : x \in \mathbb{R} \right\}.$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}.\checkmark^2$$

For the eigenspace corresponding to the eigenvalue 2 we solve

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

for $x, y, z \in \mathbb{R}$. *The resulting equations are* $x + z = 2x$, $2y = 2y$ *and* $x + z = 2z\checkmark^2$, *i.e.* $x = z$ *and* $y = y$. *Thus we find the 2-dimensional eigenspace*

$$\left\{ \begin{bmatrix} x \\ y \\ x \end{bmatrix} : x, y \in \mathbb{R} \right\} = \left\{ x \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} : x, y \in \mathbb{R} \right\}.$$

Thus a basis is given by

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.\checkmark^2$$

(3.3) Is A diagonalizable? Motivate your answer. (2)

Yes, because the geometric and algebraic multiplicities are equal for each eigenvalue$\checkmark^2$ *(i.e. 1 for* $\lambda = 0$ *and 2 for* $\lambda = 2$*).*

Alternative:
Yes, since A *is symmetric.*

(3.4) Is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ diagonalizable? Motivate your answer. (5)

Since the matrix is upper triangular, the eigenvalues are the diagonal entries, i.e. 0 (twice)$\checkmark^2$*. Solving the eigenvalue equation*

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for $x, y \in \mathbb{R}$ *yields* $y = 0$*. Thus the 1-dimensional eigenspace is given by*

$$\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} : x \in \mathbb{R} \right\} \checkmark^2$$

and the algebraic multiplicity (i.e. 2) is not equal to the geometric multiplicity (i.e 1). Thus the matrix is not diagonalizable.$\checkmark$

Question 4: 23 Marks

Let $T : M_{22} \to M_{22}$ be the transpose operation

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

where $a, b, c, d \in \mathbb{R}$.

(4.1) Show that T is a linear transformation. (4)

Let $k, a, b, c, d, \alpha, \beta, \gamma, \delta \in \mathbb{R}$*. Using the definition of* M_{22} *and* T *we find*

- $$T\left(k\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = T\left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix}\right) = \begin{bmatrix} ka & kc \\ kb & kd \end{bmatrix} = k\begin{bmatrix} a & c \\ b & d \end{bmatrix} = kT\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right). \checkmark^2$$

- $$\begin{aligned} T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right) &= T\left(\begin{bmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{bmatrix}\right) = \begin{bmatrix} a+\alpha & c+\gamma \\ b+\beta & d+\delta \end{bmatrix} \\ &= \begin{bmatrix} a & c \\ b & d \end{bmatrix} + \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) + T\left(\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}\right). \checkmark^2 \end{aligned}$$

(4.2) Find the matrix representation of T relative to the standard basis (8)

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

using the coefficient ordering

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}.$$

From

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$
$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + 1 \cdot \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

the coefficients of the basis elements in each equation provide the columns of the matrix representation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \checkmark^2 \\ \checkmark^2 \\ \checkmark^2 \\ \checkmark^2 \end{matrix}.$$

(4.3) Find the kernel of T. (4)

Solving

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark^2$$

we find $a = b = c = d = 0$ *so that*

$$\ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}. \checkmark^2$$

(4.4) Is T invertible? (2)

Explain using **only your answer for 4.3**.

Yes, because $T : M_{22} \rightarrow M_{22}$ *and*

$$\ker(T) = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\} \checkmark^2$$

consists only of the zero vector in M_{22}.

(4.5) The transform T has the eigenvalue 1. Find the corresponding eigenspace. (5)

The corresponding eigenvalue equation is

$$T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \checkmark^2$$

where $a, b, c, d \in \mathbb{R}$ *from which follows* $b = c\checkmark$. *The eigenspace is*

$$\left\{ \begin{bmatrix} a & b \\ b & d \end{bmatrix} \,:\, a, b, d \in \mathbb{R} \right\} \checkmark^2$$

(the 2×2 *symmetric matrices.)*

F.9 2010 Semester 2: Exam

Question paper

UNIVERSITY EXAMINATIONS

UNIVERSITEITSEKSAMENS

UNISA university of south africa

MAT2611

October/November 2010

LINEAR ALGEBRA

Duration 2 Hours

100 Marks

EXAMINERS .
FIRST PROF I NAIDOO
SECOND PROF JD BOTHA

This paper consists of 3 pages

ANSWER ALL THE QUESTIONS.

