Bonus Questions for MAT2611

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Bonus Questions? What is it?

A series of *Video Lecture*s shall be uploaded to the **MAT2611-18-S1** web site. The purpose of these *Video Lectures* would be to aid understanding material of the course at your own pace in your own pre-defined sets of piecemeals.

The *Video Lecture*s would go unnoticed unless it is added with some reward system. This is augmented by the **Bonus Questions**. Each *Video Lecture* shall come paired with a **Bonus Questions** section which would test you on the material of the *Video Lecture*. The posting and submission dates, as well as the weight in the calculation of the *Year Mark*, of the **Bonus Questions** is given in the table below. Each **Bonus Question**, with the exception of the last one, has a period of fourteen days from the date of posting for submission.

The last one, i.e., **Bonus Questions 8**, has a submission time of eleven days and has lesser weight. This shall not be linked to any particular *Video Lecture*, but instead would be connected to all of them. This set of questions would depend on the full material of the course and would give you a judgement of the level of examination.

Now regarding the computation of the *Year Mark*. If A_1, A_2, A_3 be the percentage marks obtained for the first, second and third assignments, respectively, and B be the percentage mark obtained from all the questions in this document then the *Year Mark* Y shall be:

$$Y = \frac{3A_1 + 2A_2 + 3A_3 + 12B}{20}.$$

Hence the **Bonus Questions** contribute to 60% of the *Year Mark* and the assignments given do the rest.

Note: The number B shall be attained as follows:

Bonus Questions No.	Posting Date	Submission Date	Percentage Weight
Bonus Questions 1	18-Feb-2018	26-Feb-2018	8%
Bonus Questions 2	05-Mar-2018	19-Mar-2018	8%
Bonus Questions 3	12-Mar-2018	26-Mar-2018	8%
Bonus Questions 4	19-Mar-2018	02-Apr-2018	8%
Bonus Questions 5	26-Mar-2018	09-Apr-2018	8%
Bonus Questions 6	02-Apr-2018	16-Apr-2018	8%
Bonus Questions 7	09-Apr-2018	23-Apr-2018	8%
Bonus Questions 8	16-Apr-2018	27-Apr-2018	4%
Total			60%

For instance, if the marks obtained by a candidate in the eight sections of Bonus Questions have percentages $11, 44, 77, 78, 99, 33, 50, 47$ then for this candidate the calculation of B is:

$$B = \frac{8 \times (11 + 44 + 77 + 78 + 99 + 33 + 50) + 4 \times 47}{60} = \frac{3324}{60} = 55.4\%.$$

Hence, his contribution from Bonus Questions to year mark is $0.6 \times 55.4 = 33.24$.

If, further, his percentage marks for the three assignments are $40, 60, 90$, then his year mark would be

$$Y = \frac{3 \times 40 + 2 \times 60 + 3 \times 90 + 12 \times 55.4}{20} = \frac{1174.8}{20} = 58.74\%$$

I hope that the calculation of the year mark should be clear and your tensions regarding when to expect the **Bonus Questions** and their deadlines should be clear. The *Video Lectures* do augment the syllabus, although **not** in the same order as the text book. However, the sections of the textbook which correspond to a *Video Lecture* shall be stated in the respective section.

Please feel free to send your queries, suggestions, opinions..., directly to me at my email address ghoshpp@unisa.ac.za. Please **do not** send your emails through the *myUnisa* website or the Microsoft Office 365 service, since there are complaints of students finding me unreachable from any of the above two services; use an independent email client instead.

4. Video Lecture 4

This video lecture conveys the following notions:

(a) Recalls that there are two kinds of vectors spaces, namely the *infinite dimensional* vector spaces and the *finite dimensional* vector spaces.

(b) For a finite dimensional vector space V with dimension $\dim V = n$ and a given basis B there is the bijective linear transformation $V \xrightarrow{\mathrm{coord}_B(-)} \mathbb{R}^n$, and it is necessary to *understand* the meaning of this.

Hence, I first explain how a bijective function is a *re-labelling* and then show how a bijective linear transformation is a *re-labeling* that preserves the vector space *structure*, leading to the fact that if $V \xrightarrow{f} W$ is a bijective linear transformation then even if the vector spaces V and W look ab initio very different — in its content of vectors, in the vector addition and scalar multiplication, they are actually the *same* as far as *being* vector spaces is concerned. Said in other words, we cannot distinguish between V and W, as far as they are vector spaces; to discern if we may, we have to look into some other aspect of them.

