Tutorial Letter 202/2/2016

DISCRETE MATHEMATICS: COMBINATORICS

MAT3707

Semester 2

Department of Mathematical Sciences

IMPORTANT INFORMATION:

This tutorial letter contains solutions of assignment 2

BAR CODE

SOLUTIONS TO ASSIGNMENT 02 (SEMESTER 2)
CLOSING DATE: 22 September 2016
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The answers to only some of the questions will be marked.

1. How many ways are there of filling in a multiple-choice questionnaire of 20 questions with 5 choices for each question? (2)

   Solution

   By the multiplication principle it is

   $$\underbrace{5 \times 5 \times 5 \times 5 \times \cdots \times 5}_{\text{20 times}} = 5^{20}.$$

2. Determine the number of 64-digit binary sequences with at most five 0's. (4)

   Solution

   The number of such sequences with exactly 5 0's is $\binom{64}{5}$, with exactly 4 0's is $\binom{64}{4}$, exactly 3 0's is $\binom{64}{3}$, exactly 2 0's is $\binom{64}{2}$, exactly 1 0'is $\binom{64}{1}$, no 0's is $\binom{64}{0} = 1$. By the addition principle the total is

   $$\binom{64}{5} + \binom{64}{4} + \binom{64}{3} + \binom{64}{2} + \binom{64}{1} + \binom{64}{0}.$$

3. How many five-digit sequences of the digits 0 to 9 are there

   (a) containing no 5? (3)

   Solution

   For each of the 5 digits there is a choice of 9 digits.

   By the multiplication principle the number of sequences is

   $$9 \times 9 \times 9 \times 9 \times 9 = 9^5.$$

   (b) containing a 5? (4)

   Solution

   This is the complement of (a). The total number of 5-digit sequences is 10^5. The difference is $10^5 - 9^5$.

4. How many ways are there to choose 20 balls from a pile of red, blue, yellow and green balls (that are identical apart from their colour)

   (a) if there must be at least three balls of each colour? (4)

   Solution

   First take three of each colour, since this has to be done in any case (exactly one way). There still remains $20 - 3 \times 4 = 8$ balls to be chosen. Think of the four colours as 4 containers, and place each of the 8 balls in the container of its colour. Since the containers determine their colour, we can now think of them as identical objects, eg.

$$\begin{array}{c|c|c|c} o \quad o & o \quad o \quad o & o \quad o & o \\ \text{red} & \text{blue} & \text{yellow} & \text{green} \end{array}$$

So this is the number of ways of distributing 8 identical objects into 4 containers, which is the same as the number of binary strings of length 11, with 3 ones. This is $\binom{11}{3}$.

(b) if there must be at least five balls of each colour? (4)

Solution

IF there must be at least 5 of each colour then there must be exactly 5 of each colour. Therefore, there is exactly 1 way (remember, balls of the same colour are considered identical).

5. How many ways are there to put

(a) five different objects into six different containers? (4)

Solution

For each object we choose its container (6 ways). By the multiplication principle the number is $6 \times 6 \times 6 \times 6 \times 6 = 6^5$.

(b) five different objects into six different containers with at most one object in each container? (4)

Solution

For the first object choose its container: 6 ways.

For the second object choose its container: 5 ways. Similarly for the third object there are 4 choices, etc. Eventually we have $6 \times 5 \times 4 \times 3 \times 2 = 6!$

(c) five identical objects into six different containers (4)

Solution

For example

$$o \mid o \mid \quad \mid \quad \mid o \quad o \mid o$$

This is the same as the number of binary strings of length 10 with 5 ones, which is $\binom{10}{5}$.

(d) five identical objects into six identical containers? (4)

Solution

This is the number of partitions of 5. There is no nice formula; just list them and make sure you don't leave out any:

$$\begin{aligned} 5 &= 1+4+2+3 \\ &= 1+1+3 = 1+2+2 \\ &= 1+1+1+2 = 1+1+1+1+1, \end{aligned}$$

7 in total.