QUESTION 1

Suppose that A is a square matrix with characteristic polynomial $p(\lambda) = \lambda^3 - \lambda$

(a) What is the order of A? (1)

(b) Is A invertible? (1)

(c) Is A diagonalizable? (2)

(d) Find the eigenvalues of A^2 (1)

Justify your answers. **[5]**

QUESTION 2

Let

$$A = \begin{bmatrix} -3 & 1 & 0 \\ -6 & 2 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

(a) Find the characteristic polynomial of A and show that the eigenvalues of A are 0 and -1 (4)

(b) Find a basis for each eigenspace of A (6)

(c) Explain why A is diagonalizable (1)

(d) Find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ (3)

(e) Use (d) to calculate A^{99} (5)

[19]

[TURN OVER]

QUESTION 3

Explain in each case whether $T : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a linear transformation. If it is, supply a proof, if not, supply a counterexample.

(a) $T(a,b) = a + b$ (4)

(b) $T(a,b) = ab$ (4)

[8]

QUESTION 4

Let $T : P_2 \longrightarrow P_2$ be the linear transformation defined by

$$\begin{aligned} T : P_2 &\longrightarrow P_2 \\ T(p(x)) &= p(x-1) \end{aligned}$$

and let

$$B = \{1, x, x^2\} \quad \text{and} \quad B' = \{1 + x + x^2, 2x + x^2, x + x^2\}$$

(a) Find the matrix of T with respect to B. (3)

(b) Verify that $[T]_B[q]_B = [T(q)]_B$ for each $q(x) = c_0 + c_1 x + c_2 x^2$ in P_2. (4)

(c) Find the transition matrix P from B' to B. (2)

(d) Write down the formula for $[T]_{B'}$ in terms of $[T]_B$ and P, and use it to calculate $[T]_{B'}$. (9)

[18]

QUESTION 5

(a) Let W be a subset of a vector space V. When is W a *subspace* of V? (3)

(b) Let $T : V \longrightarrow W$ be a linear transformation.

(i) Define ker(T), the *kernel* of T. (1)

(ii) Show that ker(T) is a subspace of V. (5)

(iii) Prove that T is one-to-one if and only if the kernel of T contains only the zero vector *i.e.* $\ker(T) = \{\mathbf{0}\}$. (6)

[15]

[TURN OVER]

QUESTION 6

Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 2 \end{bmatrix}$

(a) Find a base for the nullspace of A (6)

(b) State and use an applicable theorem to determine the rank of A (4)

[10]

QUESTION 7

Consider the bases $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ for $\mathbb{R}^2$ where

$$S \ : \ \mathbf{v}_1 = (1,0), \mathbf{v}_2 = (0,1)$$
$$T \quad \mathbf{w}_1 = (1,1), \mathbf{w}_2 = (2,1).$$

(a) Find the transition matrix $P_{S \longleftarrow T}$ from the T-base to the S-base. (4)

(b) Find the transition matrix $Q_{T \longleftarrow S}$ from the S-base to the T-base (4)

(c) Use your answer in (a) to find $[v]_S$ if $[v]_T = \begin{bmatrix} -3 \\ 5 \end{bmatrix}$ (2)

[10]

QUESTION 8

(a) Let $\mathbf{p} = p(x)$ and $\mathbf{q} = q(x)$ be polynomials in P_2. Show that

$$\langle \mathbf{p}, \mathbf{q} \rangle = p(0)q(0) + p\left(\frac{1}{2}\right)q\left(\frac{1}{2}\right) + p(1)q(1)$$

is an inner product on P_2. (8)

(b) Let $\mathbb{R}^3$ have the Euclidean inner product. Find an *orthonormal* basis for the subspace spanned by $\{(0,1,2), (-1,0,1)\}$. (7)

[15]

TOTAL: [100]

Solution

Q1 From the characteristic polynomial $p(\lambda) = \lambda^3 - \lambda = \lambda(\lambda - 1)(\lambda + 1)$ we find that A has the eigenvalues 0, 1 and -1.

(a) The order of A is 3 (degree of $p(\lambda)$).

(b) No, since 0 is an eigenvalue of A.

(c) Yes, since the eigenvalues are all different.

