Bonus Questions for MAT2611

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Bonus Questions? What is it?

A series of *Video Lecture*s shall be uploaded to the **MAT2611-18-S1** web site. The purpose of these *Video Lectures* would be to aid understanding material of the course at your own pace in your own pre-defined sets of piecemeals.

The *Video Lecture*s would go unnoticed unless it is added with some reward system. This is augmented by the **Bonus Questions**. Each *Video Lecture* shall come paired with a **Bonus Questions** section which would test you on the material of the *Video Lecture*. The posting and submission dates, as well as the weight in the calculation of the *Year Mark*, of the **Bonus Questions** is given in the table below. Each **Bonus Question**, with the exception of the last one, has a period of fourteen days from the date of posting for submission.

The last one, i.e., **Bonus Questions 8**, has a submission time of eleven days and has lesser weight. This shall not be linked to any particular *Video Lecture*, but instead would be connected to all of them. This set of questions would depend on the full material of the course and would give you a judgement of the level of examination.

Now regarding the computation of the *Year Mark*. If A_1, A_2, A_3 be the percentage marks obtained for the first, second and third assignments, respectively, and B be the percentage mark obtained from all the questions in this document then the *Year Mark* Y shall be:

$$Y = \frac{3A_1 + 2A_2 + 3A_3 + 12B}{20}.$$

Hence the **Bonus Questions** contribute to 60% of the *Year Mark* and the assignments given do the rest.

Note: The number B shall be attained as follows:

Bonus Questions No.	Posting Date	Submission Date	Percentage Weight
Bonus Questions 1	18-Feb-2018	26-Feb-2018	8%
Bonus Questions 2	05-Mar-2018	19-Mar-2018	8%
Bonus Questions 3	12-Mar-2018	26-Mar-2018	8%
Bonus Questions 4	19-Mar-2018	02-Apr-2018	8%
Bonus Questions 5	26-Mar-2018	09-Apr-2018	8%
Bonus Questions 6	02-Apr-2018	16-Apr-2018	8%
Bonus Questions 7	09-Apr-2018	23-Apr-2018	8%
Bonus Questions 8	16-Apr-2018	27-Apr-2018	4%
Total			60%

For instance, if the marks obtained by a candidate in the eight sections of Bonus Questions have percentages $11, 44, 77, 78, 99, 33, 50, 47$ then for this candidate the calculation of B is:

$$B = \frac{8 \times (11 + 44 + 77 + 78 + 99 + 33 + 50) + 4 \times 47}{60} = \frac{3324}{60} = 55.4\%.$$

Hence, his contribution from Bonus Questions to year mark is $0.6 \times 55.4 = 33.24$.

If, further, his percentage marks for the three assignments are $40, 60, 90$, then his year mark would be

$$Y = \frac{3 \times 40 + 2 \times 60 + 3 \times 90 + 12 \times 55.4}{20} = \frac{1174.8}{20} = 58.74\%$$

I hope that the calculation of the year mark should be clear and your tensions regarding when to expect the **Bonus Questions** and their deadlines should be clear. The *Video Lectures* do augment the syllabus, although **not** in the same order as the text book. However, the sections of the textbook which correspond to a *Video Lecture* shall be stated in the respective section.

Please feel free to send your queries, suggestions, opinions. . . , directly to me at my email address ghoshpp@unisa.ac.za. Please **do not** send your emails through the *myUnisa* website or the Microsoft Office 365 service, since there are complaints of students finding me unreachable from any of the above two services; use an independent email client instead.

5. (Video) Lecture 5

As you have guessed by now, there is no Video Lecture 5; however, there are still some questions. The material is based on all what we have done so far as well as the material in [**Ant, 2005**, Chapter1 & 2].

There is a lot of commentary here and some questions, most of which should be familiar to you. However, the purpose of this is to show how instead of remembering these familiar facts using *strange mnemonics* you can easily *know* them from the background of what we have learnt about linear transformations and vector spaces so far.

The following questions are to be answered and answers submitted to ghoshpp.unisa.ac.za on or before **April 10, 2018**.

The Drop Box service has been added to the *myUnisa* site for **MAT2611**. This would enable you to upload your submissions. This would guarantee recording your submission and also letting me immediately know when you submit.

No submission to the questions in this section shall be entertained after the due date.

Problem 11 (Two Almost Missed, yet Essential Facts About Matrices). Given a matrix of order $m \times n$, say:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1j} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2j} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{i1} & a_{i2} & \dots & a_{ij} & \dots & a_{in} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mj} & \dots & a_{mn} \end{pmatrix} = \left(a_{ij}\right)_{i,j=1}^{m,n},$$

its rows are $\mathbf{row}_i[A]$, for $i = 1, 2, \dots, m$ are determined by elements of $\mathbb{R}^n$ and columns are $\mathbf{col}_j[A]$, for $j = 1, 2, \dots, n$ are determined by elements of $\mathbb{R}^m$, where:

$$\mathbf{row}_i[A] = (a_{i1}, a_{i2}, \dots, a_{in}) \in \mathbb{R}^n, i = 1, 2, \dots, m$$

and

$$\mathbf{col}_j[A] = (a_{1j}, a_{2j}, \dots, a_{mj}) \in \mathbb{R}^m, j = 1, 2, \dots, n.$$

As a convention we shall always denote an element of $\mathbb{R}^n$ by a $n \times 1$ matrix, i.e., as a column vector.