6. (a) How many ways are there to arrange the letters of the word NGORONGORO? (3)

Solution

This is a famous crater valley in Tanzania. There are $2N$'s, $2G$'s, $2R$'s, $4O$'s:

$$\frac{10!}{2!2!2!4!}$$

(b) How many arrangements are there with the constants in alphabetical order? (5)

Solution

Replace each consonant by a C. (Because the order of the consonants is now fixed, they can be placed back in only one way.) Now there are $6C$'s and $4O$'s: choose positions for either, the the other positions must be filled by the other letter: choose 6 positions for the C's

$$\binom{10}{6} = \frac{10!}{6!4!}.$$

7. (a) Find the number of integer solutions to the equations

$$a + b + c + d = 20$$

with constraints $a > 2$, $b \geq 6$, $c \geq 3$, $d > 5$. (4)

Solution

We rewrite the constraints: $a \geq 3$, $b \geq 6$, $c \geq 3$, $d \geq 6$.

Let $x = a - 3$, $y = b - 6$, $z = c - 3$, $t = d - 6$. Then the constraints become $x, y, z, t \geq 0$, can the equation

$$x + y + z + t = 2.$$

This is the same as placing 2 identical objects into 4 distinct containers:

$$\text{o} \mid \text{o} \mid \quad \mid$$

Which is the same as the number of 5-digit binary strings with 2 0's: $\binom{5}{2} = 10$.

(b) Find all the integer solutions of the equation with constraints in (a). (3)

Solution

We just have to find 10, and then we know we are finished.

	1	2	3	4	5	6	7	8	9	10
a	3	3	3	3	3	3	4	4	4	6
b	6	6	6	7	7	8	6	6	7	6
c	3	4	5	3	4	3	3	4	3	3
d	8	7	6	7	6	6	7	6	6	6

8. Find the number of integer solutions to

$$x + y + z \leq 10, \quad x, y, z \geq 1.$$

(4)

Solution

By letting $a = x - 1$, $b = y - 1$, $c = z - 1$, this is the same as $a + b + c \leq 7$, $a, b, c \geq 0$. This can be converted to an equation by letting $d = 7 - a - b - c$:

$$a + b + c + d = 7, \ a, d, c, d \geq 0.$$

We know have 7 balls in 4 boxes: $\binom{10}{3}$.

9. Build a generating function for a_r, the number of

   (a) integer solutions to
$$e_1 + e_2 + e_3 + e_4 = r$$
   with constraints
$$0 < e_1 < 4,\ e_2 \geq 3,\ 1 \leq e_3 < 3,\ 0 \leq e_4 \leq 1. \qquad (6)$$
   Solution
   We have $e_1 \in \{1, 2, 3\}$, $e_2 \in \{3, 4, 5 \ldots\}$, $e_3 \in \{1, 2\}$ and $e_4 \in \{0, 1\}$. Therefore, the generating function is
$$(x + x^2 + x^3)(x^3 + x^4 + x^5 + \cdots)(x + x^2)(1 + x).$$
   (b) selections of r balls from a pile of 3 red, 10 blue and an unlimited supply of yellow balls. (4)
   Solution
   As an equation this is
$$e_{\text{red}} + e_{\text{blue}} + e_{\text{yellow}} = r,$$
   with constraints
$$0 \leq e_{\text{red}} \leq 3, \quad 0 \leq e_{\text{blue}} \leq 10, \quad e_{\text{yellow}} \geq 0.$$
   This gives the generating function
$$(1 + x + x^2 + x^3)(1 + x + x^2 + \cdots + x^{10})(1 + x + x^2 + \cdots).$$
   (c) distributions of r identical objects into 10 boxes with at most 2 in each box. (4)
   Solution
   The equation is
$$e_1 + e_2 + \cdots + e_{10} = r,$$
   with constraints $0 \leq e_i \leq 2$ for each $i = 1, 2, 3, \ldots, 10$. Each factor in the generating function is therefore $(1 + x + x^2)$, and there are 10 of them, giving the generating function
$$(1 + x + x^2)^{10}.$$

10. Find the number of ways of distributing 20 identical objects into 5 boxes if there may be at most two objects in the first box, and at least two objects in the second box. (8)

    Solution

    The generating function is
$$\begin{aligned}
g(x) &= (1 + x + x^2)(x^2 + x^3 + x^4 + \cdots)(1 + x + x^2 + \cdots)^3 \\
&= (1 + x + x^2)x^2(1 + x + x^2 + \cdots)(1 + x + x^2 + \cdots)^3 \\
&= (1 + x + x^2)x^2(1 + x + x^2 + \cdots)^4 \\
&= \frac{x^2 + x^3 + x^4}{(1 - x)^4} \\
&= (x^2 + x^3 + x^4)\left(\sum_{r=0}^{\infty} \binom{4 + r - 1}{r} x^r\right).
\end{aligned}$$

We want the coefficient of x^{20}, so we have to multiply x^2 with x^{18}, x^3 with x^{17}, and x^4 with x^{16}.