(d) The eigenvalues of A^2 are $0^2 = 0$, $1^2 = 1$ and $(-1)^2 = 1$.

Q2 (a) The characteristic polynomial of A in λ is

$$\det(\lambda I - A) = \begin{vmatrix} \lambda + 3 & -1 & 0 \\ 6 & \lambda - 2 & 0 \\ 3 & -1 & \lambda \end{vmatrix} = (\lambda + 3)(\lambda - 2)\lambda + 6\lambda = \lambda^2(\lambda + 1)$$

so that the eigenvalues of A are 0 (twice) and -1.

(b) First we consider the eigenvalue 0. Thus we solve

$$0I - A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 6 & -2 & 0 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $x, y, z \in \mathbb{R}$. Row reduction of the augmented matrix yields

$$\begin{bmatrix} 3 & -1 & 0 & : & 0 \\ 6 & -2 & 0 & : & 0 \\ 3 & -1 & 0 & : & 0 \end{bmatrix} \begin{matrix} \sim \\ R_2 - 2R_1 \\ R_3 - R_1 \end{matrix} \begin{bmatrix} 3 & -1 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

so that $3x = y$. Thus the eigenspace is

$$\left\{ \begin{bmatrix} x \\ 3x \\ z \end{bmatrix} : x, z \in \mathbb{R} \right\} = \left\{ x \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : x, z \in \mathbb{R} \right\}$$

A basis for the eigenspace corresponding to the eigenvalue 0 is given by

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Don't forget to verify that these vectors are linearly independent. If they are not, use row reduction of the matrix with these vectors **as rows** and the non-zero **rows** of the reduced matrix will form a basis (i.e. rewrite the non-zero rows as column vectors).

(c) We found 3 linearly independent eigenvectors of A.

(d) We construct P directly from the eigenvectors we found:

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

then the eigenvalues in D appear in the same order as the corresponding eigenvectors in P

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Now check that $P^{-1}AP = D$ (exercise).

(e) From $P^{-1}AP = D$ we find

$$P^{-1}A^{99}P = D^{99} = \begin{bmatrix} 0^{99} & 0 & 0 \\ 0 & 0^{99} & 0 \\ 0 & 0 & (-1)^{99} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = D$$

so that $P^{-1}A^{99}P = D$ and $A^{99} = PDP^{-1} = A$.

Q3 (a) Let $\mathbf{x} = (a, b), \mathbf{y} = (\alpha, \beta) \in \mathbb{R}^2$ with $k, a, b, \alpha, \beta \in \mathbb{R}$. Now

$$\begin{aligned} T(k\mathbf{x}) &= T(ka, kb) = (ka) + (kb) = k(a + b) = kT(a, b) = kT(\mathbf{x}) \\ T(\mathbf{x} + \mathbf{y}) &= T(a + \alpha, b + \beta) = (a + \alpha) + (b + \beta) = (a + b) + (\alpha + \beta) \\ &= T(a, b) + T(\alpha, \beta) = T(\mathbf{x}) + T(\mathbf{y}) \end{aligned}$$

so that T is linear.

(b) Let $\mathbf{x} = (a, b) \in \mathbb{R}^2$ with $k, a, b \in \mathbb{R}$. Now

$$T(k\mathbf{x}) = T(ka, kb) = (ka)(kb) = k^2(ab) = k^2T(a, b) = k^2T(\mathbf{x}).$$

Now consider $a = b = 1$ and $k = 2$, then $T(k\mathbf{x}) = 4$ and $kT(\mathbf{x}) = 2$. This provides a counter example for $T(k\mathbf{x}) = kT(\mathbf{x})$. Thus T is not linear.