(a) Show that for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$:

$$\mathbf{y} = A\mathbf{x} = x_1\mathbf{col}_1[A] + x_2\mathbf{col}_2[A] + \cdots + x_n\mathbf{col}_n[A] \in \operatorname{span}[\mathbf{col}_1[A], \mathbf{col}_2[A], \ldots, \mathbf{col}_n[A]].$$

(b) Hence show that for any matrix $B = \left(b_{ij}\right)_{i,j=1}^{n,p}$ of order $n \times p$ the columns of AB are $A\mathbf{col}_1[B], A\mathbf{col}_2[B], \ldots, A\mathbf{col}_p[B]$.

Problem 12 (Rank and Nullity for Matrices). Recall from **Video Lecture 4** the finite dimensional vector spaces are, under coordinatisation, precisely some $\mathbb{R}^n$ ($n \geq 0$) and from Problem 10(b) (see page 12) that the linear transformations $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ are precisely the $m \times n$ matrices $F = \left(\phi_{ij}\right)_{i,j=1}^{m,n}$ with real entries. The same problem also illustrates the near match between the matrix operations and operations on linear transformations.

Thus, the linear transformation corresponding to a matrix F of order $m \times n$ is $\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m$ defined by $f(x_1, x_2, \ldots, x_n) = F\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, where the point $(x_1, x_2, \ldots, x_n)$, in matrix form, shall be represented by the $n \times 1$ column matrix $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

(a) Given the $m \times n$ matrix F, show that the kernel of the associated linear transformation is the vector subspace:

$$K_F = \left\{ (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : F\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \mathbf{0} \right\}$$

The vector subspace K_F is:

(i) The set of all solutions of the homogeneous system of linear equations determined by the matrix F.

(ii) Often called the *null space* of F.

(iii) The dimension of K_F is often called the *nullity* of the matrix F.

(b) Given a $m \times n$ matrix show that the image of the associated linear transformation is:

$$R_F = \left\{ F \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ v_n \end{pmatrix} : (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n \right\}.$$

The vector subspace R_F is:

(i) Often called the *range space* of F or also the *column space* of F (see Problem 11 for an explanation of this name).

(ii) The dimension of R_F is often called the *rank* of the matrix F.

(iii) Thus, we now know that the sum of the rank and nullity of a matrix is equal to the number of columns of the matrix.

PROBLEM 13 (Solutions of Linear Equations). Given $V \xrightarrow{f} W$ a linear transformation from the vector space V to the vector space W, a fixed vector $b \in W$, the equation:

$$f(x) = b, \tag{10}$$

and the set $S = \{x \in V : f(x) = b\}$ of solutions of (10).

(a) Show that $S \neq \varnothing$, if and only if, $b \in f[V]$, the image of f.

(b) If $x_0 \in S$ then $x \in S$, if and only if, there exists a $t \in \operatorname{Ker} f$ such that $x = x_0 + t$.

The special element $x_0 \in S$ is often called a *particular solution* of the equation in (10).

Thus, the Moral is: solutions of linear equations, if they exist, are just a *translate* of a particular solution of the kernel of the linear transformation — observe the geometric language in use here, *translations* meaning as if the *kernel is pushed by the particular solution* thereby getting *translated* to the particular solution.

(c) Now suppose that both V and W are finite dimensional vector spaces with $\dim V = n$, $\dim W = m$; the linear transformation is then determined by a matrix F of order $m \times n$, and is given by $(x_1, x_2, \ldots, x_n) \mapsto F \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

The linear equation in (10) is then obtained on fixing an element $\mathbf{b} = (b_1, b_2, \ldots, b_m) \in \mathbb{R}^m$ and trying to obtain the n-tuples $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that $F\mathbf{x} = \mathbf{b}$. The set of solutions is then $S = \{\mathbf{x} \in \mathbb{R}^n : F\mathbf{x} = \mathbf{b}\}$.

(i) Show that $S \neq \varnothing$, if and only if, the rank of the matrix F and the matrix $A = \begin{pmatrix} F & \mathbf{b} \end{pmatrix}$ are equal, where A is a matrix of order $m \times (n+1)$ with the first n columns as F and the last column as $\mathbf{b}$.

(ii) Show that if $\mathbf{x_0} \in S$ then $\mathbf{x} \in S$, if and only if, there exists a $\mathbf{t} \in \mathbb{R}^n$ such that $F\mathbf{t} = \mathbf{0}$ and $\mathbf{x} = \mathbf{x_0} + \mathbf{t}$.

Note how this provides a computational justification to the process of solving linear systems of equations.

References

[Ant, 2005] (2005). *Elementary Linear Algebra with Applications.* John Wiley & Sons, 9th edition.

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