The coefficient of X^{18} on the right is $\binom{4+18-1}{18} = \binom{21}{18}$, of x^{17} is $\binom{4+17-1}{17} = \binom{20}{17}$, and of x^{16} is $\binom{4+16-1}{16} = \binom{19}{16}$.

Adding up we obtain

$$\binom{21}{18} + \binom{20}{17} + \binom{19}{16}.$$

11. How many ways are here to get a sum of 26 when 10 dice are rolled? (8)

Solution

The equation is

$$e_1 + e_2 + e_3 + \cdots + e_{10} = r, \quad 1 \le e_i \le 6 \quad \text{for each} \quad i = 1, 2, \ldots, 6$$

and the specific problem is when $r = 26$.

The generating function is

$$\begin{aligned} g(x) &= (x + x^2 + x^3 + x^4 + x^5 + x^6)^{10} \\ &= x^{10}(1 + x + x^2 + x^3 + x^4 + x^5)^{10} \\ &= x^{10}\left(\frac{1 - x^6}{1 - x}\right)^{10} \\ &= x^{10}\frac{(1 - x^6)^{10}}{(1 - x)^{10}}. \end{aligned}$$

We have to calculate the coefficient of x^{26} in $g(x)$, which equals the coefficient of x^{16} in

$$\begin{aligned} \frac{(1 - x^6)^{10}}{(1 - x)^{10}} &= \frac{\sum_{k=0}^{10} \binom{10}{k}(-1)^k x^{6k}}{(1 - x)^10} \\ &= \left(\sum_{k=0}^{10} \binom{10}{k}(-1)^k x^{6k}\right) \sum_{r=0}^{10} \binom{10 + r - 1}{r} x^r. \end{aligned}$$

The terms to consider are:

$$\begin{aligned} k = 0,\ r = 16: &\quad \binom{10}{0}(-1)^0\binom{10 + 16 - 1}{16} = \binom{10}{0}\binom{25}{16} \\ k = 1,\ r = 10: &\quad \binom{10}{1}(-1)^1\binom{10 + 10 - 1}{10} = -\binom{10}{1}\binom{19}{10} \\ k = 2,\ r = 4: &\quad \binom{10}{2}(-1)^2\binom{10 + 4 - 1}{4} = \binom{10}{2}\binom{13}{4}. \end{aligned}$$

Adding up we obtain

$$\binom{10}{0}\binom{25}{16} - \binom{10}{1}\binom{19}{10} + \binom{10}{2}\binom{13}{4}.$$

12. Find the number of ways of placing 20 persons into three rooms with at most 2 persons in the first room. (8)

Solution

Persons are distinct so we have to use exponential generating functions. The generating function is

$$\begin{aligned}
& \left(1+x+\frac{x^2}{2!}\right)\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots\right)^2 \\
= & \left(1+x+\frac{x^2}{2!}\right)e^{2x} \\
= & e^{2x}+xe^{2x}+\frac{1}{2}x^2e^{2x}
\end{aligned}$$

and we have to determine the coefficient of $\frac{x^{20}}{20!}$.

In e^{2x}, this is 2^{20}.

In xe^{2x}, it is a bit more complicated. You can either write out the whole expansion, or you can reason as follows: We first find the coefficient of x^{20}. This is the same as the coefficient of x^{19} in e^{2x}, which is $\frac{2^{19}}{19!}$. Therefore, the x^{20} term in xe^{2x} is

$$\frac{2^{19}}{19!}x^{20} = 2^{19}\cdot 20\frac{x^{20}}{20\cdot 19!} = 2^{19}\cdot 20\frac{x^{20}}{20!}$$

giving the $\frac{x^{20}}{20!}$ coefficient as $2^{19}\cdot 20$.