Q4 (a) Applying T to the basis B we find

$$\begin{aligned} T(1) &= 1\Big|_{x \to x-1} = 1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 \\ T(x) &= x\Big|_{x \to x-1} = x - 1 = -1 \cdot 1 + 1 \cdot x + 0 \cdot x^2 \\ T(x^2) &= x^2\Big|_{x \to x-1} = (x - 1)^2 = x^2 - 2x + 1 = 1 \cdot 1 - 2 \cdot x + 1 \cdot x^2 \end{aligned}$$

and writing the coefficients of the basis vectors for each equation as the column vectors in the matrix representation we find

$$[T]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) We have

$$T(q(x)) = c_0 + c_1x + c_2x^2\Big|_{x \to x-1} = c_0 + c_1(x-1) + c_2(x-1)^2 = (c_0 - c_1 + c_2) + (c_1 - 2c_2)x + c_2x^2$$

so that

$$[T(q(x))]_B = \begin{bmatrix} c_0 - c_1 + c_2 \\ c_1 - 2c_2 \\ c_2 \end{bmatrix}.$$

We also have

$$[q(x)]_B = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

so that

$$[T]_B[q(x)]_B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_0 - c_1 + c_2 \\ c_1 - 2c_2 \\ c_2 \end{bmatrix} = [T(q(x))]_B.$$

(c) We must express each element of B' in terms of the elements of B, i.e. set $b(x) = \alpha \cdot 1 + \beta \cdot x + \gamma \cdot x^2$ and solve for $\alpha, \beta, \gamma \in \mathbb{R}$ for each $b(x) \in B'$. In this case the calculation is straightforward:

$$\begin{aligned} 1 + x + x^2 &= 1 \cdot 1 + 1 \cdot x + 1 \cdot x^2 \\ 2x + x^2 &= 0 \cdot 1 + 2 \cdot x + 1 \cdot x^2 \\ x + x^2 &= 0 \cdot 1 + 1 \cdot x + 1 \cdot x^2 \end{aligned}$$

Each equation's coefficients provides the columns for the transition matrix

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(d) To calculate T in the basis B' we first convert B' to B using P, then we apply $[T]_B$ in B, and then convert back from B to B' using P^{-1}:

$$[T]_{B'} = P_{B' \leftarrow B}[T]_B P_{B \leftarrow B'} = P^{-1}[T]_B P.$$

We apply row-reduction to the matrix P augmented with the identity to find P^{-1}.

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 1 & 2 & 1 & : & 0 & 1 & 0 \\ 1 & 1 & 1 & : & 0 & 0 & 1 \end{bmatrix} \begin{matrix} \sim \\ R_2 - R_1 \\ R_3 - R_1 \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 0 & 2 & 1 & : & -1 & 1 & 0 \\ 0 & 1 & 1 & : & -1 & 0 & 1 \end{bmatrix} \\ \begin{matrix} \sim \\ R_2 - R_3 \\ \sim \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & 0 & 1 & -1 \\ 0 & 1 & 1 & : & -1 & 0 & 1 \end{bmatrix} \\ \begin{matrix} \sim \\ R_3 - R_2 \\ \sim \end{matrix} & \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 0 \\ 0 & 1 & 0 & : & 0 & 1 & -1 \\ 0 & 0 & 1 & : & -1 & -1 & 2 \end{bmatrix} \end{aligned}$$

Thus (exercise: check that $P^{-1}P = PP^{-1} = I$)

$$P^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

Now

$$[T]_{B'} = P^{-1}[T]_B P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -3 \\ -1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & -1 & -2 \\ 2 & 3 & 3 \end{bmatrix}.$$

Q5 (a) ∗ W must be non-empty.

∗ For all $\mathbf{x}, \mathbf{y} \in W$, $\mathbf{x} + \mathbf{y} \in W$ (closure under vector addition)

∗ For all $k \in \mathbb{R}$ and $\mathbf{x} \in W$, $k\mathbf{x} \in W$ (closure under scalar multiplication)

(b) Let $\mathbf{0}_V$ be the zero vector in V and $\mathbf{0}_W$ be the zero vector in W.

(i) The kernel of T is

$$\ker(T) := \{\, \mathbf{v} \in V \;:\; T(\mathbf{v}) = \mathbf{0}_W \,\}.$$

(ii) · Since T is linear and $0\mathbf{x} = \mathbf{0}_W$ for all $\mathbf{x} \in W$ we have

$$T(\mathbf{0}_V) = T(0\mathbf{y}) = 0T(\mathbf{y}) = \mathbf{0}_W$$

for all $\mathbf{y} \in V$. Thus $\mathbf{0}_V \in \ker(T)$, and $\ker(T)$ is non-empty.

· Let $k \in \mathbb{R}$ and $\mathbf{a} \in \ker(T)$. Then $T(k\mathbf{a}) = kT(\mathbf{a}) = k\mathbf{0}_W = \mathbf{0}_W$. Thus $k\mathbf{a} \in \ker(T)$.

