GIVEN: $X_p(t) = \int_0^t \Phi(t-s)F(s)ds$

With Φ a normalized fundamental matrix of $\dot{X} = AX$ at $t = 0$

TO PROVE: $\dot{X}_p(t) = AX_p(t) + F(t)$

$X_p(t) = \Phi(t)\int_0^t \Phi(-s)F(s)ds$ ...by Theorem 4.27

$= \Phi(t)\int_0^t \Phi^{-1}(s)F(s)ds$ ...by Corollary 4.29

$\dot{X}_p(t) = \dot{\Phi}(t)\int_0^t \Phi^{-1}(s)F(s)ds + \Phi(t)\frac{d}{dx}\left(\int_0^t \Phi^{-1}(s)F(s)ds\right)$ ...by Product rule

$= \dot{\Phi}(t)\int_0^t \Phi^{-1}(s)F(s)ds + \Phi(t)\Phi^{-1}(t)F(t)$

$= \dot{\Phi}(t)\int_0^t \Phi^{-1}(s)F(s)ds + F(t)$

But since the column vectors of $\Phi(t)$ are solutions to $\dot{X} = AX$ it must be that $\dot{\Phi}(t) = A\Phi(t)$ thus:

$$\dot{X}_p(t) = A\Phi(t)\int_0^t \Phi^{-1}(s)F(s)ds + F(t)$$

$= A\Phi(t)\int_0^t \Phi(-s)F(s)ds + F(t)$ ...by Corollary 4.29

$= A\int_0^t \Phi(t-s)F(s)ds + F(t)$ ...by Theorem 4.27

$= AX_p(t) + F(t)$