Bonus Questions for MAT2611

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Bonus Questions? What is it?

A series of *Video Lecture*s shall be uploaded to the **MAT2611-18-S1** web site. The purpose of these *Video Lectures* would be to aid understanding material of the course at your own pace in your own pre-defined sets of piecemeals.

The *Video Lecture*s would go unnoticed unless it is added with some reward system. This is augmented by the **Bonus Questions**. Each *Video Lecture* shall come paired with a **Bonus Questions** section which would test you on the material of the *Video Lecture*. The posting and submission dates, as well as the weight in the calculation of the *Year Mark*, of the **Bonus Questions** is given in the table below. Each **Bonus Question**, with the exception of the last one, has a period of fourteen days from the date of posting for submission.

The last one, i.e., **Bonus Questions 8**, has a submission time of eleven days and has lesser weight. This shall not be linked to any particular *Video Lecture*, but instead would be connected to all of them. This set of questions would depend on the full material of the course and would give you a judgement of the level of examination.

Now regarding the computation of the *Year Mark*. If A_1, A_2, A_3 be the percentage marks obtained for the first, second and third assignments, respectively, and B be the percentage mark obtained from all the questions in this document then the *Year Mark* Y shall be:

$$Y = \frac{3A_1 + 2A_2 + 3A_3 + 12B}{20}.$$

Hence the **Bonus Questions** contribute to 60% of the *Year Mark* and the assignments given do the rest.

Note: The number B shall be attained as follows:

Bonus Questions No.	Posting Date	Submission Date	Percentage Weight
Bonus Questions 1	18-Feb-2018	26-Feb-2018	8%
Bonus Questions 2	05-Mar-2018	19-Mar-2018	8%
Bonus Questions 3	12-Mar-2018	26-Mar-2018	8%
Bonus Questions 4	19-Mar-2018	02-Apr-2018	8%
Bonus Questions 5	26-Mar-2018	09-Apr-2018	8%
Bonus Questions 6	02-Apr-2018	16-Apr-2018	8%
Bonus Questions 7	09-Apr-2018	23-Apr-2018	8%
Bonus Questions 8	16-Apr-2018	27-Apr-2018	4%
Total			60%

For instance, if the marks obtained by a candidate in the eight sections of Bonus Questions have percentages $11, 44, 77, 78, 99, 33, 50, 47$ then for this candidate the calculation of B is:

$$B = \frac{8 \times (11 + 44 + 77 + 78 + 99 + 33 + 50) + 4 \times 47}{60} = \frac{3324}{60} = 55.4\%.$$

Hence, his contribution from Bonus Questions to year mark is $0.6 \times 55.4 = 33.24$.

If, further, his percentage marks for the three assignments are $40, 60, 90$, then his year mark would be

$$Y = \frac{3 \times 40 + 2 \times 60 + 3 \times 90 + 12 \times 55.4}{20} = \frac{1174.8}{20} = 58.74\%$$

I hope that the calculation of the year mark should be clear and your tensions regarding when to expect the **Bonus Questions** and their deadlines should be clear. The *Video Lectures* do augment the syllabus, although **not** in the same order as the text book. However, the sections of the textbook which correspond to a *Video Lecture* shall be stated in the respective section.

Please feel free to send your queries, suggestions, opinions. . . , directly to me at my email address ghoshpp@unisa.ac.za. Please **do not** send your emails through the *myUnisa* website or the Microsoft Office 365 service, since there are complaints of students finding me unreachable from any of the above two services; use an independent email client instead.

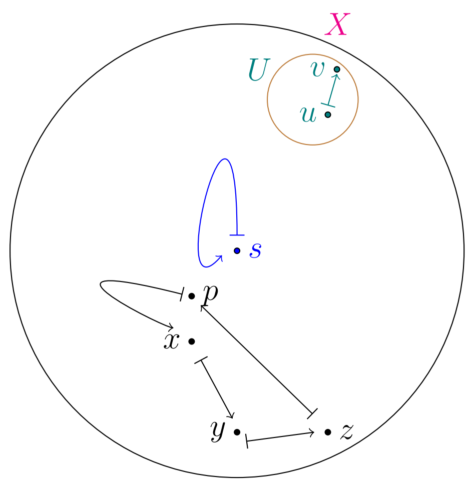
6. (Video) Lecture 6

This (video) lecture conveys the following:

(a) First starts with the idea of an *invariant subspace*, an idea which is of paramount importance in the study of linear algebra.

To say it in brief: given a function $X \xrightarrow{f} X$ on a set X (often such functions are called *colorblue endomaps* or *endomorphisms*) it is interesting to find a *fixed point* of such a function, which is an element $x \in X$, such that $f(x) = x$.

Every endomorphism on a set X ushers in a *dynamical picture*, a notion of a *motion*, in which we say that f *shifts* $x \in X$ to $y \in X$, if $y = f(x)$. This suggests the following diagram:



wherein you could think of possible *orbits* like $x \xmapsto{f} y \xmapsto{f} z \xmapsto{f} p \xmapsto{f} x$ which might be *finite*, or which could carry on and on…, and also *loops* like $s \xmapsto{f} s$, which are exactly the *fixed points*.

You could also have a subset $U \subseteq X$ such that: for all $u \in U$, $v = f(u) \in U$; in other words, $f[U] \subseteq U$. Thus, once if we land up in such a set as U, we never *come out* of it — such sets U are obviously called *invariant subsets*.