· Let $\mathbf{a}, \mathbf{b} \in \ker(T)$. Then $T(\mathbf{a} + \mathbf{b}) = T(\mathbf{a}) + T(\mathbf{b}) = \mathbf{0}_W + \mathbf{0}_W = \mathbf{0}_W$. Thus $\mathbf{a} + \mathbf{b} \in \ker(T)$.

(iii) T is one-to-one
if and only if
$\forall \mathbf{x}, \mathbf{y} \in V$: $T(\mathbf{x}) = T(\mathbf{y}) \Leftrightarrow \mathbf{x} = \mathbf{y}$
if and only if
$\forall \mathbf{x}, \mathbf{y} \in V$: $T(\mathbf{x}) - T(\mathbf{y}) = \mathbf{0}_W \Leftrightarrow \mathbf{x} - \mathbf{y} = \mathbf{0}_V$
if and only if
$\forall \mathbf{x}, \mathbf{y} \in V$: $T(\mathbf{x} - \mathbf{y}) = \mathbf{0}_W \Leftrightarrow \mathbf{x} - \mathbf{y} = \mathbf{0}_V$
if and only if
$\forall \mathbf{z} \in V$: $T(\mathbf{z}) = \mathbf{0}_W \Leftrightarrow \mathbf{z} = \mathbf{0}_V$
if and only if
$\ker(T) = \{\, \mathbf{0}_V \,\}$; where we used the linearity of T and substituted $\mathbf{z} := \mathbf{x} - \mathbf{y}$.

Q6 (a) We solve

$$A \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & -2 & 2 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

for $a, b, c \in \mathbb{R}$. Applying row-reduction we find

$$\begin{bmatrix} 1 & 1 & 1 & : & 0 \\ 1 & 0 & 1 & : & 0 \\ 2 & -2 & 2 & : & 0 \end{bmatrix} \begin{matrix} R_1 - R_2 \\ \sim \\ R_3 - 2R_1 \end{matrix} \begin{bmatrix} 0 & 1 & 0 & : & 0 \\ 1 & 0 & 1 & : & 0 \\ 0 & -2 & 0 & : & 0 \end{bmatrix} \begin{matrix} R_2 \\ R_1 \\ R_3 + 2R_1 \end{matrix} \begin{bmatrix} 1 & 0 & 1 & : & 0 \\ 0 & 1 & 0 & : & 0 \\ 0 & 0 & 0 & : & 0 \end{bmatrix}$$

so that $c = -a$ and $b = 0$. Thus

$$\text{nullspace(A)} = \left\{ \begin{bmatrix} a \\ 0 \\ -a \end{bmatrix} : a \in \mathbb{R} \right\} = \left\{ a \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} : a \in \mathbb{R} \right\}.$$

Since there is only one free parameter we easily find the basis

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

for nullspace(A).

(b) Since $\text{rank}(A) + \text{nullity}(A) = 3$ (the number of columns of A) we find

$$\text{rank}(A) = 3 - \text{nullity}(A) = 3 - 1 = 2.$$

Q7 (a) We solve

$$\mathbf{w}_1 = a\mathbf{v}_1 + b\mathbf{v}_2, \qquad \mathbf{w}_2 = c\mathbf{v}_1 + d\mathbf{v}_2$$

for $a, b, c, d \in \mathbb{R}$ to find $a = b = d = 1$ and $c = 2$. We have

$$P_{S \leftarrow T} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}.$$

(b) We solve

$$\mathbf{v}_1 = \alpha\mathbf{w}_1 + \beta\mathbf{w}_2, \qquad \mathbf{v}_2 = \gamma\mathbf{w}_1 + \delta\mathbf{w}_2$$

for $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ to find $\alpha = -1$, $\beta = 1$, $\gamma = 2$ and $\delta = -1$. We have

$$Q_{T \leftarrow S} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}.$$

Notice that $Q_{T \leftarrow S} = P_{S \leftarrow T}^{-1}$ which provides another way to find $Q_{T \leftarrow S}$.

(c) We have

$$[\mathbf{v}]_S = P_{S \leftarrow T}[\mathbf{v}]_T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

Q8 (a) Let $k \in \mathbb{R}$ and $\mathbf{p} = p(x), \mathbf{q} = q(x), \mathbf{r} = r(x) \in P_2$.