We do the same with $\frac{1}{2}x^2e^{2x}$. The coefficient of x^{20} is the same as the coefficient of x^{18} in $\frac{1}{2}e^{2x}$, which is $\frac{1}{2}\frac{2^{18}}{18!} = \frac{2^{17}}{18!}$. Therefore, the x^{20} term in $\frac{1}{2}x^2e^{2x}$ is

$$\frac{2^{17}}{18!}x^{20} = 2^{17}\cdot 20\cdot 19\frac{x^{20}}{20\cdot 19\cdot 18!} = 2^{17}\cdot 20\cdot 19\frac{x^{20}}{20!},$$

which gives the $\frac{x^{20}}{20!}$ coefficient as $2^{17}\cdot 20\cdot 19$.

Adding up, we obtain the coefficient of $\frac{x^{20}}{20!}$ in $e^{2x}+xe^{2x}+\frac{1}{2}x^2e^{2x}$ to be

$$2^{20}+2^{19}\cdot 20+2^{17}\cdot 20\cdot 19.$$

13. (a) How many 9-digit sequences of the digits 1, 2, 3, 4, 5, 6 are there with each even digit occurring an even number of times and simultaneously each odd digit an odd number of times? (8)

Solution

Constructing 9-digit sequences out of $\{1,2,3,4,5,6\}$ is the same as distributing the 9 (distinct) positions into 6 containers. It follows that the generating function is

$$\begin{aligned}
& \left(1+\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)^3\left(x+\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)^3 \\
= & \left(\frac{1}{2}(e^x+e^{-x})\right)^3\left(\frac{1}{2}(e^x-e^{-x})\right)^3 \\
= & \frac{1}{2^6}\left((e^x+e^{-x})(e^x-e^{-x})\right)^3 \\
= & \frac{1}{2^6}\left(e^{2x}-e^{-2x}\right)^3 \\
= & \frac{1}{2^6}\left(e^{6x}-3e^{2x}+3e^{-2x}-e^{-6x}\right).
\end{aligned}$$

The coefficient of $\frac{x^9}{9!}$ is

$$\begin{aligned}
& \frac{1}{2^6}(6^9 - 3\cdot 2^9 + 3(-2)^9 - (-6)^9) \\
= & \frac{1}{2^6}(2\cdot 6^9 - 2\cdot 3\cdot 2^9) \\
= & \frac{1}{2^5}(6^9 - 3\cdot 2^9).
\end{aligned}$$

(b) Repeat questions (a) with 10-digit sequences. Discuss your answer. (4)

Solution

We already have the generating function. Now we only have to calculate the coefficient of $\frac{x^{10}}{10!}$. Similar as in (a), it is

$$\begin{aligned}
& \frac{1}{2^6}(6^{10} - 3\cdot 2^{10} + 3(-2)^{10} - (-6)^{10}) \\
= & \frac{1}{2^6}(6^{10} - 3\cdot 2^{10} + 3\cdot 2^{10} - 6^{10}) \\
= & 0.
\end{aligned}$$

Discussion: The answer of 0 means there are no 10-digit sequences with each even digit occurring an even number of times and each odd digit occurring an odd number of times. Does this make sense?

If we count how many times each digit occurs, we will get 6 numbers of which three must be even (2, 4 and 6 each occur an even number of times) and three must be odd (1,3 and 5 each occur an odd number of times.) These 6 numbers have to add up to 10 (the length of the sequence). However, the sum of three even numbers and three odd numbers is odd. So the only possible lengths for such sequences are odd numbers. Since 10 is not odd, there are no such sequences of length 10.

14. Let a_n be the number of ternary sequences of length n with no two consecutive 1's.

(a) Find a recurrence relation for a_n. (4)

Solution

Consider an arbitrary such sequence. It ends in a 0, 1 or 2. We look at each case in turn.

If it ends in a 0, then the first $n-1$ digits is an arbitrary ternary sequence of length $n-1$ with no two consecutive 1's. The number of such sequences is a_{n-1}.

If it ends in a 1, we have to be careful. Then the 2nd last digit cannot be a 1, so in this case the first $n-1$ is not an arbitrary ternary sequence of length $n-1$ with no two consecutive 1's. Then second last digit can be a 0 or a 2. In each case the first $n-2$ digits now form a ternary sequence without "11". So in this case the number of sequences is $a_{n-2} + a_{n-2}$.