The situation in vector spaces is similar, but we consider linear transformations $X \xrightarrow{f} X$ instead, and then we have invariant vector subspaces.

Every endomorphism of a vector space has at least the two trivial subspaces as its invariant subspaces. However, not every endomorphism of vector spaces have a non-trivial invariant subspace. Starting with the existence of an invariant subspace of dimension one of a finite dimensional vector space leads to the notion of an eigenvalue, eigenvector and eigenspaces. The eigenspaces turn out to be invariant subspaces.

(b) The notion of an eigenvalue of a square matrix (= linear transformation on a finite dimensional vector space) is closely connected to the characteristic polynomial.

Since every odd degree polynomial function in $\mathbb{R}[x]$ has a root, every odd order square matrix has an invariant subspace, while even order square matrices may or may not.

(c) The eigenvectors corresponding to distinct eigenvalues are linearly independent. This gives rise to the notion of *diagonalisation* of a square matric, which is then completely described in terms of its eigenvalues and eigenvectors.

(d) The notions of algebraic and geometric multiplicities are required in the complete formulation of diagonalisability, which are also discussed.

The following questions are to be answered and answers submitted to ghoshpp.unisa.ac.za on or before **April 18, 2018**.

The Drop Box service has been added to the *myUnisa* site for **MAT2611**. This would enable you to upload your submissions. This would guarantee recording your submission and also letting me immediately know when you submit.

No submission to the questions in this section shall be entertained after the due date.

PROBLEM 14. Two square matrices A and B are said to be *similar*, if there exists an invertible square matrix P such that $B = P^{-1}AP$.

(a) Interprete the relation of similarity of matrices in terms of *change of basis* and deduce that similar matrices represent the same linear transformation.

(b) Show that any two similar matrices A and B have the same characteristic polynomial, and hence the same eigenvalues.

(c) If $B = P^{-1}AP$ for some invertible square matrix P, show that for any eigenvalue λ of A, $\mathcal{E}_\lambda[B] \xrightarrow{P} \mathcal{E}_\lambda[A]$ sets up a bijective linear transformation between the eigenspaces.

[5 + 5 + 5 = 15 marks]

PROBLEM 15. Let A be a square matrix of order n and $\lambda \in \mathbb{R}$ be an eigenvalue of A of geometric multiplicity k.

Choose a basis $B_0 = \langle \mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_k} \rangle$ of $\mathcal{E}_\lambda[A]$ and extend this to a basis B of $\mathbb{R}^n$.

(a) Show that:

$$\mathrm{Mat}_{B\to B}(A) = \begin{pmatrix} \lambda I_k & P \\ \mathbf{0}_{n-k,k} & Q \end{pmatrix},$$

where P is a matrix of order $k \times (n-k)$, Q is a matrix of order $(n-k)\times(n-k)$ and $\mathbf{0}_{n-k,k}$ is the zero matrix of order $(n-k)\times k$.

(b) Show that the characteristic polynomial of $S = \begin{pmatrix} \lambda I_k & P \\ \mathbf{0}_{n-k,k} & Q \end{pmatrix}$ is:

$$\chi_S(x) = (x-\lambda)^k \chi_Q(x),$$

where for any square matrix T, $\chi_T \in \mathbb{R}[x]$ is the characteristic polynomial of the matrix T.

(c) Hence, otherwise, deduce that the algebraic multiplicity of λ is at least k its geometric multiplicity.

Hint: Use Problem 14.

[10 + 10 + 10 = 30 marks]

PROBLEM 16. Given any polynomial function $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ $(a_n \neq 0)$ in $\mathbb{R}[x]$ and any square matrix A of order n, define:

$$f(A) = a_0I_n + a_1A + a_2A^2 + \cdots + a_nA^n.$$

Obviously, $f(A)$ is a square matrix of order n and hence a linear transformation on $\mathbb{R}^n$.

(a) If $f(x) = -1 + 2x + 3x^3$, evaluate $f(A)$, where:

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.$$

(b) Show that for any $f \in \mathbb{R}[x]$, any eigenvalue $\lambda \in \mathbb{R}$ of a square matrix A and any $\mathbf{v} \in \mathcal{E}_\lambda[A]$:

$$f(A)\mathbf{v} = f(\lambda)\mathbf{v}.$$

(c) Hence, or otherwise, show that if A be a diagonalisable square matrix and $\chi_A \in \mathbb{R}[x]$ be its characteristic polynomial then $\chi_A(A) = \mathbf{0}_{n,n}$ the zero matrix of order $n \times n$, i.e., the zero linear transformation on $\mathbb{R}^n$.

Hint: Use the fact that a linear transformation $V \xrightarrow{f} W$ on a finite dimensional vector space V is identically zero, if and only if, for any basis B if V, $v \in B \Rightarrow f(v) = 0$. (Can you prove this using the Method of Linear Extension?)

(d) If A be a diagonalisable matrix, then compute $f(A)$ for any $f \in \mathbb{R}[x]$. Can you give an algorithm to find A^{-1} in such a case?

[5 + 5 + 10 + (5 + 10) = 35 marks]

References

[Ant, 2005] (2005). *Elementary Linear Algebra with Applications*. John Wiley & Sons, 9th edition.

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