* $\begin{aligned}\langle \mathbf{q}, \mathbf{p} \rangle &= q(0)p(0) + q\left(\tfrac{1}{2}\right)p\left(\tfrac{1}{2}\right) + q(1)p(1) \\ &= p(0)q(0) + p\left(\tfrac{1}{2}\right)q\left(\tfrac{1}{2}\right) + p(1)q(1) = \langle \mathbf{p}, \mathbf{q} \rangle\end{aligned}$
* $\begin{aligned}\langle k\mathbf{p}, \mathbf{q} \rangle &= (kp)(0)q(0) + (kp)\left(\tfrac{1}{2}\right)q\left(\tfrac{1}{2}\right) + (kp)(1)q(1) \\ &= k\left(p(0)q(0) + p\left(\tfrac{1}{2}\right)q\left(\tfrac{1}{2}\right) + p(1)q(1)\right) = k\langle \mathbf{p}, \mathbf{q} \rangle\end{aligned}$
* $\begin{aligned}\langle \mathbf{p} + \mathbf{q}, \mathbf{r} \rangle &= (p+q)(0)r(0) + (p+q)\left(\tfrac{1}{2}\right)r\left(\tfrac{1}{2}\right) + (p+q)(1)r(1) \\ &= p(0)r(0) + q(0)r(0) + p\left(\tfrac{1}{2}\right)r\left(\tfrac{1}{2}\right) + q\left(\tfrac{1}{2}\right)r\left(\tfrac{1}{2}\right) + p(1)r(1) + q(1)r(1) \\ &= p(0)r(0) + p\left(\tfrac{1}{2}\right)r\left(\tfrac{1}{2}\right) + p(1)r(1) + q(0)r(0) + q\left(\tfrac{1}{2}\right)r\left(\tfrac{1}{2}\right) + q(1)r(1) \\ &= \langle \mathbf{p}, \mathbf{r} \rangle + \langle \mathbf{q}, \mathbf{r} \rangle.\end{aligned}$

* $\langle \mathbf{p}, \mathbf{p} \rangle = p(0)p(0) + p\left(\frac{1}{2}\right)p\left(\frac{1}{2}\right) + p(1)p(1) = p(0)^2 + p\left(\frac{1}{2}\right)^2 + p(1)^2 \geq 0$
  Now let $\mathbf{p} = p(x) = p_0 + p_1x + p_2x^2 \in P_2$ where $p_0, p_1, p_2 \in \mathbb{R}$.
  Then $\langle \mathbf{p}, \mathbf{p} \rangle = p_0^2 + \left(p_0 + \frac{p_1}{2} + \frac{p_2}{4}\right)^2 + (p_0 + p_1 + p_2)^2 = 0$
  if and only if $p_0 = p_0 + \frac{p_1}{2} + \frac{p_2}{4} = p_0 + p_1 + p_2 = 0$
  if and only if $p_0 = 0, 2p_1 + p_2 = 0, p_1 + p_2 = 0$
  if and only if $p_0 = 0, p_1 = 0, p_2 = 0$
  where we subtracted the last equation from the middle equation.

(b) We apply the Gram-Schmidt process

$$\begin{aligned}
\mathbf{v}_1 &= (0,1,2) \\
\mathbf{v}_2 &= (-1,0,1) - \frac{\langle \mathbf{v}_1, (0,1,2) \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 = (-1,0,1) - \frac{2}{5}(0,1,2) \\
&= \left(-1, -\frac{2}{5}, \frac{1}{5}\right) = \frac{1}{5}(-5,-2,1).
\end{aligned}$$

Normalizing $\mathbf{v}_1$ and $\mathbf{v}_2$ we find the orthonormal basis

$$\left\{ \frac{\mathbf{v}_1}{\sqrt{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}}, \frac{\mathbf{v}_2}{\sqrt{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}} \right\} = \left\{ \frac{\mathbf{v}_1}{\sqrt{5}}, \frac{\mathbf{v}_2}{\frac{\sqrt{30}}{5}} \right\} = \left\{ \frac{1}{\sqrt{5}}(0,1,2), \frac{1}{\sqrt{30}}(-5,-2,1) \right\}.$$