This illustrates the coordinatisation principle — every finite dimensional vector space V with $\dim V = n$ is *same as* the familiar vector space $\mathbb{R}^n$ with its usual coordinatewise addition and scalar multiplication. Hence the vector spaces $\mathbb{R}^n$ are the *only* prototypes of finite dimensional vector spaces as we vary n over the natural numbers.

(c) I know provide the first principle of defining linear transformations $V \xrightarrow{f} W$ from a finite dimensional vector space V to any other vector space — the principle of *Method of Linear Extension.* This assures us that we can create such linear transformations with as much ease as we can create functions on a finite set.

(d) This lead us to the *Rank plus Nullity* theorem for a linear transformation.

(e) If both V and W are finite dimensional vector spaces then every linear transformation $V \xrightarrow{f} W$ gives rise to the *matrix presentation* $\mathrm{Mat}_{B\to C}(f)$ of f.

(f) As a special case of this matrix presentation, I show that if V is a finite dimensional vector space, B and C are any two bases for V then the *change of basis* matrix $\boldsymbol{P}_{B\to C} = \mathrm{Mat}_{B\to C}(\mathbf{1}_V)$, where $V \xrightarrow{\mathbf{1}_V} V$ is the *identity* linear transformation defined by $\mathbf{1}_V(x) = x$.

The following questions are to be answered and answers submitted to ghoshpp.unisa.ac.za on or before **April 05, 2018**.

The Drop Box service has been added to the *myUnisa* site for **MAT2611**. This would enable you to upload your submissions. This would guarantee recording your submission and also letting me immediately know when you submit.

No submission to the questions in this section shall be entertained after the due date.

PROBLEM 8. (a) Show that there exists a unique linear transformation $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}[x]_{\deg\le 5}$ such that:

$$f(1,0,1) = 1 + x^2, f(1,1,0) = 1 + x \text{ and } f(0,1,1) = x^3 + x^4.$$

Find $\operatorname{Ker} f$, $\mathrm{I}m[f]$ and their dimensions.

(b) Show that there exists a unique linear transformation $\mathbb{R}[x]_{\deg\le 2} \xrightarrow{f} \mathbb{R}^2$ such that:

$$f(1 - x + x^2) = (0,2) = f(2x^2) \text{ and } f(1 + x + x^2) = (2,2).$$

Find $\operatorname{Ker} f$, $\mathrm{I}m[f]$ and their dimensions.

(c) Does there exist a linear transformation $\mathbb{R}^3 \xrightarrow{f} \mathbb{R}^3$ such that:

$$f(1,0,1) = (1,2,1), f(0,1,1) = (2,1,1) \text{ and } f(-1,-1,-2) = (1,1,2)?$$

If it does, what is the formula for $f(x_1,x_2,x_3)$? what is $\operatorname{Ker} f$ and $\mathrm{I}m[f]$?

If it does not, then explain why does it not?

[5 + 5 + 5 = 15 marks]

PROBLEM 9. (a) Given the linear transformation $\mathbb{R}[x]_{\deg\le 2} \xrightarrow{f} \mathbb{R}^2$ in Problem 8.(b), find the matrix presentation $\mathrm{Mat}_{B\to C}(f)$ of f, given the basis $B = \langle 1-x, 1+x, x^2\rangle$ of $\mathbb{R}[x]_{\deg\le 2}$ and the basis $C = \langle (1,2),(2,1)\rangle$ of $\mathbb{R}^2$.

(b) It is easy to see that $B' = \langle 1, x, x^2\rangle$ and $C' = \langle (1,0),(0,1)\rangle$ are bases of $\mathbb{R}[x]_{\deg\le 2}$ and $\mathbb{R}^2$, respectively, and are possibly much more simpler.

Can you use change of basis matrices to give an elegant solution to the question in (a)?

[15 + 15 = 30 marks]

PROBLEM 10. (a) Given the linear transformations $V \overset{f}{\underset{g}{\rightrightarrows}} W \xrightarrow{h} P$ between finite dimensional vector spaces, where $\dim V = m$, $\dim W = n$ and $\dim P = p$, show that for any given bases B, C, D of the vector spaces V, W, P respectively, and for any $\lambda \in \mathbb{R}$:

$$\mathrm{Mat}_{B\to C}(f+g) = \mathrm{Mat}_{B\to C}(f) + \mathrm{Mat}_{B\to C}(g) \tag{4}$$

$$\mathrm{Mat}_{B\to C}(\lambda f) = \lambda \mathrm{Mat}_{B\to C}(f) \tag{5}$$

and

$$\mathrm{Mat}_{B\to D}(h \circ f) = \mathrm{Mat}_{C\to D}(h)\mathrm{Mat}_{B\to C}(f). \tag{6}$$

> **Hint:**
> Recall from Problem 4, equations (1) & (2) on page 5:
> $$(f+g)(v) = f(v) + g(v),$$
> and
> $$(\lambda f)(v) = \lambda f(v).$$

(b) Show that the set $\mathcal{GL}_{m,n}(\mathbb{R})$ of all $m \times n$ matrices with the usual matrix addition and scalar multiplication is a finite dimensional vector space with $\dim \mathcal{GL}_{m,n}(\mathbb{R}) = mn$.