If it ends in a 2, we again have a_{n-1} sequences (the same as when it ends in a 0).

Adding up the three cases we obtain a_n:

$$\begin{aligned}
a_n &= a_{n-1} + (a_{n-2} + a_{n-2}) + a_{n-1} \\
&= 2a_{n-1} + 2a_{n-2}.
\end{aligned}$$

(b) Determine a_6. In order to use the recurrence relation we need some initial values

$$\begin{aligned}
a_1 &= 3 \text{ (3 sequences of length 1, namely 0, 1, 2).}\\
a_2 &= 8 \text{ (}3^2\text{ sequences of length 2, and we have to subtract 1 for the sequence 11)}\\
a_3 &= 2a_2 + 2a_1 = 2 \cdot 8 + 2 \cdot 3 = 22\\
a_4 &= 2a_3 + 2a_2 = 2 \cdot 22 + 2 \cdot 8 = 60\\
a_5 &= 2a_4 + 2a_3 = 2 \cdot 60 + 2 \cdot 22 = 164\\
a_6 &= 2a_5 + 2a_4 = 2 \cdot 164 + 2 \cdot 60 = 448.
\end{aligned}$$

15. Find a recurrence relation (with initial conditions) for the number of sequences of the digits 1, 2, 3 and 4 that sum to n. (4)

Solution

Denote this number by a_n. There are four cases, depending on whether the last term in the sequence is 1, 2, 3 or 4.

If it is 1, then the sequence with the last term removed, adds up to $n-1$. So in this case there are a_{n-1} such sequences.

Similarly there are a_{n-2} sequences with last term 2, a_{n-3} with last term 3, and a_{n-4} with last term 4.

Adding up we obtain that

$$a_n = a_{n-1} + a_{n-2} + a_{n-3} + a_{n-4}.$$

The initial conditions are given by

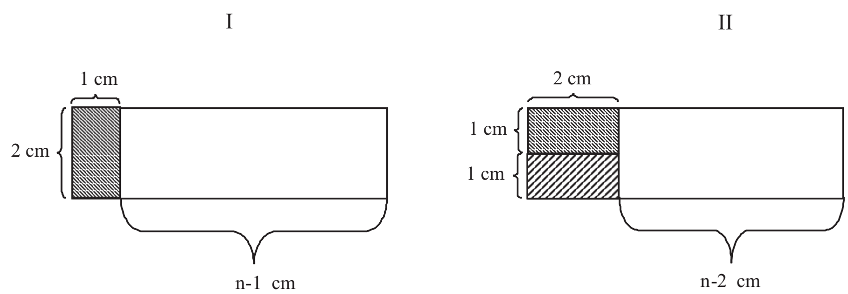
$$\begin{aligned}
a_1 &= 1 \text{ (1)}\\
a_2 &= 2 \text{ (1, 1 and 2)}\\
a_3 &= 4 \text{ (1, 1, 1; 1, 2; 2, 1; 3)}\\
a_4 &= 8 \text{ (1, 1, 1, 1; 2, 1, 1; 1, 2, 1; 1, 1, 2; 1, 3; 2, 2; 3, 1; 4)}
\end{aligned}$$

16. Determine the number of ways that a $2 \text{ cm} \times 8 \text{ cm}$ rectangle can be filled by non-overlapping $1 \text{ cm} \times 2 \text{ cm}$ rectangles. (5)

Solution

There is no simple way to do this with elementary counting techniques. Instead of $2\text{cm} \times 8\text{cm}$ we consider $2\text{cm} \times n\text{cm}$. Let a_n be the number of ways that a $2\text{cm} \times n\text{cm}$ rectangle can be filled by $1\text{cm} \times 2\text{cm}$ (i.e. dominoes). We try to find a recurrence relation for a_r.

The first column of the rectangle can be filled in one of two ways:

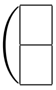
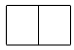


In case I the remaining $2\mathrm{cm} \times (n-1)\mathrm{cm}$ rectangle can be filled in a_{n-1} ways, and in case II, the remaining $2\mathrm{cm} \times (n-2)\mathrm{cm}$ rectangle can be filled in a_{n-2} ways.

Therefore $a_n = a_{n-1} + a_{n-2}$.

To answer the question we have to determine a_8. In order to use the relation, we have to first find initial conditions:

$$a_1 = 1$$

$a_2 = 2$ ( and )

$$\begin{aligned}
a_3 &= 2+1 = 3\\
a_4 &= 3+2 = 5\\
a_5 &= 5+3 = 8\\
a_6 &= 8+5 = 13\\
a_7 &= 13+8 = 21\\
a_8 &= 21+13 = 34.
\end{aligned}$$

(Note that these are the Fibonacci numbers.)

17. Solve the following recurrence relations:

   (a) $a_n = 3a_{n-1} + a_{n-2},\ a_0 = 2,\ a_1 = 4.$ (5)

   Solution

   The characteristic question is

$$\alpha^2 - 3\alpha - 1 = 0 \text{ i.e. } \alpha = \frac{3 \pm \sqrt{13}}{2},$$

   and the general solution is

$$a_n = A\left(\frac{3+\sqrt{13}}{2}\right)^n + B\left(\frac{3-\sqrt{13}}{2}\right)^n.$$

   Use the initial conditions to calculate A and B:

$$\begin{aligned}
n = 0: &\quad 2 = A + B \qquad (1)\\
n = 1: &\quad 4 = A\left(\frac{3+\sqrt{13}}{2}\right) + B\left(\frac{3-\sqrt{13}}{2}\right). \qquad (2)
\end{aligned}$$

   Multiply (1) by $(3+\sqrt{13})/2$:

$$3+\sqrt{13} = A\left(\frac{3+\sqrt{13}}{2}\right) + B\left(\frac{3+\sqrt{13}}{2}\right). \qquad (3)$$

   Subtract (2) from (3):

$$\begin{aligned}
\sqrt{13} - 1 &= 3B\\
\text{i.e. } B &= \frac{\sqrt{13}-1}{3}.
\end{aligned}$$

From (1)

$$A = 2 - B = \frac{6}{3} - \frac{\sqrt{13}-1}{3} = \frac{7-\sqrt{13}}{3}$$
$$\text{i.e.} \qquad a_n = \frac{7-\sqrt{13}}{3}\left(\frac{3+\sqrt{13}}{2}\right)^n + \frac{\sqrt{13}-1}{3}\left(\frac{3-\sqrt{13}}{2}\right)^n.$$

(b) $a_n = a_{n-1} + n(n+1),\ a_1 = 1.$ (5)

Solution

Since a_{n-1} is on its own on the right hand side, this can be solved by iteration:

$$\begin{aligned}
a_1 &= 1\\
a_2 &= a_1 + 2\cdot 3 = 1 + 2\cdot 3\\
a_3 &= a_2 + 3\cdot 4 = 1 + 2\cdot 3 + 3\cdot 4\\
a_4 &= a_3 + 4\cdot 5 = 1 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5\\
&\vdots\\
a_n &= 1 + 2\cdot 3 + 3\cdot 4 + 4\cdot 5 + \cdots + n(n+1)\\
&= 1 + 2\left(\frac{2\cdot 3}{2} + \frac{3\cdot 4}{2} + \cdots + \frac{n(n+1)}{2}\right)\\
&= 1 + 2\left(\binom{3}{2} + \binom{4}{2} + \cdots + \binom{n+1}{2}\right).
\end{aligned}$$

By (8) of section 5.5 of Tucker,

$$\begin{aligned}
\binom{2}{2} + \binom{3}{2} + \cdots \binom{n+1}{2} &= \binom{n+2}{3}\\
\text{i.e. } \binom{3}{2} + \cdots \binom{n+1}{2} &= \binom{n+2}{3} - \binom{2}{2} = \binom{n+2}{3} - 1\\
\text{i.e. } a_n &= 1 + 2\left(\binom{n+2}{3} - 1\right) = 2\binom{n+2}{3} - 1\\
&= 2\frac{(n+2)(n+1)n}{3\cdot 2\cdot 1} - 1\\
&= \frac{(n+2)(n+1)n}{3} - 1.
\end{aligned}$$

18. How many five-digit sequences of the digits 0 to 9 are there containing a 4 and a 5? (4)

Solution

Let A be the set of sequences not containing a 4, and B the set of sequences not containing a 5. We want $N(\overline{A} \cap \overline{B})$. By the inclusion-exclusion formula for two sets we have

$$N(\overline{A} \cap \overline{B}) = N - N(A) - N(B) + N(A \cap B).$$

$N = 10^5$ (all 5-digit sequences)
$N(A) = 9^5$ (5-digit sequences not containing a 4)
Similarly, $N(B) = 9^5$.