Show that if V and W be finite dimensional vector spaces with $\dim V = m$ and $\dim W = n$, B a basis for V and C a basis for W then $\hom(V, W) \xrightarrow{\mathrm{Mat}_{B\to C}(-)} \mathcal{GL}_{m,n}(\mathbb{R})$ is a bijective linear transformation.

Hence, or otherwise, obtain $\dim \hom(V, W)$.

Observe, in this context, something very interesting: there is a *multiplication*, $\mathcal{GL}_{p,n}(\mathbb{R}) \times \mathcal{GL}_{n,m}(\mathbb{R}) \to \mathcal{GL}_{p,m}(\mathbb{R})$ which is the usual matrix multiplication $(A, B) \mapsto AB$.

Alongside there is the *composition* of linear transformations $\hom(W, P) \times \hom(V, W) \xrightarrow{\circ} \hom(V, P)$ given by $(h, f) \mapsto h \circ f$.

None of these maps — neither the multiplication of matrices nor the composition of linear transformations, are linear transformations: say for matrices, it is known that $(A + C)(B + D) \neq AC + BD$ (check: this equation is exactly the demand for linearity of matrix multiplication). However, (6) on page 12 provides a connection between the bijective linear transformations $\mathrm{Mat}_{B\to D}(-)$, $\mathrm{Mat}_{C\to D}(-)$ and $\mathrm{Mat}_{B\to C}(-)$.

In the special case when $V = W = P$ and thence $m = n = p$, one has the matrix multiplication to be a binary operation on the vector space $\mathcal{GL}_{n,n}(\mathbb{R})$ (which is shortened to $\mathcal{GL}_n(\mathbb{R})$). This binary operation distributes over addition and scalar multiplication:

$$A(B + C) = AB + AC \qquad (B + C)A = BA + CA \tag{7}$$

and

$$A(\lambda B) = \lambda(AB) \qquad (\lambda A)B = \lambda(AB). \tag{8}$$

A vector space with an associative multiplication which distributes over addition and scalar multiplication as in (7) & (8) and which also possesses an identity is often called an *real unital algebra*, and $\mathcal{GL}_n(\mathbb{R})$ as well as $\hom(V, V)$ are examples of such. Furthermore, $\mathrm{Mat}_{B\to B}(-)$ then shows that $\hom(V, V)$ and $\mathcal{GL}_n(\mathbb{R})$ are *same* as real unital algebras. However, we shall not be discussing algebras any further, it just came up incidentally with this problem.

Furthermore, the operations of matrix multiplication or composition of linear transformations are *not very far* from being linear transformations. For instance, each of the following maps:

- (i) for each fixed matrix A, the map $B \mapsto AB$,
- (ii) for each fixed matrix B, the map $A \mapsto AB$,
- (iii) for each fixed linear transformation h, the map $f \mapsto h \circ f$,
- (iv) for each fixed linear transformation f, the map $h \mapsto h \circ f$,

are indeed linear transformations (check it...). Such maps are often called *bilinear maps* and we shall come back to them later on.

If you continue with your studies with mathematics, you shall soon learn that *bilinear* maps correspond to linear transformations, but **not** from the product vector space, but from a vector space very closely related to it, called the *tensor product* of the two vector spaces.

(c) Given the finite dimensional vector space V with $\dim V = n$ and the bases P, Q, R, show that:

(9) $$\boldsymbol{P}_{P\to R} = \boldsymbol{P}_{Q\to R}\boldsymbol{P}_{P\to Q}.$$

Hence, or otherwise, show that $\boldsymbol{P}_{P\to Q}$ is an invertible matrix and ${\boldsymbol{P}_{P\to Q}}^{-1} = \boldsymbol{P}_{Q\to P}$.

[[5 + 5 + 5] + [5 + 3 + 2] + [8 + 2] = 35 marks]

References

[Ant, 2005] (2005). *Elementary Linear Algebra with Applications*. John Wiley & Sons, 9th edition.

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