$N(A \cap B) = 8^5$ (5-digit sequences not containing 4 or 5)
Hence

$$N(\overline{A} \cap \overline{B}) = 10^5 - 2 \cdot 9^5 + 8^5.$$

19. How many ways are there to choose 20 balls from a pile of red, blue, yellow and green balls (that are identical apart from their colour)

    (a) if there must be at most six balls of each colour? (4)

    Solution

    Call the colours 1, 2, 3 and 4 and let A_i be the set of selections with more than 6 of colour i (i.e. at least 7 balls of colour i).

    $$N = \binom{20+4-1}{20} = \binom{23}{20} \text{ (total number of selections without any restriction).}$$

    For each $i = 1, 2, 3, 4$,

    $$N(A_i) = \binom{13+4-1}{13} = \binom{16}{13} \text{ (number of selections with at least 7 of colour } i\text{, and no other res}$$

    Therefore $S_1 = \binom{16}{13}$.
    For any two i, j, $N(A_i \cap A_j) = \binom{6+4-1}{6} = \binom{9}{6}$ (number of selections with at least 7 of colour i and at least 7 of colour j, and no other restrictions).
    Therefore, $S_2 = \binom{4}{2}\binom{9}{6}$.
    For any three i, j, k, $N(A_i \cap A_j \cap A_k) = 0$ (impossible to choose only 20 with at least 7 of each colours i, j, k.)
    Therefore $S_3 = 0$. Similarly $S_4 = 0$.
    We want

    $$\begin{aligned} N(\overline{A}_1 \cap \overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3 \cap \overline{A}_4) &= N - S_1 + S_2 - S_3 + S_4 \\ &= \binom{23}{20} - 4\binom{16}{13} + \binom{4}{2}\binom{9}{6}. \end{aligned}$$

    (b) if there must be at least five balls of each colour? (2)

    Solution

    If there must be at least five of each colour, there is exactly one way.

20. Find the number of 10-letter sequences of a, b, c, d with each letter occurring at least once, using

    (a) exponential generating functions; (6)

    Solution

    Each such sequence corresponds to a distribution of 10 distinct objects (the positions) into 4 boxes labelled a, b, c and d. Therefore the generating function is

    $$\begin{aligned} g(x) &= \left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)^4 \\ &= (e^x - 1)^4 \\ &= e^{4x} - 4e^{3x} + \binom{4}{2}e^{2x} - \binom{4}{3}e^x + 1, \end{aligned}$$

where we have used the binomial theorem in the last step.
The coefficient of $\frac{x^{10}}{10!}$ in $g(x)$ is

$$4^{10} - 4 \cdot 3^{10} + \binom{4}{2} 2^{10} - \binom{4}{3}.$$

(b) inclusion-exclusion. (6)

Solution

Let A be the set of 10-letter sequences not using a, B the set not using b, C the set not using c and D the set not using d. Then

$N = 4^{10}$ (all sequences).

$N(A) = N(B) = N(C) = N(D) = 3^{10}$ (sequences using 3 letters).

Hence $S_1 = 4 \cdot 3^{10}$.

$N(A \cap B) = \cdots = N(C \cap D) = 2^{10}$ (sequences using 2 letters).

Hence $S_2 = \binom{4}{2} 2^{10}$.

$N(A \cap B \cap C) = \cdots = N(B \cap C \cap D) = 1$ (only one sequence using only one letter).

Hence $S_3 = \binom{4}{3} \cdot 1$.

$N(A \cap B \cap C \cap D) = 0$ (no sequence using no letters).

Hence $S_4 = 0$.

It follows that

$$\begin{aligned} N(\overline{A} \cap \overline{B} \cap \overline{C} \cap \overline{D}) &= N - S_1 + S_2 - S_3 + S_4 \\ &= 4^{10} - 4 \cdot 3^{10} + \binom{4}{2} 2^{10} - \binom{4}{3} \cdot 1. \end{aligned}$$